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Classical Radiation of Accelerated Electrons. II. A Quantum Viewpoint*

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The known classical radiation spectrum of a high-energy charged particle in a homogeneous magnetic field is rederived. The method applies, and illuminates, an exact (to order α) expression for the inverse propagation function of a spinless particle in a homogeneous field. An erratum list for paper I is appended.

For a long time I have wanted to reexamine a classic situation of classical electrodynamics, that of high-energy charged particles radiating in a homogeneous magnetic field, from the modern quantum viewpoint that employs the machinery of propagation (Green's) functions. Since the electromagnetic and relativistic aspects of the problem are quite transparent, the comparison should be instructive in giving the more abstract quantum procedure a concrete interpretation in a particular instance. And, as an added bonus, the necessary ability to treat motion in magnetic fields that goes beyond the lowest orders in a perturbative expansion should be helpful in answering questions about very strong fields, to which recent astrophysical speculations have directed attention. This paper is devoted to describing one such procedure, and applying it to rederive (for a spin-0 particle) the known classical radiation result.[1] Another method is indicated in a separate paper of Yildiz. A subsequent joint paper will contain the analogous spin-$\frac{1}{2}$ calculation, and a discussion of the anomalous magnetic moment in strong fields.

The language and methodology of source theory[2] will be used (which should not seriously impede readers who are untutored in this art). The initial action expression of spin-0 charged particles with mass m,

$$\int(dx)[K(x)\phi(x)-\tfrac{1}{2}\phi(x)(\Pi^2+m^2)\phi(x)]\,,$$

$$\Pi=(1/i)\partial-eqA\,, \qquad (1)$$

is supplemented by the action contribution associated with the exchange of one virtual photon [cf. Eq. (4-14.2) of PSF II[2]],

$$-\tfrac{1}{2}\int(dx)(dx')\phi(x)M(x,x')\phi(x')\,. \qquad (2)$$

Here, written in a symbolic notation, we have

$$M = ie^2 \int \frac{(dk)}{(2\pi)^4}(2\Pi - k)\frac{1}{k^2}\frac{1}{(\Pi - k)^2 + m^2}(2\Pi - k) + \text{c.t.} \ , \tag{3}$$

where the contact term (c.t.) is a linear function of Π^2 that is designed to satisfy the normalization conditions. These require that M and its first derivative with respect to Π^2 vanish, in the null-field situation, at $\Pi^2 + m^2 = 0$. The stationary action principle, applied to the sum of (1) and (2), yields the field solution that is conveyed symbolically by

$$\phi = \bar{\Delta} K \ , \tag{4}$$

where

$$\bar{\Delta}^{-1} = \Pi^2 + m^2 + M \tag{5}$$

is the inverse of the modified (to order α) propagation function.

The exact evaluation of M for an applied homogeneous electromagnetic field has two basic ingredients. The first is the exponential representation of the particle and photon propagation functions,

$$\frac{1}{(\Pi - k)^2 + m^2 - i\epsilon} = i\int_0^\infty ds_1 e^{-is_1[(\Pi - k)^2 + m^2]} \ ,$$
$$\frac{1}{k^2 - i\epsilon} = i\int_0^\infty ds_2 e^{-is_2 k^2} \ , \tag{6}$$

together with their product:

$$\frac{1}{k^2}\frac{1}{(\Pi - k)^2 + m^2} = -\int_0^\infty ds\, s \int_0^1 du\, e^{-ism^2 u} e^{-isH}. \tag{7}$$

In the latter we have used the parametrization

$$s_1 = su, \quad s_2 = s(1-u) \tag{8}$$

and introduced the "Hamiltonian"

$$H = u(\Pi - k)^2 + (1-u)k^2 = (k - u\Pi)^2 + u(1-u)\Pi^2 \ . \tag{9}$$

The second one is the replacement of the k integration by an algebraic procedure associated with the vector ξ_μ that is complementary to k_μ,

$$[\xi_\mu, k_\nu] = i g_{\mu\nu} \ . \tag{10}$$

Then, on using the four-dimensional transformation functions (primes to designate eigenvalues are omitted)

$$\langle \xi | k \rangle = \frac{1}{(2\pi)^2} e^{i\xi k} \ ,$$
$$\langle k | \xi \rangle = \frac{1}{(2\pi)^2} e^{-i\xi k} \ , \tag{11}$$

we have

$$\langle \xi = 0 | f(k) | \xi = 0 \rangle = \int \langle \xi = 0 | k \rangle (dk) f(k) \langle k | \xi = 0 \rangle$$
$$= \int \frac{(dk)}{(2\pi)^4} f(k) \ . \tag{12}$$

