(3) In the sixth line after Eq. (II.9), read (II.2) for (II.7).

(4) For the fractional power occurring in the denominator of the unnumbered equation preceding (II.18), read $\frac{5}{2}$ instead of $\frac{7}{3}$.

(5) For the fractional power appearing in the denominator at the end of Eq. (II.19), read $\frac{1}{2}$ instead of $\frac{1}{3}$.

(6) In Eq. (II.20), insert

$$\frac{\Gamma(\frac{1}{3})}{2}\left(\frac{\omega}{2\omega_c}\right)^{2/3} \text{ instead of } \frac{\Gamma(\frac{2}{3})}{2}\left(\frac{\omega}{2\omega_c}\right)^{1/3} .$$

(7) In Eq. (II.37), read $\frac{\pi^2}{8}$ instead of $\frac{\pi^2}{4}$.

(8) In Eq. (III.31), read $\frac{n}{2n_c}$ instead of $\frac{n}{n_c}$.

(9) The denominator of Eq. (III.32) should contain π^3 instead of π^2.

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[1] J. Schwinger, Phys. Rev. 75, 1912 (1949), referred to as paper I.

[2] A systematic development of this new approach to particle theory is described in J. Schwinger, *Particles, Sources, and Fields I* (Addison-Wesley, Reading, Mass., 1970) and Particles, Sources, and Fields II (to be published).

[3] This procedure resembles that introduced in an earlier paper [J. Schwinger, Phys. Rev. 82, 664 (1951)], but is here applied to the system of charged particle and photon.

[4] Retaining u here gives the essence of the first quantum correction [J. Schwinger, Proc. Nat. Acad. Sci. U. S. 40, 132 (1954), and the Russian literature cited in *Synchrotron Radiation,* A. A. Sokolov and I. M. Ternov (Pergamon, New York, 1968)].

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Anharmonic Oscillator

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We solve the anharmonic-oscillator problem as given by Bender and Wu to include the $\hbar^4$ terms for the energy eignevalues and the $\hbar^2$ terms for the eigenfunctions by means of the Miller and Good modified WKB method. This is done by accepting the harmonic oscillator as a solved problem. We see that in doing so, not only can we get better energy eigenvalues, but also we can get improved eigenfunctions.

I. INTRODUCTION

We will discuss the familiar anharmonic-oscillator problem. We are concerned here with the simple one-dimensional oscillator with real and positive oscillator strengths ($k > 0$, $a > 0$) as given by the potential

$$V(x) = \tfrac{1}{2}kx^2 + ax^4, \quad k > 0 \text{ and } a > 0 . \tag{1}$$

When perturbation theory is used to solve the anharmonic-oscillator problem based on the solved problem of the harmonic oscillator, the perturbation series for the energy diverges and even changes sign. A clear account of this is illustrated by Chan, Stelman, and Thompson[1] in their Table III.

Such a simple but important problem has attracted the attention of both the field theorists and the chemists. The former are interested in it because they desire to build a model field theory on it. The latter are interested mainly due to the anharmonic bonding problem. The reader is referred to Bender and Wu[2] as well as the references listed there for the detailed reason why the field theorists are interested in the problem. As for the chemists, Chen, Stelman, and Thompson[1] have given a good account.

Of course an evaluation of the energy eigenvalues is important, since the validity of the method used can be tested. It is even more important to obtain the corresponding eigenfunctions, since, as an example, the transition matrix elements could then be determined. What we work out in this paper are the energy eigenvalues to terms of the order of $\hbar^4$ based on the known values of the harmonic oscillator, namely, $E=(n+\frac{1}{2})\hbar\omega$, $n=0, 1, 2, \ldots$, and we also work out the corresponding wave functions. We use the Miller and Good[3] modified WKB method which was extended to include higher-order terms of the bound state by Lu.[4] We will see that by including higher-order terms, the energy eigenvalues converge faster than the ordinary WKB result[5] and improve the previous results reported.[6] The eigenfunctions are better than those given by Bender and Wu in the sense that there is no need to divide the boundaries into several regions so that the problem is less complicated and much easier to handle.

