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# Runaway Modes in Dipole Electrodynamics with Shadow States*

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In this paper we study the model in which a nonrelativistic harmonically bound electron interacts with the dipole part of the electromagnetic field. With the introduction of a shadow electromagnetic field, we find that, in contrast with the conventional treatment, the unpleasant runaway modes do not occur. The scattering states of a physical photon scattered by an electron are also given, and it is shown that these physical states form a set of orthonormal states.

## I. INTRODUCTION

Certain classical field theories, in particular classical electron theory, have long been known to suffer from the unphysical "runaway modes" – solutions of the equation of motion which display exponential time development.[1] Runaway modes have also been observed in many exactly solvable model field theories whose Hamiltonians are quadratic in the dynamical variables.[2] In classical theory, the runaway modes are usually removed by imposing suitable boundary conditions. Similar techniques for treating these runaway modes have also been suggested for quantum theory. In this case the runaway modes are simply not included in the eigenfunction expansion of the field operators. It is clear that these *ad hoc* prescriptions, namely, imposing boundary condition or truncating certain eigenmodes, would destroy the self-consistency of the Hamiltonian dynamics of the system.

It is well understood that the existence of the runaway modes in quantum theory is closely related to the infinite self-energy of the particle in the point particle limit. This can easily be seen in the model field theories. In the process of renormalization, the infinite positive self-energy requires an infinite negative bare mass (for the nonrelativistic case) in order to obtain a finite physical mass. Consequently, the Hamiltonian is not a positive-definite operator and admits imaginary eigenvalues. The existence of the runaway modes implies that this class of Hamiltonian is either non-Hermitian or does not have eigenstates. In fact, the renormalization operation in quantum field theory is often beset by the inherent mathematical ambiguities in manipulating divergent expression. Furthermore, to have a more realistic field theory, one would prefer to have the unrenormalized quantities such as masses and coupling constants be finite. Therefore, a fundamental change in the conventional quantum field theory seems to be necessary.

One way of eliminating the divergence is to introduce states with negative norms.[3] In a sense the introduction of these states with indefinite metric is to take care of the nonlocal interaction, which experimentally manifests itself only in high-energy scattering processes, and yet to make the theory manifestly local such that the Hamiltonian formalism is still suitable. It is well known that the direct extension of the conventional field theory to include the states with indefinite metric encounters the fundamental difficulty of negative probabilities. This difficulty has been circumvented by the attractive idea of "shadow states" introduced by Sudarshan.[4] Shadow states differ from the ordinary states in that they "propagate" with half retarded and half advanced Green's functions. This ensures that they never become physical. In other words, the shadow states would contribute to the dynamics but do not alter the unitarity property of the theory.

To obtain some more insight about the shadow-state formalism, we study here the model in which a nonrelativistic harmonically bound electron interacts with the dipole part of the electromagnetic field. It is found in Sec. II that with the shadow electromagnetic field, the energy spectrum of the Hamiltonian, in contrast with the result of the conventional treatment, does not have the unpleasant runaway modes. In Sec. III, following the idea of shadow "states," we obtain the scattering states of one physical photon scattered by the electron. It is also shown in Sec. IV that these scattering states form a set of orthonormal states.

## II. ENERGY SPECTRUM

As mentioned in the Introduction, in addition to the ordinary electromagnetic field $A^\mu$, we introduce a satellite field $A_s^\mu$. For nomenclature purpose, we will call this satellite field a shadow electromagnetic field. This is in contrast with the auxiliary field which gives the "nonelectromagnetic" force, as proposed by Poincaré. The shadow field is a massive vector field with mass $M$ and is realized in a generalized Hilbert space with indefinite metric. The Hamiltonian for the model we are interested in is

$$H=\frac{\{\vec{P}-e\int[\rho(\vec{r}-\vec{R})\vec{A}(\vec{r})+\rho_s(\vec{r}-\vec{R})\vec{A}_s(\vec{r})]d^3r\}^2}{2m_0}+\tfrac{1}{2}mK^2R^2$$
$$+\frac{1}{8\pi}\int\left[\left(\frac{\partial\vec{A}(\vec{r})}{\partial t}\right)^2+[\vec{\nabla}\times\vec{A}(\vec{r})]^2\right]d^3r-\frac{1}{8\pi}\int\left[\left(\frac{\partial\vec{A}_s(\vec{r})}{\partial t}\right)^2+[\vec{\nabla}\times\vec{A}_s(\vec{r})]^2+M^2\vec{A}^2(\vec{r})\right]d^3r, \tag{1}$$

where $m_0$ is the bare mass and $m$ is the observable mass of the electron; $mK^2$ is the spring coupling constant. $\vec{P}$ and $\vec{R}$ are the momentum and position operators of the electron. When the shadow field is removed, the remaining part of the Hamiltonian is the same as the conventional one.