The two devices transform M into

$$M = -ie^2 \int ds\, s\, du\, e^{-is m^2 u} \langle (2\Pi - k) e^{-isH} (2\Pi - k) \rangle + \text{c.t.} \ , \tag{13}$$

where the expectation value refers to the $\xi = 0$ state.

The "time" development described by H is made explicit by introducing quantities such as

$$\xi(s) = e^{isH} \xi e^{-isH} \ , \tag{14}$$

which obey the equations of motion

$$\frac{d\xi(s)}{ds} = \frac{1}{i}[\xi(s), H] \ . \tag{15}$$

The full set of equations of motion is

$$\frac{dk(s)}{ds} = 0 \ ,$$
$$\frac{d\xi(s)}{ds} = 2[k - u\Pi(s)] \ , \tag{16}$$
$$\frac{d\Pi(s)}{ds} = 2ueqF[\Pi(s) - k] \ ,$$

where the last equation applies the commutator

$$[\Pi_\mu, \Pi_\nu] = [-i\partial_\mu - eqA_\mu, -i\partial_\nu - eqA_\nu]$$
$$= ieqF_{\mu\nu} \ . \tag{17}$$

The simplicity of the homogeneous field situation is the linearity of the equations of motion, which permits their exact solution.[3] Thus, writing the last equation of (16) as

$$\frac{d}{ds}[e^{-2ueqFs}\Pi(s)] = \frac{d}{ds}[e^{-2ueqFs}k] \ , \tag{18}$$

we get

$$\Pi(s) = e^{2ueqFs}\Pi + (1 - e^{2ueqFs})k \ . \tag{19}$$

This is followed by the integration of the ξ equation:

$$\xi(s) = \xi + 2ks - \frac{e^{2ueqFs} - 1}{eqF}\Pi$$
$$- 2uks + \frac{e^{2ueqFs} - 1}{eqF}k \ , \tag{20}$$

or

$$eqF[\xi(s) - \xi] = Dk - A\Pi \ , \tag{21}$$

where

$$A = e^{2ueqFs} - 1 \tag{22}$$

and

$$D = e^{2ueqFs} - 1 + 2(1-u)eqFs \ . \tag{23}$$

The solution of the equations of motion is used to rewrite the expectation value in (13) as

$$\begin{aligned}\langle(2\Pi - k)e^{-isH}(2\Pi - k)\rangle &= \langle e^{-isH}[2\Pi(s) - k](2\Pi - k)\rangle \\ &= \langle e^{-isH}\rangle 4\Pi(1 + A^T)\Pi - \langle e^{-isH}k\rangle 2(2 + A + 2A^T)\Pi \\ &\quad + \langle e^{-isH}k(1+2A)k\rangle \ ,\end{aligned} \tag{24}$$

where the transposed form of A is

$$A^T = e^{-2ueqFs} - 1 \ , \tag{25}$$

according to the antisymmetry of $F_{\mu\nu}$. The expectation values with one and two additional factors of k are reduced to the basic expectation value, $\langle e^{-isH}\rangle$, as follows. We first note that

$$\begin{aligned}0 &= \langle[\xi, e^{-isH}]\rangle \\ &= \langle e^{-isH}[\xi(s) - \xi]\rangle \ ,\end{aligned} \tag{26}$$

which implies [Eq. (21)]

$$\langle e^{-isH}k\rangle = \langle e^{-isH}\rangle \frac{A}{D}\Pi \ . \tag{27}$$