The essence of the Miller and Good method is that instead of starting with the free-particle wave function, the exact solution of a known potential that is as similar as possible to the unknown potential is used as the trial wave function. In this case, the known potential is chosen as the harmonic-oscillator potential. The result of Miller and Good using the zeroth-order terms in $\hbar^2$ only indicates that this choice gives the same answer as the ordinary WKB method. This situation becomes greatly improved if we include the $\hbar^2$ terms. Here the approximation includes the $\hbar^4$ terms. By including the $\hbar^4$ terms, the results obtained in the modified WKB method are better than those of both the modified WKB method to order $\hbar^2$ and the ordinary WKB method.

The method for obtaining the eigenfunctions is illustrated to order $\hbar^2$ only. The technique developed is quite general and there is no difficulty in going to higher orders in $\hbar^2$. Throughout we discuss the case where the coupling constant a is real and positive. We feel that it is easy to generalize it to complex values but it is not necessary to do so here.

II. EIGENVALUES TO THE ORDER $\hbar^2$

We are going to use the notations of Chan, Stelman, and Thompson by defining the following relations:

$$S=(4mk'/\hbar^2)^{1/4}x\,, \tag{2a}$$

$$k'=k+(a^2\hbar^2/m)^{1/3}\,. \tag{2b}$$

The Schrödinger equation under consideration is

$$\frac{d^2}{dx^2}\psi+\frac{2m}{\hbar^2}(E-\tfrac{1}{2}kx^2-ax^4)\psi=0\,. \tag{3}$$

By substitution, we can rewrite Eq. (3) to be

$$\frac{d^2}{dS^2}\psi+\tfrac{1}{4}[\lambda-(1-\alpha)S-\alpha^{3/2}S^4]\,\psi=0\,, \tag{4}$$

where

$$\lambda=4Em^{1/2}/(k'^{1/2}\hbar) \tag{5a}$$

and

$$\alpha=\frac{(a^2\hbar^2/m)^{1/2}}{k+(a^2\hbar^2/m)^{1/3}}\,. \tag{5b}$$

To the order $\hbar^2$, we see from previous work[4] that the following formula holds:

$$\oint p_1\,dx-\tfrac{1}{8}\hbar^2\oint p_1'^2p_1^{-3}\,dx =\oint p_2\,dx-\tfrac{1}{8}\hbar^2\oint p_2'^2p_2^{-3}\,dx\,, \tag{6}$$

where p_1 and p_2 are defined in the equations

$$[(d^2/dx^2)+p_1^2/\hbar^2]\psi(x)=0\,, \tag{7a}$$

$$[(d^2/dx^2)+p_2^2/\hbar^2]\,\phi(x)=0\,. \tag{7b}$$

Here the unknown part is represented by $p_1^2=2m(E-\frac{1}{2}kx^2-ax^4)$ and the known part is represented by $p_2^2=2m(E_{\text{H.O}}-\frac{1}{2}kx^2)$, where $E_{\text{H.O.}}=(n+\frac{1}{2})\times\hbar(k/m)^{1/2}$ with $n=0, 1, 2, \ldots$ are the energy eigenvalues of (7b). The contour integration is around the two turning points in both cases. Now it is easy to see from (5a) that for a pure harmonic oscillator,

$$\lambda'\equiv\lambda_{\text{H.O.}}=4n+2 \quad\text{with}\quad n=0, 1, 2, \ldots\,. \tag{8}$$

The modified WKB method, as used by Miller and Good, coincides with the ordinary WKB method so far as the lowest-order eigenvalues are concerned. We see that the discrepancy is immediately compensated for by including the next-higher-order terms. Now Eq. (6) is rewritten as

$$\oint[(\lambda'-S^2)^{1/2}+\tfrac{1}{6}(\lambda'-S^2)^{-3/2}]\,dS=\oint[\lambda-(1-\alpha)S^2-\alpha^{3/2}S^4]^{1/2}dS+\tfrac{1}{6}(1-\alpha)\oint dS/[\lambda-(1-\alpha)S^2-\alpha^{3/2}S^4]^{3/2} +\alpha^{3/2}\oint S^2dS/[\lambda-(1-\alpha)S^2-\alpha^{3/2}S^4]^{3/2} \tag{9}$$

and

TABLE I. Reduced energy levels of the anharmonic oscillator. (a) The exact result as given by Chan, Stelman, and Thompson. (b) The modified WKB result to second order in $\hbar$. (c) The modified WKB result to fourth order in $\hbar$. (d) The WKB result as given by Handelsman and Lew to second order in $\hbar$. Notice that (b) and (c) are given to five decimal points mainly because the improvement can only be seen there.

