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New Approach to Field Theory*

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A field theory is quantized covariantly on Lorentz-invariant surfaces. Dilatations replace time translations as dynamical equations of motion. This leads to an operator formulation for Euclidean quantum field theory. A covariant thermodynamics is developed, with which the Hagedorn spectrum can be obtained, given further hypotheses. The Virasoro algebra of the dual resonance model is derived in a wide class of 2-dimensional Euclidean field theories.

I. INTRODUCTION

We present a new method of quantizing a field theory. In conventional quantization, time is selected as the direction of propagation and the quantization conditions (commutators) are imposed on the spacelike surface t = constant. The recent lightlike quantization allows the system to develop along the $x^+ = (x^0 + x^3)/\sqrt{2}$ coordinate,[1] and commutators are given on the surface x^+ = constant.[2] The former scheme has the advantage of closely following physical intuition based on the nonrelativistic Schrödinger equation, and has gained wide acceptance. The latter technique appears particularly useful in discussions of high-energy behavior, and commutators on lightlike surfaces have been profitably employed in this connection.[3] However, neither method is Lorentz-invariant, though of course the complete theory possesses this property.

The approach which we have developed selects the spacelike surface x^2 = positive constant = τ^2 as the surface of quantization, and propagation takes place in the "perpendicular" direction, i.e., along x^μ. Clearly the technique is Lorentz-invariant. In a sense it is intermediate between time quantization and lightlike quantization: at $x^2 = \infty$ our surface can also satisfy t = constant; at $x^2 = 0$, our surface can also coincide with x^+ = constant.[4]

Although covariant quantization can be carried out in Minkowski space or in Euclidean space, it appears to be more useful in the latter than in the former. The reason is that in Minkowski space the hyperboloids $x^2 = \tau^2$ do not span all of space-time: as τ^2 varies from 0 to ∞, the region outside the light-cone $x^2 < 0$ is not reached. Consequently the propagation of the system in this domain must be examined separately. A related problem is that translation generators – the momentum operators – are hard to define.[5] In Euclidean space this problem does not exist. In this paper we confine our attention to Euclidean field theory.[6]

Of course the physical content of a theory is not changed by the choice of quantization surface. Indeed, as Schwinger and Tomonaga have shown, *any* spacelike surface may be used for quantization purposes. (However, since our surface is not asymptotically flat one cannot directly make use of the general Schwinger-Tomonaga result.) Thus we do not expect to obtain a different Feynman-Dyson expansion for the S matrix. Nevertheless, as with lightlike quantization, we hope that our technique, by organizing the theory in a novel fashion, will prove itself convenient for analyzing certain problems and will provide new insights into the structure of field theories.

For example, there is considerable interest in relating the dual resonance model to a conventional field theory.[7] We believe that our quantization technique will provide a bridge for the formalism of these two theoretical ideas – indeed as will be seen, our method, when applied to 2-dimensional models, is very similar to that used in dual resonance models. As an application of the general formalism, it will be shown that *any* 2-dimensional field theory which possesses a traceless symmetric energy-momentum tensor gives rise to the Virasoro algebra,[7] up to a c number; a result which has been observed earlier in special cases.[8] This c number is infinite in Minkowski space, while in Euclidean space it is finite. Furthermore, a covariant thermodynamics is invented and with additional hypotheses the Hagedorn mass spectrum is obtained.

The paper is organized as follows. In Sec. II we give a preliminary, general discussion of our quantization procedure, comparing and contrasting it with the conventional approaches. Also we explain how a field theory in Minkowski space is continued to Euclidean space. Explicit covariant quantization of a field theory in Euclidean space is carried out in Sec. III, where also a covariant thermodynamics is developed. 2-dimensional models are discussed in Sec. IV, and the Virasoro algebra is derived. Concluding remarks comprise Sec. V, and various technical computations are relegated to three Appendixes.

II. GENERALITIES

A. Methods of Quantization

The general approach of Schwinger and Tomonaga is based on quantization of a field theory on an arbitrary spacelike surface. In this context one studies transformations under an infinitesimal deformation of the quantization surface. The customary way to apply this general scheme to physical problems is by singling out a (1-dimensional) family of 3-dimensional surfaces, and by introducing one *evolution operator* which transforms the state vector from one surface to the next one.

The best known example is when one considers the family of planes t=constant. In this case the Hamiltonian operator

$$H=\int d^3x\,\Theta^{00}(x) \tag{2.1}$$

describes the evolution between different times.

It is useful to consider the most general class of surfaces represented by the equation

$$F(x^\mu)=\tau \tag{2.2}$$

and to discuss the properties of the different operators in this general case. In particular we shall concentrate on the operators which can be obtained from the new, improved energy-momentum tensor[9] $\Theta^{\mu\nu}$ by means of a surface integral

$$A_f=\int d\sigma_\mu f_\nu(x)\Theta^{\mu\nu}(x), \tag{2.3}$$

where

$$d\sigma_\mu=d^4x\frac{\partial F(x)}{\partial x^\mu}\delta(F(x)-\tau)\,. \tag{2.4}$$

The operator A_f corresponds to the infinitesimal transformation

$$f^\mu(x)\frac{\partial}{\partial x^\mu}\;,\quad \delta_f x^\mu=f^\mu(x), \tag{2.5}$$

where $f^\mu(x)$ is an arbitrary function of x. This of course, for arbitrary f^μ, will not lead to a symmetry of the problem. Once a well-defined surface is chosen, the operators A_f can be separated into two classes depending on the form of the function f^μ.

We shall call the members of the first-class *kinematical operators* if the surface is left invariant by the transformation in Eq. (2.5), i.e., when

$$\delta_f F(x)=f^\mu(x)\frac{\partial}{\partial x^\mu}F(x)=0\,. \tag{2.6}$$

When this does not happen, i.e., when

$$f^\mu(x)\frac{\partial}{\partial x^\mu}F(x)\neq 0, \tag{2.7}$$

we shall call A_f a *dynamical operator*. Dynamical operators comprise the second class.

It is easy to see that kinematical operators have a very general significance since their expression in terms of fundamental fields is independent of the interaction Lagrangian $\mathcal{L}^{\rm int}$. Indeed in the case in which

$$\Theta_{\mu\nu}=\Theta^{(0)}_{\mu\nu}-g_{\mu\nu}\mathcal{L}^{\rm int} \tag{2.8}$$

the interaction part of A_f is

$$A_f^{\rm int}=-\int d^4x\,\frac{\partial F(x)}{\partial x^\mu}\,\delta(F(x)-\tau)f^\mu(x)\mathcal{L}^{\rm int}(x)$$
$$=0\,. \tag{2.9}$$

We thus see that, when Eq. (2.6) holds, $A_f^{\rm int}$ vanishes.

Although there is no *a priori* limitation in the choice of the family of surfaces, in practical application one is guided by the desire to give a fundamental role to the operators of the Poincaré group:

$$P_\alpha \text{ corresponding to } f^\mu_\alpha(x)=g^\mu_\alpha, \tag{2.10a}$$

$$M_{\alpha\beta} \text{ corresponding to } f^\mu_{\alpha\beta}(x)=x_\alpha g^\mu_\beta-x_\beta g^\mu_\alpha. \tag{2.10b}$$

Let us examine the choice of surfaces which have been used in the past literature.

(1) The family of planes

$$F(x)=t=\tau\,.$$

The six operators, 3-dimensional momenta P^i, and 3-dimensional rotations M^{jk} are kinematical, whereas $P^0=H$ and the Lorentz boosts M^{0i} are dynamical. We have already seen that $H=P^0$ is the evolution operator between different planes.

(2) The family of lightlike planes

$$F(x)=x^+=\frac{1}{\sqrt{2}}(x^0+x^3)=\tau .$$

The six operators

$$P^1, P^2, P^+=\frac{1}{\sqrt{2}}(P^0+P^3),$$

$$M^{12}, M^{+i}=\frac{1}{\sqrt{2}}(M^{0i}+M^{3i})$$

are kinematical; the remaining four are dynamical. The evolution operator is now

$$P^-=\frac{1}{\sqrt{2}}(P^0-P^3).$$

(3) A third possibility, which will be the basis of this paper, is a family of hyperboloids which are of course invariant under any Lorentz transformation.

$$F(x)=x^2=\tau^2 .$$

In this case all six $M^{\mu\nu}$ will be kinematical operators, whereas all four translations will be dynamical. This means that we shall have explicit contribution from the interaction Lagrangian to both energy and momentum. This is the price we shall have to pay for the Lorentz invariance of the scheme.

In our theory it will be important to discuss all 15 operators of the general conformal group. We shall thus also consider in addition to the generators given in (2.10)

$$D \text{ corresponding to } f^\mu(x)=x^\mu, \tag{2.11a}$$

$$K_\alpha \text{ corresponding to } f^\mu_\alpha(x)=2x^\mu x_\alpha-g^\mu_\alpha x^2. \tag{2.11b}$$

A special feature of our surface is now easy to see. Since the hyperboloid $x^2=\tau^2$ is invariant under the transformation

$$x^2\partial^\mu-(2x^\mu x_\alpha-g^\mu_\alpha x^2)\partial^\alpha, \tag{2.12a}$$

the operator

$$x^2P^\mu-K^\mu \tag{2.12b}$$

is a kinematical operator and its form does not depend on the interaction Hamiltonian.

In our case the dilatation operator D will be the evolution operator in τ. In other words if one wishes to proceed from one surface at $x^2=\tau_1{}^2$ to another at $x^2=\tau_2{}^2$, this is done by a scale transformation. It is important to realize that no assumption of scale invariance is being made. Of course in a scale-noninvariant theory D will depend explicitly on τ^2, just as P^0 depends explicitly on t if the theory is not time-translation invariant. This in no way changes the fact that D (correspondingly P^0) generates the "equations of motion." Of course the formalism is simplest if scale invariance *is* a symmetry so that D *is* independent of τ^2. We shall assume therefore that no scale-breaking terms (masses, dimensional coupling constants) are present in the Lagrangian. This simplifying assumption does not put into question the general approach; one can treat mass terms and other scale-breaking interactions perturbatively.

Another object of great importance in the discussion of the evolution of the system is the propagator function. This is given by the vacuum expectation value of the ordered product of two fields, where the ordering is along the direction of propagation: time ordering or x^+ ordering in the familiar examples. For us the ordering will be along the surfaces $x^2=\tau^2$. One may define a space conjugate to position space – momentum space is traditional. The propagator has a spectral representation in that space: It is the sum of poles at the eigenvalues of the evolution operator. Since we shall be systematically using polar (instead of Cartesian) coordinates, the conjugate space in our theory will be determined by angular momentum and dilatation (instead of linear momentum). We shall define our conjugate space in detail below. Nevertheless the propagator retains its spectral form: It is a sum of poles at the eigenvalues of our evolution operator – of the dilatation generator. Evidently in free-field theory the eigenvalues are integers or half-integers, given by the scale dimension of appropriate quantities.

B. Euclidean Field Theory

Since this paper will concern itself with the application of our quantization procedure to a field theory which has been continued from Minkowski space to Euclidean space, we indicate here how this continuation is performed. We replace x_0 by ix_4. For every field in Minkowski space $\phi(x_0,\vec{x})$ we define a corresponding field in Euclidean space $\hat{\phi}(x_4,\vec{x})$

$$\hat{\phi}(t,\vec{x})=\phi(it,\vec{x}). \tag{2.13}$$

The d'Alembert operator

$$\Box\phi(x)=(\partial_0{}^2-\vec{\nabla}^2)\phi(x) \tag{2.14a}$$

now becomes[10]

$$\hat{\Box}\hat{\phi}(x)=(\partial_4{}^2+\vec{\nabla}^2)\hat{\phi}(x). \tag{2.14b}$$

Similarly the Dirac operator

$$i\gamma^\mu \partial_\mu \psi(x) = (i\gamma^0 \partial_0 + i\vec{\gamma}\cdot\vec{\nabla})\psi(x) \tag{2.15a}$$

becomes

$$\hat{\gamma}^\mu \partial_\mu \hat{\psi}(x) = (\hat{\gamma}^\mu \partial_\mu + \hat{\vec{\gamma}}\cdot\vec{\nabla})\hat{\psi}(x) . \tag{2.15b}$$

The Dirac matrices in Euclidean space are given by $\hat{\gamma}^4 = \gamma^0$, $\hat{\vec{\gamma}} = i\vec{\gamma}$. They satisfy $\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2\delta^{\mu\nu}$.

Evidently a real (Hermitian) field in Minkowski space is not real in Euclidean space as is seen from (2.13). The question of Hermitian conjugation in Euclidean space will be considered in detail below. Here we merely remark that if the field ϕ is complex in Minkowski space, we define by (2.13) *separate* continuations into Euclidean space for ϕ and ϕ^*. (It is of course always possible though not necessarily opportune to decompose a complex field into real fields.)

One may verify that the following definitions insure that the conformal algebra in Euclidean space is of the same form as in Minkowski space with $g^{\mu\nu}$ replaced by $\delta^{\mu\nu}$.

$$\begin{aligned} &\hat{P}^4 = iP^0, \quad \hat{P}^j = -P^j ,\\ &\hat{M}^{j4} = iM^{j0}, \quad \hat{M}^{jk} = -M^{jk} ,\\ &\hat{D} = D,\\ &\hat{K}^4 = -iK^0, \quad \hat{K}^j = K^j . \end{aligned} \tag{2.16}$$

The transformation law for the fields is

$$\begin{aligned} &i[\hat{P}^\mu, \hat{\phi}(x)] = \partial^\mu \hat{\phi}(x) ,\\ &i[\hat{M}^{\mu\nu}, \hat{\phi}(x)] = (x^\mu \partial^\nu - x^\nu \partial^\mu)\hat{\phi}(x) + \hat{\Sigma}^{\mu\nu}\hat{\phi}(x) ,\\ &i[\hat{D}, \hat{\phi}(x)] = x_\mu \partial^\mu \hat{\phi}(x) + d\hat{\phi}(x) ,\\ &i[\hat{K}^\mu, \hat{\phi}(x)] = (2x^\mu x^\alpha - x^2\delta^{\mu\alpha})\partial_\alpha \hat{\phi}(x)\\ &\qquad + 2x_\alpha(d\,\delta^{\alpha\mu} - \hat{\Sigma}^{\alpha\mu})\hat{\phi}(x) . \end{aligned} \tag{2.17}$$

The spin matrix for the fermion field is $\hat{\Sigma}^{\mu\nu} = \frac{1}{4}[\hat{\gamma}^\mu, \hat{\gamma}^\nu]$. In Sec. III we shall show how the generators (2.16) are constructed from the fields $\hat{\phi}$.

Finally we postulate that in Euclidean space the free, massless propagators are given by the continuation of the Minkowski propagators, and furthermore that the propagator is ordered along the direction of propagation, that is along the radial direction.[11] (It will be seen later that this radially ordered product is a familiar object: It is the 4-dimensional operator generalization of the well-known multipole expansion of the Coulomb potential.)

$$\begin{aligned} D(x,y) &= \langle 0|R\hat{\phi}(x)\hat{\phi}(y)|0\rangle\\ &= \theta(x^2-y^2)\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle\\ &\quad + \theta(y^2-x^2)\langle 0|\hat{\phi}(y)\hat{\phi}(x)|0\rangle\\ &= \frac{1}{4\pi^2}\frac{1}{(x-y)^2} , \end{aligned} \tag{2.18a}$$

$$\begin{aligned} S(x,y) &= \langle 0|R\hat{\psi}(x)\hat{\bar{\psi}}(y)|0\rangle\\ &= \theta(x^2-y^2)\langle 0|\hat{\psi}(x)\hat{\bar{\psi}}(y)|0\rangle\\ &\quad - \theta(y^2-x^2)\langle 0|\hat{\bar{\psi}}(y)\hat{\psi}(x)|0\rangle\\ &= -\frac{1}{4\pi^2}\hat{\gamma}^\mu \partial_\mu \frac{1}{(x-y)^2} . \end{aligned} \tag{2.18b}$$

The explicit expressions for the propagators are true in 4 dimensions. Our entire quantization procedure is based on the requirement that Eqs. (2.18) are valid. In the following, we shall omit the caret whenever this causes no confusion.