Similarly,

$$\begin{aligned}0 &= \langle[\xi_\mu, [\xi_\nu, e^{-isH}]]\rangle \\ &= \langle e^{-isH}[\xi_\mu(s)\xi_\nu(s) - \xi_\mu(s)\xi_\nu - \xi_\nu(s)\xi_\mu + \xi_\mu\xi_\nu]\rangle \\ &= \langle e^{-isH}[\xi_\mu(s) - \xi_\mu][\xi_\nu(s) - \xi_\nu]\rangle + \langle e^{-isH}[\xi_\mu, \xi_\nu(s)]\rangle\end{aligned} \tag{28}$$

leads to

$$\begin{aligned}0 &= \langle e^{-isH}(Dk - A\Pi)_\mu (Dk - A\Pi)_\nu\rangle \\ &\quad + \langle e^{-isH}\rangle i(eqFD^T)_{\mu\nu}\end{aligned} \tag{29}$$

and then

$$\langle e^{-isH}k_\mu k_\nu\rangle = \langle e^{-isH}\rangle\left[\left(\frac{A}{D}\Pi\right)_\nu\left(\frac{A}{D}\Pi\right)_\mu - i\left(\frac{eqF}{D}\right)_{\mu\nu}\right] \ . \tag{30}$$

One can verify that the right-hand side matches the left-hand side in its symmetry in μ and ν. The implied algebraic property is

$$AA^T + D + D^T = 0 \ . \tag{31}$$

It is confirmed by noting, first, that

$$AA^T + A + A^T = 0 \tag{32}$$

and, then, that $D - A$ is an antisymmetrical matrix.

The material for the main task, the evaluation of $\langle e^{-isH}\rangle$, is now at hand. We construct a differential equation,

$$\begin{aligned}i\frac{\partial}{\partial s}\langle e^{-isH}\rangle &= \langle e^{-isH}H\rangle \\ &= \langle e^{-isH}\rangle u\Pi^2 - \langle e^{-isH}k\rangle 2u\Pi + \langle e^{-isH}k^2\rangle \ ,\end{aligned} \tag{33}$$

from which, with the aid of Eqs. (27) and (30), it immediately follows that

$$\begin{aligned}i\frac{\partial}{\partial s}\ln\langle e^{-isH}\rangle &= u\Pi^2 - 2u\Pi\frac{A^T}{D^T}\Pi \\ &\quad + \Pi\frac{A^T}{D^T}\frac{A}{D}\Pi - i\,\mathrm{tr}\left(\frac{eqF}{D}\right) \ ,\end{aligned} \tag{34}$$

where the trace refers only to the vector indices. In order to have a symmetrical matrix in the Π quadratic form, we rewrite this structure as

$$u\Pi^2 + \Pi\left[\frac{A^T}{D^T}\frac{A}{D} - u\left(\frac{A}{D} + \frac{A^T}{D^T}\right)\right]\Pi - ieq\,\mathrm{tr}\left(F\frac{uA+1}{D}\right) \ , \tag{35}$$

which makes use of the commutator (17). Now,

$$\frac{A^T}{D^T}\frac{A}{D} - u\left(\frac{A}{D} + \frac{A^T}{D^T}\right) = -\frac{1}{2eqF}\left(\frac{\partial D/\partial s}{D} - \frac{\partial D^T/\partial s}{D^T}\right), \tag{36}$$

since

$$\begin{aligned}\frac{\partial D}{\partial s} &= 2eqF(uA + 1) \ , \\ \frac{\partial D^T}{\partial s} &= -2eqF(uA^T + 1) \ ,\end{aligned} \tag{37}$$

while A and D obey (31), in which A and A^T are commutative. Accordingly, (35) becomes

$$u\Pi^2 + \Pi\left(-\frac{1}{2eqF}\right)\frac{\partial}{\partial s}\ln\left(-\frac{D}{D^T}\right)\Pi - \tfrac{1}{2}i\frac{\partial}{\partial s}\mathrm{tr}\ln D \ , \tag{38}$$

and the integrated version is

$$\begin{aligned}\langle e^{-isH}\rangle &= C\frac{1}{[\det(D/2eqF)]^{1/2}} \\ &\quad \times \exp\left\{-i\left[su\Pi^2 - \Pi\frac{1}{2eqF}\ln\left(-\frac{D}{D^T}\right)\Pi\right]\right\} \ .\end{aligned} \tag{39}$$

The F-dependent factor is inserted into the determinant in order to simplify the form of the integration constant C.

