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Dimensions and Equal-Time Commutators of the Current Components

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We make use of the usual assumption that the strong-interaction Hamiltonian density contains no parts with dimension larger than four to show that the *a priori* possibly different dimensions of the time and space components of any four-vector, in particular the currents, must in fact be the same. Furthermore, under slightly more restrictive, although conventional, assumptions on scale-symmetry breaking, it is shown that the usual equal-time commutators between the charges and the space components of the currents hold if and only if the equal-time Gell-Mann commutators between the charges and the time components of the currents are satisfied. Moreover, the equal-time Gell-Mann commutator algebra of the time components of the currents implies that at most a first derivative of the δ function is contained in the equal-time commutators between their time and space components. The conventional symmetry property of the first-order Schwinger terms in these commutators furthermore follows. Incidentally, our results also imply that the local generalization of the σ terms does not contain any derivatives of the δ function. In deriving our results we have made no assumptions on the equal-time commutators of the Hamiltonian density and the currents.

I. INTRODUCTION

After some initial discussion[1,2] on the question of "what are the dimensions (if any) of the space components of the SU(3)$\otimes$SU(3) currents" it has been agreed upon in the literature that scaling in deep-inelastic scattering can certainly not be understood[2] from the scale properties of the currents if their time and space components have different dimensions. Therefore, it has become customary to assume that these dimensions are identical. Furthermore, there is agreement that the strong-interaction Hamiltonian density $T_{00}(x)$ has only parts with dimensions not exceeding four.[3,4,5] It is the first purpose of this paper to show that the latter hypothesis implies the former one.

We would like to mention already at this point that the algebra of fields does not provide a counterexample to our theorem. Although in that model the dimension of the space components of the currents is one (as compared to the dimension three for their time components) the Hamiltonian density of that model contains a part with dimension six. This is in contradiction with the basic assumption that $T_{00}(x)$ should only contain parts with dimensions not exceeding four.[3,4]

As our second point we shall apply the basic ideas of the theory of broken scale invariance in order to derive restrictions on the equal-time commutators of the SU(3)$\otimes$SU(3) currents. The results are summarized at the end of the paper. We obtain the full current-algebra information[6] on the equal-time commutators

$$[J_0^a(x)\ ,J_\mu^b(0)]$$

and

$$[J_0^a(x)\ ,\partial^\mu J_\mu^b(0)]$$

from assuming

$$[J_0^a(x)\ ,J_0^b(0)]=if^{abc}J_0^c(0)\delta(\vec{x})\ .$$

In the above we have denoted by J_0^a $(a=1-16)$ the vector and axial-vector currents $J_\mu^a=V_\mu^a$ for $a=1-8$ and $J_\mu^a=A_\mu^{a-8}$ for $a=9-16$. The generalization of f^{abc} to the axial-vector indices $a=9-16$ is obvious.

Our results have frequently been conjectured in discussions on current algebras. As a matter of fact, some of them have already been derived previously, assuming, however, that the equal-time commutators $[iT_{00}(x), J_0^a(0)]$ do not contain any noncanonical terms.[7] The most essential exception is our result that the Gell-Mann expression for $[Q^a, J_0^b(0)]$ is equivalent to

$$[Q^a, J_m^b(0)]=if^{abc}J_m^c(0)\ .$$

Previous derivations[7] of the preceding relation have always assumed that

$$\int d^3y\, y_m[J_0^b(0), \partial^\mu J_\mu^a(y)] = 0 \ .$$

It is, however, well known[7,8] that both of these relations are equivalent.