In dipole approximation $\vec{A}(\vec{R}+\vec{r})\approx\vec{A}(\vec{r})$ and $\vec{A}_s(\vec{R}+\vec{r})\approx\vec{A}_s(\vec{r})$, and thus

$$\int[\rho(\vec{r}-\vec{R})\vec{A}(\vec{r})+\rho_s(\vec{r}-\vec{R})\vec{A}_s(\vec{r})]d^3r\approx\int[\rho(\vec{r})\vec{A}(\vec{r})+\rho_s(r)\vec{A}_s(\vec{r})]d^3r.$$

When $\vec{A}(\vec{r})$ and $\vec{A}_s(\vec{r})$ are expanded in multipole waves, only the dipole parts interact with the electron. Let us expand the fields inside a large sphere of radius $L$. The dipole parts of the fields in this expansion can be written

$$\vec{A}(\vec{r})=T\sum_{n=1}^{\infty}\left(\frac{3}{L}\right)^{1/2}\vec{q}_n\frac{\sin k_n r}{r},$$
$$\vec{A}_s(\vec{r})=T\sum_{n=1}^{\infty}\left(\frac{3}{L}\right)^{1/2}\vec{q}_n^{\,s}\frac{\sin k_n r}{r},\quad k_n=\frac{n\pi}{L}, \tag{2}$$

where $T$ means "transverse part."[5] Similarly, the electric fields are given by

$$-\vec{\mathcal{E}}(\vec{r})=T\sum_{n=1}^{\infty}\left(\frac{3}{L}\right)^{1/2}\vec{p}_n\frac{\sin k_n r}{r},$$
$$-\vec{\mathcal{E}}_s(\vec{r})=T\sum_{n=1}^{\infty}\left(\frac{3}{L}\right)^{1/2}\vec{p}_n^{\,s}\frac{\sin k_n r}{r}, \tag{3}$$

where the $\vec{p}_n$ and $\vec{q}_n$, and the $\vec{p}_n^{\,s}$ and $\vec{q}_n^{\,s}$ are, respectively, canonical conjugates and satisfy the commutation relations

$$\left.\begin{aligned}[][q_{n,i},p_{n',j}]&=i\delta_{nn'}\delta_{ij}\\ [q^s_{n,i},p^s_{n',j}]&=-i\delta_{nn'}\delta_{ij}\end{aligned}\right\}\quad i,j=1,2,3$$

and

$$[\vec{q}_n,\vec{q}^{\,s}_{n'}]=[\vec{p}_n,\vec{p}^{\,s}_{n'}]=[\vec{q}_n,\vec{p}^{\,s}_{n'}]=[\vec{q}^{\,s}_n,\vec{p}_{n'}]=0. \tag{4}$$

Assuming $\rho(\vec{r})$ and $\rho_s(\vec{r})$ to be spherically symmetric, one obtains

$$\int [\rho(\vec{r})\vec{A}(\vec{r})+\rho_s(\vec{r})\vec{A}_s(r)]d^3r=\left(\frac{4}{3L}\right)^{1/2}\sum_n \vec{q}_n\int \sin(k_n r)\rho(\vec{r})4\pi r dr+\left(\frac{4}{3L}\right)^{1/2}\sum_n \vec{q}_n^s\int \sin(k_n r)\rho_s(\vec{r})4\pi r dr$$
$$=\left(\frac{4}{3L}\right)^{1/2}\sum_n k_n(\rho_n^s\vec{q}_n^s+\rho_n\vec{q}_n)\,. \qquad (5)$$

$\rho_n$ and $\rho_n^s$ are the form factors in $k$ space. We shall be ultimately interested in the point electron limit $\rho_n\to 1$ and $\rho_n^s\to 1$.

In terms of the canonical conjugate variables $q_n$, $p_n$, $q_n^s$, and $p_n^s$, the Hamiltonian can be written

$$H=\frac{\vec{P}^2}{2m_0}+\tfrac{1}{2}mK^2\vec{R}^2+\tfrac{1}{2}\sum_n(\vec{p}_n{}^2+\omega_n{}^2\vec{q}_n{}^2)-\tfrac{1}{2}\sum_n(\vec{p}_n^{s\,2}+\omega_n^{s\,2}\vec{q}_n^{s\,2})-\frac{e}{m_0}\sum_i\frac{2}{(3L)^{1/2}}k_i\rho_i\vec{P}\cdot\vec{q}_i-\frac{e}{m_0}\sum_i\frac{2}{(3L)^{1/2}}k_i\rho_i^s\vec{P}\cdot\vec{q}_i^s$$
$$+\frac{e^2}{2m_0}\sum_{ij}\frac{4}{3L}k_ik_j\rho_i\rho_j\vec{q}_i\cdot\vec{q}_j+\frac{e^2}{2m_0}\sum_{ij}\frac{4}{3L}k_ik_j\rho_i^s\rho_j^s\vec{q}_i^s\cdot\vec{q}_j^s+\frac{e^2}{m_0}\sum_{ij}\frac{4}{3L}k_ik_j\rho_i\rho_j^s\vec{q}_i\cdot\vec{q}_j^s\,, \qquad (6)$$