Starting from the expressions (2.18) for the R-ordered products it is now possible to obtain the equal-r algebra of our fields. It is convenient to introduce the following notation. Define the angular derivative

$$l^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \tag{2.19a}$$

and its norm

$$L^2 = \tfrac{1}{2} l^{\mu\nu} l_{\mu\nu} . \tag{2.19b}$$

The d'Alembertian may now be represented as

$$\Box = \frac{1}{r^3}\left[\left(\frac{d}{d\ln r}\right)^2 - (L^2+1)\right] r . \tag{2.20}$$

(For details see Appendix A.) Applying the $\Box$ operator on both sides of Eq. (2.18a) we get

$$\Box_1 \langle 0|R\phi(x_1)\phi(x_2)|0\rangle = -\delta^4(x_1-x_2) . \tag{2.21}$$

Recalling that $\phi(x)$ obeys the free massless equation

$$\Box\phi(x) = 0 \tag{2.22}$$

and introducing polar coordinates

$$\begin{aligned} &r^2 = x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 ,\\ &\alpha^\mu = x^\mu / r , \end{aligned} \tag{2.23}$$

it is easy to obtain

$$\begin{aligned} &\langle 0|[\chi(r,\alpha_1), \chi(r,\alpha_2)]|0\rangle = 0 ,\\ &\langle 0|[\dot{\chi}(r,\alpha_1), \chi(r,\alpha_2)]|0\rangle = -\delta^3(\alpha_1-\alpha_2) ,\\ &\langle 0|[\dot{\chi}(r,\alpha_1), \dot{\chi}(r,\alpha_2)]|0\rangle = 0 . \end{aligned} \tag{2.24}$$

Here we have introduced the notation

$$\begin{aligned} &\chi(r,\alpha) = r\phi(x) ,\\ &\dot{\chi}(r,\alpha) = \frac{d}{d\ln r}\chi(r,\alpha) . \end{aligned} \tag{2.25}$$

The δ function is defined on the sphere

$$\int d\alpha_1 f(\alpha_1)\delta^3(\alpha_1-\alpha_2)=f(\alpha_2).$$

Similarly applying the Dirac operator $\gamma^\mu\partial_\mu$ on both sides of Eq. (2.18b) and defining

$$\xi(r,\alpha)=r^{3/2}\psi(x), \tag{2.26}$$

one gets

$$\langle 0|\{\xi(r,\alpha_1),\bar\xi(r,\alpha_2)\}|0\rangle=\gamma^\mu\alpha_\mu\delta^3(\alpha_1-\alpha_2). \tag{2.27}$$

(In Appendix A we give the details.) If the natural assumption is made that the equal-r commutators (anticommutators) are c numbers we finally obtain

$$[\chi(r,\alpha_1),\chi(r,\alpha_2)]=0=[\dot\chi(r,\alpha_1),\dot\chi(r,\alpha_2)], \tag{2.28}$$
$$[\dot\chi(r,\alpha_1),\chi(r,\alpha_2)]=-\delta^3(\alpha_1-\alpha_2),$$

$$\{\xi(r,\alpha_1),\xi(r,\alpha_2)\}=0=\{\bar\xi(r,\alpha_1),\bar\xi(r,\alpha_2)\}, \tag{2.29}$$
$$\{\xi(r,\alpha_1),\bar\xi(r,\alpha_2)\}=\gamma^\mu\alpha_\mu\delta^3(\alpha_1-\alpha_2).$$

We wish now to discuss the properties of our field under Hermitian conjugation. Let us concentrate on the boson field $\chi(r,\alpha)$. We notice that the right-hand side of Eq. (2.28) is real. This means that $\chi(r,\alpha)$ cannot be an Hermitian operator. We postulate that the field $\chi(r,\alpha)$ satisfies the relation

$$\chi^\dagger(r,\alpha)=\chi(1/r,\alpha) \tag{2.30}$$

which is consistent with (2.28). This postulate will have far-reaching consequences. As we shall see in Sec. III the momentum operator P^μ will be the negative of the Hermitian conjugate of the conformal operator K^μ.

Let us observe that the angular variables α_μ here play the role of space variable of ordinary quantization, whereas the radius r is the analog of the time variable – propagation occurs along r. To make the analogy more precise, introduce the proper time σ defined as

$$r=e^{i\sigma}. \tag{2.31}$$

If we consider the operator χ as a function of σ and α then the nonvanishing commutator in (2.28) becomes

$$\left[\frac{d}{d\sigma}\chi(\sigma,\alpha_1),\chi(\sigma,\alpha_2)\right]=-i\delta^3(\alpha_1-\alpha_2) \tag{2.32}$$

which is formally similar to the equal-time commutator of conventional quantization. Notice now that the Eq. (2.30) simply says that the operator

$$\chi(r,\alpha)=\chi(e^{i\sigma},\alpha)$$

is Hermitian for real σ. Another advantage of the further continuation (2.31) will emerge when we consider in Sec. III matrix elements of products of our Euclidean field $\phi(x)$. It will be seen that in our theory it is possible to define simply only R ordered products, while unordered products do not exist for all r. However, on the unit circle $r=e^{i\sigma}$ both ordered and unordered products may be unambiguously defined.

Finally we remark that the commutators (2.28) and (2.29) can be obtained directly from the usual canonical formalism. For example for the scalar field, we consider the action

$$I=\tfrac{1}{2}\int d^4x\,\partial_\mu\phi\partial^\mu\phi=-\tfrac{1}{2}\int d^4x\,\phi\Box\phi. \tag{2.33}$$

When the d'Alembertian is represented as in (2.20), the action is

$$I=-\tfrac{1}{2}\int_0^\infty\frac{dr}{r}\int d\alpha\,\chi(r,\alpha)\left[\left(\frac{d}{d\ln r}\right)^2-(L^2+1)\right]\chi(r,\alpha)$$
$$=\tfrac{1}{2}\int_{-\infty}^\infty d\ln r\int d\alpha\left[\left(\frac{d}{d\ln r}\chi\right)^2+\chi(L^2+1)\chi\right]. \tag{2.34}$$

Clearly we should identify the canonical coordinate to be χ and the canonical momentum as $d\chi/d\ln r$. This then leads to the commutators (2.28), up to the constant normalizing the δ function. That constant is a matter of convention. Settling upon the convention of (2.28), we must insert an additional factor of i in the conventional (Noether) definition of infinitesimal generators.

III. LORENTZ-INVARIANT QUANTIZATION OF A SCALAR FIELD

A. Expansion of the Free Massless Field

Let us consider a free, massless scalar field $\phi(x)$ obeying the d'Alembert equation in 4 dimensions

$$\Box\phi(x)=0. \tag{3.1}$$

As before we shall introduce polar coordinates

$$\alpha^\mu=x^\mu/r,$$
$$\alpha^1=\cos\theta,$$
$$\alpha^2=\sin\theta\cos\psi, \tag{3.2}$$
$$\alpha^3=\sin\theta\sin\psi\cos\phi,$$
$$\alpha^4=\sin\theta\sin\psi\sin\phi,$$

and define

$$\chi(r,\alpha)=r\phi(x). \tag{3.3}$$

It is possible to rewrite Eq. (3.1) with the help of the angular derivatives $l^{\mu\nu}$ and $L^2=\frac{1}{2}l^{\mu\nu}l_{\mu\nu}$ given in (2.19)

$$\left[\left(\frac{d}{d\ln r}\right)^2-(L^2+1)\right]\chi(r,\alpha)=0\,. \tag{3.4}$$

We can now expand $\chi(r,\alpha)$ in a series of 4-dimensional spherical harmonics $Y_{lnm}(\alpha)$

$$\chi(r,\alpha)=\sum_{l=0}^{\infty}\sum_{n=0}^{l}\sum_{m=-n}^{n}g_{lnm}(r)Y_{lnm}(\alpha)\,. \tag{3.5}$$

The $Y_{lnm}(\alpha)$ are a complete, orthonormal set of eigenfunctions of L^2.

$$L^2Y_{lnm}(\alpha)=l(l+2)Y_{lnm}(\alpha)\,. \tag{3.6}$$

They are given explicitly by

$$Y_{lnm}(\theta,\psi,\phi)=N_{lnm}e^{im\phi}(\sin\theta)^nC_{l-n}^{n+1}(\cos\theta)(\sin\psi)^mC_{n-m}^{m+1/2}(\cos\psi)\,, \tag{3.7}$$

where $C_\nu^\lambda(z)$ is a Gegenbauer polynomial and N_{lnm} is a normalization factor. In Appendix A we list various properties of $Y_{lnm}(\alpha)$ and define N_{lnm}. The spherical functions obey the completeness relations

$$\int d\alpha\, Y^*_{lnm}(\alpha)Y_{l'n'm'}(\alpha)=\delta_{ll'}\delta_{nn'}\delta_{mm'}, \tag{3.8a}$$

$$\sum_{lnm}Y^*_{lnm}(\alpha)Y_{lnm}(\alpha')=\delta^3(\alpha-\alpha')\,. \tag{3.8b}$$

and the addition formula

$$\sum_{nm}Y^*_{lnm}(\alpha)Y_{lnm}(\alpha')=\frac{l+1}{2\pi^2}C_l^1(\alpha\cdot\alpha')\,. \tag{3.9}$$

Here $C_l^1(w)$ is the Gegenbauer polynomial, defined by the expansion

$$\sum_{l=0}^{\infty}t^lC_l^1(w)=(1-2wt+t^2)^{-1},\qquad |t|<1\,. \tag{3.10}$$

Let us now substitute (3.5) into (3.4). Using (3.6) we obtain for $g_{lnm}(r)$ the simple equation

$$\left[\left(\frac{d}{d\ln r}\right)^2-(l+1)^2\right]g_{lnm}(r)=0 \tag{3.11}$$

whose general solution is of course

$$g_{lnm}(r)=g_{lnm}^{(1)}r^{l+1}+g_{lnm}^{(2)}r^{-(l+1)}\,. \tag{3.12}$$

Therefore we can cast (3.5) into the form

$$\chi(r,\alpha)=\sum_{lnm}\left[a_{lnm}^{(-)}\frac{r^{-(l+1)}}{(2l+2)^{1/2}}Y^*_{lnm}(\alpha)+a_{lnm}^{(+)}\frac{r^{l+1}}{(2l+2)^{1/2}}Y_{lnm}(\alpha)\right], \tag{3.13}$$

where Y and Y^* are introduced in order to insure simple Hermitian and algebraic properties for $a_{lnm}^{(\pm)}$. If we compare (3.13) with our definition (2.30) of Hermiticity we readily see that $a_{lnm}^{(+)}$ is the Hermitian conjugate of $a_{lnm}^{(-)}$

$$a_{lnm}^{(+)}=[a_{lnm}^{(-)}]^\dagger\,. \tag{3.14}$$

We also notice that the expansion (3.13) for $\chi(r,\alpha)$ has the form of a Laurent expansion. This means that the definition of $\chi(r,\alpha)$ as a function of r is a very delicate one. We shall see later that only the R-ordered product has a clear, unambiguous meaning.

A very convenient way of handling the r dependence is to use the time variable $\sigma=-i\ln r$, defined in Sec. II. In the variable σ (3.13) takes the form of a Fourier expansion

$$\chi(r,\alpha)=\sum_{lnm}\left[a_{lnm}^{(-)}\frac{e^{-i\sigma(l+1)}}{(2l+2)^{1/2}}Y^*_{lnm}(\alpha)+a_{lnm}^{(+)}\frac{e^{i\sigma(l+1)}}{(2l+2)^{1/2}}Y_{lnm}(\alpha)\right]. \tag{3.15}$$

Equations (3.15) together with (3.14) show explicitly that $\chi(r,\alpha)$ is Hermitian for real values of σ.

Let us now investigate the algebraic properties of our operators. We shall postulate the fundamental commutators

$$[a_{lnm}^{(\pm)},a_{l'n'm'}^{(\pm)}]=0\,,\qquad [a_{lnm}^{(-)},a_{l'n'm'}^{(+)}]=\delta_{ll'}\delta_{nn'}\delta_{mm'}\,. \tag{3.16}$$

Using the completeness relation (3.8) it is easy to obtain from (3.16) the fundamental equal-r commutation relations of Sec. II.

$$[\chi(r,\alpha_1),\chi(r,\alpha_2)]=0=[\dot\chi(r,\alpha_1),\dot\chi(r,\alpha_2)]\,,$$
$$[\dot\chi(r,\alpha_1),\chi(r,\alpha_2)]=-\delta^3(\alpha_1-\alpha_2)\,, \tag{3.17}$$
$$\dot\chi(r,\alpha)=\frac{d}{d\ln r}\chi(r,\alpha)\,.$$

The operators of the conformal group Eqs. (2.16) are given by conventional Noether formulas in terms of the new improved energy-momentum tensor $\Theta^{\mu\nu}$, with an additional factor of i which is necessary to ensure proper commutation relations. The integrations are over the quantization surface $x^2=r^2$. Since the generators are surface-independent, r^2 may be set to 1. Thus for example the dilatation generator is

$$D=i\int d^4x\,\delta(x^2-1)2x_\mu x_\nu\Theta^{\mu\nu}(x)$$
$$=i\int d\alpha\,\alpha_\mu\alpha_\nu\Theta^{\mu\nu}(1,\alpha)\,. \tag{3.18a}$$

In terms of the creation and annihilation operators, D is diagonal.

$$D=-i\sum_{lnm}(l+1)a^{(+)}_{lnm}a^{(-)}_{lnm}\,. \tag{3.18b}$$

For the remaining generators, the formulas are

$$P^\mu=i\int d\alpha\,\alpha_\nu\Theta^{\mu\nu}(1,\alpha)\,, \tag{3.19}$$

$$K^\mu=i\int d\alpha[2\alpha^\mu\alpha_\omega\alpha_\phi\Theta^{\omega\phi}(1,\alpha)-\alpha_\omega\Theta^{\mu\omega}(1,\alpha)]\,, \tag{3.20}$$

$$M^{\mu\nu}=i\int d\alpha[\alpha^\mu\alpha_\omega\Theta^{\nu\omega}(1,\alpha)-\alpha^\nu\alpha_\omega\Theta^{\mu\omega}(1,\alpha)]\,. \tag{3.21}$$

Notice that the special kinematical operator introduced in Sec. II, Eq. (2.12) has a particularly simple form

$$\tfrac{1}{2}(P^\mu-K^\mu)=i\int d\alpha(\delta^{\mu\nu}-\alpha^\mu\alpha^\nu)\alpha^\omega\Theta_{\nu\omega}(1,\alpha)\,. \tag{3.22}$$

Also we construct the complementary combination

$$\tfrac{1}{2}(P^\mu+K^\mu)=i\int d\alpha\,\alpha^\mu\alpha^\omega\alpha^\phi\Theta_{\omega\phi}(1,\alpha)\,. \tag{3.23}$$

The expressions for the generators (3.19) to (3.23) in terms of creation and annihilation operators are extremely complicated in our basis; we do not present them here. (We do display these formulas for the 2-dimensional case, see Sec. IV.) It is easy to show however from the definitions (3.18) to (3.21), as well as from the formula for $\Theta^{\mu\nu}$

$$\Theta^{\mu\nu}=\partial^\mu\phi\partial^\nu\phi-\delta^{\mu\nu}\tfrac{1}{2}\partial^\alpha\phi\partial_\alpha\phi+\tfrac{1}{6}(\delta^{\mu\nu}\Box-\partial^\mu\partial^\nu)\phi^2 \tag{3.24}$$

that the Hermiticity condition (2.30) implies that

$$(D)^\dagger=-D,\quad (M^{\mu\nu})^\dagger=M^{\mu\nu}\,,$$
$$(P^\mu)^\dagger=-K^\mu,\quad (K^\mu)^\dagger=-P^\mu\,. \tag{3.25}$$

Since D is anti-Hermitian, it is useful to define the Hermitian operator Δ

$$\Delta=iD=\sum_{lnm}(l+1)a^{(+)}_{lnm}a^{(-)}_{lnm}\,. \tag{3.26}$$

B. Vacuum Matrix Elements

Next we define the vacuum state $|0\rangle$, as being annihilated by all $a^{(-)}_{lnm}$ operators.

$$a^{(-)}_{lnm}|0\rangle=0\,. \tag{3.27a}$$

With our definition (3.14) of Hermitian conjugation we also have

$$\langle 0|a^{(+)}_{lnm}=0\,. \tag{3.27b}$$

We now compute the vacuum expectation value of the product of two operators. Using Eqs. (3.13), (3.16), and (3.27) it is true that

$$\langle 0|\chi(r_1,\alpha_1)\chi(r_2,\alpha_2)|0\rangle=\frac{r_2}{r_1}\sum_{lnm}\left(\frac{r_2}{r_1}\right)^l\frac{1}{2l+2}Y^*_{lnm}(\alpha_1)Y_{lnm}(\alpha_2)\,. \tag{3.28a}$$

By the addition theorem (3.9) it follows

$$\langle 0|\chi(r_1,\alpha_1)\chi(r_2,\alpha_2)|0\rangle=\frac{1}{4\pi^2}\frac{r_2}{r_1}\sum_{l=0}^{\infty}\left(\frac{r_2}{r_1}\right)^l C^1_l(\alpha_1\cdot\alpha_2)\,. \tag{3.28b}$$

Notice that the above expressions are convergent only when $r_2/r_1<1$. This means that the matrix element $\langle 0|\chi(r_1,\alpha_1)\chi(r_2,\alpha_2)|0\rangle$ is defined only for $r_1>r_2$. Under this condition the series in (3.28b) can be summed [see (3.10)] and we obtain

$$\begin{aligned}\langle 0|\chi(r_1,\alpha_1)\chi(r_2,\alpha_2)|0\rangle&=\frac{r_1r_2/4\pi^2}{r_1^{\,2}-2r_1r_2\alpha_1\cdot\alpha_2+r_2^{\,2}}\,,\\ \langle 0|\phi(x_1)\phi(x_2)|0\rangle&=\frac{1}{4\pi^2}\frac{1}{(x_1-x_2)^2}\,,\qquad r_1>r_2\,.\end{aligned} \tag{3.28c}$$

The product of the two operators in the reverse R order can be simply obtained by interchanging $r_1\rightleftarrows r_2$, $\alpha_1\rightleftarrows\alpha_2$ in the above argument. We thus get

$$\begin{aligned}\langle 0|\chi(r_2,\alpha_2)\chi(r_1,\alpha_1)|0\rangle&=\frac{r_1r_2/4\pi^2}{r_1^{\,2}-2r_1r_2\alpha_1\cdot\alpha_2+r_2^{\,2}}\,,\\ \langle 0|\phi(x_2)\phi(x_1)|0\rangle&=\frac{1}{4\pi^2}\frac{1}{(x_1-x_2)^2}\,,\qquad r_2>r_1\,.\end{aligned} \tag{3.28d}$$

Equations (3.28) can be combined to give

$$\theta(r_1-r_2)\langle 0|\phi(x_1)\phi(x_2)|0\rangle+\theta(r_2-r_1)\langle 0|\phi(x_2)\phi(x_1)|0\rangle=\frac{1}{4\pi^2}\frac{1}{(x_1-x_2)^2} \tag{3.29}$$

which is the fundamental R-ordered product postulated in Sec. II. We see that whereas the product of two operators can be defined only under certain conditions, the R-ordered product can be defined unambiguously everywhere.

