all three variables x, y, and z).

[15]For a discussion of this point and its plausibility on physical grounds see, for example, A. S. Wightman, Rev. Mod. Phys. 34, 845 (1962) and references, regarding it, cited therein. This condition has been called the regularity condition there. Although this condition is not postulated by Wightman for photon fields, it becomes necessary in the extension of Wightman's idea of localizability to that of weak localizability of Ref. 11 above. Once the $\hat{\phi}$ is assumed to be continuous, the definition that the field ϕ is source-free if and only if $F(p, q; \omega) = 0$ almost everywhere may be restated without the almost everywhere condition. This is because the Lebesgue measure has support on the whole translation group, and any function which is continuous and zero almost everywhere, with respect to such a measure, is necessarily zero everywhere.

[16]See, for example, B. A. Fuks, *Introduction to the Theory of Analytic Functions of Several Complex Variables* (Am. Math. Soc., Providence, R. I., 1963).

[17]A mathematically rigorous definition of fields is given in R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (Benjamin, New York, 1964).

[18]Here the distribution $\langle \phi_1(x_1)\phi_2(x_2)\rangle$ is meant.

[19]Certain aspects of the distribution theory of the angular spectrum are developed in Part III of Ref. 5 and in Ref. 10.

[20]See, for example, E. M. Henley and W. Thirring, *Elementary Quantum Field Theory* (McGraw-Hill, New York, 1962).

[21]See, for example, I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1964), Vol. I.

[22]One ought to point out, however, that for a point charge M is infinite, as is obvious from Eq. (4.4).

PHYSICAL REVIEW D VOLUME 7, NUMBER 6 15 MARCH 1973

New Dual Models of Currents*‡

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Through a real and a complex linear combination of two conformal scalar fields, we obtain the rubber-band field of the Virasoro-Shapiro model (VSM) and a complex field suited for dual current models. A Möbius-invariant pionic off-shell extension of the VSM is then presented. For $k_i^2 = 1$, it has conformal invariance, and the associated sets of gauge as well as supergauge identities are shown to be operative. For other specific values of k_i^2, the new amplitude gives a correct continuation away from $k_i^2 = 1$. By combined use of the complex field and the Sakita-Gervais fields, we construct SU(1,1) and/or conformal-invariant spinor generalizations of the Rebbi-Drummond model. Their analytic structure is studied. In the particular case of the simple quark model with unitary spin of Bardakci and Halpern, we find two conformal-invariant conserved vector currents. One of these is akin to an axial-vector current as it exhibits a dual analog of partial conservation of axial-vector current.

I. INTRODUCTION

The duality principle and its elegant mathematical realizations, embodied in the N-point Veneziano amplitude[1] and subsequent generalizations,[2-6] have proved rich in dynamical structure. Though these discoveries in "axiomatic" duality are still far from presenting a realistic picture of hadronic reactions, they may yet provide the basic elements for building a correct theory of strong interactions.

These models share a serious deficiency at the level of the Born term, namely the problem of off-shell extension. It is closely tied to the problem of proper determination of electromagnetic and weak currents in the context of a dual theory of hadrons.[7] As photons and leptons usually are assumed to couple to hadronic currents, this connection is particularly marked if the currents (be they scalar, pseudoscalar, vector, or axial-vector) reflect the meromorphic structure of the hadronic amplitudes. Indeed, in a world of resonance-dominated hadronic S matrices, one would expect that currents also be dominated by resonances. The infinite sequence of vector mesons in the dual hadronic spectrum thus furnishes a framework for a generalized vector dominance. The importance of dual electromagnetic and weak currents is further underscored by their roles as probes into strong-interaction dynamics in the testing of current algebra, chiral constraints, and scaling laws. Therefore, a solution to the problem of constructing dual off-shell amplitudes, conserved vector currents (CVC), and partially conserved axial-vector currents (PCAC) stands as a necessary step toward a more complete dual theory.

Since Veneziano's classic paper,[8] there have been many attempts to build dual current amplitudes.[9] These models strive to achieve a set of minimal requirements desirable for the currents:

(1) all singularities as simple poles in k_i^2, the invariant current momentum transfer squared, and in the subenergy variables

$$s_{1,\ldots,l} = -\left(\sum_{i=1}^{l} k_i\right)^2 ;$$

(2) complete factorization, with a particle spectrum preferably identical to the spectrum of the on-shell amplitude;

(3) good asymptotic behavior in k_i^2 and $s_{1,\ldots,l}$;

(4) divergence conditions from CVC, PCAC, and current algebra,

(5) duality, planar or nonplanar, though planar duality seems desirable if internal symmetries are to be grafted on easily in the manner of Paton and Chan,[10] and if exotic amplitudes are to be avoided.

The off-shell dual model proposed by Rebbi[11] and Drummond[12] made a promising beginning in view of these requirements. Clavelli and Ramond[13] used a new SU(1, 1) coupling scheme and presented a unified treatment of the various nonplanar models both on and off the mass shell, notably of the off-shell Virasoro-Shapiro model[14] (VSM) and the Rebbi-Drummond model (RDM). This paper is a spinor generalization of the work of Clavelli and Ramond.

Through a real and a complex linear combination of two real conformal scalar fields, with Neumann and Dirichlet boundary conditions, respectively, we find the rubber-band field of Yoshimura[15] and Del Giudice and Di Vecchia[16] on the one hand, and the complex field needed for dual currents on the other. Equipped with these fields and the conformal spinor fields of Sakita and Gervais (SG),[5] we build a new class of dual models of currents.[12,17]

The combining of a Möbius-invariant volume element, two commuting Neveu-Schwartz (NS) fields,[4] and the rubber-band field leads to a new currentlike vertex. For the mass shell limit $k^2 = 1$, the corresponding amplitude yields a conformal invariant NS-like extension of the fully symmetric VSM. For other specific values of k^2, it provides a correct continuation in k^2 for the new amplitude.

Using the SG spinor fields and the complex field with its finite exponential vertex transforming as a conformal scalar for *any* k^2, we find off-shell amplitudes for a subclass of the generalized conformal dual models of Sakita and Gervais.[5] Except for one conformal scalar current, our models possess SU(1, 1) invariance only and constitute spinor generalizations of the Rebbi-Drummond model. If the spinors carry quarklike SU(3) labels, an off-shell extrapolation for the simple Bardakci and Halpern (BH) dual quark model[3] results. However, if they are Lorentz spinors, an off-shell BH dual pion model[18] is obtained. For the first model, we also find two conformally invariant conserved vector currents. One of these is akin to an axial-vector current as it gives a dual analog of PCAC.

Our paper is organized as follows. In Sec. II we present the dual apparatus necessary for our study. The formalism of conformal transformations and their associated irreducible fields is briefly reviewed. The conformal spin-0 and -$\frac{1}{2}$ fields receive particular attention with respect to their quantized forms and transformation laws under the action of the conformal group.

In Sec. III, after a discussion of the mechanism of conformal invariance in dual models, we construct the rubber-band field and the complex field. To achieve economy of presentation, we strike a middle course between the Lagrangian[5] and the group-theory[19] methods. Thus, while the Nambu-Susskind[20] picture is used to motivate dual emission vertices, the group-theoretical rules are invoked for the construction of manifestly cyclic-symmetric dual amplitudes.

In Sec. IV, the currentlike pionic extension of the VSM is presented. Explicit factorization is carried out, and in the $k_i^2 = 1$ case, two sets of Virasoro identities, as well as two sets of supergauge identities, are shown to be operative.

In Sec. V we construct spinor generalizations to the Rebbi-Drummond model. Feynman-like rules are derived. To illustrate the general analytic structure of our models, for a simple case we compute explicitly the two-point function, elastic form factor, and photoproduction amplitude. For the latter, we study its Regge and fixed-pole structures, as well as its Bjorken scaling limit.

In Sec. VI, we present two conserved dual vector currents exhibiting generalized vector dominance for the simple dual quark model with unitary spin.[3]

II. CONFORMAL-INVARIANCE GROUP AND IRREDUCIBLE FIELDS

Duality as realized in dual models is a precise concept. Thus the SU(1,1) invariance[21] of an amplitude guarantees its dual character while its conformal invariance[22] allows the elimination of all ghosts. In the Lagrangian approach of Virasoro, Sakita, and Gervais, an elegant theorem is reached: Given any conformal-invariant two-dimensional local field theory, one can deduce a dual model. In addition to providing a unifying viewpoint for the investigation of dual amplitudes, this method best reveals the basic relevance of conformal transformations in two dimensions. Let $Z'=x'+iy'=F(Z)$ be a conformal mapping; then

$$\frac{dZ'}{dZ}=\frac{\partial Z'}{\partial Z}$$

$$=\left(\frac{\partial Z'^*}{\partial Z^*}\right)^* \tag{2.1}$$

$$=\mu(x,y)e^{-i\theta(x,y)}\,. \tag{2.2}$$

Equation (2.2) can be construed as a local $D\times O_2$ group; D and O_2 are the dilatation and the two-dimensional rotation groups with their respective parameters $\mu(x,y)$ and $\theta(x,y)$.

Since in dual field theory, conformal transformations usurp the role of Lorentz transformations in two-space, the basic objects to consider are the irreducible fields of the conformal group[5] $D\times O_2$,

$$\Psi'_{d,j}(x',y')=\left(\frac{\partial Z'}{\partial Z}\right)^{(d-j)/2}\left(\frac{\partial Z'^*}{\partial Z^*}\right)^{(d+j)/2}\Psi_{d,j}(x,y). \tag{2.3}$$

The covariant field $\Psi_{d,j}$ is characterized by its signature d (a complex number) and j ($2j$ is an integer), d being its dimension and j its conformal spin. Moreover, Ψ remains irreducible under the subgroup SL(2C) and can also be a representation of space-time and internal symmetry groups.

Dual models are built[5,6] from conformal actions $I=\int_D d^2Z\,\mathcal{L}$. $\mathcal{L}$, the Lagrangian density, is made of tensorial products and contractions of irreducible fields (2.3). D may be the Koba-Nielsen unit disk or any other conformal equivalent domain. We find it convenient to work on the upper half plane $Z=X+iY$ and the strip $w=\nu+i\theta$ configurations with $-\infty\leqslant\nu\leqslant+\infty$, $0\leqslant\theta\leqslant\pi$. These are related by the simple mapping $Z=e^w$. The Lagrangian method provides ready-made conformal-invariant current vertices in its actions I and more importantly the quantized fields and conserved expressions which form the building blocks of our current models. However, we choose the use of group-theoretical rules[19] in the subsequent construction of dual amplitudes.

Given that d^2Z has $d=2$ and $j=0$, if the free Lagrangian density is to be bilinear in the fields, only two cases are possible:

$$\text{(a)}\quad \mathcal{L}=\frac{\partial\Phi}{\partial Z}\frac{\partial\Phi}{\partial Z^*}\,, \tag{2.4}$$

where Φ has $d=j=0$;

$$\text{(b)}\quad \mathcal{L}=i\overline{\Psi}\sigma_i\frac{\partial\Psi}{\partial x_i}\qquad (x_1=\mathrm{Re}\,Z,\ x_2=\mathrm{Im}\,Z), \tag{2.5}$$

where

$$\Psi=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix},$$

with $d=\mp j=-\frac{1}{2}$. The σ_i's are the 2×2 Pauli matrices. Notably σ_3 is the two-dimensional counterpart of γ_5. If interaction Lagrangians for the above fields are considered, again only two forms result,

$$\mathcal{L}_{\text{int}}=g_1(\overline{\Psi}\sigma_i(a+b\sigma_3)\Psi)(\overline{\Psi}\sigma_i(a'+b'\sigma_3)\Psi)$$
$$+g_2(\overline{\Psi}\sigma_i(c+d\sigma_3)\Psi)\partial_i\Phi\,. \tag{2.6}$$

We shall refer to the sum of (2.4), (2.5), and (2.6) as the dual Thirring model.[5]

When D is chosen to be the strip W, θ and $t=+i\nu$ are interpreted as space and pure imaginary time coordinates. Consequently, the mechanical system described by the action I must be of finite extent, such as a string of length π. One now allows the conformal scalar and spinor fields to propagate along the string. These fields obey their respective conformal-invariant wave equations with appropriate boundary conditions. For the system (2.4) we have

$$\frac{\partial^2\Phi_\mu}{\partial\theta^2}+\frac{\partial^2\Phi_\mu}{\partial\nu^2}=0\,, \tag{2.7}$$

the Laplace equation or Klein-Gordon equation with imaginary time. For the purpose of building dual current models, we shall need both the field $\Phi_{1\mu}$ associated with a string with free ends $\partial\Phi_{1\mu}/\partial\theta|_{\theta=0,\pi}=0$ and $\Phi_{2\mu}$ associated with a string with fixed ends $\Phi_{2\mu}|_{\theta=0,\pi}=0$. Standard canonical quantization applied to (2.4) yields in the Heisenberg picture

$$\Phi_{1\mu}(\nu,\theta)=x_\mu+ip_\mu\ln(ZZ^*)$$
$$+\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n}}\left[(Z^n+Z^{*n})c_{n\mu}+(Z^{-n}+Z^{*-n})c_{n\mu}^\dagger\right], \tag{2.8}$$

with

$$[c_{n\mu},c_{m\lambda}^\dagger]=\delta_{\mu\lambda}\delta_{nm}, \tag{2.9}$$

$\mu,\lambda=1,2,3,4$, a Lorentz vector label,[24] and $m,n=1,2,3,\ldots,\infty$. (2.8) is the familiar Nambu-Susskind[20] field with its zeroth model $x_\mu+ip_\mu\ln(ZZ^*)$ identified with the center-of-mass motion of the string in space-time such that $[x_\mu,p_\lambda]=i\delta_{\mu\lambda}$. Similarly $\Phi_{2\mu}$ is given by

$$\Phi_{2\mu}(Z,Z^*)=i\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n}}\left[(Z^{*n}-Z^n)d_{n\mu}\right.$$
$$\left.+(Z^{-n}-Z^{*-n})d_{n\mu}^\dagger\right], \tag{2.10}$$

with

$$[d_{n\mu}, d^\dagger_{m\lambda}] = \delta_{\mu\lambda}\delta_{nm} . \tag{2.11}$$

Clearly (2.10) does not act at the boundary ($Z = Z^*$), but it may be relevant for the description of processes taking place within the strip.