II. THE DIMENSIONS OF THE CURRENT COMPONENTS

Our first essential assumption[3,4] is that the hadronic Hamiltonian density $T_{00}(x)$ can be written as

$$T_{00}(x) = \sum_{j=1}^{N_T} T_{00}^j(x) \ , \tag{1}$$

with (Q_D is the dilatation charge)

$$[Q_D, T_{00}^j(x)] = -i(l_j + x^\mu \partial_\mu)T_{00}^j(x) \tag{2}$$

and

$$l_j \leq 4 \quad (j = 1, \ldots, N_T) \ . \tag{3}$$

It will be assumed throughout that $l_i \neq l_j$ for $i \neq j$, since this may always be achieved by recombinations of the terms in Eq. (1). The scale-invariant part of T_{00} (i.e., that part of T_{00} with dimension 4) will be denoted by T_{00}'. As we shall see, such a part must necessarily be present. The assumption in Eqs. (1)–(3) has a remarkable consequence for an arbitrary four-vector with time and space components having dimensions l_t and l_{sp}, respectively. Namely, it implies

$$l_t = l_{sp} \ . \tag{4}$$

The simple reason for this is covariance. On these grounds we may write

$$\int d^3x\, x_m \left[i \sum_{j=1}^{N_T} T_{00}^j(x), J_0(0) \right] = -J_m(0) \tag{5a}$$

and

$$\int d^3x\, x_m \left[i \sum_{j=1}^{N_T} T_{00}^j(x), J_n(0) \right] = g_{mn} J_0(0) \ . \tag{5b}$$

Equations (5) follow since the boost operators M_{0m} are given by

$$M_{0m} = -\int d^3x\, x_m T_{00}(x_0 = 0, \vec{x}) \ .$$

We now observe that each of the terms

$$\int d^3x\, x_m [iT_{00}^j(x), J_0(0)]$$

in Eq. (5a) has a dimension, namely $l_t + l_j - 4$. This is for example seen by commuting the expression with Q_D and using the Jacobi identity. Since we presently assume J_m, i.e., the right-hand side of (5a), to have a dimension, only one of the terms

$$\int d^3x\, x_m [iT_{00}^j(x), J_0(0)] \quad \text{for } j = 1, \ldots, N_T$$

is nonvanishing. Say that the dimension of T_{00} for which this is the case is l_1. We may then write

$$l_{sp} = l_t + l_1 - 4 \leq l_t \ . \tag{6a}$$

A completely analogous argument applied to (5b) yields

$$l_t \leq l_{sp} \tag{6b}$$

and thus $l_t = l_{sp}$. This is the desired result. The argument also shows that the terms

$$\int d^3x\, x_m [iT_{00}^j(x), J_0(0)] \tag{7a}$$

and

$$\int d^3x\, x_m [iT_{00}^j(x), J_n(0)] \tag{7b}$$

all vanish except for the ones involving T_{00}^1. From (6a) we have $l_1 = 4$ and thus

$$T_{00}^1 = T_{00}' \ ,$$

with T_{00}' defined to be the scale-invariant part of T_{00}. For these,

$$\int d^3x\, x_m [iT_{00}'(x), J_0(0)] = -J_m(0) \tag{8}$$

and

$$\int d^3x\, x_m [iT_{00}'(x), J_n(0)] = g_{mn} J_0(0) \ . \tag{9}$$

Therefore, in particular, T_{00}' is nonvanishing.

III. RESULTS FOR CURRENT ALGEBRAS

It is gratifying to learn that some of the classical questions in current algebras have an obvious solution in a theory of broken scale invariance. The present chapter is devoted to this topic. Equations (1) and (3) are assumed throughout together with[3,4] chiral invariance of the scale-invariant part T_{00}', i.e.,

$$[Q^a, T_{00}'(x)] = 0 \ . \tag{10}$$

The components of the J_μ^a are assumed to have dimensions and our theorem then implies that $l_t = l_{sp}$. With an obvious generalization of the f^{abc} to the axial indices $a = 9$–16 the charge-charge density commutators are

$$[Q^a, J_0^b(0)] = if^{abc} J_0^c(0) \ . \tag{11}$$

These commutators now imply that $l_t = l_{sp} = 3$. It is obvious from Eqs. (8)–(10) that these equal-time commutators are equivalent to the charge-current commutators