If we perform the further continuation $r\to e^{i\sigma}$, whose virtues we have appreciated already, it becomes possible to define matrix elements of products of fields for all real σ. This merely reflects the fact that on the unit circle $r=1$, our products (3.28) can be defined. We hope to explore this further continuation of field theory in the future.[12]

C. Spectral Representation of Propagator

Equations (3.27) to (3.29) can be combined to give a beautiful expression for the propagator. Let us start from the expansion

$$D(x_1,x_2)=\frac{1}{4\pi^2}\frac{1}{(x_1-x_2)^2}=\sum_{lnm}\frac{1}{2l+2}Y^*_{lnm}(\alpha_1)Y_{lnm}(\alpha_2)\left[\frac{\theta(r_1-r_2)}{r_1^{\,2}}\left(\frac{r_2}{r_1}\right)^l+\frac{\theta(r_2-r_1)}{r_2^{\,2}}\left(\frac{r_1}{r_2}\right)^l\right]\,. \tag{3.30}$$

(The analogy of this to the expansion of the Coulomb potential in multipoles is clear.) Using the identity

$$\frac{1}{\pi}\int_{-\infty}^{\infty}d\gamma\frac{e^{i\gamma\ln z}}{\gamma^2+(l+1)^2}=\frac{z^{l+1}\theta(1-z)+z^{-(l+1)}\theta(z-1)}{l+1}\,,\qquad z>0,\quad l>-1\,, \tag{3.31}$$

we can cast (3.30) in the form

$$D(x_1,x_2)=\frac{1}{4\pi^2}\frac{1}{(x_1-x_2)^2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\gamma\sum_{lnm}\frac{F^*_{\gamma lnm}(x_1)F_{\gamma lnm}(x_2)}{\gamma^2+(l+1)^2}\,, \tag{3.32}$$

where the functions

$$F_{\gamma lnm}(x) = r^{i\gamma - 1} Y_{lnm}(\alpha) \tag{3.33}$$

define the transformation between the coordinate space and the conjugate (γlnm) space in which the *dimensionality* γ and the angular momentum variables (lnm) are diagonalized. In this new space, particularly convenient for dilatation-invariant problems, the propagator is diagonal. It is

$$D(\gamma, l, n, m) = \frac{1}{\gamma^2 + (l+1)^2} . \tag{3.34}$$

This 4-dimensional (γlnm) space plays the same role in the present context as 4-dimensional momentum space in the usual quantization schemes.

The pole structure of the propagator at $\gamma = \pm i(l+1)$ is easily understood. We note from (3.16) and (3.18) that $a^{(+)}_{lnm}$ has the scale dimension $l+1$

$$i[D, a^{(+)}_{lnm}] = (l+1) a^{(+)}_{lnm} . \tag{3.35}$$

Hence the "single-particle" state

$$|lnm\rangle = a^{(+)}_{lnm} |0\rangle \tag{3.36}$$

is an eigenstate of D with eigenvalue $-i(l+1)$. Thus the propagator in our conjugate space has poles at the eigenvalues of the evolution operator, in complete analogy with the situation in momentum space for ordinary quantization.

D. Development of the System in the Presence of Interactions

When interactions are present, the equation of motion for the field $\Phi(x)$ is

$$\Box\Phi(x) = \eta(x), \tag{3.37}$$

where $\eta(x)$ is the source. One may impose the free-field canonical commutation relations (3.17) on $\Phi(x)$ at equal r. The field $\chi = r\Phi$ is expanded as in (3.5)

$$\chi(r, \alpha) = \sum_{lnm} \frac{A^m_{ln}(r) Y_{lnm}(\alpha)}{(2l+2)^{1/2}} . \tag{3.38}$$

As a consequence of the field commutators, the operators $A^m_{ln}(r)$ satisfy the following commutation relations at equal r:

$$\begin{aligned} &[A^m_{ln}(r), A^{m'}_{l'n'}(r)] = 0 = [\dot{A}^m_{ln}(r), \dot{A}^{m'}_{l'n'}(r)] , \\ &[\dot{A}^m_{ln}(r), A^{-m'}_{l'n'}(r)] = (-1)^{m+1} (2l+2) \delta_{ll'} \delta_{nn'} \delta_{mm'} , \\ &\dot{A}^m_{ln}(r) = \frac{d}{d\ln r} A^m_{ln}(r) . \end{aligned} \tag{3.39}$$

The equation of motion for $A^m_{ln}(r)$ is

$$\left[\left(\frac{d}{d\ln r}\right)^2 - (l+1)^2\right] A^m_{ln}(r) = \eta^m_{ln}(r) , \tag{3.40}$$

where $\eta^m_{ln}(r)$ is the expansion coefficient of the source in the radial basis

$$\eta^m_{ln}(r) = r^3 (2l+2)^{1/2} \int d\alpha\, Y^*_{lnm}(\alpha) \eta(x) .$$

These formulas may be cast into a form closely resembling the free field case. Define

$$A^{(+)}_{lnm}(r) = \frac{1}{2}\left[A^m_{ln}(r) + \frac{1}{l+1} \frac{d}{d\ln r} A^m_{ln}(r)\right] , \tag{3.41a}$$

$$A^{(-)}_{lnm}(r) = \frac{(-1)^m}{2}\left[A^{-m}_{ln}(r) - \frac{1}{l+1} \frac{d}{d\ln r} A^{-m}_{ln}(r)\right] . \tag{3.41b}$$

Thus we have

$$\chi(r, \alpha) = \sum_{lnm} \frac{A^{(-)}_{lnm}(r)}{(2l+2)^{1/2}} Y^*_{lnm}(\alpha) + \frac{A^{(+)}_{lnm}(r)}{(2l+2)^{1/2}} Y_{lnm}(\alpha) , \tag{3.42}$$

$$[A^{(\pm)}_{lnm}(r), A^{(\pm)}_{l'n'm'}(r)] = 0\,,$$
$$[A^{(-)}_{lnm}(r), A^{(+)}_{l'n'm'}(r)] = \delta_{ll'}\delta_{nn'}\delta_{mm'}\,, \qquad (3.43)$$

$$\frac{d}{d\ln r} A^{(\pm)}_{lnm}(r) = \pm(l+1)A^{(\pm)}_{lnm}(r) \pm \frac{(-1)^{(m\mp m)/2}}{2l+2}\,\eta^{\pm m}_{ln}(r)\,. \qquad (3.44)$$

It is not difficult to convince oneself that in a scale-invariant theory $\eta^m_{ln}(r)$ does not depend explicitly on r. Consequently there is *no* region of r in which scale-invariant, renormalizable interactions can be ignored in Eq. (3.44). This is the origin of the well-known "anomalous" breakdown of formal scale invariance in renormalizable theories.[13] If the interactions are not scale-invariant but all are superrenormalizable then $\eta^m_{ln}(r)$ will have an explicit r dependence involving *positive* powers of r. Nonrenormalizable interactions lead to explicit *negative* powers of r in $\eta^m_{ln}(r)$. Consequently for superrenormalizable interactions there should be no anomalous scale-symmetry violation since $\eta^m_{ln}(r)$ vanishes at $r=0$.

The role of the dilatation operator as the evolution operator is again apparent. From the canonical commutation relations it follows that[14]

$$i[D, \Phi(x)] = x^\mu \partial_\mu \Phi(x) + \Phi(x)\,. \qquad (3.45)$$

[When the theory is not scale-invariant, D will depend explicitly on r, and (3.45) is valid only at equal r.] Also we have

$$\begin{aligned} i[D, A^{(\pm)}_{lnm}(r)] &= \frac{d}{d\ln r} A^{(\pm)}_{lnm}(r) \\ &= \pm(l+1)A^{(\pm)}_{lnm}(r) \pm \frac{(-1)^{(m\mp m)/2}}{2l+2}\,\eta^{\pm m}_{ln}(r)\,. \end{aligned} \qquad (3.46)$$

To solve the equations of motion (3.46) we define the operator $U(r, r_0)$ by the equations

$$\frac{\partial}{\partial \ln r} U(r, r_0) = -iD^{\text{int}}(r)U(r, r_0)\,, \qquad U(r_0, r_0) = 1\,, \qquad (3.47)$$

where $D^{\text{int}}(r)$ is the interaction part of the dilatation generator, constructed from the *free* fields ϕ. In a familiar fashion, one proves that $U(r, r_0)A^{(\pm)}_{lnm}(r)U^{-1}(r, r_0)$ satisfies the free-field version of (3.46) and that $U(r, r_0)\Phi(x)U^{-1}(r, r_0)$ is a free field.

Integration of (3.47) gives

$$U(r, r_0) = R\exp\left[-i\int_{r_0}^{r} \frac{dr'}{r'} D^{\text{int}}(r')\right], \qquad (3.48)$$

where the R ordering is along the spheres. Recalling the definition of D^{int}, we see that

$$\begin{aligned} -i\int_{r_0}^{r}\frac{dr'}{r'} D^{\text{int}}(r') &= -i\int_{r_0}^{r}\frac{dr'}{r'}\int d\sigma^\mu \mathfrak{D}^{\text{int}}_\mu(x) \\ &= -i\int_{r_0}^{r}\frac{dr'}{r'}\int d^4x\,\delta(x^2 - r'^2)2x^\mu[ix^\nu\Theta^{\text{int}}_{\mu\nu}(x)] \\ &= -\int_{r_0}^{r}\frac{dr'}{r'}\int d^4x\,\delta(x^2 - r'^2)2r'^2\mathcal{L}^{\text{int}}(x) \\ &= -\int_{r_0}^{r} d^4x\,\mathcal{L}^{\text{int}}(x)\,, \end{aligned}$$
$$U(r, r_0) = R\exp\left[-\int_{r_0}^{r} d^4x\,\mathcal{L}^{\text{int}}(x)\right]. \qquad (3.49)$$

The integration is over all space bounded by the spheres $r^2 > x^2 > r_0^2$.

Thus we have regained the Schwinger-Tomonaga result that the evolution of the system from $x^2 = r_0^2$ to $x^2 = r^2$ is governed by the ordered exponential of the integral of the interaction Lagrangian. Clearly the S matrix is $U(\infty, 0)$.

$$S = R\exp\left[-\int d^4x\,\mathcal{L}^{\text{int}}(x)\right]$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int d^4x_1\cdots d^4x_n\,R\mathcal{L}^{\text{int}}(x_1)\cdots\mathcal{L}^{\text{int}}(x_n)\,. \tag{3.50}$$

Applying Wick's theorem to this, we regain the familiar Feynman-Dyson perturbation theory. Thus we see that our quantization procedure is entirely successful and consistent with the conventional approach.

The evaluation of the perturbation series may be carried out in position space, or in our conjugate (γlnm) space. Calculations in the (γlnm) space are especially convenient for conformally invariant interactions. We hope to return to the questions of divergences and renormalization in this conjugate space in a future publication.[15]

E. Nature of the Level Spectrum – Covariant Thermodynamics

It is clear that the dilatation operator D plays a fundamental role in our theory: It substitutes for the Hamiltonian as the evolution operator, and the states are eigenstates of it. Consequently if we wish to study the spectrum of states and to develop statistical methods based on our quantization technique, we are led to the problem of counting the number of levels corresponding to any given eigenvalue of $\Delta = iD$.

We have seen already that the "single-particle" state $|lnm\rangle = a_{lnm}^{(+)}|0\rangle$ is an eigenstate of Δ with eigenvalue $(l+1)$. Since the range of the "magnetic" quantum numbers n, m is $l \geq n \geq 0$, $n \geq m \geq -n$, the degeneracy is

$$\sum_{n=0}^{l}\sum_{m=-n}^{n} = \sum_{n=0}^{l}(2n+1)$$
$$= (l+1)^2\,. \tag{3.51}$$

It will be instructive to carry out the treatment in a space with arbitrary number of dimensions δ. In this case it is easy to see that Δ will have the form

$$\Delta = \sum_{l=0}^{\infty}\sum_{\mu}\left(l+\frac{\delta-2}{2}\right)a_{l\mu}^{(+)}a_{l\mu}^{(-)}\,, \tag{3.52}$$

where l is the total angular momentum quantum number, μ represents the ensemble of magnetic quantum numbers, and $a_{l\mu}^{(\pm)}$ are the expansion operators for the *dimensionless* boson field in δ dimensions, $\chi(r,\alpha) = r^{(\delta-2)/2}\phi(x)$. (For details, see Appendix B.) The application of the above formula to the 2-dimensional case requires further modifications, due to the appearance of infrared divergences, related to the zero dimensionality of the field. This will be further discussed in a future publication.[16] We anticipate however that if we disregard the contribution to the 2-dimensional Δ of the zero-dimensional solutions – constants and logarithms – of the Laplace equation, then the above formula is valid. Note in that case the sum begins at $l=1$. (For 2-dimensional fermion theories, discussed in detail in Sec. IV, this problem does not arise.)

Let us denote by $\alpha_\delta(l)$ the single-particle-state degeneracy in the δ-dimensional case, i.e., the number of independent operators $a_{l\mu}^{(+)}$ corresponding to fixed value of l. The formula for $\alpha_\delta(l)$, which generalizes (3.51) to δ dimensions, is derived in Appendix B. It is

$$\alpha_\delta(l) = \frac{2l+\delta-2}{l+\delta-2}\,\frac{(l+\delta-2)!}{l!(\delta-2)!} \tag{3.53}$$

which reduces to the usual results for $\delta = 2, 3,$ and 4

$$\alpha_2(l) = 2,\quad l>0$$
$$\alpha_3(l) = 2l+1\,, \tag{3.54}$$
$$\alpha_4(l) = (l+1)^2\,.$$

We now wish to study the general problem of the degeneracy of the "many-particle" states. This can be done by means of a generating function.

$$F(\beta) = \mathrm{Tr}\,e^{-\beta\Delta}\,. \tag{3.55}$$

If we denote by d_N the different eigenvalues of Δ in the many-particle states and by $g_\delta(N)$ the degeneracy corresponding to each eigenvalue, (3.55) is equivalent to

$$F(\beta) = \sum_{N=0}^{\infty} g_\delta(N)e^{-\beta d_N}\,. \tag{3.56}$$

Alternate expressions for $F(\beta)$ may be obtained from its definition, see Appendix B.

$$\ln F(\beta) = -\sum_{l=0}^{\infty}\alpha_\delta(l)\ln\{1-e^{-\beta[l+(\delta-2)/2]}\}\,, \tag{3.57}$$

$$\ln F(\beta) = \sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{e^{\beta n/2}+e^{-\beta n/2}}{(e^{\beta n/2}-e^{-\beta n/2})^{\delta-1}}\right]\,. \tag{3.58}$$

When $\delta = 2$, the sum in (3.57) begins with $l=1$, while the term in square brackets in (3.58) is decreased by 1.

Before proceeding, we observe from (3.55) that $F(\beta)$ is a partition function where β plays the role of inverse temperature, $\beta \to 1/T$, and Δ is the analog of the energy operator in the usual theory. This is as it should be – we have repeatedly emphasized that the dilatation generator replaces the Hamiltonian in our method. We are therefore in a position to develop a thermodynamics, which is manifestly covariant. The thermodynamical technique can then be applied to an evaluation of $g_\delta(N)$. An alternate, analytic evaluation of $g_\delta(N)$ is given in Appendix B.