As for the spin-$\frac{1}{2}$ field, (2.5) leads to the Dirac equations

$$\begin{aligned} &\sigma_i \partial_i \Psi = 0 , \\ &\partial_i \bar{\Psi} \sigma_i = 0 , \end{aligned} \tag{2.12}$$

which are the Cauchy-Riemann equations in disguise. Ψ and $\bar{\Psi}$ appear in on-shell meson amplitudes [3-6] as well as in models of dual fermions.[23] We confine ourselves to the meson sector with the relevant boundary conditions

$$\begin{aligned} &\Psi_1(\nu, 0) = \Psi_2(\nu, 0) , \\ &\Psi_1(\nu, \pi) = -\Psi_2(\nu, \pi) , \\ &\left.\frac{\partial \Psi_1}{\partial \theta}\right|_{\theta=0} = -\left.\frac{\partial \Psi_2}{\partial \theta}\right|_{\theta=0} , \\ &\left.\frac{\partial \Psi_1}{\partial \theta}\right|_{\theta=\pi} = \left.\frac{\partial \Psi_2}{\partial \theta}\right|_{\theta=\pi} . \end{aligned} \tag{2.13}$$

The associated Heisenberg fields [5] are

$$\begin{aligned} &\Psi_1 = \sum_{m=-\infty}^{\infty} Z^{m+1/2} a_m , \\ &\Psi_2 = \sum_{m=-\infty}^{\infty} Z^{*m+1/2} a_m , \\ &\bar{\Psi}_1 = \sum_{m=-\infty}^{\infty} Z^{*-(m+1/2)} a^\dagger_m , \\ &\bar{\Psi}_2 = \sum_{m=-\infty}^{\infty} Z^{-(m+1/2)} a^\dagger_m , \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} &\{a_m, a^\dagger_n\} = \delta_{mn} , \\ &\{a_m, a_n\} = \{a^\dagger_m, a^\dagger_n\} = 0 . \end{aligned} \tag{2.15}$$

The connection with the BH notation [3] is accomplished via

$$\begin{aligned} &a_m = b_m , \\ &a_{-1-m} = d^\dagger_m , \quad m = 0, 1, \ldots . \end{aligned} \tag{2.16}$$

The conformal transformations which leave D invariant are of crucial significance to the existence of infinite set(s) of ghost-eliminating gauge identities in dual resonance models.[22] The corresponding Virasoro gauge operators L_n are derived from the Schwinger action principle with the variations (to first order in ϵ)

$$\begin{aligned} &Z' = (1+\epsilon) Z, \quad n = 0 , \\ &Z'^n = Z^n + n\epsilon , \quad n \neq 0, \ n \gtrless 0 . \end{aligned} \tag{2.17}$$

These transformations leave the boundary conditions invariant. The L_n's associated with the field $\Phi_{1\mu}$ are[25]

$$\begin{aligned} L_0^{(c)} &= -p^2 - \sum_{n=1}^{\infty} n c^\dagger_n \cdot c_n , \\ L_n^{(c)} &= i\sqrt{2n}\, p \cdot c_n - \sum_{m=1}^{\infty} [m(n+m)]^{1/2} c_{m+n} \cdot c^\dagger_m \\ &\quad + \tfrac{1}{2} \sum_{n=1}^{n-1} [m(n-m)]^{1/2} c_{n-m} \cdot c_m , \end{aligned}$$

$$\text{for } n > 0. \tag{2.18}$$

Similar expressions are obtained for $L_0^{(d)}$ and $L_n^{(d)}$ connected to the field $\Phi_{2\mu}$ except the zeroth-mode p-dependent terms are absent.

For the spinor case, we have [5]

$$L_n^{(a)} = -\sum_{-\infty}^{\infty} a^\dagger_m a_{n+m} [m + \tfrac{1}{2}(n+1)] . \tag{2.19}$$

The $L_n^{(i)}$ ($i = a, c, d$) are such that $L_{-n} = L_n^\dagger$ ($n > 0$) and they satisfy the conformal algebra

$$[L_n^{(i)}, L_m^{(i)}] = (m-n) L_{n+m}^{(i)} , \tag{2.20}$$

except when $m + n = 0$, a c-number term irrelevant to our work has to be added to the commutator. $L_0^{(i)}$, $L_{\pm 1}^{(i)}$ hold special interest as they are the generators of the important subgroup SU(1, 1), the mathematical earmark of duality.

We can check that the rigorous conformal transformation laws for the above scalar and spinor fields are in their differential forms

$$[L_n^{(c,d)}, \Phi_{1,2}] = \left(Z^{-n+1} \frac{\partial}{\partial Z} + Z^{*-n+1} \frac{\partial}{\partial Z^*} \right) \Phi_{1,2} , \tag{2.21}$$

$$\left[L_n^{(a)}, \begin{pmatrix} \Psi_1 \\ \bar{\Psi}_1 \end{pmatrix} \right] = \begin{pmatrix} Z^{-n}\left(Z \frac{\partial}{\partial Z} - \frac{1}{2} n \right) \Psi_1 \\ Z^{*-n}\left(Z^* \frac{\partial}{\partial Z^*} - \frac{1}{2} n \right) \bar{\Psi}_1 \end{pmatrix} , \tag{2.22}$$

with $\bar{\Psi}_2$ and Ψ_2 transforming like Ψ_1 and $\bar{\Psi}_1$, respectively. By iteration, these equations give explicit forms of

$$\exp(\epsilon L_n^{(c,d)}) \Phi_{1,2}(Z, Z^*) \exp(\epsilon L_n^{(c,d)}) = \Phi_{1,2}(Z', Z'^*) , \tag{2.23}$$

$$\exp(\epsilon L_n^{(a)}) \begin{pmatrix} \dfrac{\Psi_1(Z)}{\sqrt{Z}} \\ \dfrac{\bar{\Psi}_1(Z)}{\sqrt{Z^*}} \end{pmatrix} \exp(-\epsilon L_n^{(a)}) = \begin{pmatrix} \left(\dfrac{\partial Z'}{\partial Z}\right)^{+1/2} \dfrac{\Psi_1(Z')}{\sqrt{Z!}} \\ \left(\dfrac{\partial Z'^*}{\partial Z^*}\right)^{+1/2} \dfrac{\bar{\Psi}_1(Z')}{\sqrt{Z'^*}} \end{pmatrix} , \tag{2.24}$$

where

$$\left(\frac{\partial Z'}{\partial Z}\right)=\left(\frac{Z'}{Z}\right)^{-n+1} \tag{2.25}$$

by (2.17), (2.23), and (2.24) concur with the assignment of $d=j=0$ for $\Phi_{1,2}$ and $-d=j=\frac{1}{2}$ for Ψ_1 and $\bar{\Psi}_2$ and $-d=-j=\frac{1}{2}$ for $\bar{\Psi}_1$ and Ψ_2.

The algebraic forms of this section are essential for our construction of current amplitudes, as are the group-theoretical rules to be discussed next.

III. DUAL ON-SHELL AND OFF-SHELL VERTICES

A. Conformal Invariance of Amplitudes

A solution to the problem of building off-shell dual amplitudes may be found in an analysis of the mechanism of conformal invariance of known on-shell dual models. Consider the generalized Veneziano model with intercept $a_0=1$ and the Virasoro-Shapiro model ($\alpha_0=2$).

In the first case, the corresponding Koba-Nielsen amplitude[21] is

$$A_N=\frac{1}{C}\prod_{i=1}^{N}\int dZ_i\prod_{i<j}^{N}|Z_i-Z_j|^{2k_i\cdot k_j}, \tag{3.1}$$

where in the real axis configuration the Z_i are real and ordered, with C the SU(1, 1) Haar measure, the infinite volume to be divided out of the integrals. It may be expressed in the factorized form of

$$A_N=\frac{1}{C}\left\langle 0\left|T\left(\prod_{i=1}^{N}\int dZ_i U(k_i,Z_2)\right)\right|0\right\rangle , \tag{3.2}$$

where the Z integrations range over the entire real axis, and T is the operator which causes the Z_i to be ordered;

$$U(k,Z)=\frac{:\exp[ik\cdot\Phi_1(Z,Z^*)]|_{Z=Z^*}:}{Z}, \tag{3.3}$$

with $\Phi_{1\mu}$ given by (2.8) and

$$:\exp(ik\cdot\Phi_1):=\exp[ik\cdot\Phi_1^{(0)}(Z)]\exp[ik\cdot\Phi_1^{(+)}(Z)]\times\exp[ik\cdot\Phi_1^{(-)}(Z)],$$

$$\Phi_{1\mu}^{(0)}(Z)=x_\mu+i2p_\mu\ln Z, \tag{3.4}$$

$$\Phi_{1\mu}^{(+)}(Z)=\sqrt{2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}c_{n\mu}^{\dagger}Z^{-n},$$

$$\Phi_{1\mu}^{(-)}(Z)=\sqrt{2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}c_{n\mu}Z^{n}.$$

The finite vertex (3.3) is obtained from a limiting procedure

$$U(k,Z)=\lim_{\epsilon\to 0}\frac{\exp[ik\cdot\Phi_1(Z)]}{E_\epsilon(Z,k^2)}, \tag{3.5}$$

where the non-normal ordered vertex $\exp(ik\cdot\Phi_1)$ is manifestly a conformal scalar but is divergent; the infinity is removed by the renormalization factor

$$E_\epsilon(Z,k^2)\underset{\epsilon\to 0}{\longrightarrow}\exp(k^2\ln|Z-Z|). \tag{3.6}$$

In the electrical analog picture of Nielsen,[26] the infinity in E_ϵ is the familiar divergence arising from the self-interaction energy of a charge k. In the limit of $\epsilon\to 0$, E transforms irreducibly as

$$E_\epsilon(Z',k^2)=E_\epsilon(Z,k^2)\left[\left|\frac{\partial Z'}{\partial Z}\right|^{k^2}+O(\epsilon)\right], \tag{3.7}$$

i.e., an object with $d=k^2$, $j=0$. Since $d=1$, $j=0$ for dZ, (3.7) dictates the mass-shell condition $k^2=1$ if A_N is to be conformally invariant.

Similarly, the Virasoro-Shapiro amplitude is

$$A_N=\frac{1}{C}\prod_{i=1}^{N}\int d^2Z\prod_{i<j}^{N}|Z_i-Z_j|^{2k_i\cdot k_j}, \tag{3.8}$$

where C is now the SL(2, C) Haar measure

$$C=\frac{d^2Z_a d^2Z_b d^2Z_c}{|Z_a-Z_b|^2|Z_b-Z_c|^2|Z_c-Z_a|^2}. \tag{3.9}$$

Equation (3.8) has in the factorized form

$$A_N=\frac{1}{C}\left\langle 0\left|T\left(\prod_{i=1}^{N}\int d^2Z_i U(k_i,Z_i,Z_i^\dagger)\right)\right|0\right\rangle, \tag{3.10}$$

$$U(k,Z,Z^*)=\frac{:\exp[ik\cdot\Phi_r(Z,Z^*)]:}{|Z|^2} \tag{3.11}$$

$$=\lim_{\epsilon\to 0}\frac{\exp[ik\cdot\Phi_r]}{E_\epsilon(Z,k^2)}, \tag{3.12}$$

where Φ_r is the rubber-band field of Yoshimura,[15] Del Giudice, and Di Vecchia.[16]

As in the $\alpha_0=1$ case, $\exp(ik\cdot\Phi_r)$ diverges and as $\epsilon\to 0$, $E_\epsilon(k,Z)$ is a $d=k^2$, $j=0$ object. Since d^2Z has $d=2$, $j=0$, conformal invariance of A_N demands the mass-shell condition $k^2=2$. With its Z and Z^* structure completely decoupled, reflecting the SU(1, 1)⊗SU(1, 1) symmetry, the VSM is just a simple "squaring" of the $a_0=1$ case.

The above discussion points out the root of past frustrations in constructing factorizable dual models of currents. The missing element had been a *finite* exponential vertex $\exp(ik\cdot\Phi)$ which transforms as a conformal scalar for arbitrary values of k^2. Next we present such an operator.

B. The Double String Fields

By inspection of the fields $\Phi_{1\mu}(Z,Z^*)$ (2.8) and $\Phi_{2\mu}(Z,Z^*)$ (2.10), associated, respectively, with a string with free ends and one with fixed ends, we can construct two new fields through the following real and complex linear combinations:

(a)
$$\Phi_{r\mu}=\Phi_{1\mu}+\Phi_{2\mu} = x_\mu+ip_\mu\ln(ZZ^*) +\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\,(A^\dagger_{n\mu}Z^{-n}+A_{n\mu}Z^n + B^\dagger_{n\mu}Z^{*-n}+B_{n\mu}Z^{*m})\,, \tag{3.13}$$

where

$$\begin{aligned} A^\dagger_{n\mu}&=\frac{1}{\sqrt{2}}(c^\dagger_{n\mu}+id^\dagger_{n\mu})\,,\\ A_{n\mu}&=\frac{1}{\sqrt{2}}(c_{n\mu}-id_{n\mu})\,,\\ B^\dagger_{n\mu}&=\frac{1}{\sqrt{2}}(c^\dagger_{n\mu}-id^\dagger_{n\mu})\,,\\ B_n&=\frac{1}{\sqrt{2}}(c_{n\mu}+id_{n\mu})\,, \end{aligned} \tag{3.14}$$

satisfying the commutation relations

$$\begin{aligned}&[A_{n\mu},A^\dagger_{m\lambda}]=\delta_{\mu\lambda}\delta_{mm}\,,\\ &[B_{n\mu},B^\dagger_{m\lambda}]=\delta_{\mu\lambda}\delta_{nm}\,,\end{aligned} \tag{3.15}$$

and

$$[A,B]=0\,.$$

(3.13) is recognized as the field of the Virasoro-Shapiro model.[15,16] Thus the dual rubber band is equivalent to a composite of two strings. We shall subsequently use (3.13) to build currentlike spinor extension of the VSM (Sec. IV B),

(b)
$$\Phi_{c\mu}=\Phi_{1\mu}+i\Phi_{2\mu} \tag{3.16}$$

$$= x_\mu+ip_\mu\ln(ZZ^*) +\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}[C^\dagger_{n\mu}Z^{*-n}+C_{n\mu}Z^n +D^\dagger_{n\mu}Z^{-n}+D_{n\mu}Z^{*-n}]\,, \tag{3.17}$$

where

$$\begin{aligned} C^\dagger_{n\mu}&=\frac{1}{\sqrt{2}}[c^\dagger_{n\mu}+d^\dagger_{n\mu}]\,,\\ C_{n\mu}&=\frac{1}{\sqrt{2}}[c_{n\mu}+d_{n\mu}]\,,\\ D^\dagger_{n\mu}&=\frac{1}{\sqrt{2}}[c^\dagger_{n\mu}-d^\dagger_{n\mu}]\,,\\ D_{n\mu}&=\frac{1}{\sqrt{2}}[c_{n\mu}-d_{n\mu}]\,, \end{aligned} \tag{3.18}$$

with

$$\begin{aligned}&[C_{n\mu},C^\dagger_{m\lambda}]=\delta_{\mu\lambda}\delta_{nm}\,,\\ &[D_{n\mu},D^\dagger_{m\lambda}]=\delta_{\mu\lambda}\delta_{nm}\,,\end{aligned} \tag{3.19}$$

and

$$[C,D]=0\,.$$

The complex field $\Phi_{c\mu}$ which reduces to the usual Fubini-Veneziano field at the strip boundary is most suitable for the construction of off-shell vertices.