n	(a)	(b)	(c)	(d)	n	(a)	(b)	(c)
		$\alpha = 0.2$					$\alpha = 0.8$	
0	2.042 70	2.034 11	2.037 20	1.989 57	0	2.516 48	2.361 65	2.341 45
1	6.510 50	6.510 326	6.510 57	6.473 73	1	8.880 62	8.900 00	8.898 18
2	11.6292	11.629 15	11.629 18	11.6011	2	17.2430	17.243 36	17.242 85
3	17.2332	17.233 17	17.233 17	17.2103	3	26.7955	26.795 96	26.795 75
4	23.2391	23.239 07	23.239 07	23.2196	4	37.3022	37.302 46	37.302 34
5	29.5930	29.593 03	29.593 03	29.5760	5	48.6119	48.612 08	48.612 01
6	36.2567	36.256 69	26.256 69	36.2414	6	60.6228	60.622 91	60.622 87
7	43.2010	43.200 98	43.200 98	43.1871	7	73.2604	73.260 49	73.260 46
8	50.4029	50.402 93	50.402 93	50.3902	8	86.4675	86.467 53	86.467 51
9	57.8438	57.843 84	57.843 84	57.8321	9	100.1983	100.198 33	100.198 32
		$\alpha = 0.4$					$\alpha = 1.0$	
0	2.192 03	2.141 136	2.145 86	2.151 81	0	2.671 95	2.471 38	2.435 85
1	7.338 24	7.342 51	7.342 22	7.339 83	1	9.574 58	9.601 43	9.598 97
2	13.6919	13.692 17	13.692 01	13.6897	2	18.7872	18.787 43	18.786 78
3	20.8389	20.839 06	20.838 98	20.8370	3	29.3429	29.343 54	29.343 27
4	28.6242	28.624 37	28.624 32	28.6226	4	40.9773	40.977 51	40.977 38
5	36.9519	36.951 98	36.951 95	36.9504	5	53.5174	53.517 51	53.517 43
6	45.7558	45.755 91	45.755 89	45.7546	6	66.8476	66.847 66	66.847 61
7	54.9874	54.987 42	54.987 41	54.9862	7	80.8835	80.883 47	80.883 43
8	64.6087	64.608 74	64.608 73	64.6076	8	95.5600	95.560 03	95.560 00
9	74.5895	74.589 54	74.589 53	74.5885	9	110.8256	110.825 61	110.825 59
		$\alpha = 0.6$						
0	2.355 26	2.251 10	2.245 64					
1	8.136 48	8.148 09	8.147 00					
2	15.5569	15.557 36	15.557 01					
3	23.9899	23.990 31	23.990 16					
4	33.2344	33.234 62	33.234 54					
5	43.1639	43.164 01	43.163 95					
6	53.6926	53.692 68	53.692 65					
7	64.7577	64.757 76	64.757 74					
8	76.3107	76.310 76	76.310 74					
9	88.3128	88.312 91	88.312 89					

$$\oint(\lambda - S^2)^{-3/2}dS = 0\,. \tag{9$'$}$$

The numerical values in Table I indicate the importance of the terms to $\hbar^2$ in Eq. (6).

The evaluation of the first integral on the right-hand side of Eq. (9) is easily done by use of the elliptic integrals with the following results:

$$\begin{aligned}\oint[\lambda - (1-\alpha)S^2 - \alpha^{3/2}S^4]^{1/2}dS &= \alpha^{3/4}\oint\left[\frac{\lambda}{\alpha^{3/2}} - \frac{1-\alpha}{\alpha^{3/2}}S^2 - S^4\right]^{1/2}dS \\ &= \alpha^{3/4}\oint[(b^2 - S^2)(a^2 + S^2]^{1/2}\,dS \\ &= \alpha^{3/4}\tfrac{4}{3}(a^2+b^2)^{1/2}\left[b^2F\left(\frac{a}{(a^2+b^2)^{1/2}}\right) + (b^2-a^2)E\left(\frac{a}{(a^2+b^2)^{1/2}}\right)\right],\end{aligned} \tag{10}$$