$$[Q^a, J_m^b(0)] = if^{abc} J_m^c(0) \ . \tag{12}$$

The derivation consists in commuting Eqs. (7) written for J_μ^b with Q^a. The classical current-algebra problem of relating the commutators in Eqs. (11) and (12) has therefore an obvious and simple solution in the context of broken scale invariance. Assuming Eq. (11) we may use the relation[7,8]

$$[Q^b, J_m^a(0)] = if^{bac}J_m^c(0) - \int d^3y\, y_m[J_0^a(0), \partial^\mu J_\mu^b(y)] \tag{13}$$

together with the result in Eq. (12) in order to see that

$$\int d^3y\, y_m[J_0^a(0), \partial^\mu J_\mu^b(y)] = 0 \;. \tag{14}$$

[A derivation of Eq. (13) is given as a by-product below.] Incidentally, another way to obtain Eqs. (12) and (14) would be to start with Eq. (13). According to a well-known argument we have

$$\partial^\mu J_\mu^a(x) = [iT_{00}(x), Q^a(x_0)] \;. \tag{15}$$

Equation (10) is seen to imply that $\partial^\mu J_\mu^a$ has only parts with dimensions *below* 4. This implies that the right-hand side of (13) has no part with dimension 3. Since the left-hand side of that relation has dimension 3, Eqs. (12) and (14) follow.

Another classical current-algebra problem to be answered in the context of scale-symmetry breaking is the question of Schwinger terms in the equal-time commutators $[J_0^a(x), J_m^b(0)]$. Our essential current-algebra input will be to assume

$$[J_0^a(x), J_0^b(0)] = if^{abc}J_0^c(0)\delta(\vec{x}) \;, \tag{16}$$

i.e., the local generalization of Eq. (11). We shall furthermore need the following result[7,9]:

$$\int d^3y\, y_m y_{n_1} y_{n_2}\cdots y_{n_\beta}[J_0^a(0), \partial^\mu J_\mu^b(y)] = -\int d^3y\,\{y_{n_1}y_{n_2}\cdots y_{n_\beta}[J_0^b(y), J_m^a(0)] + y_m y_{n_2}\cdots y_{n_\beta}[J_0^a(0), J_{n_1}^b(y)] + \cdots + y_m y_{n_1} y_{n_2}\cdots y_{n_\beta - 1}[J_0^a(0), J_{n_\beta}^b(y)]\} \;. \tag{17}$$

A derivation starts from the formula[7]

$$i[T_{00}(x), J_0^a(0)] = \partial^\mu J_\mu^a(0)\delta(\vec{x}) + J_k^a(x)\frac{\partial}{\partial x_k}\delta(\vec{x}) + \sum_{\alpha=2}^{M} j_{l_1\cdots l_\alpha}^a(0)\frac{\partial}{\partial x_{l_\alpha}}\delta(\vec{x}) \;, \tag{18}$$

which is a simple consequence of covariance, i.e., Eq. (5a) and the Heisenberg equation of motion $i[H, J_0^a(x)] = \partial^0 J_0^a(x)$. Our assumptions imply that the Schwinger terms $j_{l_1\cdots l_\alpha}^a(0)$ have only parts with dimension less than or equal to $4-\alpha$. A well-known argument of Wilson then implies[3] that $j_{l_1 l_2 l_3 l_4}$ is a c number whereas the $j_{l_1 l_2 l_3 l_4 l_5\cdots l_N}$ all vanish. From (15) we obtain

$$\partial^{l_1}\partial^{l_2}j_{l_1 l_2}^a + \partial^{l_1}\partial^{l_2}\partial^{l_3}j_{l_1 l_2 l_3}^a = 0 \;. \tag{19}$$

This relation does not imply the vanishing of $j_{l_1 l_2}^a$ and $j_{l_1 l_2 l_3}^a$. This, for example, could be seen in a theory (see Ref. 10) containing a vector-gluon field B_μ. Since the space components of such a field have dimension one, the expressions

$$j_{l_1 l_2}^a = (\partial_{l_1}B_{l_2} + \partial_{l_2}B_{l_1})$$

and

$$j_{l_1 l_2 l_3}^a = -\tfrac{2}{3}(g_{l_1 l_2}B_{l_3} + g_{l_1 l_3}B_{l_2} + g_{l_2 l_3}B_{l_1})$$

have the correct dimensions. They evidently also fulfill Eq. (19).