To evaluate C, we consider the limit of small s, where

$$\begin{aligned}\frac{D}{2eqF} &= s + u^2 eqFs^2 + \cdots, \\ -\frac{D}{D^T} &= 1 + 2u^2 eqFs + \cdots \ .\end{aligned} \tag{40}$$

Then, (39) exhibits the dominant behavior

$$\langle e^{-isH}\rangle \sim C\frac{1}{s^2} , \tag{41}$$

in view of the four-dimensional nature of the determinant. The singularity at $s=0$ arises from the large values of k that are increasingly demanded, as $s\to 0$, by complimentarity with $\xi=0$. Accordingly, the limiting structure is given by the elementary k integral

$$\int \frac{(dk)}{(2\pi)^4} e^{-isk^2} = \frac{1}{(4\pi)^2}\frac{1}{is^2} , \tag{42}$$

and

$$C = -\frac{i}{(4\pi)^2} . \tag{43}$$

We present the result as

$$\langle e^{-isH}\rangle = -\frac{i}{(4\pi)^2}\frac{e^{-isu(1-u)\Pi^2}}{s^2}\left(\det\frac{2eqFs}{D}\right)^{1/2} \times \exp\left\{i\Pi\left[\frac{1}{2eqF}\ln\left(-\frac{D}{D^T}\right) - u^2 s\right]\Pi\right\} , \tag{44}$$

which is so written that the last two factors approach unity as $F\to 0$. The remainder, the structure of $\langle e^{-isH}\rangle$ for $F=0$, is immediately evident from the second version of H in Eq. (9) and the integral (42).

We now have before us all the ingredients to construct M as the double parametric integral of Eq. (13). It is, however, not necessary to display M in detail in order to make the principal application of this paper – the derivation of the classical radiation spectrum. Since the properties of the real charged particle are essentially characterized by $\Pi^2 + m^2 = 0$, we only need M for this circumstance. And, since radiative decay is the question of interest, it is only the imaginary part of M that is required. There is, furthermore, a simplification associated with the concentration on classical radiation. To appreciate it, let us note that in the classical limit the k integral of e^{-isH} should be dominated by the point of stationary phase,

$$\frac{\delta H}{\partial k} = 2(k - u\Pi) = 0 . \tag{45}$$

The value of k thus selected, $k = u\Pi$, is not that of a real photon, in general,

$$-k^2 = -u^2\Pi^2 = u^2 m^2 , \tag{46}$$

but becomes so if u is sufficiently small. In this circumstance, we can express the energy of the radiated photon, $k^0=\omega$, relative to the energy of the particle, $\Pi^0 = E$, by

$$\frac{\omega}{E} = u \ll 1 , \tag{47}$$

which is evidently a classical restriction. With this identification, the u integral of $\operatorname{Im} M$ becomes a spectral integral for the radiation.

There is yet another simplification associated with the restriction to high particle energy,

$$E \gg m . \tag{48}$$

We first remark that the periodicity of motion in the magnetic field $\vec{H}$ has its representation in the exponential function e^{2ueqFs}, where the nonzero eigenvalues of F are $\pm iH$. [Recall that $\operatorname{tr} F^2 = -F^{\mu\nu}F_{\mu\nu} = 2(\vec{E}^2 - \vec{H}^2)$.] This gives the identification

$$2eHus = \omega_0 \tau , \tag{49}$$

where τ is a time variable and ω_0 is the rotational frequency. According to the classical equations of motion [$\dot{\vec{p}} = e\vec{v}\times\vec{H}$], the high-energy form of ω_0 is

$$\omega_0 = \frac{eH}{E} , \tag{50}$$

and one can write (49) as

$$2Eus = \tau . \tag{51}$$

Now, the point about high energies is this. Only a small fraction of the orbit, $\sim m/E$, is involved in classical radiation toward a particular direction. We therefore expect that the dominant contributions will come from values of us such that

$$2eHus \sim \frac{m}{E} \ll 1 . \tag{52}$$

Under these circumstances the logarithmic function in (44) has the leading terms

$$\frac{1}{2eqF}\ln\left(-\frac{D}{D^T}\right) \simeq u^2 s + \tfrac{1}{3}u^4(1-u)^2(eqF)^2 s^3 , \tag{53}$$

which are comparable, since

$$\Pi\frac{1}{2eqF}\ln\left(-\frac{D}{D^T}\right)\Pi \simeq -m^2u^2s - \tfrac{1}{3}u^4(1-u)^2(eHE)^2s^3 = -m^2u^2s[1+\tfrac{1}{3}(1-u)^2(eHusE/m)^2] . \tag{54}$$