with

$$a^2 = \frac{1}{2}\left[\frac{1-\alpha}{\alpha^{3/2}} + \left(\frac{(1-\alpha)^2}{\alpha^3} + \frac{4\lambda}{\alpha^{3/2}}\right)^{1/2}\right] \tag{11a}$$

and

$$b^2=\frac{1}{2}\left[-\frac{1-\alpha}{\alpha^{3/2}}+\left(\frac{(1-\alpha)^2}{\alpha^3}+\frac{4\lambda}{\alpha^{3/2}}\right)^{1/2}\right]. \tag{11b}$$

The F and E are the complete elliptic integrals of the first and second kind. They can be approximated by the polynomial approximation to within an error of 10^{-8} by the formulas given on p. 591 of Ref. 7.

We will make use of the fact that the integrals involved are contour integrals. The evaluation of them is greatly facilitated by the use of the external differentiation method. The same result can also be obtained by using a table of integrals. We illustrate the former method by performing differentiation on both sides of

$$\oint\frac{dx}{[(a^2+x^2)(b^2-x^2)]^{1/2}}=\frac{4}{(a^2+b^2)^{1/2}}F(\tfrac{1}{2}\pi,K)$$

with respect to b^2 and using the relations

$$\frac{dF}{dK}=\frac{E-(1-K^2)F}{K(1-K^2)}\quad\text{and}\quad\frac{dE}{dK}=\frac{E-F}{K}$$

with $K=b/[a^2+b^2]^{1/2}$. We then obtain

$$\oint\frac{dx}{(a^2+x^2)^{1/2}(b^2-x^2)^{3/2}}=\frac{4(F-E)}{b^2(a^2+b^2)^{1/2}}. \tag{12}$$

The table of integrals[8] gives

$$\int_0^u\frac{dx}{(a^2+x^2)^{1/2}(b^2-x^2)^{3/2}}=\frac{4(F-E)}{b^2(a^2+b^2)^{1/2}}[F(\gamma,K)-E(\gamma,K)]+\frac{u}{b^2[(a^2+u^2)(b^2-u^2)]^{1/2}}, \tag{13}$$

with

$$\gamma=\sin^{-1}\left(\frac{(a^2+b^2)^{1/2}}{(a^2+u^2)^{1/2}}\,\frac{u}{6}\right).$$

Also notice that

$$E(m\pi\pm\phi,K)=2mE(\tfrac{1}{2}\pi,K)\pm E(\phi,K),$$

$$F(m\pi\pm\phi,K)=2mF(\tfrac{1}{2}\pi,K)\pm F(\phi,K).$$

Here m is an integer and $\phi=0$ gives $E(0,K)=0$ and $F(0,K)=0$. Therefore, it gives

$$\oint\frac{dx}{(a^2+x^2)^{1/2}(b^2-x^2)^{3/2}}=\frac{4}{b^2(a^2+b^2)^{1/2}}[F(\tfrac{1}{2}\pi,K)-E(\tfrac{1}{2}\pi,K)]. \tag{14}$$

Notice here that the contour integration leads from $-b$ to 0 and 0 to b in the third and first quadrant and from b to 0 and 0 to $-b$ in the second and fourth quadrant, respectively. This gives the number 4 in front, which is the same as before [Eq. (12)]. This external diffraction method is important in finding some of the more complicated contour integrals used in Sec. III where we just list the results obtained and do not go into details.

Similarly we get

$$\oint\frac{dx}{(a^2+x^2)^{3/2}(b^2-x^2)^{3/2}}=\frac{4}{a^2b^2(a^2+b^2)^{3/2}}[a^2F(\tfrac{1}{2}\pi,K)-(a^2-b^2)E(\tfrac{1}{2}\pi,K)] \tag{15a}$$

and

$$\oint\frac{x^2\,dx}{(a^2+x^2)^{3/2}(b^2-x^2)^{3/2}}=\frac{4}{(a^2+b^2)^{3/2}}[F(\tfrac{1}{2}\pi,K)-2E(\tfrac{1}{2}\pi,K)]. \tag{15b}$$

With the identification a^2 and b^2 as given in Eq. (11), Eq. (9) is readily evaluated.