As one of our main points we obtain restrictions on the Schwinger terms in the current commutators independent of the unknown noncanonical terms in Eq. (18).

Commuting (18) with $J_0^b(y)$ at equal times and using (16) a somewhat lengthy formula is obtained. It reads

$$\begin{aligned}
&[J_0^a(0), \partial^\mu J_\mu^b(y)]\delta(\vec{x}-\vec{y}) - [J_0^b(y), \partial^\mu J_\mu^a(0)]\delta(\vec{x}) + [J_0^a(0), J_k^b(x)]\frac{\partial}{\partial x_k}\delta(\vec{x}-\vec{y}) - [J_0^b(y), J_k^a(x)]\frac{\partial}{\partial x_k}\delta(\vec{x})\\
&\quad = if^{abc}\,\partial^\mu J_\mu^c(x)\delta(\vec{x})\delta(\vec{y}) + if^{abc}J_k^c(x)\delta(y)\frac{\partial}{\partial x_k}\delta(\vec{x}) + \sum_{\alpha=2}^{N}[J_0^b(0), j_{k_1\cdots k_\alpha}^a(0)]\frac{\partial}{\partial x_{k_1}}\cdots\frac{\partial}{\partial x_{k_\alpha}}\delta(\vec{x})\\
&\quad\quad - \sum_{\alpha=2}^{N}[J_0^a(0), j_{k_1\cdots k_\alpha}^b(y)]\frac{\partial}{\partial x_{k_1}}\cdots\frac{\partial}{\partial x_{k_\alpha}}\delta(\vec{x}-\vec{y}) + \sum_{\alpha=2}^{N} if^{abc}j_{k_1\cdots k_\alpha}^c(0)\delta(\vec{y})\frac{\partial}{\partial x_{k_1}}\cdots\frac{\partial}{\partial x_{k_\alpha}}\delta(\vec{x}) \;.
\end{aligned}$$

This relation has the important feature that upon multiplication with $x_m y_{n_1}\cdots y_{n_\beta}$ and integration over $\vec{x}$

and $\vec{y}$ the unknown terms involving $j_{l_1\cdots l_\alpha}$ drop out. The result of this procedure is Eq. (17). The reader should notice that for $\beta=1$ Eq. (17) reduces to Eq. (13).

It is next our aim to derive restrictions on the Schwinger terms in the equal-time commutators

$$[J_0^a(x), J_m^b(0)] = if^{abc}J_m^c(0)\delta(\vec{x}) + B_{m;l}^{ab}(0)\frac{\partial}{\partial x_l}\delta(\vec{x}) + C_{m;lk}^{ab}(0)\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_k}\delta(\vec{x}) + D_{m;lkn}^{ab}(0)\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_n}\delta(\vec{x}) . \tag{20}$$

In this relation, the dimensions of B, C, and D are 2, 1, and 0, respectively. According to Wilson's argument[3] as mentioned above, there are no Schwinger terms of higher order in Eq. (20). Furthermore, $D_{m;lkn}^{ab}$ has dimension 0 and thus must be a c number. Moreover, it is symmetric in the indices lkn and covariant under rotations. Counting dimensions we obtain from Eq. (17) that both sides of that relation vanish. This implies immediately that no gradient terms are contained in $[J_0^a(x), \partial^\mu J_\mu^b(0)]$. In addition,

$$D_{m;n_1n_2n_3}^{ba} = D_{n_1;mn_2n_3}^{ab} + D_{n_2;mn_1n_3}^{ab} + D_{n_3;mn_1n_2}^{ab} , \tag{21}$$