While $\exp(ik\cdot\Phi_r)$ [Eq. (3.12)] is divergent, $\exp[ik\cdot\Phi_c(Z,Z^*)]$ is a *finite* conformal scalar for *any* k^2. Though the conformal scalarity of $\Phi_{c\mu}$ and $\exp(ik\cdot\Phi_c)$ is manifest by construction, we can easily check its equivalent differential statement in

$$\begin{aligned}&[L_n,\Phi_{c\mu}]=\left(Z^{-n+1}\frac{\partial}{\partial Z}+Z^{*-n+1}\frac{\partial}{\partial Z^*}\right)\Phi_{c\mu}\,,\\ &[L_n,\exp(ik\cdot\Phi_c]=\left(Z^{-n+1}\frac{\partial}{\partial Z}+Z^{*-n+1}\frac{\partial}{\partial Z^*}\right)\times\exp(ik\cdot\Phi_c)\,,\end{aligned} \tag{3.20}$$

where the L_n are obtained as the sum of the $L_n^{(c)}$ (2.18) and the $L_n^{(d)}$ taking into account the relations (3.18). By virtue of the Baker-Hausdorff formula we can cast $\exp(ik\cdot\Phi_c)$ in an obviously finite and useful normal ordered form

$$\exp(ik\cdot\Phi_c)=\frac{|Z-Z^*|^{k2}}{|Z|^{k2}}:\exp(ik\cdot\Phi_c):\,, \tag{3.21}$$

with $:\exp(ik\cdot\Phi_c):$ defined as in (3.4). The finiteness of the vertex is achieved at the high price of introducing an imaginary coupling factor i for the field $\Phi_{2\mu}$. The net effect is to endow $i\Phi_{2\mu}$ with a wrong metric relative to $\Phi_{1\mu}$. Amplitudes resulting from (3.17) consequently have ghosts, even when they are conformal-invariant. This new feature is to be contrasted with the VSM, constructed, from $:\exp(ik\cdot\Phi_r):$ which is ghost-free. However the connected general gauge problem is beyond the purpose of our study.

C. Rules for Constructing Dual Current Amplitudes

Having found the desirable current vertex (3.21), we next discuss a few rules necessary for the construction of our dual amplitudes.

The group-theoretical rules of Clavelli and Ramond[19] offer a systematic procedure for constructing manifestly cyclic-symmetric and dual amplitudes. Since we shall deal strictly with fully symmetric Möbius- (conformal-) invariant planar and nonplanar on-shell models and their off-shell extensions, it is possible to reformulate the rules for writing on-shell and off-shell amplitudes in a unified manner, as follows:

(1) The factorizable dual amplitude for the scattering of N on-shell or off-shell particles reads

$$A_N=\frac{1}{C}\left\langle 0\left|T\left(\prod_{i=1}^{N}V(k_i\{\lambda_i\})\right)\right|0\right\rangle\,. \tag{3.22}$$

C is the integrated infinite SU(1,1) or SL(2C) Haar measure, depending on which of the two is the Möbius invariance group of the amplitude. Except for C, A_N is the projective vacuum expectation value of "time" ($t=i\ln|Z|$) ordered products of $d=j=0$ vertices $V(k_i,\{\lambda_i\})$.

(2) $V(k,\{\lambda\})$ is the emission vertex for a particle of momentum k with quantum numbers collectively labeled by $\{\lambda\}$.

Thus in the planar on-shell case, we recall

$$V(k)=\int_B dZ\, U(k,Z)\,. \tag{3.23}$$

B is the boundary of the strip, the unit circle, or the real axis. Without allowing for space-time or internal symmetry labels for the conformal spinors, the general conformally invariant vertex $U(k,Z)$ (Ref. 5) for a meson of mass $m_i^2=-k_i^2=-1+\frac{1}{2}(p_i+q_i)$ is

$$U(k_i,Z_i)=\frac{:\Psi_1(Z_i)^{p_i}\bar{\Psi}_1(Z_i)^{q_i}:}{Z_i^{(p_i+q_i)/2}}\,\frac{:\exp[ik_i\cdot\Phi_1(Z_i)]:}{Z_i^{k_i^2}}\,, \tag{3.24}$$

where $\Psi_1(Z)$, $\bar{\Psi}_1(Z)$, and $\Phi_{1\mu}(Z)$ are given by (2.14) and (3.4), respectively, with $Z=Z^*$. The above mass quantization condition easily follows from the transformation laws (2.22), (3.5), and (3.7). Appropriate selections of p_i and q_i reduce to various known meson models.[5]

In the nonplanar case, we have

$$V(k)=\int_D d^2Z\, U(k,Z,Z^*)\,. \tag{3.25}$$

D extends over the entire strip, the half plane, or the whole complex plane as the case may be. Equation (3.11) is an example of a covariant $U(k,Z,Z^*)$ which yields the VSM.[14-16] Indeed, the current vertices built from the composite fields $\Phi_{r\mu}$ (3.13) and $\Phi_{c\mu}$ (3.17) are necessarily of this two-dimensional type.

The emission vertex $V(k,\{\lambda\})$ has an intuitive dynamical interpretation. Taking D as the strip $w=\nu+i\theta=\ln Z$, Möbius invariance of (3.22) allows the choice of a projective frame defined by the fixed points $|Z_1|=0$, $|Z_{N-1}|=1$, and $|Z_N|=\infty$. In this multiperipheral configuration we recover the Nambu-Susskind[19] picture (Fig.1). In brief, the strip identified with the Harari-Rosner diagram, is conceived as a world sheet swept in time $t=i\nu$ by a string of length π. The string is presumably made up of quark-antiquark pairs labeled by $0\leq\theta\leq\pi$. For on-shell hadronic processes only the end quarks at 0 and π can emit mesons at various t_i [Fig. 1(a)]. For off-shell objects the coupling mechanism is unconstrained [Fig. (1(b)]. Any quark of the string (or rather the double-string) can emit a current. Its position in Lorentz space-time is described by either the real field $\Phi_{r\mu}$ or the complex field $\Phi_{c\mu}$. The scalar current emission vertex, local in Minkowski space, then has the general form

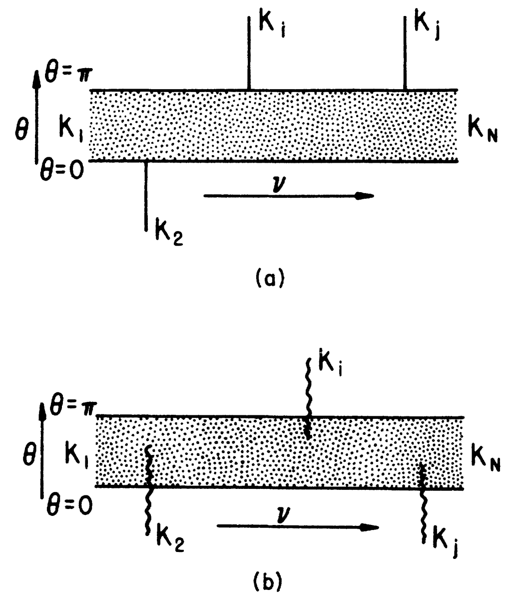

FIG. 1. Dual emmission in the Nambu-Susskind picture: (a) on-mass-shell, (b) off-mass-shell.

$$V(x,\{\lambda\})=\int_D d^2Z\, J(\{\lambda\},Z,Z^*)\delta^4(x_\mu-\Phi_\mu) \tag{3.26}$$

or in momentum space

$$V(k,\{\lambda\})=\int_D d^2Z\, J(\{\lambda\},Z,Z^*)\exp[ik\cdot\Phi(Z,Z^*)]\,, \tag{3.27}$$

where $\Phi_\mu=\Phi_r,\Phi_c$ and D spans the strip or any other conformally equivalent domain. The $J(\{\lambda\},Z,Z^*)$ c-number or q-number function must be such that $V(k,\{\lambda\})$ transforms like an SU(1,1) scalar or better as a $d=j=0$ conformal representation. If quantum numbers are present, J must also transform under the groups generating $\{\lambda\}$ as the field of the emitted meson if the correct selection rules are to be enforced. As is clear from (3.26), J represents a yet unspecified charge or current density distribution in the internal two-space; for instance, it could describe the flow of unitary spin currents in the world sheet (Sec. VI). In our current models to be discussed next, the vertices (3.11) and (3.24) will be shown to be particular on-shell limits of specific realization of the current emission vertex (3.27), which can also be readily generalized for vector currents.

IV. CURRENTLIKE GENERALIZATIONS OF THE VSM

A. A Currentlike Extension of the VSM

In hyperbolic geometry,[27] we can consider the interior of the unit circle or the upper half-plane as the conformal image of the non-Euclidean plane. The $|Z|=1$ circle or the real axis then corresponds to the infinitely distant horizon. The Lobachevsky motions or conformal self-mappings of the half-plane are represented by Möbius transformations of the form

$$Z'=\frac{aZ+\beta}{\gamma Z+\delta} \quad \text{and} \quad \frac{aZ+\beta}{\gamma Z^*+\delta}=Z'^* \tag{4.1}$$

with a, β, γ, δ real and $a\delta-\beta\gamma=1$. The above group of transformations leaves invariant the non-Euclidean element area $d\sigma=d^2Z/(\mathrm{Im}Z)^2$ which provides the necessary integration measure for the construction of SU(1, 1) invariant current vertices. Indeed under a Möbius mapping, ImZ transforms as $|dZ|$, though it is not an irreducible object under the action of general conformal transformations (2.3).

Let us first consider a Möbius-invariant vertex made out of the rubber-band field (3.14):

$$V(k)=\int_D d^2Z\,(\mathrm{Im}\,Z)^{k^2-2}\,\frac{:\exp[ik\cdot\Phi_r(Z,Z^*)]:}{|Z|^{k^2}}\,. \tag{4.2}$$

D extends over the whole complex plane. Equation (4.2) with the rule (3.22) gives the N-point amplitude

$$A_N=\frac{1}{C}\left\langle 0\left|T\left(\prod_{i=1}^{N}\int_D d^2Z_i\,(\mathrm{Im}Z_i)^{k_i{}^2-2}\,\frac{:\exp[ik_i\cdot\Phi_r(Z_i,Z_i^*)]:}{|Z|^{k_i{}^2}}\right)\right|0\right\rangle . \tag{4.3}$$

Möbius invariance is easily checked first by using the explicit transformation laws

$$\mathrm{Im}\,Z'=|\gamma Z+\delta|^{-2}\,\mathrm{Im}\,Z\,, \tag{4.4}$$

$$e^{i\vec{\xi}\cdot\vec{L}}\,\frac{:\exp[ik\cdot\Phi_r(Z,Z^*)]:}{|Z|^{k^2}}\,e^{-i\vec{\xi}\cdot\vec{L}}=|\gamma Z+\delta|^{-2k^2}\,\frac{:\exp[ik\cdot\Phi_r(Z',Z'^*)]:}{|Z'|^{k^2}}\,, \tag{4.5}$$

where the vector $\vec{\xi}$ is a finite Möbius transformation and the components L_0, $L_{\pm 1}$, of $\vec{L}$ are the sum of the generators made out of A_n and B_n oscillators, and second by using the fact that the L_i $(i=0,\pm1)$ annihilate the bra and ket vacua.

By (3.8) and (3.10), the Koba-Nielsen amplitude for (4.3) is

$$A_N=\frac{1}{C}\prod_{i=1}^{N}\int d^2Z_i\,(\mathrm{Im}Z_i)^{k_i{}^2-2}\prod_{i<j}^{N}|Z_i-Z_j|^{2k_ik_j}\,, \tag{4.6}$$

which as $Z_i\to Z_j$, allows singularities in $k_i\cdot k_j$ to occur off the real axis. Thus (4.2) does not qualify as a current vertex in view of the desired properties for off-shell amplitudes listed in Sec. I. However, it does provide a good continuation in the intercept α_0 of the VSM for $\alpha_0\neq 1,-1,-3\ldots$.[13] Of special interest is the case of $k^2=2$, where, apart from the SU(1, 1) Haar measure, (4.6) is the VSM. As the expectation value in (4.3) for $k_i{}^2=2$ acquires the larger Möbius symmetry SL(2, C) [SU(1, 1)⊗SU(1, 1)], for a finite A_N, C must now be the SL(2, C) measure (3.9).