III. EIGENVALUES TO THE ORDER $\hbar^4$

Equations (3) and (4) of Ref. 4 are the equations to be used.

Equation (3) of Ref. 4 reads

$$\oint p_1 dx - \tfrac{1}{8}\hbar^2 \oint (p_1')^2 p_1^{-3} dx - \tfrac{1}{32}\hbar^4 \oint (p_1'')^2 p_1^{-5} dx + \tfrac{15}{128}\hbar^4 \oint (p_1')^4 p_1^{-7} dx$$
$$= \oint p_2 dx - \tfrac{1}{8}\hbar^2 \oint (p_2')^2 p_2^{-3} dx - \tfrac{1}{32}\hbar^4 \oint (p_2'')^2 p_2^{-5} dx + \tfrac{15}{128}\hbar^4 \oint (p_2')^4 p_2^{-7} dx \tag{16}$$

and Eq. (4) of Ref. 4 reads

$$\oint\left(R_1 - \frac{1}{48}\frac{T_1''}{T_1^{3/2}} - \frac{7}{1536}\frac{T_1''^2}{T_1^{7/2}} + \frac{5}{1536}\frac{T_1' T_1'''}{T_1^{7/2}}\right) dx = \oint\left(R_2 - \frac{1}{48}\frac{T_2''}{T_2^{3/2}} - \frac{7}{1536}\frac{T_2''^2}{T_2^{7/2}} + \frac{5}{1536}\frac{T_2' T_2'''}{T_2^{7/2}}\right) dx \tag{17}$$

with

$$p_{1,2} \equiv \hbar R_{1,2} \equiv \hbar (T_{1,2})^{1/2}.$$

Equation (17) is obtained from Eq. (16) after integrating by parts several times. Equation (16) can be directly seen as an extension of Eq. (6) by including the $\hbar^4$ terms. Equation (17) is the equation to work with here. We list in the following the formulas used in evaluating the integrals, which are obtained by means of differentiation before the contour integrals are evaluated. In Table I we list the results and it can be seen that the contributions of the $\hbar^4$ terms are in the right direction. They are converging and they give us a feeling that this modified WKB approximation is essentially correct, in spite of the complexity of the higher-order terms.

The following formulas shown here are important:

$$\oint \frac{dx}{(a^2+x^2)^{5/2}(b^2-x^2)^{3/2}} = (-\tfrac{2}{3})\frac{1}{a^4 b^2 (a^2+b^2)^{5/2}}[F(\tfrac{1}{2}\pi, K)(-6a^4 + 2a^2b^2) + E(\tfrac{1}{2}\pi, K)(6a^4 - 14a^2b^2 - 4b^4)], \tag{18}$$

$$\oint \frac{dx}{(a^2+x^2)^{5/2}(b^2-x^2)^{5/2}} = \frac{4}{9}\frac{1}{a^4 b^4 (a^2+b^2)^{7/2}}[(6a^6 + 27a^4b^2 - 3a^4b^2)F(\tfrac{1}{2}\pi, K) + (-6a^6 - 30a^4b^2 + 30a^2b^4 + 6b^6)E(\tfrac{1}{2}\pi, K)], \tag{19}$$

$$\oint \frac{dx}{(a^2+x^2)^{5/2}(b^2-x^2)^{7/2}} = -\frac{4}{45}\frac{1}{a^4 b^6 (a^2+b^2)^{9/2}}[(-24a^8 - 117a^6b^2 - 270a^4b^4 + 15a^2b^6)F(\tfrac{1}{2}\pi, K)$$
$$+ (24a^8 + 129a^6b^2 + 324a^4b^4 - 195a^2b^6 - 30b^8)E(\tfrac{1}{2}\pi, K)], \tag{20}$$

and

$$\oint \frac{dx}{(a^2+x^2)^{7/2}(b^2-x^2)^{7/2}} = \frac{4}{225}\frac{1}{a^6 b^6 (a^2+b^2)^{11/2}}[a^2(120a^8 + 735a^6b^2 + 2250a^4b^4 - 345a^2b^2 - 60b^8)F(\tfrac{1}{2}\pi, K)$$
$$+ (-120a^{10} - 795a^8b^2 - 2595a^6b^4 + 2595a^4b^6 + 795a^2b^8 + 120b^{10})E(\tfrac{1}{2}\pi, K)]. \tag{21}$$