with

$$\omega_n=k_n\,,\quad \omega_n^s=(k_n{}^2+M^2)^{1/2}\,.$$

Define the creation and annihilation operators as usual:

$$\left.\begin{aligned} a_{ni}&=\frac{1}{(2\omega_n)^{1/2}}(\omega_n q_{ni}+ip_{ni})\,, & a_{ni}^\dagger&=\frac{1}{(2\omega_n)^{1/2}}(\omega_n q_{ni}-ip_{ni})\,,\\ a_{ni}^s&=\frac{1}{(2\omega_n)^{1/2}}(\omega_n q_{ni}^s+ip_{ni}^s)\,, & a_{ni}^{s\dagger}&=\frac{1}{(2\omega_n)^{1/2}}(\omega_n q_{ni}^s-ip_{ni}^s)\,,\end{aligned}\right\}\ \text{where } i=1,2,3. \qquad (7)$$

It is easy to see that the creation and annihilation operators defined in (7) satisfy the commutation relations

$$[a_{ni},a_{n'j}^\dagger]=\delta_{nn'}\delta_{ij}\,,\qquad [a_{ni}^s,a_{n'j}^{s\dagger}]=-\delta_{nn'}\delta_{ij}\,,\qquad i=1,2,3 \qquad (8)$$

and the rest of the commutators vanish. To simplify the notation, all vector notations will be suppressed hereafter.

In terms of the creation and annihilation operators, the Hamiltonian becomes

$$H=\frac{P^2}{2m_0}+\tfrac{1}{2}mK^2R^2+\sum_i(a_i^\dagger a_i+\tfrac{1}{2})\omega_i-\sum_i(a_i^{s\dagger}a_i^s-\tfrac{1}{2})\omega_i^s-\frac{e}{m_0}P\left(\frac{2}{3L}\right)^{1/2}\sum_i\frac{k_i}{(\omega_i)^{1/2}}\rho_i(a_i^\dagger+a_i)$$
$$-\frac{e}{m_0}P\left(\frac{2}{3L}\right)^{1/2}\sum_i\frac{k_i}{(\omega_i^s)^{1/2}}(a_i^{s\dagger}+a_i^s)+\frac{e^2}{3m_0L}\sum_{ij}\frac{k_ik_j}{(\omega_i\omega_j)^{1/2}}\rho_i\rho_j(a_i^\dagger+a_i)(a_j^\dagger+a_j)$$
$$+\frac{e^2}{3m_0L}\sum_{ij}\frac{k_ik_j}{(\omega_i^s\omega_j^s)^{1/2}}\rho_i^s\rho_j^s(a_i^{s\dagger}+a_i^s)(a_j^{s\dagger}+a_j^s)+\frac{2e}{3m_0L}\sum_{ij}\frac{k_ik_j}{(\omega_i\omega_j^s)^{1/2}}\rho_i\rho_j^s(a_i^\dagger+a_i)(a_j^{s\dagger}+a_j^s)\,. \qquad (9)$$

Let us denote the ground state of $H$ with eigenvalue $E_0$ by $|0\rangle$ and the eigenstate with eigenvalue $E=E_0+\Delta E$ by $|\Delta E\rangle$,

$$H|0\rangle=E_0|0\rangle\,,$$
$$H|\Delta E\rangle=(E_0+\Delta E)|\Delta E\rangle\,. \qquad (10)$$

From the relations

$$\langle\Delta E|(E-H)a_i^\dagger|0\rangle=0\,,$$
$$\langle\Delta E|(E-H)a_i|0\rangle=0\,,$$
$$\langle\Delta E|(E-H)a_i^{s\dagger}|0\rangle=0\,, \qquad (11)$$
$$\langle\Delta E|(E-H)a_i^s|0\rangle=0\,,$$

we obtain

$$0=(E-E_0)\langle\Delta E|a_i^\dagger|0\rangle+\langle\Delta E|[a_i^\dagger,H]|0\rangle\,,$$
$$=(E-E_0-\omega_i)\langle\Delta E|a_i^\dagger|0\rangle$$
$$+\frac{e}{m_0}\left(\frac{2}{3L}\right)^{1/2}\frac{k_i}{(\omega_i)^{1/2}}\rho_i\langle\Delta E|P|0\rangle$$
$$-\frac{2e^2}{3m_0L}\frac{k_i}{(\omega_i)^{1/2}}\rho_i\langle\Delta E|F+F^s|0\rangle\,, \qquad (12)$$