B. A Spinor Extension of the VSM

One possible modification[28] of the vertex (4.2) is achieved when we introduce two mutually *commuting* Neveu-Schwarz (NS) type fields[4]

$$H(Z)=\sum_{-\infty}^{\infty}b_{m\mu}Z^m,\quad \bar{H}(Z^*)=\sum_{-\infty}^{\infty}\bar{b}_{m\mu}Z^{*m},\quad [b_{m\mu},\bar{b}_{n\mu}]=0,\quad m=\pm\tfrac{1}{2},\pm\tfrac{3}{2},\ldots, \tag{4.7}$$

with the anti-commutation relations

$$\{b_{m\mu},b_{n\nu}\}=\delta_{\mu\nu}\delta_{m+n,0},\qquad \{\bar{b}_{m\mu},\bar{b}_{n\nu}\}=\delta_{\mu\nu}\delta_{m+n,0}\,.$$

The associated Virasoro generators are

$$L_n^{(b)}=-\tfrac{1}{4}:\sum_{-\infty}^{\infty}(n+2m)b_{-m}\cdot b_{n+m}:\,,\quad L_n^{(\bar{b})}=-\tfrac{1}{4}:\sum_{-\infty}^{\infty}(n+2m)\bar{b}_{-m}\cdot\bar{b}_{n+m}:\,. \tag{4.8}$$

$H(Z)$ and $\bar{H}(Z^*)$ transform like Ψ_1 and $\bar{\Psi}_1$ (2.22), respectively.

According to (2.22), (4.4), and (4.5) we can write a new SU(1, 1) invariant vertex in

$$V(k)=\int_D d^2Z\,(\mathrm{Im}Z)^{k^2-1}\frac{k\cdot H(Z)k\cdot\overline{H}(Z^*)}{|Z|^{k^2+1}}:\exp[ik\cdot\Phi_r(Z,Z^*)]:. \tag{4.9}$$

Just as for the orbital model (4.2), the vertex of interest occurs when $k^2=1$. Again we have SL(2C) as well as conformal invariance of the amplitude A_N resulting from (4.9). For $k^2=1$, the vertex (4.9) consists of the product of a NS vertex and its mirror counterpart with respect to the real axis. This is apparent as we rewrite (4.9) as

$$V(k)=\int_D dZdZ^*\frac{k\cdot H(Z)}{Z^{(k^2+1)/2}}:\exp[ik\cdot Q(Z)]:\frac{k\cdot\overline{H}(Z^*)}{Z^{*(k^2+1)/2}}:\exp[ik\cdot\overline{Q}(Z^*)]:, \tag{4.10}$$

where we define in the manner of Di Vecchia and Del Giudice[16]

$$\Phi_{r\mu}=Q_\mu(Z)+\overline{Q}_\mu(Z^*); \tag{4.11}$$

here

$$\begin{aligned}Q_\mu(Z)&=q_{0\mu}+ip_{0\mu}\ln Z+\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(A_{n\mu}Z^n+A^\dagger_{n\mu}Z^{-n}),\\ \overline{Q}_\mu(Z^*)&=\overline{q}_{0\mu}+i\overline{p}_{0\mu}\ln Z^*+\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(B_{n\mu}Z^{*n}+B^\dagger_{n\mu}Z^{*-n}),\end{aligned} \tag{4.12}$$

with

$$[q_{0\mu},p_{0\nu}]=i\delta_{\mu\nu},\quad[\overline{q}_{0\mu},\overline{p}_{0\nu}]=i\delta_{\mu\nu} \tag{4.13}$$

and q_0, p_0 commuting with $\overline{q}_0$, $\overline{p}_0$. The associated Virasoro operators $L_i^{(A)}$ ($L_i^{(B)}$) are given by (2.18), where the operators p and C_n are replaced by p_0 ($\overline{p}_0$) and A_n (B_n).

Expressed in terms of (4.10), the N-point function

$$A_N=\frac{1}{C}\left\langle 0\left|T\left(\prod_{i=1}^{N}\int_D dZ_idZ_i^*\frac{k\cdot H(Z_i):\exp[ik\cdot Q(Z_i)]}{Z_i}:\frac{k\cdot\overline{H}(Z_i^*):\exp[ik\cdot\overline{Q}(Z_i^*)]}{Z_i^*}:\right)\right|0\right\rangle, \tag{4.14}$$

[where C is now given by (3.9)] has its Z and Z^* dependence entirely decoupled and is just the NS amplitude "squared". An obvious consequence of the above property is that the NS "G-parity" selection rule[4] is also obeyed as $A_N=0$ for N odd. To complete the analogy with the VSM[16] and NSM,[4] we compose the sum of generators

$$L_i^{(A)}+L_i^{(b)}\equiv L_i, \tag{4.15}$$

$$L_i^{(B)}+L_i^{(\tilde{b})}\equiv\overline{L}_i. \tag{4.16}$$

We then have

$$[L_n,U(k,Z)]=Z^{-n}\left[Z\frac{\partial}{\partial Z}-n\left(\frac{k^2+1}{2}\right)\right]U(k,Z), \tag{4.17}$$

$$[\overline{L}_n,\overline{U}(k,Z^*)]=Z^{*-n}\left[Z^*\frac{\partial}{\partial Z^*}-n\left(\frac{k^2+1}{2}\right)\right]\overline{U}(k,Z^*), \tag{4.18}$$

where

$$U(k,Z)=k\cdot H(Z):\exp[ik\cdot Q(Z)]:,\qquad \overline{U}(k,Z^*)=k\cdot\overline{H}(Z^*):\exp[ik\cdot\overline{Q}(Z^*)]:. \tag{4.19}$$

Among the Virasoro operators (4.15) and (4.16), L_0, $L_{\pm1}$ and $\overline{L}_0$, $\overline{L}_\pm$ form the SU(1, 1)⊗SU(1, 1) algebra underlying the new model (4.14).

If we now define a total gauge operator $L_i^T=L_i+\overline{L}_i$ and the vertex $V(k,Z,Z^*)\equiv U\overline{U}$, from (4.17) and (4.18) we get

$$[L_n^T,V(k,Z,Z^*)]=\left[Z^{-n+1}\frac{\partial}{\partial Z}+Z^{*-n+1}\frac{\partial}{\partial Z^*}-n(Z^{-n}+Z^{*-n})\tfrac{1}{2}(k^2+1)\right]V(k,Z,Z^*), \tag{4.20}$$

which is the differential statement that the dimension and conformal spin of $V(k,Z,Z^*)/|Z|^{k^2+1}$ are $d=-(k^2+1)$ and $j=0$. Since d^2Z has $d=2$, $j=0$, conformal invariance of (4.14) results for $k^2=1$.

Using the same techniques as in Ref. 16, complete factorization of (4.14) in the multiperipheral configuration $|Z_1|=0$, $|Z_{N-1}|=1$, $|Z_N|=\infty$ is achieved.

We define the bra and ket states

$$\lim_{|Z_1|\to 0} |Z_1|^{-2}\langle 0|\, V(k_1, Z_1, Z_1^*) = \langle 0|\exp[ik_1\cdot(q_0+\bar{q}_0)]k_1\bar{b}_{1/2}k_1 b_{1/2} \equiv \langle k_1, \varphi_1|, \tag{4.21}$$

$$\lim_{|Z_N|\to\infty} |Z_N|^2 V(k_N, Z_N, Z_N^*)|0\rangle = k_N\cdot\bar{b}_{-1/2}k_N\cdot b_{-1/2}\exp[ik_N\cdot(q_0+\bar{q}_0)]|0\rangle \equiv |\varphi_N, k_N\rangle. \tag{4.22}$$

Owing to the SL(2, C) group properties [(4.17) and (4.18)] we have

$$U(k, Z)\bar{U}(k, Z^*) = Z^{L_0}Z^{*\bar{L}_0}U(k,1)\bar{U}(k,1)Z^{-L_0}Z^{*-\bar{L}_0}. \tag{4.23}$$

Substituting the above relations in the amplitude (4.14), and remembering that L_0 and $\bar{L}_0$ annihilate the vacuum, we obtain

$$A_N = \prod_{i=2}^{N-2}\int_0^\infty\int_0^{2\pi}\frac{dr_i d\theta_i}{\prod_{j=3}^{N-2} r_j^{-1}}\left(\frac{r_i}{r_{i+1}}\right)^{-3}$$
$$\times\left\langle k_1, \Phi_1\right| T\left[V(k_2,1)\left(\frac{r_2}{r_3}\right)^{-L_0-\bar{L}_0}\exp[-i(\theta_2-\theta_3)(L_0-\bar{L}_0)]\cdots\right.$$
$$\left.\times V(k_{N-2},1)\left(\frac{r_{N-2}}{r_{N-1}}\right)^{-L_0-\bar{L}_0}\exp[-i(\theta_{N-2}-\theta_{N-1})(L_0-\bar{L}_0)]V(k_{N-1},1)\right]\left|\Phi_N, k_N\right\rangle, \tag{4.24}$$

with $r_{N-1}=1$, $\theta_{N-1}=0$. Performing the change of variable $x_i=r_i/r_{i+1}$ $(0\le x_i\le 1)$ and $\Omega_i=\theta_i-\theta_{i+1}$, we then can do both the x_i and Ω_i integrations which finally yield

$$A=\sum_P\langle k_1, \Phi_1|v(k_2)\Delta v(k_3)\cdots\Delta v(k_{N-1})|\Phi_N, k_N\rangle, \tag{4.25}$$

where the sum is over the $(N-2)!$ permutations of the momenta $k_2, k_3, \ldots, k_{N-1}$, and the Feynman-like rules are

$$v(k) = k\cdot H(1):\exp[ikQ(1)]\exp[ik\cdot\bar{Q}(1)]:k\cdot\bar{H}(1), \tag{4.26}$$

$$\Delta=\frac{2}{L_0+\bar{L}_0+2}\,\frac{\sin\pi(L_0-\bar{L}_0)}{(L_0-\bar{L}_0)}. \tag{4.27}$$

The contribution to the amplitude of a pole in the subenergy $s=-\pi^2=-(\sum_{i=1}^{l}k_i)^2$ then has the form

$$A_N=\langle p|\Delta|q\rangle, \tag{4.28}$$

with

$$\langle p| = \sum_P\langle k_1, \Phi_1|v(k_2)Dv(k_3)\cdots Dv(k_l),$$
$$|q\rangle = \sum_P v(k_{l+1})D\cdots v(k_{N-1})|\Phi_N, k_N\rangle. \tag{4.29}$$

Just as in the VSM, upon insertion of a complete set of intermediate states $|\lambda\rangle$, and because $p_0^2=\bar{p}_0^2=\pi^2$ on a state $|\lambda\rangle$, the projection operator $[\sin\pi(L_0-\bar{L}_0)]/(L_0-\bar{L}_0)$ singles out as contributing states only those $|\lambda\rangle$ obeying the constraint equation

$$\sum_{l=0}^\infty lA_l^\dagger\cdot A_l+\sum_{m=1/2}^\infty mb_m^\dagger\cdot b_m = \sum_{n=0}^\infty nB_n^\dagger\cdot B_n+\sum_{r=1/2}^\infty r\bar{b}_r\cdot\bar{b}_r. \tag{4.30}$$

For $k^2=1$, making use of the commutators

$$[L_0-L_{-n}, U(k,1)] = -nU(k,1)$$
and
$$[\bar{L}_0-\bar{L}_{-n}, \bar{U}(k,1)] = -n\bar{U}(k,1), \tag{4.31}$$

we can show for any positive l that

$$\langle p|W_l = \langle p|\bar{W}_l = 0, \tag{4.32}$$

where

$$W_l = L_0-L_l^\dagger-(l-1),\quad \bar{W}_l=\bar{L}_0-\bar{L}_l^\dagger-(l-1) \tag{4.33}$$

are the Ward operators.

Following the method of Ref. 4, we define two sets of supergauge operators

$$G_m=\frac{1}{\sqrt{2}\pi}\oint\frac{dZ}{Z^{m+1}}H(Z)\frac{dQ(Z)}{dZ},$$
$$\bar{G}_m=\frac{1}{\sqrt{2}\pi}\oint\frac{dZ}{Z^{m+1}}\bar{H}(Z)Z\frac{d\bar{Q}(Z)}{dZ}, \tag{4.34}$$

where the integral is evaluated along a small circle of the complex Z plane with the center at the origin. The new operators obey the commutation relations

$$[L_{-m}, G_n] = -(\tfrac{1}{2}m - n)G_{n+m}\ ,$$
$$[\bar{L}_{-m}, \bar{G}_n] = -(\tfrac{1}{2}m - n)\bar{G}_{n+m}\ , \qquad (4.35)$$

$$\{G_n, G_m\} = 2L_{-(n+m)}\ ,$$
$$\{\bar{G}_n, \bar{G}_m\} = 2\bar{L}_{-(n+m)}\ . \qquad (4.36)$$

Then we can express the state $|\Phi_N, k_N\rangle$ as $G_{-1/2}\tilde{G}_{-1/2}|k_N, 0\rangle$. We note that the propagator (4.27) has an alternative form

$$\Delta = \int_{\odot} d^2Z\, Z^{-L_0-2} Z^{*-\bar{L}_0-2}\ , \qquad (4.37)$$

where $\odot$ is the unit disk. Utilizing (4.36) and (4.37), (4.25) becomes

$$A_N = \sum_P \prod_{i=2}^{N-2} \int_{\odot} d^2Z_i \langle O_{A,b}, k_1|G_{1/2}U(k_2)Z_2{}^{-L_0-2}\, U(k_3)\cdots U(k_{N-1})G_{-1/2}|k_N, O_{A,b}\rangle$$
$$\times\langle O_{B,\tilde{b}}, k_1|\bar{G}_{1/2}\bar{U}(k_2)Z^*_2{}^{-\bar{L}_0-2}\bar{U}(k_3)\cdots\bar{U}(k_{N-1})\bar{G}_{-1/2}|k_N, O_{B,\tilde{b}}\rangle. \qquad (4.38)$$

Making use of the following relations,

$$(L_0+\tfrac{1}{2})|k, O_{A,b}\rangle = (\bar{L}_0+\tfrac{1}{2})|k, O_{B,\tilde{b}}\rangle = 0\ , \qquad (4.39)$$

$$0 = \langle 0|G_{-1/2} = \langle 0|\bar{G}_{-1/2} \qquad (4.40)$$

and

$$[L_0, G_{-1/2}] = -\tfrac{1}{2}G_{-1/2}\ ,$$
$$[\bar{L}_0, \bar{G}_{-1/2}] = -\tfrac{1}{2}\bar{G}_{-1/2}\ , \qquad (4.41)$$

we can immediately show the decoupling of the ground state $\langle 0, k| \equiv \langle G|$ from any number of the odd-parity "pions" $\langle 0, k|G_{1/2}\bar{G}_{1/2}$, i.e.,

$$A_{G-(N-1)} = \sum_P \langle 0, k_1|v(k_2)\Delta v(k_3)\cdots$$
$$\times\,\Delta v(k_{N-1})G_{-1/2}\bar{G}_{-1/2}|k_N, 0\rangle$$
$$= 0. \qquad (4.42)$$

Similarly, we can define a new Fock space F_2 just as in the dual pion model and write the N-point amplitude (4.25) as

$$A_N = \sum_P \langle 0, k_1|v(k_2)\Delta v(k_3)\cdots\Delta v(k_{N-1})|k_N, 0\rangle\ , \qquad (4.43)$$

where the "pion" is now the ground state of the new space of oscillators,

$$\Delta = \int_{\odot} d^2Z\, Z^{-L_0-3/2} Z^{*-\bar{L}_0-3/2}$$
$$= \frac{2}{L_0+\bar{L}_0+1}\,\frac{\sin\pi(L_0-\bar{L}_0)}{(L_0-\bar{L}_0)}\ , \qquad (4.44)$$

and $v(k)$ is given by (4.26). The adoption of F_2 space formulation also permits an easier analysis of the spectrum of physical states of the model.