The additional terms added to the right-hand side of Eq. (6) through Eq. (17) are

$$-\frac{7}{1536}\oint \frac{T_1''^2}{T_1^{7/2}} dx + \frac{5}{1536}\oint \frac{T_1' T_1'''}{T^{7/2}} dx$$
$$= \frac{-1}{1536}\left(224(1-\alpha^2)\oint \frac{dS}{[\lambda - (1-\alpha)S^2 - \alpha^{3/2}S^4]^{7/2}} + 768(1-\alpha)\alpha^{3/2}\oint \frac{S^2 dS}{[\lambda - (1-\alpha)S^2 - \alpha^{3/2}S^4]^{7/2}}\right.$$
$$\left. + 4224\,\alpha^3 \oint \frac{S^4 dS}{[\lambda - (1-\alpha)S^2 - \alpha^{3/2}S^4]^{7/2}}\right)$$
$$= \frac{1}{1536}\left[\frac{4224}{\alpha^{9/4}}\oint \frac{dS}{(b^2-S^2)^{5/2}(a^2+S^2)^{5/2}} - \left(\frac{768(1-\alpha)}{\alpha^{15/4}} + \frac{4224(b^2-a^2)}{\alpha^{9/4}}\right)\oint \frac{dS}{(b^2-S^2)^{7/2}(a^2+S^2)^{7/2}}\right.$$
$$\left. + \left(\frac{-224(1-\alpha^2)}{\alpha^{21/4}} + \frac{768(1-\alpha)}{\alpha^{15/4}}a^2 - \frac{4224}{\alpha^{9/4}}a^4\right)\oint \frac{dS}{(b^2-S^2)^{7/2}(a^2+S^2)^{7/2}}\right], \tag{22}$$

and the higher terms of Eq. (9) in $\hbar^2$ are zero [e.g., Eq. (9a)]. The definitions of the parameters α, a, and b are the same as in Eq. (5b), Eq. (11a), and Eq. (11b). They are easily handled by a computer. The results obtained for different α values up to the order of $\hbar^4$ are given in Table I. An important fact of this calculation is that the contribution due to $\hbar^4$ terms above is very small but in the right direction.

IV. EIGENFUNCTIONS TO THE ORDER OF $\hbar^2$

Assume that r is the one-dimensional space coordinate of the anharmonic oscillator. Since we know that

$$\oint u\,dv = -\oint v\,du ,$$

then by several integrations by parts, Eq. (6) can be written as

$$\oint\left[R_1-\frac{1}{24T_1^{1/2}}\left(\frac{T_1'''}{T_1'}-\frac{T_1''^2}{T_1'^2}\right)\right]dr = \oint\left[R_2-\frac{1}{24T_2^{1/2}}\left(\frac{T_2'''}{T_2'}-\frac{T_2''^2}{T_2'^2}\right)\right]dr , \tag{23}$$

where as before

$$R_{1,2}\equiv T_{1,2}^{1/2}\equiv p_{1,2}/\hbar .$$

Because there is no longer any apparent divergence in the second-order term in $\hbar^2$, we can evaluate this contour integral as follows:

$$\int_{r_1}^{r}\left[R_1-\frac{1}{24}\frac{1}{T_1^{1/2}}\left(\frac{T_1'''}{T_1'}-\frac{T_1''^2}{T_1'^2}\right)\right]dr = \int_{x_1}^{x}\left[R_2-\frac{1}{24}\frac{1}{T_2^{1/2}}\left(\frac{T_2'''}{T_2'}-\frac{T_2''^2}{T_2'^2}\right)\right]dx , \tag{24}$$

where r_1 and x_1 are the corresponding turning points. We see that the above gives us the relation

$$x=x(r) , \tag{25}$$

where x is the one-dimensional space coordinate of the harmonic oscillator. The wave function of the harmonic oscillator is already known and given by

$$\phi_n(x)=H_n(\xi)e^{-\xi^2/2} \tag{26}$$

with $\xi=(mk/\hbar^2)^{1/4}x$ and H_n being the Hermite polynomials. Therefore, the wave function of the anharmonic oscillator is[3]