The evaluation used here,

$$\Pi(eqF)^2\Pi \simeq -(eHE)^2 , \tag{55}$$

assumes zero momentum parallel to the magnetic field, confining the motion to the plane perpendicular to the field. In the strict classical limit under consideration, we should also replace $1-u$ by unity,[4] in (54). As for the determinant of (Eq. 44), the expansion

$$\frac{D}{2eqFs}=1+u^2eqFs+\tfrac{2}{3}u^3(eqFs)^2+\cdots \qquad (56)$$

and the evaluation

$$\left(\det\frac{D}{2eqFs}\right)^2=\det\frac{D}{2eqFs}\left(\frac{D}{2eqFs}\right)^T \simeq\det[1+\tfrac{4}{3}u^3(eqFs)^2] =1-\tfrac{2}{3}u(2eHus)^2 \qquad (57)$$

show that the determinant reduces to unity in the classical high-energy limit, where both u and $2eHus$ are small quantities.

The terms of (24) that have one or two additional factors of k clearly become relatively negligible in the classical limit (as one can verify). In the high-energy limit, we also have

$$A^T\simeq-2ueqFs+2(eqFus)^2\ , \qquad (58)$$

where the antisymmetrical F term, which introduces the commutator $[\Pi,\Pi]$, is a negligible quantum correction. Accordingly,

$$\frac{\langle(2\Pi-k)e^{-isH}(2\Pi-k)\rangle}{\langle e^{-isH}\rangle}\simeq-4[m^2+2(eHEus)^2] =-4E^2[(m^2/E^2)+\tfrac{1}{2}(\omega_0\tau)^2]\ , \qquad (59)$$

where the last version begins the process of introducing the classical time variable τ, for comparison with the known result. The other basic combination is (54), which now reads

$$\Pi\frac{1}{2eqF}\ln\left(-\frac{D}{D^T}\right)\Pi\simeq-\omega\tau[\tfrac{1}{2}(m^2/E^2)+\tfrac{1}{24}(\omega_0\tau)^2]\ . \qquad (60)$$

Putting together the limiting forms of the various parts of M gives

$$M\to\frac{\alpha}{\pi}E\int d\omega\int_0^\infty\frac{d\tau}{\tau}\left\{\left(\frac{m^2}{E^2}+\tfrac{1}{2}\omega_0^2\tau^2\right)\exp\left[-i\omega\left(\frac{1}{2}\frac{m^2}{E^2}\tau+\tfrac{1}{24}\omega_0^2\tau^3\right)\right]-\frac{m^2}{E^2}\exp\left(-i\omega\frac{1}{2}\frac{m^2}{E^2}\tau\right)\right\}, \qquad (61)$$

which now incorporates the contact term that is required to make M vanish in the absence of the magnetic field ($\omega_0=0$). What is needed for the description of radiative decay is

$$-\frac{1}{E}\mathrm{Im}M=\int d\omega\frac{1}{\omega}P(\omega)\ , \qquad (62)$$

where

$$P(\omega)=\frac{\alpha}{\pi}\omega\left[\int_0^\infty d\tau\left(\frac{m^2}{E^2}+\tfrac{1}{2}\omega_0^2\tau^2\right)\frac{\sin\omega[\frac{1}{2}(m^2/E^2)\tau+\frac{1}{24}\omega_0^2\tau^3]}{\tau}-\tfrac{1}{2}\pi\frac{m^2}{E^2}\right] =\frac{\alpha}{\pi}\omega\frac{m^2}{E^2}\left[\int_0^\infty dx(1+2x^2)\frac{\sin\frac{3}{2}\xi(x+\frac{1}{3}x^3)}{x}-\tfrac{1}{2}\pi\right]; \qquad (63)$$

the last form introduces the variables

$$x=\tfrac{1}{2}\omega_0\tau\frac{E}{m}\ ,\qquad \xi=\tfrac{2}{3}\frac{\omega}{\omega_0}\left(\frac{m}{E}\right)^3 . \qquad (64)$$

The physical identification of $P(\omega)$ follows on writing the inverse propagation function (omitting $\mathrm{Re}M$) as

$$\vec{\Pi}^2+m^2-\left[E+\tfrac{1}{2}i\left(-\frac{1}{E}\mathrm{Im}M\right)\right]^2\ , \qquad (65)$$

which displays (62) as the damping constant of the system. Therefore $\omega^{-1}P(\omega)$ is the probability per unit time for radiation into a unit ω interval, and $P(\omega)$ is the spectral distribution of the radiated power. The results stated in Eq. (63) coincide with the classically derived ones contained in Eqs. (II-5) and (II-7) of paper I.