In all respects, the new nonplanar model duplicates the properties of the dual pion model. It stands as a first example of a fully symmetric model with supergauges. For other specific values of k^2 ($k^2 \neq 1$), the N-point function resulting from (4.9) provides a correct continuation in k^2 for the new amplitude.

V. OFF-SHELL AMPLITUDES

A. The Rebbi-Drummond Model (RDM)

In this section we make use of the exponential vertex $\exp[ik\cdot\Phi_c(Z, Z^*)]$ and the proper bilinears in the SG conformal spinor fields in building off-shell models.

Expressed in the variable of the conformal frame $Z = X + iY$ of the upper-half plane, the simplest current vertex has the form

$$V(k) = \int_D \frac{d^2Z}{(\mathrm{Im}\,Z)^2}\exp[ik\cdot\Phi_c(Z, Z^*)]\ , \qquad (5.1)$$

where D is the upper-half plane.

Its Möbius-scalar character is obvious from previous considerations (4.4) and (3.20). Equation (5.1) corresponds to choosing a c-number density $J(Z, Z^*) = (\mathrm{Im}\,Z)^{-2}$. According to the rule (3.22), the N-point current amplitude for (5.1) is then

$$A_N = \frac{1}{C}\left\langle 0\left|T\left(\prod_{i=1}^{N}\int_D d^2Z_i\,(\mathrm{Im}\,Z_i)^{-2}\right.\right.\right.$$
$$\left.\left.\left.\times\exp[ik_i\cdot\Phi_c(Z_i, Z_i^*)]\right)\right|0\right\rangle\ . \qquad (5.2)$$

As in the VSM, the $|Z|$-ordering of the vacuum expectation value in (5.2) is required for the convergence of the commutator

$$[\Phi^{(+)}_{c\mu}(Z_i, Z_i^*), \Phi^{(-)}_{c\lambda}(Z_j, Z_j^*)] = 2\delta_{\mu\lambda}\ln|1 - Z_j^*/Z_i|, \qquad (5.3)$$

$$|Z_i| > |Z_j| \quad \text{when} \quad i > j$$

where $\Phi^{(+)}_{c\mu}$ and $\Phi^{(-)}_{c\mu}$ are defined in the same way as (3.4). The normal-ordered form (3.21) and the commutation relations (5.3), together with the usual conditions

$$\Phi_{c\mu}^{(+)}|0\rangle = \langle 0|\Phi_{c\mu}^{(-)} = 0 \;,$$

$$p_\mu |0\rangle = 0 \;,$$

and

$$[x_\mu , p_\lambda] = i\delta_{\mu\lambda}$$

are sufficient to show that

$$\left\langle 0 \left| T\left(\prod_{i=1}^{N} \exp[ik_i \cdot \Phi_C(Z_i, Z_i^*)]\right) \right| 0 \right\rangle = \prod_{i,j}^{N} |Z_i - Z_j^*|^{k_i \cdot k_j} . \tag{5.4}$$

Consequently

$$A_N = \frac{1}{C} \prod_{l=1}^{N} \int_D \frac{d^2 Z_l}{(\mathrm{Im}\, Z_l)^2} \prod_{i,j}^{N} |Z_i - Z_j^*|^{k_i \cdot k_j} \tag{5.5}$$

is just the Rebbi off-shell amplitude.[11]

By inspection, the zeros and infinities of the integrand in (5.5) are observed to occur only when $Z_i \to Z_j^*$ at the real axis, the boundary of the half plane. They thus ensure the correct singularity structure in the external masses k_i^2 as well as in the subenergies $S_{1,\ldots,l} = -(\sum_{i=1}^{l} k_i)^2$.

The Drummond amplitude[12] results from a modified vertex differing from (5.1) by a multiplicative factor $(\frac{1}{2})^{k^2}$, an entire function of k^2. This rather *ad hoc* alteration[29] has the virtue of eliminating the Gaussian behavior of the elastic form factor, a bad feature of the Rebbi model which renders it unsuitable for analysis of scaling limits ($k^2 \to \infty$). In subsequent discussions we shall always deal with the modified vertex as it gives rise to a falling form factor $F(k^2) \propto 1/|k|$ as $k^2 \to \infty$.

B. Spinor Off-Shell Dual Models

When all the currents are set on their tachyon mass shell ($k_i^2 = 1$), the RDM reduces to the symmetrized sum of the conventional Veneziano amplitude.[12] Can the fully symmetric amplitudes of the generalized SG dual models be extrapolated analogously off-shell? The answer is affirmative for a subclass of these models.

The new SU(1, 1) invariant vertices for dual currents can be induced from a simple observation. Under a Möbius transformation

$$j(Z', Z'^*) = |\gamma Z + \delta|^{+2} \exp(i\vec{\xi} \cdot \vec{\mathrm{L}}^{(a)}) \times j(Z, Z^*) \exp(-i\vec{\xi} \cdot \vec{\mathrm{L}}^{(a)}) \;, \tag{5.6}$$

where $j(Z, Z^*) = \overline{\Psi}\Psi / |Z|$,

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

is given by (2.14) and transforms according to (2.22). $j(Z, Z^*)$, being a conformal vector, transforms like $(\mathrm{Im} Z)^{-1}$ (4.4) under the SU(1, 1) subgroup. This suggests a whole new class of Möbius-invariant emission vertices of the kind

$$V^p(k) = \int_D d^2 Z (\mathrm{Im} Z)^{p-2} [:j(Z, Z^*):]^p \times (\tfrac{1}{2})^{k^2} \exp[ik \cdot \Phi_c(Z, Z^*)] \quad (p = 0, 1, 2, \ldots), \tag{5.7}$$

where the normal ordering of the bilinears prevents contraction of a spinor field at Z and its conjugate field at Z^*. Such contraction would give rise to infinite residues for the poles of (5.7) which occur at the boundary ($Z = Z^*$). Furthermore, it can be verified that only one component of the spinor field $\Psi_1(Z)$ and $\overline{\Psi}_1(Z^*)$ [or $\Psi_2(Z^*)$ and $\overline{\Psi}_2(Z)$] can be used to make up $j(Z, Z^*)$ if factors of the form $|Z_i - Z_j|^{-m_{ij}}$, m_{ij} being some positive integer, are to be avoided. The latter terms would cause unwanted singularities of the Drummond type[12] to occur *inside* the strip when $Z_i \to Z_j$. So we choose $j(Z, Z^*) = \overline{\Psi}_1(Z^*)\Psi_1(Z)/|Z|$. Due to the symmetry between Z and Z^* basic to our coupling scheme (a mirror symmetry guaranteeing both the occurence of the correct singularities at the boundary and the Möbius invariance of the emission vertex $V^p(k)$, off-shell extrapolation of the SG generalized dual vertices (3.24) is possible only for $p_i = q_i$. Therefore our scheme precludes an off-shell extrapolation of the Neveu-Schwarz model where the vertex is linear in the conformal spinor field. This situation has its parallel in ordinary field theory where the currents are always composed of spinor bilinears.

Just as in the RDM ($p = 0$), the vertex $V^p(k)$ has poles at $k^2 = 1 - p, -p, -1-p, \ldots$; this is seen directly by an expansion of the integrand of (5.7) about the real axis:

$$V^p(k) = \int_{-\infty}^{\infty} \frac{dX}{X^{k^2+p}} \sum_{n=0}^{\infty} \frac{R_n(X, k)}{\alpha_p(k^2) - n} \,, \qquad \alpha_p(k^2) = (1 - p) - k^2 \;. \tag{5.8}$$

The residue of the first pole $k^2 = 1 - p$,

$$\tilde{V}^p(k) = \int_{-\infty}^{\infty} \frac{dX}{X} \,(:\overline{\Psi}_1(X)\Psi_1(X):)^p : \exp[ik\Phi_1(X)]: \tag{5.9}$$

is identical to the on-shell vertex for a subclass of the SG generalized dual models (3.24).

When all the external legs are put on their mass-shell values $k_i^2 = 1 - p_i$ the amplitude

$$A_N = \frac{1}{C} \left\langle 0 \left| T\left(\prod_{i=1}^{N} V^{p_i}(k_i)\right) \right| 0 \right\rangle \tag{5.10}$$

becomes in the notations of Ref. 5.

$$A_N \propto \prod_{i=1}^{N} \frac{1}{k_i^2 - 1 + p_i} \sum_{\text{all graphs}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{dX_i}{C} \prod_{i \neq j}^{N} |X_i - X_j|^{-m_{ij}} |X_i - X_j|^{k_i \cdot k_j}, \tag{5.11}$$

the residue of which is the generalization of the conformal-invariant N-point function with unit intercept. Equation (5.11) can be partitioned in the manner of Fairlie and Jones[17] into $\frac{1}{2}(N-1)!$ terms corresponding to all noncyclic and nonanticyclic permutations of the N legs.

Due to the Möbius invariance, we can reduce (5.10) to an explicit factorized form by the Fubini-Veneziano technique[30] and obtain a set of Feynman-like rules for off-shell processes. To factorize A_N in a multiperipheral configuration, we must pay the price of putting at least two currents on their tachyon mass shells. Thus the choice of the standard set of fixed points $X_1 = 0$, $X_{N-1} = 1$, $X_N = \infty$ and the corresponding Haar measure

$$C = \frac{dX_1 dX_{N-1} dX_N}{|X_1 - X_{N-1}||X_{N-1} - X_N||X_N - X_1|} \tag{5.12}$$

leads directly to the requirement

$$k_i^2 = 1 - p_i, \qquad i = 1, N-1, N. \tag{5.13}$$

By the SU(1, 1) group property (3.20), we get

$$|Z|^{L_0} \exp[ik\cdot\Phi_c(\theta)]|Z|^{-L_0} = \exp[ik\cdot\Phi_c(Z, Z^*)], \tag{5.14}$$

where

$$\Phi_{c\mu}(\theta) = X_\mu + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} [(C_{n\mu}^\dagger + C_{n\mu})e^{in\theta} + (D_{n\mu}^\dagger + D_{n\mu})e^{-in\theta}] \tag{5.15}$$

and $Z = re^{i\theta}$. With the on-shell ground-state bra and ket defined as

$$\lim_{X_1 \to 0} \lim_{k_1^2 \to (1-p_1)} k_1^2 \langle 0 | V^{p_1}(k_1) = \int_{-\infty}^{\infty} dX_1 \langle k_1 | (b_0 d_0)^{p_1}, \tag{5.16}$$

$$\lim_{X_N \to 0} \lim_{k_N^2 \to (1-p_N)} X_N^2 k_N^2 V^{p_N}(k_N) |0\rangle = \int_{-\infty}^{\infty} dX_N (b_0^\dagger d_0^\dagger)^{p_N} |k_N\rangle, \tag{5.17}$$

the residual amplitude A_N for N-3 currents and three tachyons can be written as

$$\begin{aligned}\tilde{A}_N = \prod_{i=2}^{N-2} \int_0^{\pi} d\theta_2 \int_0^{\infty} \frac{dr_i}{\prod_{j=3}^{N-2} r_j^{-1}} \left(\frac{r_i}{r_{i+1}}\right)^{-2} |\sin\theta_i|^{k_i^2-2} \\ \times \left\langle k \middle| T\left(:\exp[ik_1\Phi_c(\theta_2)]: [:\bar{\Psi}(\theta_2)\Psi(\theta_2):]^{p_2} \left(\frac{r_2}{r_3}\right)^{-L_0} :\exp[ik_3\cdot\Phi_c(\theta_3)]: \cdots \right.\right. \\ \left.\left. \times \left(\frac{r_{N-2}}{r_{N-1}}\right)^{-L_0} :\exp[ik_{N-1}\cdot\Phi_c(1)]: [:\bar{\Psi}(\theta_{N-1})\Psi(\theta_{N-1}):]^{p_{N-1}} \right) \middle| k_N \right\rangle,\end{aligned} \tag{5.18}$$

where we have used (5.16), (5.17), and the fact that L_0 annihilates the bra or ket vacuum. Performing the usual change of variables $u_i = r_i/r_{i+1}$, $0 \le u_i \le 1$ with the Jacobian

$$J = \frac{d(u)}{d(r)} = \prod_{i=3}^{n-2} r_i^{-1}, \tag{5.19}$$

and carrying out all the u_i integrations, we arrive at the final form

$$\tilde{A}_N = \sum_P \langle k_1 | (b_0 d_0)^{p_1} \Gamma^{p_2}(k_2) \Delta \Gamma^{p_3}(k_3) \cdots \Gamma^{p_{N-1}}(k_{N-1}) (b_0^\dagger d_0^\dagger)^{p_N} | k_N \rangle, \tag{5.20}$$

where the sum is over all the $(n-2)!$ permutations of the momenta $k_2, k_3, \ldots, k_{N-1}$,

$$\Delta = \frac{1}{H + p^2 - 1}, \tag{5.21}$$

with

$$H=\sum_{n=0}^{\infty}\left[n(C_n^\dagger\cdot C_n+D_n^\dagger\cdot D_n)+(n+\tfrac{1}{2})(b_n^\dagger\cdot b_n+d_n^\dagger\cdot d_n)\right] \tag{5.22}$$

and

$$\Gamma^{p_i}(k_i)=\int_0^\pi d\theta_i\,|\sin\theta_i|^{k_i^2+p_i-2}[:\bar\Psi_1(\theta_i)\Psi_1(\theta_i):]^{p_i}:\exp[ik_i\cdot\Phi_c(\theta_i)]:\quad (i=2,3,\ldots,N-2), \tag{5.23}$$

with

$$\begin{aligned}\bar\Psi_1(\theta)&=\sum_{m=0}^{\infty}b_m^+\exp[i(m+\tfrac{1}{2})\theta]+d_m\exp[-i(m+\tfrac{1}{2})\theta]\;,\\ \Psi_1(\theta)&=\sum_{m=0}^{\infty}b_m\exp[i(m+\tfrac{1}{2})\theta]+d_m^+\exp[-i(m+\tfrac{1}{2})\theta]\;.\end{aligned} \tag{5.24}$$

$\Gamma^p(k)$ is meromorphic in k^2 with the poles occuring at the boundary ($\theta=0,\pi$). It is now apparent that only two currents, k_1 and k_N need to be put on their tachyon mass shells since the Feynman-like rules [(5.21)–(5.23)] give the amplitude for $N-2$ currents and two hadrons to be (5.20) where *all* the $\Gamma^{p_i}(k_i)$ ($i=2,3,\ldots,N-1$) are now of the form (5.23). While we deduce this result from the complete factorizability of (5.20), we could obtain it directly from (5.10). Instead of fixing three vertical lines ($X_1=0,\ X_{N-1}=1,\ X_N=\infty$) which correspond to the measure (5.12), we could choose two vertical lines ($X_1=0,\ X_N=\infty$) and one fixed circle $r_{N-1}=1$.[12]

Having two hadrons in the amplitude is in no way a drawback. From a physical standpoint, (5.20) suffices for all purposes since at least one hadron must be present in each side of a reaction to act as a source for the currents.