$$C_{m;n_1n_2}^{ba} + C_{n_1;mn_2}^{ab} + C_{n_2;mn_1}^{ab} = 0 , \tag{22}$$

and

$$-B_{m;n}^{ba} + B_{n;m}^{ab} + 2\partial^l C_{n;ml}^{ab} = 0 . \tag{23}$$

Equation (21) implies that $D_{m;n_1n_2n_3}^{ab}=0$. In order to see this we note that due to rotational invariance and symmetry in the last three indices $D_{m;n_1n_2n_3}^{ab}$ can be written as

$$D_{m;n_1n_2n_3}^{ab} = D^{ab}(g_{mn_1}g_{n_2n_3} + g_{mn_2}g_{n_1n_3} + g_{mn_3}g_{n_1n_2})$$

and thus $D^{ab}=9D^{ab}$ immediately follows from twice applying (21). Therefore, there are no third-order Schwinger terms in the current commutators.

It remains to be shown that $C_{m;n_1n_2}^{ab}$ vanishes. To see this we first of all interchange a and b together with m and n in Eq. (23) and add the result to Eq. (23) itself. We obtain

$$\partial^l(C_{n;ml}^{ab} + C_{m;nl}^{ba}) = 0 . \tag{24}$$

With the help of (22) this may be rewritten as

$$\frac{\partial}{\partial x_k} C_{k;mn}^{ab}(x) = 0 . \tag{25}$$

The result $C_{k;mn}^{ab}=0$ then follows from the preceding equation. To show this one first recalls that $C_{k;mn}^{ab}$ has dimension one. Incidentally, the example,

$$C_{k;mn}^{ab} = \epsilon_{mkl}\,\partial^l\partial_n\,\phi^{ab} + \epsilon_{nkl}\,\partial^l\partial_m\phi^{ab}$$

(with ϕ^{ab} a scalar operator) shows that in the derivation of $C_{k;mn}^{ab}=0$ from the above, information on the dimension of $C_{k;mn}^{ab}$ is actually needed. One then proceeds to note that no part of $\partial_0 C_{k;mn}^{ab}$ has a dimension above two (since $\dim H \le 1$). Therefore, one concludes that (the dependence of e on k', m', and n' is suppressed)

$$[\partial_0 C_{k';m'n'}^{ab}(x), C_{k;mn}^{ab}(y)] = e_{k;mn}^{ab}\,\delta(\vec{x}-\vec{y}) \tag{26}$$

with a c number $e_{k;mn}^{ab}$. Operating with $\int d^3y\, y_l\,\partial/\partial y_k$ on Eq. (26) we then immediately see that $e_{k;mn}^{ab}=0$. Thus, in particular,

$$\langle\Omega|\,[\partial_0 C_{k;mn}^{ab}(x), C_{k;mn}^{ab}(0)]\,|\Omega\rangle = 0 . \tag{27a}$$

We therefore also have that

$$\int_0^\infty dp_0 p_0 \int d^3p \sum_\alpha |\langle\Omega|\, C_{k;mn}^{ab}(0)|\,p;\alpha\rangle|^2 (e^{-i\vec{p}\cdot\vec{x}} + e^{i\vec{p}\cdot\vec{x}}) = 0 . \tag{27b}$$

This follows upon introducing a complete set of intermediate states in (27). Upon applying $\int d^3x\, e^{i\vec{q}\cdot\vec{x}}$ to (27b) we obtain

$$\int_0^\infty dq_0 q_0 \sum_\alpha \{|\langle\Omega|\, C_{k;mn}^{ab}(0)|\,q_0, \vec{q};\alpha\rangle|^2 + |\langle\Omega|\, C_{k;mn}^{ab}(0)|\,q_0, -\vec{q};\alpha\rangle|^2\} = 0 . \tag{27c}$$

Both terms in the curly parentheses are positive and thus the local operators $C^{ab}_{k;mn}$ annihilate the vacuum. Therefore, the desired result $C^{ab}_{k;mn}=0$ follows.