We are thus led to define the "free energy" by

$$A = -\frac{1}{\beta}\ln F(\beta)\,. \tag{3.59}$$

A is related to the "entropy" S and to the "internal energy" U by the well-known equations

$$S = \beta^2 \frac{\partial}{\partial\beta} A\,, \tag{3.60}$$

$$U = A + \frac{1}{\beta} S$$

$$= -\frac{\partial}{\partial\beta}\ln F(\beta)\,. \tag{3.61}$$

Since we seek $g_\delta(N)$ only for large N, we are effectively in the high-temperature limit $T\to\infty$ or $\beta\to 0$. In this region the "entropy" should be identified with $\ln g_\delta(N)$ and the "internal energy" with the dilatation eigenvalue d_N. Furthermore the asymptotic form for $F(\beta)$ as $\beta\to 0$ is easy to obtain from (3.58)

$$\ln F(\beta) \to C(\delta)\beta^{1-\delta}\,, \tag{3.62}$$

where $C(\delta)$ is twice the Riemann zeta function

$$C(\delta) = 2\sum_{n=1}^{\infty} n^{-\delta}\,, \tag{3.63}$$

$$C(2) = \tfrac{1}{3}\pi^2\,,$$

$$C(3) = 2.40\ldots, \tag{3.64}$$

$$C(4) = \tfrac{1}{45}\pi^4\,.$$

Hence we get from (3.59)–(3.62)

$$A \to -C(\delta)\beta^{-\delta}\,, \tag{3.65a}$$

$$S \to \delta C(\delta)\beta^{1-\delta}\,, \tag{3.65b}$$

$$U \to (\delta-1)C(\delta)\beta^{-\delta}\,. \tag{3.65c}$$

Eliminating the temperature $1/\beta$ between (3.65b) and (3.65c) gives the final result

$$S \to \delta(\delta-1)^{-1+1/\delta}[C(\delta)]^{1/\delta}U^{1-1/\delta}\,. \tag{3.66}$$

With our physical interpretation for S and U we finally derive

$$\ln g_\delta(N) \to \delta(\delta-1)^{-1+1/\delta}[C(\delta)]^{1/\delta}d_N^{\,1-1/\delta} \tag{3.67}$$

which for the usual cases $\delta = 2, 3, 4$ is

$$g_2(N) \to \exp\left(\frac{2\pi}{\sqrt{3}}\,d_N^{\,1/2}\right)\,, \tag{3.68a}$$

$$g_3(N) \to \exp(2.5 d_N^{\,2/3})\,, \tag{3.68b}$$

$$g_4(N) \to \exp(2.1 d_N^{\,3/4})\,. \tag{3.68c}$$

This same formula is obtained in Appendix B by an analytic evaluation. The agreement between the two methods shows that our development of covariant thermodynamics is consistent and successful.

F. Discussion and Speculation

The degeneracy formulas, which we have derived, correctly describe the level density of states in our quantum theory. It is tempting to compare them to the Hagedorn level density, which is realized in dual models,

$$N = \exp(c\sqrt{m^2}) \tag{3.69}$$

where m^2 is the mass of the levels.

It has previously been observed that one can obtain Eq. (3.69) in a field-theoretic context if the following two steps are taken[17]:

(a) The field theory should be confined to two space-time dimensions.

(b) A new "law of nature" is postulated relating mass with dimensionality.

$$d_N = cm^2\,. \tag{3.70}$$

It is clear from (3.68) and (3.70) that the Hagedorn formula follows. Of course both assumptions are *ad hoc* and cannot be justified at the present time. Nevertheless these two postulates may be the starting point of a new theoretical scheme for particle physics.

Pursuing this line of speculation, we wish to emphasize that if (3.70) is accepted, then only the 2-dimensional distribution function (3.68a) can reproduce the Hagedorn spectrum, and the constant c in (3.70) becomes proportional to the limiting temperature.

The circumstance that 2-dimensional models are in some way relevant to particle theory of the 4-dimensional physical world seems to be a very important though not understood feature of nature. Historically the first observation of the relevance of a 2-dimensional transverse space occurred in the analysis of multiparticle production in the "pionization" region.[18] A consistent, satisfactory description of single-particle production in this region is obtained by cutting off large, transverse momenta and using simply the longitudinal, 2-dimensional phase-space factor dE/E. More recently, the success of the parton model in interpreting many of the important features of high-energy reactions has called attention to 2-dimensional field theories as a means of describing the parton distribution functions.[17] Indeed it was in this context that the relation (3.70) was first suggested. Finally, recent investigations of dual resonance models make essential use of the general formalism of 2-dimensional field theories.[7]

This collection of facts leads us to exhibit a detailed application of our techniques to 2-dimensional field theories.

IV. TWO-DIMENSIONAL FIELD THEORIES

Field theories in 2 dimensions have been first introduced as an interesting theoretical laboratory where explicitly solvable models can be studied. More recently, however, the interest in such models has become much less academic, for reasons that were explained at the end of Sec. III. We apply our formalism to 2-dimensional field theories and demonstrate that our Lorentz-invariant quantization rules in *Euclidean* space lead directly to the operator structure present in dual models.

It has been recently shown that the higher moments of the energy-momentum tensor, in a large class of 2-dimensional field theories, obey essentially the Virasoro algebra.[7,8] However, with a conventional approach, some difficulties persist in the interpretation of various integrals. With our formalism, we can obtain a clear, unambiguous derivation of the Virasoro algebra, without the difficulties previously encountered.

Another point of contact between the present approach and dual models is seen in the structure of the R-ordered product appearing in our Euclidean perturbation theory, Eq. (3.50). This is identical to that of the operator form for the integrals in the n-point Veneziano formulas.

A. Two-Dimensional Conformally Invariant Field Theories in Minkowski Space

We review the quantum properties of conformally invariant 2-dimensional field theories in Minkowski space, with special emphasis on the large symmetry group which is related to the Virasoro algebra. We show that there exist inevitable infinities in this formalism which are removed by our Euclidean quantization. Some of these results are well known; we include them here for completeness.

Examples of conformally invariant field theories in 2 dimensions are the following:

1. Free boson field.

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi,\qquad \Theta^{\mu\nu}=\partial^\mu\phi\partial^\nu\phi-\tfrac{1}{2}g^{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi. \tag{4.1}$$

2. Free fermion field.

$$\mathcal{L}=\tfrac{1}{2}i\bar\psi\gamma^\mu\overleftrightarrow{\partial}_\mu\psi,\qquad \Theta^{\mu\nu}=\tfrac{1}{4}i(\bar\psi\gamma^\mu\overleftrightarrow{\partial}^\nu\psi+\bar\psi\gamma^\nu\overleftrightarrow{\partial}^\mu\psi). \tag{4.2}$$

3. Gradient coupling model.

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi+\tfrac{1}{2}i\bar\psi\gamma^\mu\overleftrightarrow{\partial}_\mu\psi-\lambda\bar\psi\gamma^\mu\psi\partial_\mu\phi,$$
$$\Theta^{\mu\nu}=\partial^\mu\phi\partial^\nu\phi+\tfrac{1}{4}i(\bar\psi\gamma^\mu\overleftrightarrow{\partial}^\nu\psi+\bar\psi\gamma^\nu\overleftrightarrow{\partial}^\mu\psi)-\tfrac{1}{2}\lambda(\bar\psi\gamma^\mu\psi\partial^\nu\phi+\bar\psi\gamma^\nu\psi\partial^\mu\phi)-\tfrac{1}{2}g^{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi. \tag{4.3}$$

4. Thirring model.

$$\mathcal{L}=\tfrac{1}{2}i\bar\psi\gamma^\mu\overleftrightarrow{\partial}_\mu\psi-\lambda\bar\psi\gamma^\mu\psi\bar\psi\gamma_\mu\psi,\qquad \Theta^{\mu\nu}=\tfrac{1}{4}i(\bar\psi\gamma^\mu\overleftrightarrow{\partial}^\nu\psi+\bar\psi\gamma^\nu\overleftrightarrow{\partial}^\mu\psi)-\lambda g^{\mu\nu}\bar\psi\gamma^\alpha\psi\bar\psi\gamma_\alpha\psi. \tag{4.4}$$

5. Generally covariant string model.[19]

$$\mathcal{L}=[\tfrac{1}{2}(\epsilon_{\mu\nu}\partial^\mu\phi^A\partial^\nu\phi^B)(\epsilon_{\alpha\beta}\partial^\alpha\phi^A\partial^\beta\phi^B)]^{1/2}$$
$$=(\det\partial_\mu\phi^A\partial_\nu\phi^A)^{1/2}. \tag{4.5a}$$

Here $\epsilon^{\mu\nu}$ is the totally antisymmetric tensor, $\epsilon^{01}=1$. The indices A, B describe some additional degree of freedom and are summed over. (In applications to the dual model, A, B refer to physical space-time[19] – one may of course allow models I–IV to possess such further degrees of freedom also. It should be emphasized that in the field-theoretic string realization of the dual model, the 2-dimensional space does not correspond directly to physical space-time.) The string model leads to an energy-momentum tensor

$$\Theta^{\mu\nu}=\frac{1}{\mathcal{L}}[\partial_\lambda\phi^A\partial^\lambda\phi^B(\partial^\mu\phi^A\partial^\nu\phi^B-\tfrac{1}{2}g^{\mu\nu}\partial_\sigma\phi^A\partial^\sigma\phi^B)-\partial_\lambda\phi^A\partial^\lambda\phi^A(\partial^\mu\phi^B\partial^\nu\phi^B-\tfrac{1}{2}g^{\mu\nu}\partial_\sigma\phi^B\partial^\sigma\phi^B)] \tag{4.5b}$$

which vanishes identically. This is best seen when the $\mu\nu$ components are explicitly set equal to + and –. The reason for this behavior will emerge below; see also Appendix C.

For all these models, the energy-momentum tensor is traceless and symmetric. This may be verified with the help of the equations of motion. (For the string Lagrangian the statement is of course vacuous since $\Theta^{\mu\nu}=0$.) We now observe that any current of the form $J_f^\mu(x)=\Theta^{\mu\nu}(x)f_\nu(x)$ is conserved, provided that f^μ satisfies[20]

$$\partial_\mu f_\nu+\partial_\nu f_\mu-g_{\mu\nu}\partial_\alpha f^\alpha=0\,. \tag{4.6}$$

The proof is simple.

$$\begin{aligned}\partial_\mu J_f^\mu&=\partial_\mu(\Theta^{\mu\nu}f_\nu)\\&=\Theta^{\mu\nu}\partial_\mu f_\nu\\&=\Theta^{\mu\nu}\tfrac{1}{2}(\partial_\mu f_\nu+\partial_\nu f_\mu)\\&=\Theta^{\mu\nu}\tfrac{1}{2}g_{\mu\nu}\partial_\alpha f^\alpha=0\,.\end{aligned} \tag{4.7}$$

In deriving (4.7) we used the conservation, symmetry and tracelessness of $\Theta^{\mu\nu}$, as well as (4.6).

Since we are in 2 dimensions, Eq. (4.6) possesses a wide class of solutions. To exhibit them, take the $++$ component and the $--$ component of (4.6). (The $+-$ component vanishes identically.)

$$\partial_+f_+=0\,,\quad \partial_-f_-=0\,. \tag{4.8a}$$

The solution clearly is (recall that $f_\pm=f^\mp$)

$$f^+=f^+(x^+)\,,\quad f^-=f^-(x^-)\,. \tag{4.8b}$$

Also it is true that

$$\Box f^\mu(x)=0\,. \tag{4.8c}$$

We see that in 2 dimensions, theories with a traceless, symmetric energy-momentum tensor possess a large symmetry. In Appendix C we show that this symmetry is a consequence of invariance under the coordinate transformation

$$\delta_f x^\mu=f^\mu(x)\,, \tag{4.9a}$$

where f^μ is taken to satisfy (4.6). Under this transformation, the field obeys

$$\delta_f\phi(x)=f^\mu(x)\partial_\mu\phi(x)+\frac{\partial_\mu f_\alpha(x)}{2}(g^{\alpha\mu}d-\epsilon^{\alpha\mu}\Sigma)\phi(x)\,. \tag{4.9b}$$

Here d is the scale dimension of the field ϕ and $\epsilon^{\alpha\mu}\Sigma$ is the spin matrix associated with ϕ. (In 2-dimensions we may set $\Sigma^{\alpha\mu}=\epsilon^{\alpha\mu}\Sigma$.) We also derive directly, in Appendix C, the conserved current J_f^μ with the help of Noether's theorem, and exhibit the general conditions which a 2-dimensional Lagrangian must satisfy in order that the symmetry (4.9) be present.

For the string model, the currents J_f^μ vanish identically since $\Theta^{\mu\nu}$ is zero. Moreover, it will be seen in Appendix C that this model is invariant under the transformation (4.9), even if f^μ does *not* satisfy (4.6). This is a gauge symmetry which does not lead to any conserved currents. In the following we consider only theories with nonvanishing $\Theta^{\mu\nu}$.

The composition law for the transformation (4.9) is

$$\begin{aligned}\delta_g\delta_f x^\mu&=f^\mu(x)+g^\mu(x)+f^\nu(x)\partial_\nu g^\mu(x)\,,\\ \delta_f\delta_g x^\mu&=g^\mu(x)+f^\mu(x)+g^\nu(x)\partial_\nu f^\mu(x)\,.\end{aligned} \tag{4.10a}$$

It is easy to see that if f^μ and g^μ satisfy (4.6), so does

$$h^\mu=g^\nu\partial_\nu f^\mu-f^\nu\partial_\nu g^\mu \tag{4.10b}$$

and the transformations form a group. By choosing

$$f^\mu:\quad f^+=0,\quad f^-=f(x^-)\,, \tag{4.11a}$$

$$\tilde f^\mu:\quad \tilde f^+=\tilde f(x^+),\quad \tilde f^-=0 \tag{4.11b}$$

we recognize the fact that one is dealing with a direct product of two groups of transformation, and it suffices to consider only one of the factors. Henceforth, we make the choice (4.11a); identical considerations apply to the choice (4.11b).

The charges can be constructed as follows[21]:

$$Q_f = \int_{-\infty}^{\infty} dx^1 \Theta^0{}_\mu(x) f^\mu$$

$$= \int_{-\infty}^{\infty} dx^1 \Theta^{0+}(x) f(x^-)$$

$$= \int_{-\infty}^{\infty} \frac{dx^1}{\sqrt{2}} [\Theta^{++}(x) + \Theta^{-+}(x)] f(x^-) . \tag{4.12a}$$

$\Theta^{+-}(x)$ vanishes since the stress tensor is traceless. Also from the conservation of the stress tensor, $\partial_\mu \Theta^{\mu+}(x) = 0 = \partial_+ \Theta^{++}(x)$, we learn that Θ^{++} depends only on x^-. Hence the charge is

$$Q_f = \int_{-\infty}^{\infty} dx^- \Theta^{++}(x) f(x^-) . \tag{4.12b}$$

From the composition law (4.10b) one would expect that the charges satisfy the commutation relations

$$i[Q_f, Q_g] = Q_h , \quad h = f'g - g'f . \tag{4.13}$$

In order to verify (4.13) we consider

$$\int_{-\infty}^{\infty} dx^- dy^- f(x^-) g(y^-) [\Theta^{++}(x), \Theta^{++}(y)] = \int_{-\infty}^{\infty} dx^- dy^- f(x^-) g(y^-) [\Theta^{++}(x), \Theta^{++}(y)]|_{x^+ = y^+} , \tag{4.14}$$

where the equality is true simply because Θ^{++} does not depend on x^+. The form of the local commutator of two energy-momentum tensors which will ensure the validity of (4.13) is

$$[\Theta^{++}(x), \Theta^{++}(y)]|_{x^+ = y^+} = i[\Theta^{++}(x) + \Theta^{++}(y)] \partial_- \delta(x^- - y^-) \tag{4.15}$$

and canonical evaluation in simple models yields this result.

However from general principles one can show that the commutator of the stress tensor with itself cannot be of the form given in (4.15). That the term proportional to the first derivative of the δ function is as indicated ensures Poincaré invariance of the theory: Indeed (4.15) is the analog of the Dirac-Schwinger commutator for the present problem. In addition positivity and Lorentz covariance ensure that a triple derivative is necessarily present. This is the analog of the usual Schwinger term in current commutators, and the failure of canonical manipulations to expose it is of course familiar.[21,22]

To see how such a Schwinger term arises, we consider the vacuum expectation value of the commutator. The most general expression for this object consistent with the symmetry, tracelessness, and conservation of $\Theta^{\mu\nu}$ is

$$\langle 0|[\Theta^{\mu\nu}(x), \Theta^{\alpha\beta}(y)]|0\rangle = a \partial^\mu \partial^\nu \partial^\alpha \partial^\beta \Delta(x-y), \tag{4.16}$$

where a is a constant which cannot vanish by positivity and $\Delta(x-y)$ is the free boson field commutator

$$\Delta(x-y) = \frac{1}{2i} \epsilon(x^0 - y^0) \Theta((x-y)^2) . \tag{4.17}$$

Hence we find that

$$\langle 0|[\Theta^{++}(x), \Theta^{++}(y)]|0\rangle|_{x^+ = y^+} = -\tfrac{1}{2} i a \partial_-{}^3 \delta(x^- - y^-) \tag{4.18}$$

and (4.15) must be replaced by

$$[\Theta^{++}(x), \Theta^{++}(y)]|_{x^+ = y^+} = i[\Theta^{++}(x) + \Theta^{++}(y)] \partial_- \delta(x^- - y^-) - \tfrac{1}{2} i a \partial_-{}^3 \delta(x^- - y^-) , \tag{4.19}$$

where we have assumed that the anomaly is given entirely by the c number. The commutator of the charges therefore is

$$i[Q_f, Q_g] = Q_h - \tfrac{1}{2} a \int_{-\infty}^{\infty} dx^- f'''(x^-) g(x^-) \tag{4.20}$$

and for arbitrary f and g the charges do not satisfy the "classical" algebra.