To go to the world sheet representation, we perform the change in variable $u_i=\exp[-i(t_i-t_{i+1})]=e^{i\tau_i}$ where τ is a relative time since the operator (5.21)

$$\Delta_i=i\int_0^\infty d\tau\exp[-i(H+p^2-1)\tau] \tag{5.25}$$

takes the familiar form of the Feynman propagator. The interesting connection with a parton model suggested by this string picture is discussed by Susskind, Nielsen *et al.*[31]

In a way analogous to the factorization of the VSM, the contribution at a pole in the variable $S_{1,\ldots,l}=-(\sum_{i=1}^{l}k_i)^2$ may be expressed as

$$\tilde A_N=\langle p|\Delta|q\rangle\,, \tag{5.26}$$

where

$$\begin{aligned}|q\rangle&=\sum_P(\Gamma^{p_{l+1}}\Delta\cdots\Gamma^{p_{N-1}})(b_0^\dagger d_0^\dagger)^{p_N}|k_N\rangle\,,\\ \langle p|&=\sum_P\langle k_1|(b_0d_0)^{p_1}(\Gamma^{p_2}\Delta\cdots\Gamma^{p_l})\,.\end{aligned} \tag{5.27}$$

Accordingly, as a consequence of SU(1, 1) invariance of the amplitude, $\langle p|$ obeys the Fubini-Veneziano Ward-like identity

$$\langle p|(L_0-L_{-1})=0\,. \tag{5.28}$$

The models discussed previously have been devoid of spin or internal symmetry degrees of freedom. Since the assignment of Lorentz spin and internal symmetry labels to the orbital oscillators C_n and D_n would lead to exotic states on the leading trajectory, the task of bearing these important quantum numbers rests entirely with the conformal spinor fields. For instance, we can thus have ($\Psi\equiv\Psi_1$)

$$j^\alpha(Z,Z^*)=\frac{(\bar\Psi\Gamma\lambda^\alpha\Psi)}{|Z|}, \tag{5.29}$$

where the conformal fields Ψ and $\bar\Psi$ are at the same time quarklike 4-spinors and unitary spinors. Γ and λ^α are a Dirac 4×4 matrix and a unitary spin 3×3 matrix, respectively. The two cases of $p=1$ and $p=2$ are of special interest among the possible current vertices (5.7). In the first instance if we allow the Ψ's to be unitary spinors, we have a Möbius-invariant off-shell vertex for the simple BH dual quark model.[3]

$$V'(k)^\alpha=\int_D d^2Z\,(\mathrm{Im}Z)^{-1}:\frac{\bar\Psi\lambda^\alpha\Psi}{|Z|}:(\tfrac{1}{2})^{k^2}\exp(ik\cdot\Phi_c). \tag{5.30}$$

Since we also succeed in constructing a conserved vector current for this model (Sec. VI), it is worth recalling the main features of the on-shell amplitude. This model has a leading vacuum trajectory with unit intercept and a quark-antiquark (octet and singlet) trajectory with an intercept one unit lower that has a massless "pion" and a massive "ρ" at $k^2=1$. Moreover, it contains exotic resonances starting three units below the vacuum trajectory.

If we allow Ψ and $\bar\Psi$ to be Lorentz spinors and introduce and additional Lorentz-scalar, one-component conformal spinor $\Psi_5(Z)$ $(\bar{\Psi}_5(Z^*))$, we can construct the Möbius-invariant vertex

$$V(k)=\int_D d^2Z\,\frac{(\mathrm{Im}Z)^{-1}}{|Z|}\,(:\bar{\Psi}\gamma^5 k\cdot\gamma\Psi:+:\bar{\Psi}_5\Psi_5:)$$
$$\times(\tfrac{1}{2})^{k^2}\exp[ik\cdot\Phi_c(Z,Z^*)]\cdot \qquad (5.31)$$

Equation (5.31) is then the off-shell vertex for the dual pion quark model,[18] which while having a richer spectrum contains the NS model. The bilinear current of a spinless quark field Ψ_5 has the property of giving at the boundary, π^5 the fifth component field appearing in the BH models.[18]

For $p=2$, in a model without spin and quantum numbers, we have

$$V^2(k)=\int_D d^2Z[:j(Z,Z^*):]^2(\tfrac{1}{2})^{k^2}\exp[ik\cdot\Phi_c(Z,Z^*)]. \qquad (5.32)$$

This current vertex is not only SU(1, 1) but also conformal-invariant. In fact, apart from $\exp(ik\cdot\Phi_c)$, its integrand is but one of the two available interaction Lagrangian densities (2.6) where Ψ $(\bar{\Psi})$ is restricted to be a *one*-component spinor. We shall reserve discussion of conformal-invariant vertices for a later section on vector currents.

C. A Few Illustrative Computations

It is sufficient to take the case of $p_i=1$ in (5.20) to illustrate the general analytic properties of the new amplitude (5.10). The photoproduction amplitude is thus given by the residue of (5.20) for $n=4$, all $p_i=1$, at the pole $k_3{}^2=0$ (Fig. 2):

$$\tilde{A}_4=\langle k_1|(b_0d_0)[\Gamma_2\Delta\Gamma_3+\Gamma_3\Delta\Gamma_2](b_0^\dagger d_0^\dagger)|k_4\rangle. \qquad (5.33)$$

Defining

$$\alpha_s=-(q+k_1)^2,$$
$$\alpha_t=-(q+k_3)^2, \qquad (5.34)$$
$$\alpha_u=-(q+k_4)^2,$$

and

$$q=k_2,$$

the *st* term is computed to be

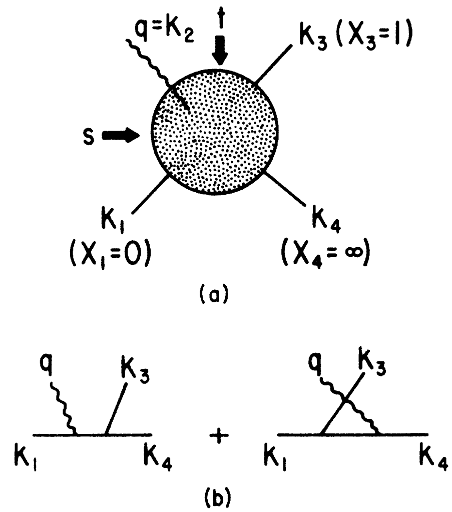

FIG. 2. Photoproduction amplitude.

$$\tilde{A}_4^{s-t}=\int_0^1 dr\int_0^\pi d\theta|\sin\theta|^{q^2-1}r^{-\alpha_s-2}|1-re^{i\theta}|^{2q\cdot k_3-2}$$
$$\times[r^2(1-4\sin^2\theta)+|1-re^{i\theta}|^2$$
$$+r^2|1-re^{i\theta}|^2], \qquad (5.35)$$

while the *ut* term after a change of variable $r'=1/r$ can be cast into a form identical to (5.35) except the r integration now ranges from 1 to ∞. Consequently the complete amplitude can be written as

$$\tilde{A}_4=\int_D d^2Z\,(\mathrm{Im}Z)^{k^2-1}(|Z|^2)^{q\cdot k_1-1}(|1-Z|^2)^{q\cdot k_3-1}$$
$$\times[|Z|^2(1-4\sin^2\theta)+|1-Z|^2+|Z|^2|1-Z|^2], \qquad (5.36)$$

where $Z=re^{i\theta}$ and D spans the entire upper half-plane. Next we evaluate an integral of the general form

$$I=\int_D d^2Z\,(\mathrm{Im}Z)^\rho(|Z|^2)^{-(\sigma/2+1)}(|1-Z|^2)^{-(\tau/2+1)}, \qquad (5.37)$$

where ρ, σ, and τ are not functions of Z. Using the formula

$$\int_0^\infty t^v e^{-pt}dt=\Gamma(v+1)p^{-(v+1)} \qquad (5.38)$$

$(\mathrm{Re}p>0, \mathrm{Re}v>1)$ with $p=|Z|^2$ and $|1-Z|^2$,

the Z integral is Gaussian and subsequent integrals are readily put into the forms of integral representations of γ and β functions. We find

$$I=\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(\rho+1))\,\Gamma(\frac{1}{2}(\tau-\rho))\,\Gamma(-\frac{1}{2}(\sigma-\rho))\Gamma(\frac{1}{2}(\sigma+\tau-\rho)+1)}{\Gamma(\frac{1}{2}\sigma+1)\Gamma(\frac{1}{2}\tau+1)\Gamma(-\frac{1}{2}(\sigma+\tau)-\rho)}. \qquad (5.39)$$

Using (5.39), we can evaluate (5.36) explicitly and find

$$A_4=\sqrt{\pi}\,\Gamma(\tfrac{1}{2}q^2)\Bigg[\frac{\Gamma(-\frac{1}{2}(\alpha_s+1))\Gamma(-\frac{1}{2}(\alpha_t-1))\Gamma(-\frac{1}{2}(\alpha_u-1))}{\Gamma(-\frac{1}{2}(\alpha_s+\alpha_t))\Gamma(-\frac{1}{2}(\alpha_t+\alpha_u)+1)\Gamma(-\frac{1}{2}(\alpha_u+\alpha_s))}$$
$$+\frac{\Gamma(-\frac{1}{2}(\alpha_t+1))\Gamma(-\frac{1}{2}(\alpha_u-1))\Gamma(-\frac{1}{2}(\alpha_s-1))}{\Gamma(-\frac{1}{2}(\alpha_t+\alpha_u))\Gamma(-\frac{1}{2}(\alpha_u+\alpha_s)+1)\Gamma(-\frac{1}{2}(\alpha_s+\alpha_t))}$$
$$+\frac{\Gamma(-\frac{1}{2}(\alpha_u+1))\Gamma(-\frac{1}{2}(\alpha_s-1))\Gamma(-\frac{1}{2}(\alpha_t-1))}{\Gamma(-\frac{1}{2}(\alpha_u+\alpha_s))\Gamma(-\frac{1}{2}(\alpha_s+\alpha_t)+1)\Gamma(-\frac{1}{2}(\alpha_t+\alpha_u))}$$
$$-2q^2\frac{\Gamma(-\frac{1}{2}(\alpha_s+1))\Gamma(-\frac{1}{2}(\alpha_t+1))\Gamma(-\frac{1}{2}(\alpha_u+1))}{\Gamma(-\frac{1}{2}(\alpha_s+\alpha_t)+1)\Gamma(-\frac{1}{2}(\alpha_t+\alpha_u)+1)\Gamma(-\frac{1}{2}(\alpha_u+\alpha_s)+1)}\Bigg]\,. \tag{5.40}$$

Therefore the resulting nonplanar amplitude is completely symmetric in α_s, α_t, and α_u. Its residue at the pole $q^2=0$ is an explicit expression for the residue of (5.11) in the particular case of $n=4$ and $p_i=1$ $(i=1,2,3,4)$.

For an analysis of the t-channel singularity structure, we need only to A^{st} (5.35). As in the Veneziano model the main contribution of the integrand to the asymptotic behavior $\alpha_s\to-\infty$ arises near $r=1$. Since by construction all singularities of the model must occur at the boundary, the connection between Regge and fixed poles can be shown by restricting ourselves to a domain of sufficiently small θ.