APPENDIX

Paper I seems to have escaped proofreading, since it contains a number of rather obvious typographical errors. Among these are the following:

(1) In the first of the three equations of (I.30), read

$$\frac{d\Omega}{4\pi}\ \text{for}\ \frac{d\Omega}{4\omega}.$$

(2) In Eq. (I.44), read $\sin^2\theta\cos^2\phi$ for $\sin^2\theta\cos^2\theta$.

(3) In the sixth line after Eq. (II.9), read (II.2) for (II.7).

(4) For the fractional power occurring in the denominator of the unnumbered equation preceding (II.18), read $\frac{5}{2}$ instead of $\frac{7}{3}$.

(5) For the fractional power appearing in the denominator at the end of Eq. (II.19), read $\frac{1}{2}$ instead of $\frac{1}{3}$.

(6) In Eq. (II.20), insert

$$\frac{\Gamma(\frac{1}{3})}{2}\left(\frac{\omega}{2\omega_c}\right)^{2/3} \text{ instead of } \frac{\Gamma(\frac{2}{3})}{2}\left(\frac{\omega}{2\omega_c}\right)^{1/3}.$$

(7) In Eq. (II.37), read $\frac{\pi^2}{8}$ instead of $\frac{\pi^2}{4}$.

(8) In Eq. (III.31), read $\frac{n}{2n_c}$ instead of $\frac{n}{n_c}$.

(9) The denominator of Eq. (III.32) should contain π^3 instead of π^2.

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[1]J. Schwinger, Phys. Rev. 75, 1912 (1949), referred to as paper I.

[2]A systematic development of this new approach to particle theory is described in J. Schwinger, *Particles, Sources, and Fields I* (Addison-Wesley, Reading, Mass., 1970) and Particles, Sources, and Fields II (to be published).

[3]This procedure resembles that introduced in an earlier paper [J. Schwinger, Phys. Rev. 82, 664 (1951)], but is here applied to the system of charged particle and photon.

[4]Retaining u here gives the essence of the first quantum correction [J. Schwinger, Proc. Nat. Acad. Sci. U. S. 40, 132 (1954), and the Russian literature cited in *Synchrotron Radiation*, A. A. Sokolov and I. M. Ternov (Pergamon, New York, 1968)].

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Anharmonic Oscillator

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We solve the anharmonic-oscillator problem as given by Bender and Wu to include the $\hbar^4$ terms for the energy eignevalues and the $\hbar^2$ terms for the eigenfunctions by means of the Miller and Good modified WKB method. This is done by accepting the harmonic oscillator as a solved problem. We see that in doing so, not only can we get better energy eigenvalues, but also we can get improved eigenfunctions.

I. INTRODUCTION

We will discuss the familiar anharmonic-oscillator problem. We are concerned here with the simple one-dimensional oscillator with real and positive oscillator strengths ($k>0$, $a>0$) as given by the potential

$$V(x)=\tfrac{1}{2}kx^2+ax^4, \quad k>0 \text{ and } a>0. \tag{1}$$

When perturbation theory is used to solve the anharmonic-oscillator problem based on the solved problem of the harmonic oscillator, the perturbation series for the energy diverges and even changes sign. A clear account of this is illustrated by Chan, Stelman, and Thompson[1] in their Table III.

Such a simple but important problem has attracted the attention of both the field theorists and the chemists. The former are interested in it because they desire to build a model field theory on it. The latter are interested mainly due to the anharmonic bonding problem. The reader is referred to Bender and Wu[2] as well as the references listed there for the detailed reason why the field theorists are interested in the problem. As for the chemists, Chen, Stelman, and Thompson[1] have given a good account.

Of course an evaluation of the energy eigenvalues