Our derivation of $C^{ab}_{k;mn}=0$ has assumed positivity of the metric and existence of the spectral integral in (27b). If this is, however, not the case, the result $C^{ab}_{k;mn}=0$ still follows if all local operators of the theory can be constructed as polynomials in space derivatives of local operators with dimensions less than three. This remark is of interest since in a vector-gluon theory (as given for example in Ref. 10) the latter hypothesis holds although the metric is not positive definite. At equal times a local operator X with dimension less than 3 has with $C^{ab}_{k;mn}$ the commutator

$$[C^{ab}_{k;mn}(x), X(0)] = d^{ab}_{k;mn}\delta(\vec{x}) \ .$$

In the above, $d^{ab}_{k;mn}$ can only be present for $\dim X=2$ and is in that case a c number. Hence, we obtain as before $d^{ab}_{k;mn}=0$ from $\partial^k C^{ab}_{k;mn}=0$. Thus, $C^{ab}_{k;mn}$ commutes with all the local operators of the theory under our present assumptions and is therefore a c number. Since, however, $\dim C^{ab}_{k;mn}=1$, a c number is also excluded, and thus $C^{ab}_{k;mn}=0$ follows.

In summary, we have shown that

$$C^{ab}_{k;mn}=0$$

and therefore [Eq. (23)]

$$B^{ab}_{m;n}=B^{ba}_{n;m} \ . \tag{28}$$

This is the current-algebra symmetry property of the first-order Schwinger terms in the current commutators.

IV. CONCLUSIONS

We made use of the usual assumption that the strong-interaction Hamiltonian density $T_{00}(x)$ contains no parts with dimension larger than four to show that the *a priori* possibly different dimensions of the time and space components of any four-vector, in particular the currents, must in fact be the same.

We should emphasize [Eq. (5b)] that the Hamiltonian density $T_{00}(x)$ of any model has a part with dimension $l_t - l_{sp}+4$. In particular, the $T_{00}(x)$ of the algebra of fields ($l_t=3$, $l_{sp}=1$) has a part with dimension six.

We then proceeded to derive constraints on current commutators from dimensional arguments. From chiral invariance of the scale-invariant part of $T_{00}(x)$ we showed that the equal-time commutators

$$[Q^a, J^b_0(0)] = if^{abc}J^c_0(0) \tag{29}$$

are valid if and only if

$$[Q^a, J^b_m(0)] = if^{abc}J^c_m(0) \ . \tag{30}$$

This equivalence was one of our new essential results. Furthermore, it was seen that no gradient terms are present in $[J^a_0(x), \partial^\mu J^b_\mu(0)]$. The argument of Wilson that no local operator except a c number can have a dimension below 1 was in addition finally adopted. The local generalization of Eq. (29), i.e. (with $a=1-16$; $J^9_\mu - J^{16}_\mu$ denote the axial-vector currents and an obvious generalization of the f tensor is involved)

$$[J^a_0(\vec{x}), J^b_0(0)] = if^{abc}J^c_0(0)\delta(\vec{x}) \ , \tag{31}$$

was then seen to imply the complete current-algebra information on the commutators of the charge densities with the current densities, i.e., it was shown that Eq. (31) implies that

$$[J^a_0(x), J^b_m(0)] = if^{abc}J^c_m(0) + B^{ab}_{m;l}(0)\frac{\partial}{\partial x_l}\delta(\vec{x}) \ , \tag{32}$$

with

$$B^{ab}_{m;l}=B^{ba}_{l;m} \ . \tag{33}$$

We should like to finish with two remarks. Firstly, no assumptions were made here on the equal-time commutators $i[T_{00}(x), J^a_0(0)]$. It is well known that assuming absence of noncanonical terms (i.e., Schwinger terms of at least second order) in this equal-time commutator some of our results are implied. We find it gratifying that the theory of broken scale invariance yields results on the chiral algebra and chiral symmetry breaking already expected on different grounds. Secondly, we evidently have assumed throughout the existence of the occurring equal-time commutators. The occurrence of nonexisting equal-time commutators [such as, for example, an infinite c number in Eq. (32)] would lead to obvious modifications.