Following Drummond,[32] we set $r=1-u$ and demand $0<u<R$ and $0<\theta<H$, both R and H be small. The pertinent piece of $\tilde{A}^{st}$ is

$$\tilde{A}^{st}=\int_0^H d\theta\,\theta^{q^2-1}\int_0^R du\,(1-u)^{-\alpha_s-2}(u^2+\theta^2)^{q\cdot k_3-1}[(1-u)^2(1-4\theta^2)+(u^2+\theta^2)+(1-u)^2(u^2+\theta^2)]\,. \tag{5.41}$$

For large values of α_s, we invoke the Mellin transform technique

$$\tilde{A}_J^{st}=\int_0^\infty d(-\alpha_s)(-\alpha_s)^{-J-1}\tilde{A}^{st}\,. \tag{5.42}$$

Then the right-most singularity of $\tilde{A}_J^{st}$ in the J plane defines the leading high-α_s behavior of $\tilde{A}^{st}$. Using the smallness of u and omitting an irrelevant factor $(-J)$, we find

$$\tilde{A}_J^{st}=\int_0^H d\theta\,\theta^{q^2-1}\int_0^R du\,u^J\,[(u^2+\theta^2)^{q\cdot k_3-1}+(u^2+\theta^2)^{q\cdot k_3}]\,. \tag{5.43}$$

After change of variable $\phi=\theta/u$ which permits the separation of the integration into two parts, we find

$$A_J^{st}=\left(\int_0^{H/R}d\phi\int_0^R du+\int_{H/R}^\infty d\phi\int_0^{H/\phi}du\right)\phi^{q^2-1}(1+\phi^2)^{q\cdot k_3}[u^{J-\alpha_t}+(1+\phi^2)^{-1}u^{J-\alpha_t-2}]\,. \tag{5.44}$$

We approximate $(1+\phi^2)^{q\cdot k_3}\propto 1$ in the first term and $\propto\phi^{2q\cdot k_3}$ in the second. We can then evaluate the partial-wave amplitude $\tilde{A}_J^{st}$ to give

$$\tilde{A}_J^{st}\simeq\frac{(H/R)}{q^2}\left[\frac{1}{J-(\alpha_t-1)}+\frac{1}{J-(\alpha_t+1)}\right]+\frac{(R/H)}{(J+1)}\left[\frac{1}{J-(\alpha_t-1)}+\frac{1}{J-(\alpha_t+1)}\right]\,. \tag{5.45}$$

(5.45) displays the main feature of the amplitude, the existence of a multiplicative fixed pole at $J=-1$ along with Regge poles at $J=\alpha_t\pm1$. The fixed pole disappears when the current is put on its mass shell value at $q^2=0$. One can also verify that the Compton amplitude shows a similar structure.

The explicit form (5.40) allows for an easy analysis of the Bjorken scaling behavior of $\tilde{A}_4$. Maintaining t fixed with

$$\alpha_s=-\omega q^2,\quad \alpha_u=(1-\omega)q^2,\quad q^2\to\infty\,, \tag{5.46}$$

we obtain

$$\tilde{A}_4=\sqrt{\pi}(\tfrac{1}{2}q^2)^{\alpha_t/2}\Gamma(-\tfrac{1}{2}(\alpha_t+1))[\omega(\omega-1)]^{(\alpha_t+1)/2}\,. \tag{5.47}$$

So scaling behavior is peculiar in that it goes with a power factor, a function of the momentum transfer. Due to the rather *ad hoc* modification of the current vertex (Sec. V A) used to get rid of Gaussian form factors,[29] any conclusion reached in scaling analysis of our current models must be taken with caution. From the group-theoretical point of view, we are free to multiply the invariant current vertex (5.7) by a suitable form factor $F(k^2)$, an entire function of k^2. As emphasized by Rebbi,[10] a definite form for the k^2 dependence of $V^p(k)$ is unlikely to be given by duality alone and may well require the postulate of current algebra. With this reservation regarding k^2 behavior in mind, we proceed to compute the form factor from $\tilde{A}_3$ to give

$$F(q^2) = \int_0^\pi |\sin\theta|^{q^2} d\theta = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}q^2+\frac{1}{2})}{\Gamma(\frac{1}{2}q^2+1)}. \tag{5.48}$$

So $F(q^2)$ has poles spaced by two units at $q^2=-1, -3, -5, \ldots,$ and behaves like $1/|q|$ as $q^2 \to \infty$.

Though the absence of a pole at $q^2=0$ in (5.48) is consistent with the decoupling of an odd number of massless on-shell particles, the vanishing of the elastic form factor which is necessary for consistency can only be implemented by the inclusion of some internal symmetry in the model. For example, the "pion" elastic form factor resulting from the off-shell vertex for the BH simple quark model (5.30) is zero owing to the completely antisymmetric SU(3) coupling factor. The above observation underscores the importance of incorporating internal quantum numbers in our generalized off-shell vertices (5.7), if consistency is to be achieved between on and off-shell amplitudes. We shall take up a particular off-shell model with SU(3) symmetry in our discussion of conserved vector currents in Sec. VI.

Finally, by adopting Drummond's method of dividing out the Haar measure, we can check that the propagator for the current in this case of $p=1$ is of the form $A_2(q^2) \propto 1/q^2$. This result is unsatisfactory in that it does not have an infinite sequence of poles, a structure to be expected of a candidate for hadronic photon propagator.[32,33]

VI. DUAL VECTOR AND "AXIAL-" VECTOR CURRENTS

A. Nambu's Method

Weak and electromagnetic processes involve vector and axial-vector rather than scalar currents. Our scheme allows for a specific realization of Nambu's ansatz[9] for the construction of conserved vector currents in a dual theory. Repeating the formal argument of Nambu, we assume the existence of a conserved current $j_i(x_1, x_2)$ $(l=1,2)$ in the internal space of the world sheet with j_i taken to be entirely decoupled from space-time degrees of freedom,

$$\partial_i j_i = 0. \tag{6.1}$$

Proceeding as in (3.28), and making use of the newly acquired complex field $\Phi_{c\mu}$ (3.17), we can construct a vector current in the external 4-space as

$$J_\mu(x) = \int_D d^2Z\, j_i \partial_i \Phi_{c\mu} \delta^4(x_\mu - \Phi_{c\mu}), \tag{6.2}$$

or in momentum space

$$J_\mu(k) = \int_D d^2Z\, j_i \partial_i \Phi_{c\mu} \exp(ik\cdot\Phi_c). \tag{6.3}$$

In consequence of (6.1),

$$ik\cdot J(k) = \int_D d^2Z\, \partial_i[j_i \exp(ik\cdot\Phi_c)]. \tag{6.4}$$

By Gauss's theorem, (6.4) results in current conservation if

$$ik\cdot J(k) = \oint_B ds\, \vec{j}\cdot\vec{n} \exp(ik\cdot\Phi_c) = 0, \tag{6.5}$$

where B is the boundary of D and $\vec{n}$ is a unit vector normal to B. Thus (6.3) is conserved whenever the internal current satisfies

$$\vec{j}\cdot\vec{n}|_B = 0, \tag{6.6}$$

provided the factor $\exp(ik\Phi_c)$ is well behaved at B. One of the merits of the complex field is that this proviso is obeyed. Indeed, while the connection (3.18) would indicate a divergent $\exp(ik\cdot\Phi_c)$ according to our discussion in Sec. IIIA, in fact we have

$$\exp(ik\cdot\Phi_c) = |2\sin\theta|^{k^2} :\exp(ik\cdot\Phi_c): \tag{6.7}$$

by formula (3.21). Therefore for spacelike k^2 $(k^2>0)$, $\exp(ik\cdot\Phi_c)$ *vanishes* at the boundary. For other values of k^2, we invoke the usual analytic continuation argument.

B. Conserved Internal Currents

The power and elegance of Nambu's method rests on the interconnection between the conservation of the external space-time current and that of the internal current. The existence of two conserved currents in the Thirring model is well known and that this theorem still holds when the model is dualized. A further elaboration of the theory of the conformal spinor field is useful beginning with an analysis of two inportant conserved currents in the fermion system (2.5). They are

$$j_i^V = :\bar{\Psi}\sigma_i\Psi:, \quad i=1,2 \tag{6.8}$$

and

$$j_i^A = :\bar{\Psi}\sigma_i\sigma_3\Psi: = -i\epsilon_{ik} j_k^V, \quad i=1,2 \tag{6.9}$$

where the σ_i are the Pauli matrices and the *covariant* fields

$$\Psi = \begin{pmatrix} \Psi_1/\sqrt{Z} \\ \Psi_2/\sqrt{Z^*} \end{pmatrix}$$

and $\bar{\Psi} = (\bar{\Psi}_1/\sqrt{Z^*}, \bar{\Psi}_2/\sqrt{Z})$ are given by (2.14) and

$$\epsilon = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}.$$

The subscripts $i = 1, 2$ can label the timelike (ν) and spacelike (θ) components of the currents on the strip.

The conservation laws

$$\partial_i j_i^{V,A} = 0 \tag{6.10}$$

then follow from the equations of motion (2.12).

An elegant and useful representation of $j_i^{V,A}$ is achieved if using (6.8), (6.9), and (2.14) we write

$$\begin{aligned} j_1^V &= -i j_2^A \\ &= [j(Z) + j(Z^*)], \end{aligned} \tag{6.11}$$

$$\begin{aligned} j_2^V &= i j_1^A \\ &= i[j(Z) - j(Z^*)], \end{aligned} \tag{6.12}$$

where

$$j(Z) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} Z^{m-n-1} : a_m^\dagger a_n :,$$

$$j(Z^*) = [j(Z)]^* = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} Z^{*m-n-1} : a_m^\dagger a_n : \tag{6.13}$$

are *solely* functions of Z and Z^*, respectively.

Inspection of the Eq. (6.12) reveals a remarkable feature of the currents, a consequence of the boundary conditions (2.12). At the boundary of the strip where $Z = Z^*$,

$$\begin{aligned} j_2^V|_{0,\pi} &= i j_1^A|_{0,\pi} \\ &= 0. \end{aligned} \tag{6.14}$$

Therefore the vector current, which flows parallel to the boundary at $\theta = 0, \pi$, does not leave the strip, while the pseudovector current, which flows across the boundary, is orthogonal to the vector current at $\theta = 0, \pi$. Due to the isogonality of conformal transformations, the above property remains invariant whether we select the strip, the unit circle, or the upper-half plane as our domain.

In consequence of the infinite degrees of freedom of the quantized conformal fermion system, the normal-ordered spinor bilinears $j(Z)$ and $j(Z^*)$ can be expressed in a new Fock space of *Bose-like* oscillators j_l where

$$j(Z) = \sum_{l=-\infty}^{\infty} Z^{l-1} j_l,$$

$$j(Z^*) = \sum_{l=-\infty}^{\infty} Z^{*l-1} j_l, \tag{6.15}$$

with

$$\begin{aligned} &[j_l, j_m] = l\delta_{l,-m}, \\ &j_m|0\rangle = 0, \quad m \geq 0. \end{aligned} \tag{6.16}$$

In terms of the old Fermi modes b_n and d_n (2.16), the j_l's have the form

$$j_l = \sum_{m=0}^{\infty} (b_m b_{m+l}^\dagger - d_m^\dagger d_{m+l}) + \sum_{m=0}^{l-1} d_m b_{l-m-1}, \tag{6.17}$$

$$j_0 = \sum_{m=0}^{\infty} (b_m^\dagger b_m - d_m^\dagger d_m). \tag{6.18}$$

We notice that j_0 is just the total charge operator for the spinor field, and as such can be expressed in the alternative form

$$\begin{aligned} j_0 &= \int_0^\pi d\theta : \bar{\Psi}\sigma_1\Psi : \\ &= Q. \end{aligned} \tag{6.19}$$

The representation (6.15) suggests that we define a composite Bose field ρ such that

$$j_i^V = \epsilon_{ik} \partial_k \rho. \tag{6.20}$$

We then find

$$\begin{aligned} \rho = \sum_{l=1}^{\infty} \frac{-1}{\sqrt{l}} [(Z^l - Z^{*l})\rho_l + (Z^{*-l} - Z^{-l})\rho_l^\dagger] \\ + iQ \ln(Z^*/Z), \end{aligned} \tag{6.21}$$

where

$$[\rho_l, \rho_m^\dagger] = \delta_{l,m}$$

and

$$\rho_l = \sqrt{l}\, j_l. \tag{6.22}$$

ρ can be interpreted as a charge density operator since

$$Q = \int d\rho. \tag{6.23}$$

More importantly, if the Ψ's are also quarklike representations of the isospin or unitary spin group, in the currents

$$\begin{aligned} j_i^{V\alpha} &= : \bar{\Psi}\sigma_i\lambda^\alpha\Psi : \\ &= \sum_{r,s} : \bar{\Psi}_r \sigma_i (\lambda^\alpha)_{rs} \Psi_s :, \end{aligned} \tag{6.24}$$

$$j_i^{A\alpha} = -i\epsilon_{ik} j_k^{V\alpha}, \tag{6.25}$$

we have the possible models for the flow of internal symmetry spin currents in the strip. In (6.24), λ^α are the usual SU(2) [SU(3)] matrices and the labels r, s run from 1 to 2 (1 to 3) corresponding to the SU(2) [SU(3)] quark representation for $\bar{\Psi}$ and Ψ.

In the case of SU(3), we obtain accordingly

$$j^{\alpha}(Z)=[j^{\alpha}(Z^*)]^* = \sum_{l=-\infty}^{\infty} Z^{l-1} j_l^{\alpha}, \tag{6.26}$$

with j_l^{α} satisfying current-algebraic commutation relations

$$[j_l^{\alpha}, j_m^{\beta}] = 2if^{\alpha\beta\gamma} j_{l+m}^{\gamma} + 2l\delta^{\alpha\beta}\delta_{l,-m} \tag{6.27}$$

and

$$\delta(\nu-\nu')[j^{\alpha}(Z), j^{\beta}(Z')] = 2if^{\alpha\beta\gamma}\delta(\nu-\nu')\delta(\theta-\theta')\frac{1}{Z}j^{\gamma}(Z) - 2iZ^{-2}\delta^{\alpha\beta}\delta(\nu-\nu')\partial_{\theta}\delta(\theta-\theta'). \tag{6.28}$$

C. Conserved External Dual Vector and "Axial-" Vector Currents

The introduction of an operator current density j_i^V from Sec. IIC allows a specific realization of the Nambu model of a conserved dual vector current:

$$J_{\mu}^V(k) = \int_D d^2Z :\bar{\Psi}\sigma_i\Psi :\partial_i\Phi_{\mu} \exp(ik\cdot\Phi_c) . \tag{6.29}$$

Apart from the operatorial factor $\exp(ik\Phi_c)$, the integrand of (6.29) is the second interaction Lagrangian density (2.6) of the dual Thirring model. Conformal invariance of (6.29) is therefore manifest.