$$\psi_n(r)=[x'(r)]^{-1/2}\phi(x(r)) . \tag{27}$$

Of course we see that for $x\geq|x_1|$ and $x\leq-|x_1|$ mapping to $r\geq|r_1|$ and $r\leq-|r_1|$, there is one-to-one correspondence. But both R_1 and R_2 simultaneously turn from real values to imaginary values. The equation is still valid outside of the turning-point regions. We just have to multiply by i to transform both sides into real quantities. There appears to be a divergence at first glance in the place where $T_1'=T_2'=0$. There are important conditions used in deriving Eq. (23), simply $T_1'\neq0$ and $T_2'\neq0$. Or we have to return back to the earlier Eq. (6), where, as r and x are close to zero, only the zeroth-order terms contribute. Or we can approximate the case by choosing a finite distance around both $r_{\min}$ and $x_{\min}$ such that only the zeroth-order terms contribute there and with adequate connections outside. Thus the eigenfunction obtained is valid throughout.

In so doing, we think that we get a better eigenfunction than Bender and Wu, for then we have avoided the necessity of dividing the space into many regions where each of the regions has its own form of valid wave function.

V. REMARKS

As we look at the results in Table I, several interesting things can be seen. First, the modified WKB method gives better over-all results than the ordinary WKB results to second order in $\hbar$ already and this leaves very little room for improvement to fourth order in $\hbar$. Although the terms of fourth order in $\hbar$ are messy, the contribution to the over-all values is small. Here we begin to consider the accuracy of the exact result which is obtained by the method of diagonalization. It could be possible that there are some errors to the fourth or fifth decimal points due to roundoff. Second, we see that the modified WKB method cannot give the eigenvalues for $n=0$ and $n=1$. However, the results indicate that the eigenvalues for $n=2$ or above are already good. We see that this statement is true too even if we include the $\hbar^4$ terms. This seems contrary to our anticipation, namely, the improvement obtained with the $\hbar^4$ terms is not drastic. However, it is in the right direction.

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Lorentz-Invariant Localization for Elementary Systems. V. General Transformations

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In parts II and III of the present series of papers, the localization problem was studied for the cases of relativistic nonzero mass (spin 0, $\frac{1}{2}$) and zero mass (spin 0, $\frac{1}{2}$, and 1). With a set of postulates, of which the basic one is the imposition of Lorentz invariance of localization (LIL) (which is a self-consistency requirement), it was possible to define the position operator uniquely (except for a constant in the spin-$\frac{1}{2}$, nonzero-mass case). For the determination of the position operators it was enough to consider only very special cases of homogeneous Lorentz transformations. One of the purposes of the present paper is to consider the effect of arbitrary inhomogeneous continuous Lorentz transformations in order to obtain general Poincaré-invariant localization (GPIL). Our main result will be that localization is LIL if and only if it is GPIL. In II and III, LIL was obtained in spite of the fact that the position operators were not 4-vector operators but only 3-vector operators. In this paper, following Fleming, we construct formally covariant position operators from the 3-vector operators. We prove that their eigenfunctions are just the localized states defined in the GPIL sense.

I. INTRODUCTION

A. General

In papers II (Ref. 1) and III (Ref. 2) of the present series, the problem of localizability of elementary systems in relativistic quantum mechanics was considered by requiring the consistency of the description of localization from different inertial frames as the only basic assumption; the position operators were uniquely defined (except for a constant in the spin-$\frac{1}{2}$, nonzero-mass case[3]), and their form was found explicitly. In II the nonzero-mass systems of spin 0 and $\frac{1}{2}$ were considered, and in III the zero-mass systems of spin 0, $\frac{1}{2}$, and 1 were considered.

The basic assumption (Postulate 4 of II) is a self-consistency one: If a state is a 1-localized state[1] (i.e., an eigenstate of only one component of the position operator $\vec{X}$) in a two-dimensional plane, e.g., $X^3=0$ at $t=0$, then, making a homogeneous continuous Lorentz transformation such that the region of localization remains invariant, the transformed state is again a localized state in the same region. This is what was called *Lorentz-invariant localization* (LIL). It is clear that only very special cases of homogeneous Lorentz transformations leave invariant those two-dimensional planes.

One of the purposes of the present paper is to consider the effect of arbitrary inhomogeneous continuous Lorentz transformations on states