The additional c number poses also the problem of convergence of the integral. If $f(x)$ or $g(x)$ are powers of x, as they are for the Virasoro algebra,

$$\int_{-\infty}^{\infty} dx^- g(x^-) f'''(x^-)$$

does not converge. Furthermore, one may question the convergence of the formulas (4.12) for the charges, when f is a positive or negative power.

B. Two-Dimensional Conformally Invariant Field Theories in Euclidean Space

The energy-momentum tensor in Euclidean space remains of course conserved, symmetric, and traceless. Hence if we can find a function $f^\mu(x)$ satisfying

$$\partial^\mu f^\nu + \partial^\nu f^\mu - \delta^{\mu\nu}\partial_\alpha f^\alpha = 0\,, \tag{4.21}$$

we can construct a family of conserved currents

$$J_f^\mu = \Theta^{\mu\nu} f_\nu\,. \tag{4.22}$$

The most general solution of (4.21) is

$$\begin{aligned} &f_1(x) + i f_2(x) = f^+(re^{i\theta})\,,\\ &f_1(x) - i f_2(x) = f^-(re^{-i\theta})\,,\\ &x_1 = r\cos\theta\,, \quad x_2 = r\sin\theta\,. \end{aligned} \tag{4.23}$$

The charges are given by

$$\begin{aligned} Q &= i\int_0^{2\pi} d\theta\, x_\mu J_f^\mu(x)\\ &= i\int_0^{2\pi} d\theta\, re^{i\theta} f^+(re^{i\theta})\tfrac{1}{2}[\Theta^{11}(x) - i\Theta^{12}(x)] + i\int_0^{2\pi} d\theta\, re^{-i\theta} f^-(re^{-i\theta})\tfrac{1}{2}[\Theta^{11}(x) + i\Theta^{12}(x)]\,. \end{aligned} \tag{4.24}$$

In deriving (4.24) we have used the tracelessness and symmetry of $\Theta^{\mu\nu}$. As before we may set f^- to zero and confine our attention to Q_f.

$$Q_f = i\int_0^{2\pi} d\theta\, re^{i\theta} f(re^{i\theta})\Theta^{++}(x)\,, \qquad \Theta^{++}(x) = \tfrac{1}{2}[\Theta^{11}(x) - i\Theta^{12}(x)]\,. \tag{4.25a}$$

It is easy to show that $\Theta^{++}(x)$ is only a function of $x^1 + ix^2 = re^{i\theta}$. Hence Q_f is also given by

$$Q_f = \oint dz\, f(z)\Theta^{++}(z)\,, \tag{4.25b}$$

where the integration contour is over the circle of radius r. Clearly Q_f is independent of r, if $f(z)\Theta^{++}(z)$ is analytic for $0 < |z| < \infty$.

We inquire whether the composition law (4.10) is realized by the Q_f in Euclidean space, i.e., whether

$$i[Q_f, Q_g] = Q_h\,, \qquad h = f'g - g'f\,. \tag{4.26}$$

This would be the case if the commutator of the relevant components of $\Theta^{\mu\nu}$ were of the following form:

$$i[\Theta^{++}(x), \Theta^{++}(x')]|_{r=r'} = \frac{1}{r^2}[e^{-2i\theta}\Theta^{++}(x) + e^{-2i\theta'}\Theta^{++}(x')]\,\partial_\theta\delta(\theta - \theta')\,. \tag{4.27}$$

This term is also required by the Poincaré algebra; it is the analog of the Dirac-Schwinger commutator in the present context.

However just as in Minkowski space, (4.27) is not correct; an additional triple derivative of the δ function is present. To expose this object in Euclidean space, we cannot proceed as before by considering the vacuum matrix element of the commutator of $\Theta^{\mu\nu}(x)$ with $\Theta^{\alpha\beta}(y)$ for $x \neq y$, since that object is not defined – only R products exist. However, the commutator can be extracted from the R product of two energy-momentum tensors. The details are given in Appendix C. The result is

$$i[\Theta^{++}(x), \Theta^{++}(x')]|_{r=r'} = \frac{1}{r^2}[e^{-2i\theta}\Theta^{++}(x) + e^{-2i\theta'}\Theta^{++}(x')]\partial_\theta\delta(\theta - \theta') + \frac{a}{2r^4}\,e^{-2i(\theta+\theta')}[\partial_\theta\delta(\theta - \theta') + \partial_\theta{}^3\delta(\theta - \theta')]\,, \tag{4.28}$$

where a is defined to be the **necessarily** nonvanishing coefficient of the tensor

$$\partial^\mu\partial^\nu\partial^\alpha\partial^\beta\left[-\frac{1}{4\pi}\ln(x-y)^2\right],$$

occurring in the vacuum matrix element of the R product of two energy-momentum tensors.

$$\langle 0|R\Theta^{\mu\nu}(x)\Theta^{\alpha\beta}(y)|0\rangle = a\partial^\mu\partial^\nu\partial^\alpha\partial^\beta\left[-\frac{1}{4\pi}\ln(x-y)^2\right]+\text{local terms}. \tag{4.29}$$

The commutator of two charges can now be correctly evaluated from (4.28).

$$i[Q_f,Q_g]=Q_h-\frac{ia}{2}\int_0^{2\pi}d\theta\, re^{i\theta}g(re^{i\theta})f'''(re^{i\theta})=Q_h-\frac{a}{2}\oint dz\,g(z)f'''(z). \tag{4.30}$$

When f and g are analytic for $0<|z|<\infty$, the integral is independent of contour. In contrast to the Minkowski space formula (4.20), the present result (4.30) is free from divergences. Also the expression for the charge, (4.25), involves finite-range integrals and no question about convergence need be raised.

To obtain the discrete Virasoro algebra, set

$$f(z)=iz^{1-n},\quad g(z)=iz^{1-m}. \tag{4.31}$$

Then from (4.30) one gets

$$[Q_n,Q_m]=(m-n)Q_{n+m}-\pi a\delta_{n+m,0}(n^3-n). \tag{4.32}$$

The presence of the c number indicates that Q_n cannot annihilate the vacuum for all n, as is well known.

C. Explicit Example

As an explicit example of the general consideration in Sec. IV B above, we shall here quantize by our method a 2-dimensional theory in Euclidean space.

One may consider boson fields or fermion fields. However in 2-dimensions, a scale-invariant, massless boson field is beset by infrared singularities.[23] There are methods for handling these; in order not to complicate the discussion, we defer the treatment of 2-dimensional boson fields to a future publication.[16] Here we shall discuss only the fermion field.

The analog of (4.2) in Euclidean space is

$$\begin{aligned}\mathcal{L}&=\tfrac{1}{2}\bar\psi\gamma^\mu\overleftrightarrow{\partial}_\mu\psi,\\ \Theta^{\mu\nu}&=\tfrac{1}{4}(\bar\psi\gamma^\mu\partial^\nu\psi+\bar\psi\gamma^\nu\overleftrightarrow{\partial^\mu}\psi).\end{aligned} \tag{4.33}$$

The symmetry (4.9) is present. With the special Virasoro choice

$$\begin{aligned}f_1(x)+if_2(x)&=i(re^{i\theta})^{1-n},\\ f_1(x)-if_2(x)&=0\end{aligned} \tag{4.34}$$

the transformation law (4.9) becomes

$$\delta_n\psi=\frac{i}{2}r^{-n}e^{-in\theta}\left[\left(\frac{d}{d\ln r}-i\frac{\partial}{\partial\theta}\right)\psi+(1-n)(\tfrac{1}{2}-i\Sigma)\psi\right], \tag{4.35}$$

where Σ is the spin matrix

$$\Sigma=\tfrac{1}{4}\epsilon_{\mu\nu}\gamma^\mu\gamma^\nu \tag{4.36}$$

and the scale dimension d, occurring in (4.9b), has been set to $\frac{1}{2}$, as is appropriate for a fermion field in 2 dimensions. We have introduced 2-dimensional "spherical" coordinates

$$x_1=r\cos\theta,\quad x_2=r\sin\theta,\quad \alpha^\mu=x^\mu/r. \tag{4.37}$$

It is convenient to dispense with the compact spinor notation and to exhibit everything explicitly. The fermion field is a two-component object

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

which satisfies the free-field equation. The γ matrices in 2 dimensions have a realization in terms of the Pauli matrices.

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}, \tag{4.38a}$$

$$\gamma^1 = \tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} . \tag{4.38b}$$

Hence the equation of motion is

$$(\partial_1 + i\partial_2)\psi_1 = 0, \quad (\partial_1 - i\partial_2)\psi_2 = 0 . \tag{4.39a}$$

The conjugate field $\bar{\psi} = \psi^* \gamma^1$ satisfies

$$(\partial_1 + i\partial_2)\psi_1^* = 0, \quad (\partial_1 - i\partial_2)\psi_2^* = 0 . \tag{4.39b}$$

The propagator is

$$S(x, y) = \langle 0 | R\psi(x)\bar{\psi}(y) | 0 \rangle$$

$$= \gamma \cdot \partial_x \frac{1}{4\pi} \ln(x-y)^2 \tag{4.40a}$$

and satisfies

$$\gamma \cdot \partial_x S(x, y) = \delta^2(x - y) . \tag{4.40b}$$

As discussed in Sec. II, we may conclude that the fields obey the following anticommutation relations:

$$\{\xi(r, \theta), \bar{\xi}(r, \theta')\} = \alpha \cdot \gamma \delta(\theta - \theta'), \tag{4.41}$$

where the fields ξ and $\bar{\xi}$ are dimensionless

$$\xi(r, \theta) = r^{1/2}\psi(x), \quad \bar{\xi}(r, \theta) = r^{1/2}\bar{\psi}(x) . \tag{4.42}$$

The expansions of the various fields are

$$\xi_1(r, \theta) = \frac{r^{1/2}}{(2\pi)^{1/2}} \sum_{l=0}^{\infty} [a_l^{(+)} z^l + b_l^{(-)} z^{-l-1}] ,$$

$$\xi_2(r, \theta) = \frac{r^{1/2}}{(2\pi)^{1/2}} \sum_{l=0}^{\infty} [A_l^{(+)} (z^*)^l + B_l^{(-)} (z^*)^{-l-1}] ,$$

$$\xi_1^*(r, \theta) = \frac{r^{1/2}}{(2\pi)^{1/2}} \sum_{l=0}^{\infty} [a_l^{(-)} z^{-l-1} + b_l^{(+)} z^l] , \tag{4.43}$$

$$\xi_2^*(r, \theta) = \frac{r^{1/2}}{(2\pi)^{1/2}} \sum_{l=0}^{\infty} [A_l^{(-)} (z^*)^{-l-1} + B_l^{(+)} (z^*)^l] ,$$

$$z = re^{i\theta}, \quad z^* = re^{-i\theta} .$$

The anticommutation relations for the various operators which reproduce (4.41) are

$$\{a_l^{(+)}, a_{l'}^{(-)}\} = \delta_{ll'}, \quad \{b_l^{(+)}, b_{l'}^{(-)}\} = \delta_{ll'} ,$$
$$\{A_l^{(+)}, A_{l'}^{(-)}\} = \delta_{ll'}, \quad \{B_l^{(+)}, B_{l'}^{(-)}\} = \delta_{ll'} . \tag{4.44}$$

All other anticommutators are zero. When the $a_l^{(+)}$, etc. are interpreted as creation operators; and the $a_l^{(-)}$, etc. as annihilation operators, one reproduces the propagator (4.40a). Hermitian conjugation is defined by $[a_l^{(+)}]^\dagger = a_l^{(-)}$, etc.

The generators of the conformal group can be computed as in (3.18) to (3.21). We find, with $\eta^\mu = (^1_i)$

$$D = -i \sum_l (l + \tfrac{1}{2})(a_l^{(+)}a_l^{(-)} + b_l^{(+)}b_l^{(-)} + A_l^{(+)}A_l^{(-)} + B_l^{(+)}B_l^{(-)}), \tag{4.45}$$

$$P^\mu = -i \sum_l l[(a_l^{(+)}a_{l-1}^{(-)} + b_l^{(+)}b_{l-1}^{(-)})\eta^\mu + (A_l^{(+)}A_{l-1}^{(-)} + B_l^{(+)}B_{l-1}^{(-)})\eta^{*\mu}] , \tag{4.46}$$

$$K^\mu = -i \sum_l l[(a_{l-1}^{(+)}a_l^{(-)} + b_{l-1}^{(+)}b_l^{(-)})\eta^{*\mu} + (A_{l-1}^{(+)}A_l^{(-)} + B_{l-1}^{(+)}B_l^{(-)})\eta^\mu] , \tag{4.47}$$

$$M = \sum_l (l + \tfrac{1}{2})(a_l^{(+)}a_l^{(-)} + b_l^{(+)}b_l^{(-)} - A_l^{(+)}A_l^{(-)} - B_l^{(+)}B_l^{(-)}) . \tag{4.48}$$

Since we are in two dimensions, we have set $M^{\mu\nu} = \epsilon^{\mu\nu}M$. It is easy to verify the conformal algebra with the help of (4.44). Also the generators (4.45) satisfy the proper commutation relations with the fields. Notice that

$$(D)^\dagger = -D, \quad M^\dagger = M, \quad (P^\mu)^\dagger = -K^\mu, \quad (K^\mu)^\dagger = -P^\mu . \tag{4.49}$$

The "single-particle" states are of four kinds: $a_l^\dagger|0\rangle$, $b_l^\dagger|0\rangle$, $A_l^\dagger|0\rangle$, $B_l^\dagger|0\rangle$. It is clear from (4.44) and (4.45) that they are all eigenstates of $\Delta = iD$ with eigenvalue $l + \frac{1}{2}$. Hence the "single-particle" degeneracy is 4-fold. The level density can be computed as in Sec. III E. We define

$$F(\beta) = \mathrm{Tr} \exp(-\beta\Delta) . \tag{4.50}$$

An analysis completely analogous to that given in Appendix B shows that

$$\ln F(\beta) = 4 \sum_{l=0}^{\infty} \ln[1 + e^{-\beta(l+1/2)}]$$

$$= 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{1}{e^{\beta n/2} - e^{-\beta n/2}} . \tag{4.51}$$

(The principal difference between the fermion and boson case is that in the former the Pauli principle dictates that $\mathrm{Tr}\, x^{a^\dagger a} = 1 + x$.) For small β (4.51) becomes

$$\ln F(\beta) \to \frac{4}{\beta}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{\pi^2}{3\beta}\,. \tag{4.52}$$

Since this is identical with the 2-dimensional boson result, Eqs. (3.62) and (3.64), we conclude that the level density $g(N)$ corresponding to d_N, a fixed eigenvalue of Δ, is for large N

$$g(N) \to \exp\left(\frac{2\pi}{\sqrt{3}}d_N\right)\,. \tag{4.53}$$

The Virasoro charges Q_n are given by

$$\begin{aligned}
&Q_n = \oint dz\, i(z^{1-n})\Theta^{++}(z)\,,\\
&Q_0 = \sum_{l=0}^{\infty}(l+\tfrac{1}{2})(a_l^{(+)}a_l^{(-)} + b_l^{(+)}b_l^{(-)})\,,\\
&Q_n = \sum_{l=0}^{\infty}(l+\tfrac{1}{2}n+\tfrac{1}{2})(a_{l+n}^{(+)}a_l^{(-)} + b_{l+n}^{(+)}b_l^{(-)}) + \sum_{l=0}^{n-1}(l-\tfrac{1}{2}n+\tfrac{1}{2})(b_l^{(+)}a_{n-1-l}^{(+)})\,,\quad n\geq 1\\
&Q_{-n} = (Q_n)^\dagger\,,\quad n\geq 1\,.
\end{aligned} \tag{4.54}$$

The well-known phenomenon on which we commented before is exhibited explicitly: For $|n|>1$, Q_n does not annihilate the vacuum. One may verify that the charges generate the transformation (4.35)

$$i[Q_n, \psi(x)] = \delta_n\psi(x)\,.$$

The commutator of the charges can be computed explicitly from (4.44) and (4.54). We prefer, however, to compute it by the general method described in Sec. IV B and in Appendix C.

Consider the covariant vacuum expectation value of the ordered product of two energy-momentum tensors

$$\langle 0|R^*\Theta^{\mu\nu}(x)\Theta^{\alpha\beta}(y)|0\rangle\,. \tag{4.55a}$$

By Wick's theorem this is

$$\begin{aligned}
\langle 0|R^*\Theta^{\mu\nu}(x)\Theta^{\alpha\beta}(y)|0\rangle &= \tfrac{1}{16}\,\mathrm{Tr}[\gamma^\mu S(x,y)\gamma^\alpha\partial_x^\nu\partial_x^\beta S(x,y) + \gamma^\alpha S(x,y)\gamma^\mu\partial_x^\nu\partial_x^\beta S(x,y)\\
&\quad - \gamma^\mu\partial_x^\nu S(x,y)\gamma^\alpha\partial_x^\beta S(x,y) - \gamma^\mu\partial_x^\beta S(x,y)\gamma^\alpha\partial_x^\nu S(x,y)]\\
&\quad + [(\mu\nu\alpha\beta) \to (\mu\nu\beta\alpha),\ (\nu\mu\alpha\beta),\ (\nu\mu\beta\alpha)]\\
&= \frac{1}{12\pi}\partial^\mu\partial^\nu\partial^\alpha\partial^\beta\left[-\frac{1}{4\pi}\ln(x-y)^2\right] + \text{local terms}.
\end{aligned} \tag{4.55b}$$

By inspection, we see that a defined by (4.29) is

$$a = 1/12\pi\,.$$

Consequently the c number in the Virasoro algebra is $-\delta_{n+m,0}(n^3-n)/12$.