$$\begin{aligned}0 &= (E-E_0)\langle \Delta E|a_i|0\rangle + \langle \Delta E|[a_i, H]|0\rangle \\ &= (E-E_0+\omega_i)\langle \Delta E|a_i|0\rangle \\ &\quad -\frac{e}{m_0}\left(\frac{2}{3L}\right)^{1/2}\frac{k_i}{(\omega_1)^{1/2}}\rho_i\langle \Delta E|P|0\rangle \\ &\quad +\frac{2e^2}{3m_0L}\frac{k_i}{(\omega_i)^{1/2}}\rho_i\langle \Delta E|F+F^s|0\rangle ,\end{aligned} \tag{13}$$

$$\begin{aligned}0 &= (E-E_0)\langle \Delta E|a_i^{s\dagger}|0\rangle + \langle \Delta E|[a_i^{s\dagger}, H]|0\rangle \\ &= (E-E_0-\omega_i^s)\langle \Delta E|a_i^{s\dagger}|0\rangle \\ &\quad -\frac{e}{m_0}\left(\frac{2}{3L}\right)^{1/2}\rho_i^s\frac{k_i}{(\omega_i^s)^{1/2}}\langle \Delta E|P|0\rangle \\ &\quad +\frac{2e^2}{3m_0L}\frac{k_i}{(\omega_i^s)^{1/2}}\rho_i^s\langle \Delta E|F+F^s|0\rangle ,\end{aligned} \tag{14}$$

$$\begin{aligned}0 &= (E-E_0)\langle \Delta E|a_i^s|0\rangle + \langle \Delta E|[a_i^s, H]|0\rangle \\ &= (E-E_0+\omega_i^s)\langle \Delta E|\omega_i^s|0\rangle \\ &\quad +\frac{e}{m_0}\left(\frac{2}{3L}\right)^{1/2}\rho_i^s\frac{k_i}{(\omega_i^s)^{1/2}}\langle \Delta E|P|0\rangle \\ &\quad -\frac{2e^2}{3m_0L}\frac{k_i}{(\omega_i^s)^{1/2}}\rho_i^s\langle \Delta E|F+F^s|0\rangle ,\end{aligned} \tag{15}$$

where

$$F = \sum_i \frac{k_i}{(\omega_i)^{1/2}}\rho_i(a_i^\dagger + a_i), \tag{16}$$

$$F^s = \sum_i \frac{k_i}{(\omega_i^s)^{1/2}}\rho_i^s(a_i^{s\dagger} + a_i^s). \tag{17}$$

Similarly,

$$\begin{aligned}0 &= \langle \Delta E|(E-H)P|0\rangle \\ &= (E-E_0)\langle \Delta E|P|0\rangle - imK^2\langle \Delta E|R|0\rangle ,\end{aligned} \tag{18}$$

$$\begin{aligned}0 &= \langle \Delta E|(E-H)R|0\rangle \\ &= (E-E_0)\langle \Delta E|R|0\rangle + \frac{i}{m_0}\langle \Delta E|P|0\rangle \\ &\quad -\frac{ie}{m_0}\left(\frac{2}{3L}\right)^{1/2}\langle \Delta E|F+F^s|0\rangle .\end{aligned} \tag{19}$$

From (11)–(15), (18), and (19) we obtain the equation

$$\left[1+\frac{4e^2E^2}{3L(mK^2-\Delta E^2 m_0)} \times \sum_i k_i{}^2\left(\frac{\rho_i{}^2}{\Delta E^2-\omega_i{}^2}-\frac{\rho_i^{s2}}{\Delta E^2-\omega_i^{s2}}\right)\right] \times \langle \Delta E|(F+F^s)|0\rangle = 0 . \tag{20}$$

Equation (20) implies that

$$1 = \frac{4e^2\Delta E^2}{3L(mK^2-\Delta E^2 m_0)}\sum_i k_i{}^2\left(\frac{\rho_i^{s2}}{\Delta E^2-\omega_i^{s2}}-\frac{\rho_i{}^2}{\Delta E^2-\omega_i{}^2}\right) \tag{21}$$

or, equivalently,

$$m_0 = \frac{mK^2}{\Delta E} + \frac{4e^2}{3L}\sum_i k_i{}^2\left(\frac{\rho_i{}^2}{\Delta E^2-\omega_i{}^2}-\frac{\rho_i^{s2}}{\Delta E^2-\omega_i^{s2}}\right). \tag{22}$$

Similar to the renormalization procedure used by several authors,[6] let us define the constant

$$\delta m = \frac{4e^2}{3L}\sum_i \rho_i{}^2 - \rho_i^{s2}$$

as the electromagnetic mass of the electron. With the definition

$$m = m_0 + \delta m , \tag{23}$$

Eq. (22) becomes

$$m = \frac{mK^2}{\Delta E^2} + \frac{4e^2}{3L}\sum_i \left(\frac{\rho_i{}^2\Delta E^2}{\Delta E^2-\omega_i{}^2}-\frac{\rho_i^{s2}(\Delta E^2-M^2)}{\Delta E^2-\omega_i^{s2}}\right). \tag{24}$$