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[1]H. Genz and J. Katz, Lett. Nuovo Cimento 4, 1103 (1971), and Phys. Rev. D 3, 1860 (1971).

[2]M. Bég, J. Bernstein, D. Gross, R. Jackiw, and A. Sirlin, Phys. Rev. Letters 25, 1231 (1971).

[3]K. Wilson, Phys. Rev. 179, 1499 (1967).

[4]M. Gell-Mann, in *Proceedings of the Hawaii Topical Conference in Particle Physics*, edited by S. Pakvasa and S. F. Tuan (Hawaii Univ. Press, Honolulu, Hawaii, 1968).

[5]H. Kastrup, Nucl. Phys. B15, 179 (1970); G. Mack, *ibid.* B5, 499 (1968); D. Gross and J. Wess, Phys. Rev. D 2, 753 (1970).

[6]S. Adler and R. Dashen, *Current Algebras* (Benjamin, New York, 1968); B. Renner, *Current Algebras and Their Applications* (Pergamon, London, 1968).

[7]D. Gross and R. Jackiw, Phys. Rev. 163, 1688 (1967); R. Jackiw, *ibid.* 175, 2058 (1968); H. Genz and J. Katz; Phys. Rev. D 2, 2225 (1970); See also Ref. 9 for a review.

[8]G. Jakob and B. Stech, Z. Physik 239, 379 (1970).

[9]H. Genz, in *Springer Tracts in Modern Physics*, edited by G. Höhler (Springer, New York, 1972), Vol. 61, p. 130.

[10]D. Gross and S. Treiman, Phys. Rev. D 4, 1059 (1971).

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Runaway Modes in Dipole Electrodynamics with Shadow States*

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In this paper we study the model in which a nonrelativistic harmonically bound electron interacts with the dipole part of the electromagnetic field. With the introduction of a shadow electromagnetic field, we find that, in contrast with the conventional treatment, the unpleasant runaway modes do not occur. The scattering states of a physical photon scattered by an electron are also given, and it is shown that these physical states form a set of orthonormal states.

I. INTRODUCTION

Certain classical field theories, in particular classical electron theory, have long been known to suffer from the unphysical "runaway modes" – solutions of the equation of motion which display exponential time development.[1] Runaway modes have also been observed in many exactly solvable model field theories whose Hamiltonians are quadratic in the dynamical variables.[2] In classical theory, the runaway modes are usually removed by imposing suitable boundary conditions. Similar techniques for treating these runaway modes have also been suggested for quantum theory. In this case the runaway modes are simply not included in the eigenfunction expansion of the field operators. It is clear that these *ad hoc* prescriptions, namely, imposing boundary condition or truncating certain eigenmodes, would destroy the self-consistency of the Hamiltonian dynamics of the system.

It is well understood that the existence of the runaway modes in quantum theory is closely related to the infinite self-energy of the particle in the point particle limit. This can easily be seen in the model field theories. In the process of renormalization, the infinite positive self-energy requires an infinite negative bare mass (for the nonrelativistic case) in order to obtain a finite physical mass. Consequently, the Hamiltonian is not a positive-definite operator and admits imaginary eigenvalues. The existence of the runaway modes implies that this class of Hamiltonian is either non-Hermitian or does not have eigenstates. In fact, the renormalization operation in quantum field theory is often beset by the inherent mathematical ambiguities in manipulating divergent expression. Furthermore, to have a more realistic field theory, one would prefer to have the unrenormalized quantities such as masses and coupling constants be finite. Therefore, a fundamental change in the conventional quantum field theory seems to be necessary.

One way of eliminating the divergence is to introduce states with negative norms.[3] In a sense the introduction of these states with indefinite metric is to take care of the nonlocal interaction, which experimentally manifests itself only in high-energy scattering processes, and yet to make the theory manifestly local such that the Hamiltonian formalism is still suitable. It is well known that the direct extension of the conventional field theory to include the states with indefinite metric encounters the fundamental dif-