This number of course also emerges if the computation is performed directly from (4.54). However the advantage of the present method is that it allows the computation of the c number in an "interacting" field theory, the Thirring model. The calculation has been performed by Georgi,[24] who finds

$$a = \frac{1}{12\pi}\,. \tag{4.57}$$

V. CONCLUSION

The aspects of this investigation that we find most interesting are the following: The development of a manifestly covariant quantization procedure which can replace the conventional noncovariant method provides a wealth of covariant canonical commutators. It will be interesting to examine current commutators in this context and to see whether previous successes of equal-time and lightlike current algebra can be extended.

We have given an operator basis for Euclidean quantum field theory. The quantization procedure follows closely the method of solving partial differential equations of the Laplacian type in Euclidean space, where the initial-value data are specified on a sphere, rather than on a plane. Our approach gives special prominence to the dilatation operator. It makes contact with the explicitly conformally covariant formalism of Johnson and Adler.[12] Furthermore it offers the intriguing possibility of studying anomalies of scale invariance as modifications of the dilatation propagator.

A covariant thermodynamics has put us closer to the goal of deriving the Hagedorn spectrum from first principles. What still is missing is an *a priori* justification of the use of 2-dimensional field theory and of the identification of mass squared with the dilatation eigenvalue. Nevertheless our approach to field theory in *Euclidean* space offers an attractive alternative to the conventional field-theoretic formulations of the dual resonance model.

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APPENDIX A

In this appendix we discuss in detail some of the formulas and manipulations occurring in 4-dimensional polar coordinates. These coordinates are defined by

$$r^2 = x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2, \tag{A1a}$$

$$\alpha^\mu = x^\mu / r, \tag{A1b}$$

$$\begin{aligned} \alpha^1 &= \cos\theta, \\ \alpha^2 &= \sin\theta\cos\psi, \\ \alpha^3 &= \sin\theta\sin\psi\cos\phi, \\ \alpha^4 &= \sin\theta\sin\psi\sin\phi. \end{aligned} \tag{A1c}$$

The 4-dimensional phase space is

$$\int d^4x = \int_0^\infty r^3 dr \int d\alpha = \int_0^\infty r^3 dr \int_0^\pi \sin^2\theta\, d\theta \int_0^\pi \sin\psi\, d\psi \int_0^{2\pi} d\phi. \tag{A2}$$

Notice that the angular volume is $2\pi^2$.

The d'Alembert and Dirac differential operators in Euclidean space can be transformed into a form appropriate to polar coordinates. To this end, we introduce the four-dimensional angular derivative operator

$$l^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \tag{A3a}$$

and its norm

$$L^2 = \tfrac{1}{2} l_{\mu\nu} l^{\mu\nu}. \tag{A3b}$$

This operator depends only on the angles (θ, ψ, ϕ) and not on r.

$$L^2 = -r^2\Box + \left(x^\mu \frac{\partial}{\partial x^\mu} + 2\right)\left(x^\nu \frac{\partial}{\partial x^\nu}\right). \tag{A4}$$

Using now the fact that the dilatation derivative operator

$$x^\mu \frac{\partial}{\partial x^\mu} = r\frac{\partial}{\partial r} \tag{A5}$$

depends only on r, we can write

$$\begin{aligned} r^2\Box &= -(L^2+1) + \frac{1}{r}\left(r\frac{d}{dr}\right)^2 r \\ &= -(L^2+1) + \frac{1}{r}\left(\frac{d}{d\ln r}\right)^2 r. \end{aligned} \tag{A6}$$

Let us now define

$$\chi(r, \alpha) = r\phi(x). \tag{A7}$$

Consequently the d'Alembert equation

$$\Box\phi(x) = 0 \tag{A8}$$

becomes

$$\frac{1}{r^3}\left[\left(\frac{d}{d\ln r}\right)^2 - (L^2+1)\right]\chi(r, \alpha) = 0. \tag{A9}$$

It will be useful to have the above formulas in an arbitrary number of dimensions δ. The analog of (A4) is

$$L^2 = -r^2\Box + \left(x^\mu \frac{\partial}{\partial x^\mu} + \delta - 2\right)\left(x^\nu \frac{\partial}{\partial x^\nu}\right). \tag{A10}$$

Equation (A10) may be rewritten as

$$r^2\Box = -\left[L^2 + \left(\frac{\delta-2}{2}\right)^2\right] + \frac{1}{r^{(\delta-2)/2}}\left(\frac{d}{d\ln r}\right)^2 r^{(\delta-2)/2}. \tag{A11}$$

We define the dimensionless field $\chi(r, \alpha)$ in δ dimensions by

$$\chi(r, \alpha) = r^{(\delta-2)/2}\phi(x). \tag{A12}$$

Hence the d'Alembert equation becomes

$$\left\{\left(\frac{1}{r^{(\delta+2)/2}} \quad \frac{d}{d\ln r}\right)^2 - \left[L^2 + \left(\frac{\delta-2}{2}\right)^2\right]\right\}\chi(r, \alpha) = 0. \tag{A13}$$

Now we devote our attention to the Dirac operator (in 4 dimensions). In analogy to (A3), we introduce the spin operator

$$\sigma^{\mu\nu} = \tfrac{1}{2} i[\gamma^\mu, \gamma^\nu]. \tag{A14}$$

The Dirac matrices are Euclidean, i.e.,

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}. \tag{A15}$$

From (A3) we have

$$\tfrac{1}{2}\sigma^{\mu\nu} l_{\mu\nu} = x^\mu \frac{\partial}{\partial x^\mu} - (\gamma^\mu x_\mu)\left(\gamma^\nu \frac{\partial}{\partial x^\nu}\right). \tag{A16}$$

The Dirac field is redefined by

$$\xi(r, \alpha) = r^{3/2}\psi(x) \tag{A17}$$

and the Dirac equation

$$\gamma \cdot \partial\psi(x) = 0 \tag{A18}$$

is equivalent to

$$\left(r\frac{\partial}{\partial r} - \tfrac{3}{2} - \tfrac{1}{2}\sigma^{\mu\nu} l_{\mu\nu}\right)\xi(r, \alpha) = 0. \tag{A19}$$

Let us use the previous formulas to deduce the field (anti) commutators from the ordered products. For the scalar field we have

$$\langle 0|R\phi(x_1)\phi(x_2)|0\rangle = \frac{1}{4\pi^2}\frac{1}{(x_1 - x_2)^2}. \tag{A20}$$

Applying the d'Alembert operator, we find

$$\Box_1\langle 0|R\phi(x_1)\phi(x_2)|0\rangle = -\delta^4(x_1 - x_2). \tag{A21a}$$

Reexpressing the left-hand side in terms of $\chi(r, \alpha)$ and (A6), we deduce

$$\begin{aligned}\left[\left(\frac{d}{d\ln r_1}\right)^2 - (L_1^2 + 1)\right]\langle 0|R\chi(r_1, \alpha_1)\chi(r_2, \alpha_2)|0\rangle &= -r_1^4\delta^4(x_1 - x_2)\\ &= -\delta(\ln r_1 - \ln r_2)\delta^3(\alpha_1 - \alpha_2).\end{aligned} \tag{A21b}$$

With (A9) and the definition of an R-ordered product, (A21b) implies

$$\langle 0|[\chi(r_2, \alpha_1), \chi(r_2, \alpha_2)]|0\rangle\delta'(\ln r_1 - \ln r_2) + \langle 0|[\dot\chi(r_2, \alpha_1), \chi(r_2, \alpha_2)]|0\rangle\delta(\ln r_1 - \ln r_2) = -\delta(\ln r_1 - \ln r_2)\delta^3(\alpha_1 - \alpha_2),$$

$$\dot\chi(r, \alpha) = \frac{d}{d\ln r}\chi(r, \alpha), \tag{A21c}$$

The conclusion is

$$\langle 0|[\chi(r, \alpha_1), \chi(r, \alpha_2)]|0\rangle = 0, \tag{A22a}$$

$$\langle 0|[\dot\chi(r, \alpha_1), \chi(r, \alpha_2)]|0\rangle = -\delta^3(\alpha_1 - \alpha_2), \tag{A22b}$$

$$\langle 0|[\dot\chi(r, \alpha_1), \dot\chi(r, \alpha_2)]|0\rangle = 0, \tag{A22c}$$

where the last equation above follows from the previous two by differentiation of (A22b) with respect to $\ln r$.

For the spin-$\frac{1}{2}$ field, the Green's function is

$$\langle 0|R\psi(x_1)\bar\psi(x_2)|0\rangle = -\gamma\cdot\partial\frac{1}{4\pi^2}\frac{1}{(x_1 - x_2)^2}. \tag{A23}$$

Again we find

$$\gamma\cdot\partial_1\langle 0|R\psi(x_1)\bar\psi(x_2)|0\rangle = \delta^4(x_1 - x_2). \tag{A24a}$$

Upon multiplication by $\gamma\cdot x_1$ and use of (A16), the above is reexpressed as

$$\begin{aligned}\left(\frac{d}{d\ln r_1} - \tfrac{1}{2}\sigma_{\mu\nu} l_1^{\mu\nu}\right)\langle 0|R\psi(x_1)\bar\psi(x_2)|0\rangle &= \gamma\cdot x_1\delta^4(x_1 - x_2)\\ &= \frac{\gamma\cdot\alpha_1}{r_1^3}\delta(\ln r_1 - \ln r_2)\delta^3(\alpha_1 - \alpha_2).\end{aligned} \tag{A24b}$$

With (A17) and the equation of motion (A18), as well as the definition of the R product, (A24b) implies

$$\langle 0|\{\xi(r, \alpha_1), \bar\xi(r, \alpha_2)\}|0\rangle = \gamma\cdot\alpha_1\delta^3(\alpha_1 - \alpha_2). \tag{A25}$$

Next we discuss the separation of variables for the boson case. The differential operator L^2 has eigenfunction $Y_{lnm}(\alpha)$

$$L^2 Y_{lnm}(\alpha) = l(l+2)Y_{lnm}(\alpha). \tag{A26}$$

The harmonics are defined as follows in terms of Gegenbauer polynomials

$$\begin{aligned}Y_{lnm}(\alpha) &= Y_{lnm}(\theta, \phi, \psi)\\ &= N_{lnm}e^{im\phi}(\sin\theta)^n C_{l-n}^{n+1}(\cos\theta)(\sin\psi)^m C_{n-m}^{m+1/2}(\cos\psi).\end{aligned} \tag{A27a}$$

The normalization constant N_{lnm} is chosen to be

$$(N_{lnm})^{-2} = 2\pi E_0(l,n)E_1(n,m)\,,$$
$$E_k(l,n) = \frac{\pi 2^{k-2n}\,\Gamma(l+n-k+2)}{(2l+2-k)(l-n)!\,[\Gamma(n+1-\frac{1}{2}k)]^2}\;. \tag{A27b}$$

With this normalization the following completeness and orthonormality relations are true:

$$\sum_{n=0}^{l}\sum_{m=-n}^{n} Y^*_{lnm}(\alpha)Y_{lnm}(\alpha') = \frac{l+1}{2\pi^2}C^1_l(\alpha\cdot\alpha')\,, \tag{A28a}$$

$$\sum_{l=0}^{\infty}\sum_{n=0}^{l}\sum_{m=-n}^{n} Y^*_{lnm}(\alpha)Y_{lnm}(\alpha') = \delta^3(\alpha-\alpha')$$
$$= \frac{\delta(\theta-\theta')\delta(\psi-\psi')\delta(\phi-\phi')}{\sin^2\theta\,\sin\psi}\;, \tag{A28b}$$

$$\int_0^{\pi} d\theta\,\sin^2\theta\int_0^{\pi} d\psi\,\sin\psi\int_0^{2\pi} d\phi\, Y^*_{lnm}(\alpha)Y_{lnm}(\alpha) = \delta_{ll'}\delta_{nn'}\delta_{mm'}\;. \tag{A29}$$

Also our phase convention is such that

$$Y^*_{lnm}(\alpha) = (-1)^m Y_{ln-m}(\alpha)\,. \tag{A30}$$

The general solution to (A9) is written as

$$\chi(r,\alpha) = \sum_{lnm} g_{lnm}(r)Y_{lnm}(\alpha)\,,$$
$$g_{lnm}(r) = \int d\alpha\, Y^*_{lnm}(\alpha)\chi(r,\alpha)\,. \tag{A31}$$

Thus $g_{lnm}(r)$ satisfies

$$\left(\frac{d}{d\ln r}\right)^2 g_{lnm}(r) = (l+1)^2 g_{lnm}(r)\,. \tag{A32a}$$

This has the solutions

$$g_{lnm}(r) = g^{(+)}_{lnm}\,r^{l+1} + g^{(-)}_{lnm}\,r^{-(l+1)}\;. \tag{A32b}$$

Equation (A31) may now be written as

$$\chi(r,\alpha) = \sum_{lnm} g^{(-)}_{lnm}\,r^{-(l+1)}(-1)^m Y^*_{ln-m}(\alpha) + g^{(+)}_{lnm}\,r^{l+1}Y_{lnm}(\alpha)\,. \tag{A33}$$

This is equivalent to (3.13) in the text.

The δ-dimensional analog of (A26) to (A33) is the following. The eigenvalues of L^2 are $l(l+\delta-2)$. Denoting the eigenfunctions of L^2 by $Y_{l\mu}(\alpha)$ where μ stands for the ensemble of "magnetic" quantum numbers, we have

$$L^2 Y_{l\mu}(\alpha) = l(l+\delta-2)Y_{l\mu}(\alpha)\,. \tag{A34}$$

The harmonics may be chosen orthonormal,

$$\int d\alpha\, Y^*_{l\mu}(\alpha)Y_{l'\mu'}(\alpha) = \delta_{ll'}\delta_{\mu\mu'}\;. \tag{A35}$$

The field $\chi(r,\alpha)$, defined in (A12), can be expanded as

$$\chi(r,\alpha) = \sum_{l=0}^{\infty}\sum_{\mu} g_{r\mu}(r)Y_{l\mu}(\alpha) \tag{A36}$$

and according to (A13), (A34), and (A35), $g_{l\mu}(r)$ satisfies

$$\left(\frac{d}{d\ln r}\right)^2 g_{l\mu}(r) = \left(l+\frac{\delta-2}{2}\right)^2 g_{l\mu}(r)\,. \tag{A37}$$

Hence (A36) becomes, for $\delta > 2$

$$\chi(r, \alpha) = \sum_{l\mu} [g_{l\mu}^{(-)} r^{-[l+(\delta-2)/2]} + g_{l\mu}^{(+)} r^{l+(\delta-2)/2}] Y_{l\mu}(\alpha) . \tag{A38}$$

The explicit form of the harmonics may be found in the literature.[25] For $\delta = 2$, the $l = 0$ part of (A37) has a constant and logarithmic solution. These cause infrared singularities[23] which shall be discussed in a separate publication.[16]

APPENDIX B

We present some details of the calculations of the level degeneracy in Sec. III E, and derive formula (3.67) by an analytic method.

1.

That the dilatation generator has the form (3.52) for $\delta > 2$

$$\Delta = iD = \sum_{l=0}^{\infty} \sum_{\mu} \left(l + \frac{\delta - 2}{2}\right) a_{l\mu}^{(+)} a_{l\mu}^{(-)} \tag{B1}$$

is established as follows. It is obvious from (A38) that the scale dimension of $a_{l\mu}^{(\pm)}$ is $\pm[l + (\delta - 2)/2]$ where the $a_{l\mu}^{(\pm)}$ are the creation and annihilation operators relevant to the expansion of the dimensionless scale field $\chi(r, \alpha)$.

$$i[D, a_{l\mu}^{(\pm)}] = \pm \left(l + \frac{\delta - 2}{2}\right) a_{l\mu}^{(\pm)} . \tag{B2}$$

Since we assume that $a_{l\mu}^{(\pm)}$ satisfy conventional commutation relations, we are led to (B1) as the form for Δ.

For 2 dimensions, the constant and logarithmic solution to the equation $\Box\phi = 0$ gives rise to additional contributions to Δ at $l = 0$. These terms cause well-known infrared difficulties[23] and will be discussed in a future paper.[16] For present purposes we shall ignore this, and use Δ as given by (B1) even for $\delta = 2$. This means that we are counting only states of the form $a_{l_1}^\dagger \cdots a_{l_n}^\dagger |0\rangle$, $l_i > 0$. For 2-dimensional spin-$\frac{1}{2}$ fields this problem does not exist.

2.