In the point electron limit $\rho_i = \rho_i^s = 1$ and $\delta m = 0$, and (24) becomes

$$m = \frac{mK^2}{\Delta E^2} + \frac{4e^2}{3L}\sum_i k_i{}^2\left(\frac{1}{\Delta E^2-\omega_i{}^2}-\frac{1}{\Delta E^2-\omega_i^{s2}}\right). \tag{25}$$

Now it is easy to show that (25) does not have complex roots for $\Delta E$. Assuming that (25) has a pair of imaginary roots,

$$\Delta E = \pm iC , \tag{26}$$

then (25) becomes

$$m = \frac{4e^2}{3L}\frac{C^2}{C^2+K^2}\sum_i k_i{}^2\frac{-M^2}{(C^2+\omega_i^{s2})(C^2+\omega_i{}^2)} . \tag{27}$$

Since $m > 0$ and all the quantities on the right-hand side are positive except the negative sign, this equation is inconsistent. Thus there is no solution to the runaway mode equation for $C$.

## III. SCATTERING STATES

The introduction of a shadow field has nicely eliminated the unpleasant runaway modes as we have seen in the previous section. However, the direct extension of the conventional theory to include this shadow field invokes a new difficulty, namely, the negative probability. This negative-probability problem has provided a deadlock in the progress of the quantum theory with indefinite metric for a long time. In the physical subspace which only includes the states with positive-definite metric, the unitarity condition is not satisfied if the causal propagation is used for the states of the entire space. A breakthrough of this difficulty is due to the introduction of the idea of shadow states.[4] Shadow states propagate with half retarded and half advanced Green's functions, which ensures that they never become physical. Therefore, the unitarity condition in the physical subspace is always satisfied. The theory with shadow states has been studied extensively in the last few years.[7] In this section, following the idea of shadow states, we will find the scattering states of a physical photon scattered by an electron.

To start with, let us again denote the ground state by $|0\rangle$ and the one-photon-scattering state by $|k_i\rangle_+$. $|k_i\rangle_+$ is an eigenstate of $H$ with eigenvalue $E_i = E_0 + \omega_i$. $|k_i\rangle_+$ can be written as

$$|k_i\rangle_+ = a_i^\dagger|0\rangle + |\chi\rangle_+ . \tag{28}$$

$|\chi\rangle_+$ consists of outgoing states and shadow states. To avoid inessential complications, let us assume that $K=0$.

By making use of the relation

$$(E_i - H)|k_i\rangle_+ = 0 , \tag{29}$$

we have

$$(E_i - H)|\chi\rangle_+ = -\frac{e}{m_0}\left(\frac{2}{3L}\right)^{1/2} \frac{k_i}{(\omega_i)^{1/2}} P\bigg|0\bigg\rangle + \frac{2e^2}{3m_0} \frac{k_i}{(\omega_i)^{1/2}} (F + F^s)\bigg|0\bigg\rangle , \tag{30}$$

where $F$ and $F^s$ were defined in (16) and (17). With the help of the relation

$$P(E_i - H)|0\rangle = 0 \tag{31}$$

and the assumption that $K=0$, it can be shown that

$$P|0\rangle = 0 . \tag{32}$$

We have, therefore,

$$|k_i\rangle_+ = a_i^\dagger|0\rangle + \frac{2e^2}{3m_0 L} \frac{k_i}{(\omega_i)^{1/2}} \frac{1}{E_i - H} (F + F^s)\bigg|0\bigg\rangle . \tag{33}$$

Here the Green's function $G = 1/(E_i - H)$ is not yet defined. Following the prescription of shadow states, we choose the boundary condition in such a way that

$$|k_i\rangle_+ = a_i^\dagger|0\rangle + \frac{2e^2}{3m_0 L} \frac{k_i}{(\omega_i)^{1/2}} \frac{1}{E_i - H + i\epsilon} F\bigg|0\bigg\rangle + \frac{2e^2}{3m_0 L} \frac{k_i}{(\omega_i)^{1/2}} \mathcal{P}\frac{1}{E_i - H} F^s\bigg|0\bigg\rangle . \tag{34}$$