An alternative form for (6.29) in terms of the ρ field (6.21) is

$$J_{\mu}^V(k) = \int_D d^2w \frac{\partial(\rho, \Phi_{c\mu})}{\partial(\nu,\theta)} \exp(ik\cdot\Phi_c) . \tag{6.30}$$

In this form, (6.29) is an explicit example of a vector current obtained from a Lagrangian density $\mathcal{L}_{\text{e.m.}}$ in Nambu's dual electrodynamics[9]:

$$\mathcal{L}_{\text{e.m.}} = \int_0^{\pi} \frac{\partial(\rho,\Phi_{\mu})}{\partial(\nu,\theta)} A_{\mu}(\Phi)d\theta , \tag{6.31}$$

where A_{μ} is the quantized electromagnetic field. The current operator (6.30) is identified via comparison of the action $S_{\text{e.m.}} = \int_{-\infty}^{+\infty} d\nu\, \mathcal{L}_{\text{e.m.}}$ with the usual $\int J_{\mu}(x)A_{\mu}(x)d^4x$. Since it is essential for the consistency of the model that the currents incorporate some internal symmetry, we take the fields Ψ and $\bar{\Psi}$ to be quarklike SU(3) representations. In

$$J_{\mu}^{V\alpha}(k) = \int_D d^2Z\, j_i^{V\alpha}\partial_i\Phi_{c\mu} \exp(ik\cdot\Phi_c), \tag{6.32}$$

we then have a model of unitary spin vector currents for the simple BH dual quark model.[3]

If in (6.2)–(6.5), instead of j_i^V we insert j_i^A (6.9), the conserved pseudovector current, we have

$$J_{\mu}^A(k) = \int_D d^2Z :\bar{\Psi}\sigma_i\sigma_3\Psi :\partial_i\Phi_{c\mu}\exp(ik\cdot\Phi_c) , \tag{6.33}$$

with

$$k\cdot J^A(k) = \oint_B ds\, \vec{J}^A\cdot\vec{n} \exp(ik\cdot\Phi_c). \tag{6.34}$$

Though $\vec{j}^A\cdot\vec{n}\,|_B = ij_1^V$ (6.11) does not vanish on the boundary, $J_{\mu}^A(k)$ is nevertheless conserved as $\exp(ik\cdot\Phi_c)$ vanishes at B (6.7). The singularity structure of the current vertices (6.29) and (6.33) is analyzed by the same method as that used for scalar currents. We begin by rewriting them in more usable forms. By virtue of (6.7), inserting in the factor $(\frac{1}{2})^{k^2}$, we find

$$\begin{aligned} J_{\mu}^{V\alpha}(k) &= \int_D d^2Z\,|(Z-Z^*)/2Z|^{k^2} j_i^{V\alpha}\partial_i\Phi_{c\mu} :e^{ik\cdot\Phi_c}:, \\ J_{\mu}^{A\alpha}(k) &= \int_D d^2Z\,|(Z-Z^*)/2Z|^{k^2} j_i^{A\alpha}\partial_i\Phi_{c\mu} :e^{ik\cdot\Phi_c}:, \end{aligned} \tag{6.35}$$

or

$$\begin{aligned} J_{\mu}^V(k) &= \int_D d^2Z\,|(Z-Z)/2Z|^{k^2} j_i^{V\alpha}[(\partial_i\Phi_{c\mu}, e^{ik\cdot\Phi_c^+})e^{ik\cdot\Phi_c^0}e^{ik\cdot\Phi_c^-} + e^{ik\cdot\Phi_c^+}\partial_i\Phi_{c\mu}e^{ik\cdot\Phi_c^0}e^{ik\cdot\Phi_c^-}], \\ J_{\mu}^A(k) &= \int_D d^2Z\,|(Z-Z)/2Z|^{k^2} j_i^{A\alpha}[(\partial_i\Phi_{c\mu}, e^{ik\cdot\Phi_c^+})e^{ik\cdot\Phi_c^0}e^{ik\cdot\Phi_c^-} + e^{ik\cdot\Phi_c^+}\partial_i\Phi_{c\mu}e^{ik\cdot\Phi_c^0}e^{ik\cdot\Phi_c^-}]. \end{aligned} \tag{6.36}$$

Using the identity

$$\begin{aligned} j_i^{V\alpha}\partial_i &= j^{\alpha}(Z)\partial/\partial Z + j^{\alpha}(Z^*)\partial/\partial Z^*, \\ j_i^{A\alpha}\partial_i &= j^{\alpha}(Z)\partial/\partial Z - j^{\alpha}(Z^*)\partial/\partial Z^*, \end{aligned} \tag{6.37}$$

after some algebra, we can express (6.35) in the revealing form

$$J_\mu^{V\alpha}(k) = \int_D d^2Z\,|(Z-Z^*)/2Z|^{k^2}\{ik_\mu[(j^\alpha(Z)-j^\alpha(Z^*))/(Z-Z^*)]:e^{ik\cdot\Phi_c}: + :[j^\alpha(Z^*)P_\mu(Z)/Z + j^\alpha(Z)\overline{P}_\mu(Z^*)/Z^*]\,e^{ik\cdot\Phi_c}:\},$$
$$J_\mu^{A\alpha}(k) = \int_D d^2Z\,|(Z-Z^*)/2Z|^{k^2}\{ik_\mu[(j^\alpha(Z)+j^\alpha(Z^*))/(Z-Z^*)]:e^{ik\cdot\Phi_c}: + :[j^\alpha(Z^*)P_\mu(Z)/Z - j^\alpha(Z)\overline{P}_\mu(Z^*)/Z^*]\,e^{ik\cdot\Phi_c}:\},\qquad(6.38)$$

with

$$\overline{P}_\mu(Z) = Z^*\frac{\partial}{\partial Z^*}\Phi_{c\mu} = ip_\mu + \sum_{n=1}^{\infty}\sqrt{n}\,(D_{n\mu}Z^{*n} - C_{n\mu}^\dagger Z^{*-n}),$$
$$P_\mu(Z) = Z\frac{\partial}{\partial Z}\Phi_{c\mu} = ip_\mu + \sum_{n=1}^{\infty}\sqrt{n}\,(C_{n\mu}Z^n - D_{n\mu}^\dagger Z^{-n}).\qquad(6.39)$$

That the vector current $J_\mu^{V\alpha}(k)$ has an infinite sequence of poles at $k^2=-1, -2, -3, \ldots$ is seen by an expansion about the real axis

$$J_\mu^{V\alpha}(k) = \int_{-\infty}^{\infty}\frac{dX}{X}\sum_{n=1}^{\infty}\frac{R_{n\mu}^{V\alpha}(k,X)}{\alpha(k^2)-n},\qquad \alpha(k^2) = -k^2.\qquad(6.40)$$

The residual vertex at the first vector-meson pole $k^2=-1$ yields

$$\tilde{J}_\mu^{V\alpha}(k) = i\int_{-\infty}^{\infty}\frac{dX}{X}:\overline{\Psi}\lambda^\alpha\Psi\Pi_\mu(X)e^{ik\cdot\Phi_1}:,\qquad(6.41)$$

where we have made use of (3.16)–(3.18) and

$$\Pi_\mu(X) = 2p_\mu + \sum_{n=1}^{\infty} i\sqrt{2n}\,(c_{n\mu}^\dagger X^{-n} - c_{n\mu}X^n).\qquad(6.42)$$

Equation (6.41) is recognized as the emission vertex for the first on-shell vector particle of the BH model.

As for the vector current $J_\mu^{A\alpha}(k)$, the poles are located at $k^2=0, -1, -2, \ldots$. The $k^2=0$ pole clearly results from the fact that the first term in the integrand of (6.38) does *not* vanish at the boundary. In fact the residual vertex at $k^2=0$ is given as

$$\lim_{k^2\to 0}[k^2 J_\mu^{A\alpha}(k)] = ik_\mu\int_{-\infty}^{\infty}\frac{dX}{X}:\overline{\Psi}\lambda^\alpha\Psi\,e^{ik\cdot\Phi_1}:,\qquad(6.43)$$

which is the dual analog of PCAC (partially conserved axial-vector current) since the integral on the right-hand side of (6.43) is just the emission vertex of the massless BH meson. This has therefore all the earmarks of doing what a dual axial-vector current should do, though it clearly does not transform as a space-time axial vector. Nevertheless, it is of great interest to probe the combined structure of $J_\mu^{V\alpha}(k)$ (6.29) and $J_\mu^{A\alpha}(k)$ (6.33) using the suggestive commutation relations [(6.27), (6.28)] as a possible realization of Gell-Mann's current algebra. An analysis of the algebra of these currents and their light-cone properties is currently under investigation.

VII. DISCUSSION

Using two NS fields and the rubber-band field, we have constructed a conformal-invariant spinor extension of the VSM with unit intercept and a Möbius-invariant currentlike model which provides a correct continuation in k^2 for the new amplitude. This model stands as the first example of a nonplanar model with supergauges and may reduce to a Yang-Mills theory of the graviton in a zero-slope limit.[34]

Capitalizing on the full two-dimensional symmetry structure of a complex scalar field and the conformal SG spinor fields, we have also constructed SU(1, 1)-invariant off-shell extensions of a subclass of the conformal-invariant SG generalized dual models. For the particular case of the BH simple dual quark model with unitary spin, we have found two conformal-invariant conserved vector currents. One of these is akin to an axial-vector current as it leads to a dual equivalent of PCAC.

We have shown that the duality and factorization of the new models are consequences of Möbius invariance and have analyzed their general analytic structure. However, we have not yet studied their spectra of physical states in any detail. As spinor generalizations of the Rebbi-Drummond model, the new amplitudes share all its defects, including an enlarged spectrum of states (resulting in an explosive degeneracy on the leading trajectory) and the presence of ghosts.

We are faced with a novel situation in the cases of the conformal-invariant scalar current (5.32) and vector currents (6.35). Because the field is complex, conformal invariance of the off-shell amplitude does not preclude the presence of ghosts. It would be beneficial to study the general gauge problem associated with these currents.

Another area of concern is the problem of eliminating the Gaussian form factors inherent in our models. If we were to interpret the conformal-invariant vector-current vertex as the momentum transform of a space-time current, then the q^2 dependence of the vertex would be highly nontrivial. In fact, if the currents $J_\mu^{V\alpha}(k)$ and $J_\mu^{A\alpha}(k)$ were made to obey current-algebra commutation relations in momentum space, the expedient modification of the current vertices by a multiplicative $F(q^2)$ (an entire function in q^2) would be impossible. The proper modification should be made in the two-dimensional field(s) $\Phi_\mu(x,y)$ of $\exp[ik\cdot\Phi(x,y)]$ itself.[29]

Finally, we note that when internal symmetries are included, the nonplanar character of our current models necessarily leads to amplitudes with exotic resonances.[35] Though the case against exotic resonances remains essentially an experimental one, it may be desirable to seek dual planar models of currents, such as the one recently proposed by Neveu and Scherk.[36]

After the completion of this work, we learned of a paper by Kikkawa and Sakita[37] which provides the functional counterpart of our work on the conserved vector currents. Beside proving the generalized Ward-Takahashi identities for these currents, these authors also computed form factor, propagator, and Compton amplitude, and checked the Fubini–Dashen–Gell-Mann sum rule. We thank Professor Y. Nambu for making the above paper available to us.

With respect to our NS-like extension of the VSM, we received a recent paper from Schwarz[38] proposing a model identical to ours in the case of $k^2=1$. He conjectures that this pionic version of the VSM is the scalar multi-Pomeranchukon amplitude (in the critical dimension $d=10$) of the dual pion model. We thank John Schwarz for discussing this interpretation with us. Aldrovandi and Neveu also treat the same on-shell model in their paper.[39]

ACKNOWLEDGMENTS

I am very grateful to Stanley Fenster for his continuous guidance throughout the course of my graduate studies. I wish to thank him and Professor Peter Freund for suggesting work in the area of dual currents and for giving me much time and helpful advice. I have also benefitted from many conversations with Dr. F. Mansouri and Dr. J. Willemsen, and wish to thank Professor Y. Nambu for a critical reading of the manuscript and for an enlightening discussion of dual models of currents and their scaling behavior.

In addition, thanks are due Dr. R. C. Arnold and the other members of the High Energy Theory Group at Argonne National Laboratory for their warm hospitality.

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*Work supported in part by the U. S. Atomic Energy Commission, Contract No. AEC AT(11-1)-264 and supported in part by the National Science Foundation, Contract No. NSF GP 32904 X.

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PHYSICAL REVIEW D VOLUME 7, NUMBER 6 15 MARCH 1973

Classical Radiation of Accelerated Electrons. II. A Quantum Viewpoint*

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(Received 27 November 1972)

The known classical radiation spectrum of a high-energy charged particle in a homogeneous magnetic field is rederived. The method applies, and illuminates, an exact (to order α) expression for the inverse propagation function of a spinless particle in a homogeneous field. An erratum list for paper I is appended.

For a long time I have wanted to reexamine a classic situation of classical electrodynamics, that of high-energy charged particles radiating in a homogeneous magnetic field, from the modern quantum viewpoint that employs the machinery of propagation (Green's) functions. Since the electromagnetic and relativistic aspects of the problem are quite transparent, the comparison should be instructive in giving the more abstract quantum procedure a concrete interpretation in a particular instance. And, as an added bonus, the necessary ability to treat motion in magnetic fields that goes beyond the lowest orders in a perturbative expansion should be helpful in answering questions about very strong fields, to which recent astrophysical speculations have directed attention. This paper is devoted to describing one such procedure, and applying it to rederive (for a spin-0 particle) the known classical radiation result.[1] Another method is indicated in a separate paper of Yildiz. A subsequent joint paper will contain the analogous spin-$\frac{1}{2}$ calculation, and a discussion of the anomalous magnetic moment in strong fields.

The language and methodology of source theory[2] will be used (which should not seriously impede readers who are untutored in this art). The initial action expression of spin-0 charged particles with mass m,

$$\int(dx)[K(x)\phi(x) - \tfrac{1}{2}\phi(x)(\Pi^2 + m^2)\phi(x)] ,$$

$$\Pi = (1/i)\partial - eqA , \qquad (1)$$

is supplemented by the action contribution associated with the exchange of one virtual photon [cf. Eq. (4-14.2) of PSF II[2]],

$$-\tfrac{1}{2}\int(dx)(dx')\phi(x)M(x, x')\phi(x') . \qquad (2)$$

Here, written in a symbolic notation, we have