Next we derive the expression (3.53) for $\alpha_\delta(l)$, the number of independent creation operators corresponding to a single value l. This is of course equal to the number of independent spherical functions $Y_{l\mu}$ corresponding to a fixed value of l. One can derive the following recurrence relation for $\alpha_\delta(l)$:

$$\alpha_\delta(l) = \sum_{l'=0}^{l} \alpha_{\delta-1}(l') . \tag{B3}$$

This follows from decomposing the δ-dimensional harmonics in terms of $\delta - 1$ harmonics, and observing that each $\delta - 1$ representation with $l' \leq l$ appears only once.

To solve (B3), define

$$\begin{aligned} A_\delta(x) &= \sum_{l=0}^{\infty} \alpha_\delta(l) x^l \\ &= \sum_{l=0}^{\infty} x^l \sum_{l'=0}^{l} \alpha_{\delta-1}(l') \\ &= \sum_{l'=0}^{\infty} \alpha_{\delta-1}(l') \sum_{l=l'}^{\infty} x^l \\ &= \sum_{l'=0}^{\infty} \alpha_{\delta-1}(l') \frac{x^{l'}}{1-x} \\ &= \frac{1}{1-x} A_{\delta-1}(x) . \end{aligned} \tag{B4}$$

Hence

$$A_\delta(x) = (1-x)^{3-\delta} A_3(x) . \tag{B5}$$

For $\delta = 3$, we have the well-known result

$$\begin{aligned} \alpha_3(l) &= 2l + 1 , \\ A_3(x) &= \sum_{l=0}^{\infty} (2l+1) x^l \\ &= \frac{1+x}{(1-x)^2} \end{aligned} \tag{B6}$$

and $A_\delta(x)$ is determined for $\delta \geq 3$

$$A_\delta(x) = (1+x)(1-x)^{1-\delta} . \tag{B7}$$

When this is expanded in powers of x we find from (B4)

$$\alpha_\delta(l) = \frac{2l + \delta - 2}{l + \delta - 2} \; \frac{(l + \delta - 2)!}{l!\,(\delta - 2)!} . \tag{B8a}$$

For $\delta = 2$, the degeneracy for $l > 0$ is 2. The case $l = 0$ is singular. We can accept (B8a) for $\delta = 2$, provided $\alpha_2(0)$ is set equal to 1. For large l, $\alpha_\delta(l)$ asymptotically becomes

$$\alpha_\delta(l) \to \frac{2l^{\delta-2}}{(\delta-2)!} . \tag{B8b}$$

3.

The formula (3.57) for $F(\beta)$ is deduced from (3.52) and (3.55) by a familiar argument. Inserting the expression for Δ into the definition of $F(\beta)$ gives

$$F(\beta) = \mathrm{Tr}\exp\left[-\beta\sum_{l\mu}\left(l+\frac{\delta-2}{2}\right)a_{l\mu}^{(+)}a_{l\mu}^{(-)}\right]$$
$$= \mathrm{Tr}\prod_{l\mu}\exp\left[-\beta\left(l+\frac{\delta-2}{2}\right)a_{l\mu}^{(+)}a_{l\mu}^{(-)}\right]. \quad \text{(B9a)}$$

Since the operator $a_{l\mu}^{(+)}a_{l\mu}^{(-)}$ does not connect states with different $l\mu$ quantum numbers (B9a) is equal to

$$F(\beta) = \prod_{l\mu}\mathrm{Tr}\exp\left[-\beta\left(l+\frac{\delta-2}{2}\right)a_{l\mu}^{(+)}a_{l\mu}^{(-)}\right]. \quad \text{(B9b)}$$

Using the identity

$$\mathrm{Tr}x^{a^\dagger a} = \frac{1}{1-x}$$

and the fact that states with the same l and different μ are $\alpha_\delta(l)$-fold degenerate, we obtain (3.57).

$$F(\beta) = \prod_{l=0}^{\infty}(1-e^{-\beta[l+(\delta-2)/2]})^{-\alpha_\delta(l)},$$
$$\ln F(\beta) = -\sum_{l=0}^{\infty}\alpha_\delta(l)\ln(1-e^{-\beta[l+(\delta-2)/2]}). \quad \text{(B10)}$$

(For $\delta=2$, the series begins with $l=1$.)

Equation (B10) may be cast into another more convenient form (3.58). We expand the logarithm in (B10)

$$\ln F(\beta) = \sum_{n=1}^{\infty}\frac{1}{n}e^{-\beta n(\delta-2)/2}\sum_{l=0}^{\infty}\alpha_\delta(l)e^{-\beta nl}. \quad \text{(B11a)}$$

From (B4) and (B7) the sum in l may be evaluated

$$\ln F(\beta) = \sum_{n=1}^{\infty}\frac{1}{n}e^{-\beta n(\delta-2)/2}\frac{1+e^{-\beta n}}{(1-e^{-\beta n})^{\delta-1}}$$
$$= \sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{e^{\beta n/2}+e^{-\beta n/2}}{(e^{\beta n/2}-e^{-\beta n/2})^{\delta-1}}\right]. \quad \text{(B11b)}$$

(When $\delta=2$, the term in the square brackets is decreased by 1, since the sum in l begins with $l=1$, rather than $l=0$.)

4.

Finally we compute the asymptotic formula for $g_\delta(N)$. From (3.56)

$$F(\beta) = \sum_{N=0}^{\infty}g_\delta(N)e^{-\beta d_N} \quad \text{(B12)}$$

it is seen that

$$g_\delta(N) = \frac{1}{2\pi i}\oint\frac{dz}{z^{1+d_N}}F(-\ln z), \quad \text{(B13)}$$

where the contour is a circle about the origin with radius $|z|<1$. We will show that the estimate of the asymptotic form for $g_\delta(N)$ is obtained by choosing the radius to be arbitrarily close to $|z|=1$, and that the leading contribution comes from the point $z\to 1$, i.e., $-\ln z\to 0$. Thus we may use the limiting form for $F(\beta)$ near $\beta=0$, derived in the text

$$\ln F(\beta)\to C(\delta)\beta^{1-\delta},$$
$$C(\delta) = 2\sum_{n=1}^{\infty}n^{-\delta}. \quad \text{(B14)}$$

Let us change variables in the integral (B13) by setting $\beta=-\ln z$. Defining the radius of the circle as $|z|=e^{-\beta_0}$, and using (B14), we get

$$g_\delta(N)\to\frac{1}{2\pi i}\int_{\beta_0-i\pi}^{\beta_0+i\pi}d\beta\,\exp[d_N\beta+C(\delta)\beta^{1-\delta}]. \quad \text{(B15)}$$

Notice that β_0 is free to be any positive number – we shall use this freedom later. Making another change of variable,

$$\beta = \left[\frac{C(\delta)}{d_N}\right]^{1/\delta}, \quad \text{(B16)}$$

we transform (B15) to

$$g_\delta(N)\to\frac{1}{2\pi i}\left[\frac{C(\delta)}{d_N}\right]^{1/\delta}\int_{y_0-i\nu}^{y_0+i\nu}dy\,e^{a_N h(y)}, \quad \text{(B17)}$$

where y_0 is an arbitrary positive number given by $y_0=\beta_0[d_N/C(\delta)]^{1/\delta}$ and

$$\nu = \pi\left[\frac{d_N}{C(\delta)}\right]^{1/\delta},$$
$$a_N = [C(\delta)]^{1/\delta}d_N^{\,1-1/\delta},$$
$$h(y) = y+y^{1-\delta}.$$

Notice that as $d_N\to\infty$, ν, and a_N approach infinity, and (B17) may be evaluated by the saddle-point method.

The location of the saddle point of $h(y)$ is given by

$$y_s = (\delta-1)^{1/\delta} \quad \text{(B18)}$$

and we chose the arbitrary constant y_0 to be equal to y_s, so that the contour passes through the saddle point. Since only the region of y near $y_s=y_0$ contributes significantly to (B17) we see from (B16) that β *does* go to zero as $d_N\to\infty$; thus our use of the approximate formula for $F(\beta)$, (B14), is justified. Moreover the only way that y_0 can remain constant and equal to y_s as $d_N\to\infty$ is if β_0 goes to zero.

We may expand $h(y)$ about y_0.

$$h(y) = h(y_0)+\tfrac{1}{2}h''(y_0)(y-y_0)^2,$$
$$h(y_0) = \delta(\delta-1)^{-1+1/\delta},$$
$$h''(y_0) = \delta(\delta-1)^{-1/\delta}. \quad \text{(B19)}$$

Introducing yet another new variable t

$$y-y_0 = it \quad \text{(B20)}$$

(B17) becomes

$$g_\delta(N) \to \frac{1}{2\pi}\left[\frac{C(\delta)}{d_N}\right]^{1/\delta} e^{a_N h(y_0)} \int_{-\nu}^{\nu} dt\, e^{-ah''(y_0)t^2/2}. \tag{B21}$$

Finally setting ν to ∞, we get

$$g_\delta(N) \to \left[\frac{C(\delta)}{d_N}\right]^{1/\delta} e^{a_N h(y_0)} [2\pi a_N h''(y_0)]^{-1/2}. \tag{B22}$$

Thus

$$\ln g_\delta(N) \to a_N h(y_0). \tag{B23}$$

Only the first, dominant term is significant. The remaining terms are unreliable since the errors made in replacing the exact expression (B10) for $F(\beta)$ by the approximation (B14) are of the same magnitude. The final result is

$$\ln g_\delta(N) \to \delta(\delta-1)^{-1+1/\delta}[C(\delta)]^{1/\delta} d_N^{\,1-1/\delta}. \tag{B24}$$

This is precisely the expression obtained by the thermodynamical method, (3.67).

The above analysis is patterned after the classical paper of Hardy and Ramanujan,[26] where the asymptotic formula for the partitions of an integer is obtained. The reader desiring greater mathematical rigor than in our analysis is referred to that work. For $\delta=2$, our formula is in agreement with these authors. Specifically we find $\ln g_2(N) = (2\pi/\sqrt{3})d_N^{1/2}$ while Hardy and Ramanujan obtain the factor $2\pi/\sqrt{6}$. The discrepancy of $\sqrt{2}$ arises from the fact that for us $F(-\ln x)$ with $\delta=2$ is given by $\prod_{l=1}(1-x^l)^{-2}$, while Hardy and Ramanujan consider $\prod_{l=1}(1-x^l)^{-1}$.

APPENDIX C

Computations relevant to the discussion of 2-dimensional fields in Sec. IV are given in detail here.

1.

We investigate under what conditions a Lagrangian $\mathcal{L}$ leads to a theory which is invariant under the transformation

$$\delta x^\mu = f^\mu(x). \tag{C1}$$

For generality the problem will be solved in δ space-time dimensions. It is required that

$$f_{\mu\nu} = \partial_\mu f_\nu + \partial_\nu f_\mu - \frac{2}{\delta} g_{\mu\nu}\partial_\alpha f^\alpha = 0. \tag{C2}$$

We shall assume that the theory is invariant under the conformal group. This places the following constraints on $\mathcal{L}$.[9]

(1) translation invariance

$$\partial_\mu \mathcal{L} = \Pi^\alpha \partial_\mu \partial_\alpha \phi + \Pi \partial_\mu \phi. \tag{C3a}$$

(2) Lorentz invariance

$$\Pi^\alpha \Sigma^{\mu\nu}\partial_\alpha\phi + \Pi\Sigma^{\mu\nu}\phi = \Pi^\nu\partial^\mu\phi - \Pi^\mu\partial^\nu\phi, \tag{C3b}$$

(3) dilatation invariance

$$\mathcal{L} = \frac{1}{\delta}\Pi^\alpha(d+1)\partial_\alpha\phi + \frac{1}{\delta}\Pi d\phi, \tag{C3c}$$

(4) conformal invariance

$$V^\mu \equiv \Pi_\alpha[g^{\alpha\mu}d - \Sigma^{\alpha\mu}]\phi = \partial_\alpha\sigma^{\alpha\mu}. \tag{C3d}$$

Here

$$\Pi^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}, \quad \Pi = \frac{\delta\mathcal{L}}{\delta\phi}.$$

V^μ is called the field virial, d is the field scale dimension and $\Sigma^{\mu\nu}$ is the spin matrix. The meaning of (C3d) is that for conformal invariance the field virial must be a total divergence.

Before attacking our problem it must be decided how the field transforms. We postulate

$$\delta_f\phi = f^\mu\partial_\mu\phi + \partial_\mu f_\alpha\left(g^{\alpha\mu}\frac{d}{\delta} - \tfrac{1}{2}\Sigma^{\alpha\mu}\right)\phi. \tag{C4}$$

This formula reduces to the conventional transformation law when f^μ corresponds to the conformal group. We are now in a position to study the change in the Lagrangian

$$\delta_f\mathcal{L} = \Pi^\mu\partial_\mu\delta_f\phi + \Pi\delta_f\phi. \tag{C5}$$

Substituting (C4) into (C5) and repeatedly using the properties (C3) we get

$$\delta_f\mathcal{L} = \partial_\mu\left(f^\mu\mathcal{L} + \frac{1}{\delta}\sigma^{\mu\nu}\partial_\nu\partial_\alpha f^\alpha\right) - \frac{1}{\delta}\sigma^{\mu\nu}\partial_\mu\partial_\nu\partial_\alpha f^\alpha + \tfrac{1}{2}\Pi^\mu\partial^\nu\phi f_{\mu\nu} - \tfrac{1}{2}\Pi^\mu\Sigma^{\nu\alpha}\phi\partial_\alpha f_{\mu\nu}. \tag{C6}$$

When f^μ satisfies (C2), that is $f^{\mu\nu}$ vanishes, (C6) becomes

$$\delta_f\mathcal{L} = \partial_\mu\left(f^\mu\mathcal{L} + \frac{1}{\delta}\sigma^{\mu\nu}\partial_\nu\partial_\alpha f^\alpha\right) - \frac{1}{\delta}\sigma^{\mu\nu}\partial_\mu\partial_\nu\partial_\alpha f^\alpha. \tag{C7}$$

The condition that the transformation (C1) and (C4) be a symmetry operation is

$$\sigma^{\mu\nu}\partial_\mu\partial_\nu\partial_\alpha f^\alpha = 0. \tag{C8}$$

Equation (C8) may be achieved in two ways. For $\delta > 2$, the only f^μ satisfying (C2) is the quadratic polynomial corresponding to the conformal group[27]; see Eq. (2.10) and (2.11).

$$f^\mu(x) = a^\mu + ax^\mu - 2\omega^{\mu\nu}x_\nu + 2x^\mu c^\nu x_\nu - c^\mu x^2.$$

Clearly (C8) is satisfied since $\partial_\mu\partial_\nu\partial_\alpha f^\alpha = 0$.

In 2 dimensions a wider class of functions satisfies (C2), and $\partial_\mu \partial_\nu \partial_\alpha f^\alpha$ need not be zero. Expanding the components we see that (C8) requires

$$\sigma^{++}\partial_+{}^3 f^+(x^+) + \sigma^{--}\partial_-{}^3 f^-(x^-) = 0\,,$$

where we have used the fact that the solution to (C2) has the property that f^+ (f^-) depends only on x^+ (x^-). Hence we have a symmetry for arbitrary f^+ and f^- in two dimensions if

$$\sigma^{\mu\nu} = g^{\mu\nu}\sigma + \epsilon^{\mu\nu}\bar{\sigma}\,. \tag{C9}$$

For the models I–IV considered in the text, (C9) is trivially satisfied since V^μ, hence $\sigma^{\mu\nu}$, vanishes identically.