Here $\mathcal{P}$ means that the principal value is to be taken. After some tedious but straightforward manipulation, (34) can be written

$$|k_i\rangle_+ = a_i^\dagger|0\rangle + \sum_j \frac{2e^2}{3m_0 L} \frac{k_i}{(\omega_i)^{1/2}} \left(\frac{\alpha_j}{\omega_i - \omega_j + i\epsilon} + \mathcal{P}\frac{\beta_j}{\omega_i - \omega_j}\right)\bigg|k_j\bigg\rangle_+ , \tag{35}$$

with

$$\alpha_j = \frac{(k_j)^{1/2}}{1 + \sum_j \frac{2e^2 k_i}{3m_0 L}\left(\frac{1}{\omega_i - \omega_j + i\epsilon} + \frac{1}{\omega_j + \omega_i} + \mathcal{P}\frac{1}{\omega_i - \omega_j}\frac{B(\omega_j)}{1 - B(\omega_j)}\right)} , \tag{36a}$$

$$\beta_j = \frac{B(\omega_j)}{1 - B(\omega_j)}\alpha_j , \tag{36b}$$

$$B(\omega_j) = \sum_i \frac{4e^2 k_i^{\,2}}{3m_0 L} \mathcal{P}\frac{1}{\omega_j^{\,2} - \omega_i^{s2}} . \tag{36c}$$

## IV. ORTHONORMALITY OF THE SCATTERING STATES

According to the theory of shadow states, one can find a complete set of orthonormal states for the physical channel and a complete set of orthonormal states for the shadow channel. These two subspaces are expected to be orthogonal to each other. In this section we shall explicitly show that the states given in (34) form a set of orthonormal states,

$$_+\langle k_i|k_j\rangle_+ = \delta_{ij} . \tag{35$'$}$$

From (34) we have

$$\begin{aligned} _+\langle k_i|k_j\rangle_+ &= \langle 0|a_i|k_j\rangle_+ + \frac{2e^2}{3mL}\frac{k_i}{(\omega_i)^{1/2}}\bigg\langle 0\bigg|\left(F\frac{1}{E_i - H - i\epsilon} + F^s\mathcal{P}\frac{1}{E_i - H}\right)\bigg|k_j\bigg\rangle_+ \\ &= \langle 0|a_i|k_j\rangle_+ + \frac{2e^2}{3mL}\frac{k_i}{(\omega_i)^{1/2}}\left(\langle 0|F|k_j\rangle_+\frac{1}{\omega_i - \omega_j - i\epsilon} + \langle 0|F^s|k_j\rangle_+\mathcal{P}\frac{1}{\omega_i - \omega_j}\right) . \end{aligned} \tag{37}$$

The first term on the right-hand side may be written as

$$\langle 0|a_i|k_j\rangle_+ = \langle 0|a_i a_j^\dagger|0\rangle + \frac{2e^2}{3mL}\frac{k_j}{(\omega_j)^{1/2}}\left(\left\langle 0\left|a_i\frac{1}{E_j-H+i\epsilon}F\right|0\right\rangle + \left\langle 0\left|a_i\mathcal{P}\frac{1}{E_j-H}F^s\right|0\right\rangle\right) . \tag{38}$$

The relation

$$a_i(E_0-H)|0\rangle = 0$$

yields

$$a_i|0\rangle = \frac{2e^2}{3mL}\frac{k_i}{(\omega_i)^{1/2}}\frac{1}{E_0-\omega_i-H}(F+F^s)\Big|0\Big\rangle . \tag{39}$$

By making use of the commutation relation of the creation and annihilation operators and Eq. (39), the first term in (38) takes the form

$$\langle 0|a_i a_j^\dagger|0\rangle = \delta_{ij} + \langle 0|a_j^\dagger a_i|0\rangle$$

$$= \delta_{ij} + \frac{2e^2}{3m_0L}\frac{k_ik_j}{(\omega_i\omega_j)^{1/2}}\left\langle 0\left|(F+F^s)\frac{1}{E_0-\omega_j-H}\,\frac{1}{E_0-\omega_i-H}(F+F^s)\right|0\right\rangle . \tag{40}$$

It can also be shown that

$$a_i\frac{1}{E_j-H} = \frac{1}{E_j-\omega_i-H}\left(a_i + \frac{2e^2}{3mL}\frac{k_i}{(\omega_i)^{1/2}}(F+F^s)\frac{1}{E_j-H}\right) \tag{41}$$