For the model V of the text, the generally covariant string model of Eq. (4.5), the transformation (C1) is a symmetry operation even if $f^{\mu\nu}$ does not vanish, i.e., f^μ can be arbitrary. This is seen as follows. Consider $\delta_f \mathcal{L}$ for arbitrary f^μ, Eq. (C6). Since the string model involves spinless, dimensionless fields, $\Sigma^{\mu\nu}$, d, and the field virial V^μ are all zero. Thus (C6) reduces to

$$\delta_f \mathcal{L} = \partial_\mu (f^\mu \mathcal{L}) + \tfrac{1}{2}\Pi^\mu \partial^\nu \phi f_{\mu\nu}\,. \tag{C10}$$

From its definition $f^{\mu\nu}$ is traceless, i.e., the only nonvanishing components are f^{++} and f^{--}. On the other hand, direct evaluation in the string Lagrangian gives $\Pi^+\partial^+\phi = 0 = \Pi^-\partial^-\phi$. Consequently x^μ may be changed into *any* function of x, without affecting the dynamics of the string model – indeed it was the desire to have this freedom to reparametrize x^μ that led to the development of this model.[19]

When the symmetry is present, the conserved canonical current in δ dimensions is given by

$$\begin{aligned} J_f^\mu &= (\Pi^\mu \partial^\nu \phi - g^{\mu\nu}\mathcal{L}) f_\nu + \Pi^\mu \left(g^{\alpha\beta}\frac{d}{\delta} - \tfrac{1}{2}\Sigma^{\alpha\beta}\right)\phi\, \partial_\beta f_\alpha - \frac{1}{\delta}\sigma^{\mu\alpha}\partial_\alpha \partial_\beta f^\beta \\ &= \Theta_c^{\mu\alpha} f_\alpha + \Pi^\mu \left(g^{\alpha\beta}\frac{d}{\delta} - \tfrac{1}{2}\Sigma^{\alpha\beta}\right)\phi\, \partial_\beta f_\alpha - \frac{1}{\delta}\sigma^{\mu\alpha}\partial_\alpha \partial_\beta f^\beta\,. \end{aligned} \tag{C11}$$

We have introduced the canonical energy-momentum tensor $\Theta_c^{\mu\nu}$. It is useful to reexpress (C11) in terms of the symmetric Belinfante tensor $\Theta_B^{\mu\nu}$.

$$\begin{aligned} \Theta_B^{\mu\alpha} &= \Theta_c^{\mu\alpha} + \tfrac{1}{2}\partial_\beta X^{\beta\mu\alpha}, \\ X^{\beta\mu\alpha} &= \Pi^\beta \Sigma^{\mu\alpha}\phi - \Pi^\mu \Sigma^{\beta\alpha}\phi - \Pi^\alpha \Sigma^{\beta\mu}\phi. \end{aligned} \tag{C12}$$

The formula for J_f^μ is

$$J_f^\mu = \Theta_B^{\mu\alpha} f_\alpha + \tfrac{1}{2}X^{\beta\mu\alpha}\partial_\beta f_\alpha + \Pi^\mu \left(g^{\alpha\beta}\frac{d}{\delta} - \tfrac{1}{2}\Sigma^{\alpha\beta}\right)\phi\, \partial_\beta f_\alpha - \frac{1}{\delta}\sigma^{\mu\alpha}\partial_\alpha \partial_\beta f^\beta - \tfrac{1}{2}\partial_\beta (X^{\beta\mu\alpha} f_\alpha)\,. \tag{C13a}$$

The last term may be dropped since it is a superpotential (divergence of an antisymmetric tensor). The second and third terms on the right-hand side combine. Thus we are led to introduce the Belinfante current

$$\begin{aligned} J_f^\mu &= \Theta_B^{\mu\alpha} f_\alpha + \frac{1}{\delta}\partial_\alpha \sigma^{\alpha\mu}\partial_\beta f^\beta - \frac{1}{\delta}\sigma^{\mu\alpha}\partial_\alpha \partial_\beta f^\beta \\ &= \Theta_B^{\mu\alpha} f_\alpha + \frac{1}{\delta}\partial_\alpha \sigma_+^{\alpha\mu}\partial_\beta f^\beta - \frac{1}{\delta}\sigma_+^{\mu\alpha}\partial_\alpha \partial_\beta f^\beta\,, \end{aligned} \tag{C13b}$$

where in the second equality $\sigma_+^{\alpha\mu}$ represents the symmetric part of $\sigma^{\alpha\mu}$. The antisymmetric part leads to a superpotential and again has been dropped.

Finally we introduce the new improved energy-momentum tensor[9] $\Theta^{\mu\alpha}$ with yet another superpotential

$$\begin{aligned} \Theta^{\mu\alpha} &= \Theta_B^{\mu\alpha} + \frac{1}{\delta - 2}\partial_\lambda \partial_\rho X^{\lambda\rho\mu\alpha}, \\ X^{\lambda\rho\mu\alpha} &= \sigma_+^{\lambda\rho} g^{\mu\alpha} + \sigma_+^{\mu\alpha} g^{\lambda\rho} - \sigma_+^{\lambda\mu} g^{\alpha\rho} - \sigma_+^{\alpha\rho} g^{\mu\lambda} - \frac{1}{\delta - 1}(g^{\lambda\rho} g^{\mu\alpha} - g^{\lambda\mu} g^{\alpha\rho})\sigma_{+\nu}^{\nu}\,. \end{aligned} \tag{C14}$$

In terms of the new stress tensor, the current (C13) is, apart from superpotentials,

$$J_f^\mu = \Theta^{\mu\nu} f_\nu\,. \tag{C15}$$

Note that when $\sigma_+^{\mu\nu}$ is simply $g^{\mu\nu}\sigma$, as it is in 2 dimensions when the symmetry (C1) is present, see (C9), then a simpler formula is valid

$$\frac{1}{\delta-2}X^{\lambda\rho\mu\alpha}=\frac{\sigma}{\delta-1}(g^{\lambda\rho}g^{\mu\alpha}-g^{\rho\alpha}g^{\mu\lambda}),$$
$$\Theta^{\mu\alpha}=\Theta_B^{\mu\alpha}+\frac{1}{\delta-1}(\Box g^{\mu\alpha}-\partial^\mu\partial^\alpha)\sigma. \tag{C16}$$

2.

The Schwinger term in the equal-r commutator of Θ^{++} with itself is derived, and the c-number modification to the commutator of the charges Q_f is found.

We consider the R product of two energy-momentum tensors. According to (4.16), continued to Euclidean space, this Green's function is

$$\langle 0|R\Theta^{\mu\nu}(x)\Theta^{\alpha\beta}(x')|0\rangle=a\partial^\mu\partial^\nu\partial^\alpha\partial^\beta\left[-\frac{1}{4\pi}\ln(x-x')^2\right] \tag{C17}$$

apart from entirely local terms proportional to $\delta^2(\vec{x}-\vec{y})$ and gradients thereof which do not concern us here. In Minkowski space a is nonvanishing; it remains nonzero in Euclidean space since the Green's function in Euclidean space is a continuation of the Minkowski space quantity. Thus it follows that

$$\langle 0|R\Theta^{++}(x)\Theta^{++}(x')|0\rangle=\frac{a}{16}(\partial_1^x-i\partial_2^x)^4\left[-\frac{1}{4\pi}\ln(x-x')^2\right]+\rho(x,x'). \tag{C18}$$

Here the local terms have been represented by $\rho(x,x')$. Since $\Theta^{++}(x)$ depends only on x^1+ix^2,

$$(\partial_1+i\partial_2)\Theta^{++}(x)=e^{i\theta}\left(\partial_r+\frac{i}{r}\partial_\theta\right)\Theta^{++}(x)=0.$$

Consequently it is true that

$$\begin{aligned}e^{i\theta}\left(\partial_r+\frac{i}{r}\partial_\theta\right)\langle 0|R\Theta^{++}(x)\Theta^{++}(x')|0\rangle&=e^{i\theta}\delta(r-r')\langle 0|[\Theta^{++}(\chi),\Theta^{++}(x')]|0\rangle\\&=\frac{a}{16}(\partial_1^x-i\partial_2^x)^3\Box\left(-\frac{1}{4\pi}\ln(x-x')^2\right)+(\partial_1^x+i\partial_2^x)\rho(x,x')\\&=-\frac{a}{16}(\partial_1^x-i\partial_2^x)^3\delta^2(x-x')+(\partial_1^x+i\partial_2^x)\rho(x,x').\end{aligned} \tag{C19}$$

The commutator of the charges can now be evaluated

$$\begin{aligned}i[Q_f,Q_g]&=-i\int_0^{2\pi}d\theta d\theta' re^{i\theta}f(re^{i\theta})re^{i\theta'}g(re^{i\theta'})[\Theta^{++}(x),\Theta^{++}(x')]|_{r=r'}\\&=-i\int_0^{2\pi}d\theta' r'e^{i\theta'}g(r'e^{i\theta'})\int d^2x\,f(re^{i\theta})e^{i\theta}\delta(r-r')[\Theta^{++}(x),\Theta^{++}(x')]\\&=-i\int_0^{2\pi}d\theta' r'e^{i\theta'}g(r'e^{i\theta'})\int d^2x\,f(re^{i\theta})\\&\qquad\times\left\{\frac{e^{i\theta}}{ir}[e^{-2i\theta}\Theta^{++}(x)+e^{-2i\theta'}\Theta^{++}(x')]\partial_\theta\delta^2(x-x')-\frac{a}{16}(\partial_1^x-i\partial_2^x)^3\delta^2(x-x')+(\partial_1^x+i\partial_2^x)\rho(x,x')\right\}\\&=Q_h-\frac{ia}{16}\int_0^{2\pi}d\theta' r'e^{i\theta'}g(r'e^{i\theta'})(\partial_1-i\partial_2)^3f(r'e^{i\theta'})\\&\quad-i\int_0^{2\pi}d\theta' r'e^{i\theta'}g(r'e^{i\theta'})\int d^2x\,f(re^{i\theta})(\partial_1^x+i\partial_2^x)\rho(x,x').\end{aligned} \tag{C20a}$$

The last term in (C20a) is zero. This is seen as follows: Since $\rho(x,x')$ is local, one may integrate by parts in the x integration, and use

$$(\partial_1+i\partial_2)f(re^{i\theta})=0.$$

Therefore we find

$$i[Q_f, Q_g] = Q_h - \tfrac{1}{2} i\, a \int_0^{2\pi} d\theta\, re^{i\theta} g(re^{i\theta}) f'''(re^{i\theta})$$
$$= Q_h - \tfrac{1}{2} a \oint dz\, g(z) f'''(z)\,. \qquad \text{(C20b)}$$

This establishes (4.30).

Since (C20b) holds for arbitrary f and g, we may also deduce the complete commutator of stress-tensor components. We obtain

$$i[\Theta^{++}(x), \Theta^{++}(y)]|_{r=r'} = \frac{1}{r^2}\left[e^{-2i\theta}\Theta^{++}(x) + e^{-2i\theta'}\Theta^{++}(x')\right]\partial_\theta \delta(\theta-\theta') - \frac{a}{2r^4} e^{-i\theta}(e^{-i\theta'}\partial_{\theta'})^3 \delta(\theta-\theta')\,. \qquad \text{(C21)}$$

It is elementary to rearrange the Schwinger term to obtain agreement with (4.28).

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[1]Our metric convention is $g^{00} = -g^{ii} = 1$. Frequently we use light-cone components defined as $\pm = (0 \pm 3)/\sqrt{2}$. Note that $g^{++} = g^{--} = 0$, $g^{+-} = 1$, and for any vector A^μ it is true that $A^{\pm} = A_{\mp}$. When we pass to Euclidean space the indices on tensors will run from 1 to 4, and no distinction is made between upper and lower indices. Occasionally we shall use the $\pm$ notation in Euclidean space. This will always be explicitly defined in the text.

[2]P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949); J. B. Kogut and D. E. Soper, Phys. Rev. D 1, 2901 (1970).

[3]For a review see: R. Jackiw, in *Springer Tracts in Modern Physics,* edited by G. Höhler (Springer, Berlin, 1972), Vol. 62, p. 1, and V. De Alfaro, S. Fubini, G. Furlan, and C. Rossetti, *Currents in Hadron Physics,* (North-Holland, Amsterdam, 1973).

[4]This method of quantization was also considered by Dirac. See Ref. 2.

[5]The reason for this is the following: It will be seen that in our method the dilatation generator D is diagonalized. Hence, if D satisfies $i[D,P^\mu] = P^\mu$, and D is Hermitian, as it is in Minkowski space, P^μ cannot be an operator defined in the Hilbert space, since it shifts the eigenvalues of D by an imaginary quantity. In Euclidean space we avoid this problem since it will be seen that D becomes anti-Hermitian. The difficulty in Minkowski space is related to the well-known fact that matrix elements of the momentum operator cannot be evaluated in the SL(2, C) reduction of the Poincaré group; see for example, W. W. MacDowell and R. Roskies, J. Math. Phys. 13, 1585 (1972).

[6]Euclidean field theory has been previously studied by Schwinger and Symanzik. For a review and further references see: J. Schwinger, Proc. Nat. Acad. Sci. 44, 956 (1958), and *Mathematical Theory of Elementary Particles,* edited by R. Goodman and I. Segal (MIT Press, Cambridge, Mass., 1966). However, the approach of these two authors is quite different from ours. In particular the Euclidean field operator of Schwinger and Symanzik is a self-commuting operator which even in the absence of interactions does not satisfy a free-field equation. As will be seen below, our field in Euclidean space does not commute with itself, but in the absence of interactions it satisfies a free-field equation.

[7]For a review see: M. A. Virasoro, in *Proceedings of the International Conference on Duality and Symmetry in Hadron Theories,* edited by E. Gotsman (Weizmann Science Press, Jerusalem, 1971), p. 224.

[8]F. Mansouri and Y. Nambu, Phys. Letters 39B, 375 (1972); S. Ferrara, R. Gatto, and A. F. Grillo, Nuovo Cimento 12A, 959 (1972).

[9]C. G. Callan, Jr., S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970). For a review see: S. Ferrara, R. Gatto, and A. F. Grillo, in Springer Tracts in Modern Physics (Springer, Berlin, to be published).

[10]We continue to use the expressions "d'Alembert operator" and "Lorentz transformation" to describe objects in Euclidean space which properly speaking are called "Laplacian operator" and "4-dimensional rotation," respectively. We hope no confusion will arise.

[11]The fermion propagator in Euclidean space has been defined with an additional minus sign, relative to the propagator in Minkowski space. This simplifies various factors in the subsequent formalism.

[12]It appears that our theory provides an operator basis for manifestly conformally invariant Feynman rules which have been developed by K. Johnson and S. Adler for purposes of studying the Gell-Mann–Low eigenvalue problem in quantum electrodynamics; see S. L. Adler, Phys. Rev. D 6, 3445 (1972).

[13]K. Wilson, Phys. Rev. D 2, 1473 (1970); 2, 1478 (1970); S. Coleman and R. Jackiw, Ann. Phys. (N.Y.) 67, 552 (1971). A similar observation in a related context was made by K. Johnson, in Proceedings of the Symposium on Basic Questions in Elementary Particle Physics, held at Max Planck Institut für Physik und Astrophysik, München, Germany, 1971 (unpublished).

[14]It may appear that anomalous commutators invalidate this relation. Nevertheless we make use of it in the same spirit as in conventional quantization, where canonical equal-time commutation relations are used to define the interaction picture, even though ultimately these canonical commutators are not verified by the interaction picture perturbative calculations.

[15]Especially intriguing is the possibility of defining anomalous dimensions as the poles of the complete propagator. Clearly interaction corrections will modify the position of the poles from their free-field values given in (3.34), just as conventional perturbation theory shifts the position of the pole in the conventional propagator.
[16]The appropriate formalism will be close to the one used for the same difficulty in the treatment of the dual resonance model. S. Fubini and G. Veneziano, Nuovo Cimento 67A, 29 (1970).
[17]E. del Giudice, P. di Vecchia, S. Fubini, and R. Musto, Nuovo Cimento 12A, 813 (1972).
[18]D. Amati, S. Fubini, and A. Stanghellini, Nuovo Cimento 26, 896 (1962); S. Fubini, in *Lectures at the 1963 Scottish Universities Summer School,* edited by R. G. Moorhouse (Oliver and Boyd, Edinburgh, Scotland, 1966).
[19]P. A. M. Dirac, Proc. Roy. Soc. (London) A268, 57 (1962); Y. Nambu, Lectures at the Copenhagen Summer Symposium (1970); O. Hara, Progr. Theoret. Phys. (Kyoto) 46, 1549 (1971); T. Goto, *ibid.* 46, 1560 (1971), L. N. Chang and F. Mansouri, Phys. Rev. D 5, 2535 (1972). F. Mansouri and Y. Nambu, Phys. Letters 39B, 375 (1972); P. Goddard, J. Goldstone, C. Rebbi and C. B. Thorn, Nucl. Phys. (to be published).
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[26]G. H. Hardy and S. Ramanujan, Proc. Lond. Math. Soc. (Ser. 2) 17, 75 (1918).
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Quantum Field Theories in the Infinite-Momentum Frame III. Quantization of Coupled Spin-One Fields*

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Light-front quantization of spin-one fields coupled to a conserved or nonconserved current constructed from a Dirac field is studied. It is shown that an operator phase transformation must be performed on the Dirac field in order to maintain simple canonical commutation relations and a simple Hamiltonian. In this formulation quantum electrodynamics emerges as the zero-mass limit of the massive gluon model. Lorentz invariance of the vector-gluon model is explicitly verified. Vacuum expectation values of operator products and Green's functions are studied and spectral sum rules are derived. The general structure of the current commutators on a light front is *formally* not altered by the interactions. Feynman's parton model for deep-inelastic electron scattering is derived from canonical light-front current commutation relations. The structure function in the Bjorken scaling limit is related to the p^+ distribution of the constituents of the hadron target in any frame of reference.

I. INTRODUCTION

In this third of a series of papers devoted to the study of quantum field theories in an infinite-momentum frame,[1] we consider the quantization of spin-one fields coupled to a Dirac field. Interacting spin-one fields possess several new features not shared by scalar and Dirac fields studied earlier.[1] In particular, the canonical commutation relations are modified by the presence of interactions. The commonly adopted procedure of imposing the commutation relations obtained from free-field theories even in the presence of interactions does not work in this case, although it does in the cases of coupled scalar and Dirac fields.

If light-front quantization is to be claimed as an alternative to the conventional equal-time quantization, it should also be applicable to physical systems involving spin-one fields. Recently, Soper[2] has succeeded in formulating a theory of a vector field coupled to a conserved current (the gluon model) in a special gauge with the introduction of an additional scalar field. In the present paper a general procedure for quantization of interacting spin-one fields in light-front coordinates will be described. Schwinger's action principle[3]