and

$$a_iF = \frac{k_i}{(\omega_i)^{1/2}} + Fa_i . \tag{42}$$

Substituting (39)–(42) into (38), we get

$$\begin{aligned}\langle 0|a_i|k_j\rangle = {} & \delta_{ij} + \left(\frac{2e^2}{3m_0L}\right)^2\frac{k_ik_j}{(\omega_i\omega_j)^{1/2}}\left\langle 0\left|(F+F^s)\frac{1}{E_0-\omega_j-H}\,\frac{1}{E_0-\omega_i-H}(F+F^s)\right|0\right\rangle \\ & + \frac{1}{\omega_j-\omega_i+i\epsilon}\left(\frac{2e^2}{3m_0L}\right)\frac{k_ik_j}{(\omega_i\omega_j)^{1/2}} + \frac{1}{\omega_j-\omega_i+i\epsilon}\left(\frac{2e^2}{3m_0L}\right)^2\frac{k_ik_j}{(\omega_i\omega_j)^{1/2}}\left\langle 0\left|F\frac{1}{E_0-\omega_i-H}(F+F^s)\right|0\right\rangle \\ & + \frac{1}{\omega_j-\omega_i+i\epsilon}\left(\frac{2e^2}{3m_0L}\right)\frac{k_ik_j}{(\omega_i\omega_j)^{1/2}}\left(\left\langle 0\left|F\frac{1}{E_j-H+i\epsilon}F\right|0\right\rangle + \left\langle 0\left|F^s\mathcal{P}\frac{1}{E_j-H}F\right|0\right\rangle\right) \\ & + \mathcal{P}\frac{1}{\omega_j-\omega_i}\left(\frac{2e^2}{3m_0L}\right)^2\frac{k_ik_j}{(\omega_i\omega_j)^{1/2}}\left\langle 0\left|F^s\frac{1}{E_0-\omega_i-H}(F+F^s)\right|0\right\rangle \\ & + \mathcal{P}\frac{1}{\omega_j-\omega_i}\left(\frac{2e^2}{3m_0L}\right)^2\frac{k_ik_j}{(\omega_i\omega_j)^{1/2}}\left\langle 0\left|(F+F^s)\mathcal{P}\frac{1}{E_j-H}F^s\right|0\right\rangle .\end{aligned} \tag{43}$$

With the help of (34) and (42), we have

$$\begin{aligned}\langle 0|F|k_j\rangle_+ = {} & \frac{k_j}{(\omega_j)^{1/2}} + \frac{2e^2}{3mL}\frac{k_j}{(\omega_j)^{1/2}}\left\langle 0\left|(F+F^s)\frac{1}{E_0-\omega_j-H}F\right|0\right\rangle \\ & + \frac{2e^2}{3mL}\frac{k_j}{(\omega_j)^{1/2}}\left(\left\langle 0\left|F\frac{1}{E_j-H+i\epsilon}F\right|0\right\rangle + \left\langle 0\left|F\mathcal{P}\frac{1}{E_j-H}F^s\right|0\right\rangle\right) ,\end{aligned} \tag{44}$$

$$\begin{aligned}\langle 0|F^s|k_j\rangle_+ = {} & \frac{2e^2}{3mL}\frac{k_j}{(\omega_j)^{1/2}}\left\langle 0\left|(F+F^s)\frac{1}{E_0-\omega_j-H}F^s\right|0\right\rangle \\ & + \frac{2e^2}{3mL}\frac{k_j}{(\omega_j)^{1/2}}\left(\left\langle 0\left|F^s\mathcal{P}\frac{1}{E_j-H}F\right|0\right\rangle + \left\langle 0\left|F^s\mathcal{P}\frac{1}{E_j-H}F^s\right|0\right\rangle\right) .\end{aligned} \tag{45}$$

Substituting (43)–(45) into (37), and making use of the fact that $F$ and $F^s$ are Hermitian operators, we obtain

$$
{}_+\langle k_i|k_j\rangle_+ = \delta_{ij} + \left(\frac{2e^2}{3m_0L}\right)^2 \frac{k_ik_j}{(\omega_i\omega_j)^{1/2}} \left\langle 0\left|(F+F^s)\frac{1}{E_0-\omega_j-H}\frac{1}{E_0-\omega_i-H}(F+F^s)\right|0\right\rangle
$$
$$
+\frac{1}{\omega_j-\omega_i+i\epsilon}\left(\frac{2e^2}{3mL}\right)^2\frac{k_ik_j}{(\omega_i\omega_j)^{1/2}}\left\langle 0\left|(F+F^s)\left(\frac{1}{E_0-\omega_i-H}-\frac{1}{E_0-\omega_j-H}\right)F\right|0\right\rangle
$$
$$
+\mathcal{P}\frac{1}{\omega_j-\omega_i}\left(\frac{2e^2}{3mL}\right)^2\frac{k_ik_j}{(\omega_i\omega_j)^{1/2}}\left\langle 0\left|(F+F^s)\left(\frac{1}{E_0-\omega_i-H}-\frac{1}{E_0-\omega_j-H}\right)F^s\right|0\right\rangle . \tag{46}
$$

But

$$
\frac{1}{\omega_j-\omega_i}\left(\frac{1}{E_0-\omega_i-H}-\frac{1}{E_0-\omega_j-H}\right)=-\frac{1}{(E_0-\omega_j-H)(E_0-\omega_i-H)}, \tag{47}
$$

hence

$$
{}_+\langle k_i|k_j\rangle_+ = \delta_{ij} . \tag{35'}
$$

## V. CONCLUSION

We have made an interesting observation that, with the introduction of the shadow electromag-magnetic field, the runaway modes do not occur in the dipole approximation model of electrodynamics. This result might not be so surprising. The runaway modes in the conventional theory are often attributed to the fact that the self-energy of the electron is infinite; consequently, the bare electron mass has to be negative-infinite, which makes the Hamiltonian of the system not positive-definite. With the introduction of the shadow field, the self-energy becomes finite and the Hamiltonian can be positive-definite; therefore, the runaway modes do not occur.

For further understanding of the physical significance of the shadow field, we also give, as an example, the physical photon scattering states and show that they form a set of orthonormal states. The boundary conditions adopted in the shadow-state theory are to ensure that the shadow states would contribute to the dynamics, but do not alter the unitarity property of the $S$ matrix.

As a physical theory, one would ask: What will be the difference between the predictions of the present theory and the usual quantum electrodynamics? This problem is closely related to the value of the shadow photon mass $M$. It is not difficult to see that, for small $M$, the predictions of the present theory will be very much different from the conventional one, but for $M\rightarrow\infty$, they will be the same. However, we know that the usual quantum electrodynamics is, except in the extremely high-energy region, in good agreement with experiments. One would therefore anticipate that $M$ will be very large. From the experimental tests of the quantum electrodynamics, we would expect $M$ to be at least in the region of GeV or larger.[8] In other words, the deviation from the conventional theory manifests itself only in high-energy experiments which in turn would give us the information about the value of the shadow photon mass $M$. The model presented in this paper is a nonrelativistic model and is applicable only in the low-energy region; therefore, the predictions of this model will not have any visible difference from those of the conventional one.

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# New Approach to Field Theory*

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A field theory is quantized covariantly on Lorentz-invariant surfaces. Dilatations replace time translations as dynamical equations of motion. This leads to an operator formulation for Euclidean quantum field theory. A covariant thermodynamics is developed, with which the Hagedorn spectrum can be obtained, given further hypotheses. The Virasoro algebra of the dual resonance model is derived in a wide class of 2-dimensional Euclidean field theories.

## I. INTRODUCTION

We present a new method of quantizing a field theory. In conventional quantization, time is selected as the direction of propagation and the quantization conditions (commutators) are imposed on the spacelike surface $t=\text{constant}$. The recent lightlike quantization allows the system to develop along the $x^+=(x^0+x^3)/\sqrt{2}$ coordinate,[1] and commutators are given on the surface $x^+=\text{constant}$.[2] The former scheme has the advantage of closely following physical intuition based on the nonrelativistic Schrödinger equation, and has gained wide acceptance. The latter technique appears particularly useful in discussions of high-energy behavior, and commutators on lightlike surfaces have been profitably employed in this connection.[3] However, neither method is Lorentz-invariant, though of course the complete theory possesses this property.

The approach which we have developed selects the spacelike surface $x^2=\text{positive constant}=\tau^2$ as the surface of quantization, and propagation takes place in the "perpendicular" direction, i.e., along $x^\mu$. Clearly the technique is Lorentz-invariant. In a sense it is intermediate between time quantization and lightlike quantization: at $x^2=\infty$ our surface can also satisfy $t=\text{constant}$; at $x^2=0$, our surface can also coincide with $x^+=\text{constant}$.[4]

Although covariant quantization can be carried out in Minkowski space or in Euclidean space, it appears to be more useful in the latter than in the former. The reason is that in Minkowski space the hyperboloids $x^2=\tau^2$ do not span all of space-time: as $\tau^2$ varies from 0 to $\infty$, the region outside the light-cone $x^2<0$ is not reached. Consequently the propagation of the system in this domain must be examined separately. A related problem is that translation generators – the momentum operators – are hard to define.[5] In Euclidean space this problem does not exist. In this paper we confine our attention to Euclidean field theory.[6]

Of course the physical content of a theory is not changed by the choice of quantization surface. Indeed, as Schwinger and Tomonaga have shown, *any* spacelike surface may be used for quantization purposes. (However, since our surface is not asymptotically flat one cannot directly make use of the general Schwinger-Tomonaga result.) Thus we do not expect to obtain a different Feynman-Dyson expansion for the $S$ matrix. Nevertheless, as with lightlike quantization, we hope that our technique, by organizing the theory in a novel fashion, will prove itself convenient for analyzing certain problems and will provide new insights into the structure of field theories.

For example, there is considerable interest in relating the dual resonance model to a conventional field theory.[7] We believe that our quantization technique will provide a bridge for the formalism of these two theoretical ideas – indeed as will be seen, our method, when applied to 2-dimensional models, is very similar to that used in dual resonance models. As an application of the general formalism, it will be shown that *any* 2-dimensional field theory which possesses a traceless symmetric energy-momentum tensor gives rise to the Virasoro algebra,[7] up to a $c$ number; a result