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Quantization and Gauge Freedom in a Theory with Spontaneously Broken Symmetry*

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We quantize a model with spontaneously broken symmetry in the asymmetrical gauge $\partial_\mu A^\mu = (\kappa^2/m)\Phi$, where A_μ is the vector meson with mass m and Φ is the gauge excitation field. We show that κ^2 can be identified as the mass parameter of the Φ field, in addition to the numerical parameter ξ defined in the text. This implies that Φ is not a Goldstone boson. A renormalization gauge corresponds to any gauge with κ^2 finite. One can formally go over to the unitarity gauge by the limit $\kappa^2 \to \infty$. However, this limit should be taken with extreme care, so as not to interfere with the limits of internal-loop integration. Several examples, including the anomalous magnetic moment of an electron, are given to illustrate that (a) physical quantities are gauge-invariant, (b) the gauge with $\xi = 1$ and $\kappa^2 = m^2$ is particularly convenient for finite calculations, (c) ambiguities may arise if the $\kappa^2 = \infty$ is taken haphazardly, and (d) renormalization constants are gauge-dependent and in some cases can be defined only for certain values of κ^2.

INTRODUCTION

Interest in renormalizable-weak-interaction theories has recently been rekindled. The further impetus was provided by Weinberg[1-4] in a model where there exists pointwise gauge symmetry, which is subsequently spontaneously broken by the vacuum. The conjecture was that this breaking mechanism may be mild enough that the system remembers its past renormalizability.

Such conjectures have been formally substantiated by the extensive work of 't Hooft[5] and Lee and Zinn-Justin,[6] in particular. Low-order direct verifications have been performed.[7-10] It is fair to conclude that all indications tend to point towards a finite renormalizable theory.

One must now build models to encompass hadrons such that they can meet all the experimental constraints. This is not our concern in this paper.

What we intend to investigate here has been partly solved by the authors in Refs. 5 and 6. This has to do with gauge freedom. Since this model possesses pointwise gauge invariance, one has the freedom to pick a convenient gauge, depending on what problems one chooses to tackle. However, at some point one must prove that the physical amplitudes obtained are identical up to normalization.

There are two popular gauges considered so far. One is the renormalization gauge $\partial_\mu A^\mu = 0$,[6] where A_μ is the vector meson(s) in the theory. In this gauge, the free boson propagators behave like $1/q^2$ for large q^2. Thus, simple power counting can be carried out to show that the theory is renormalizable. There are unwelcome features accompanying this gauge. They all stem from the fact that the gauge excitation (i.e., one of the scalar bosons in the theory which eventually cancels out the scalar ghost mode in the vector field) is massless. One must show that this singularity does not appear in the physical amplitudes. This has been done.[6] Even so, in practical calculation, this nonphysical infrared pole can still be a source of complication, as any reader who has experience in calculations in quantum electrodynamics (QED) may bear witness to this criticism.

The other commonly used gauge is the unitarity gauge. It is obtained if one makes a cylindrical transformation[11] to make the gauge excitation disappear completely. Therefore, there is no spurious pole *ab initio*. The disadvantage here is that the "apparent" order of primitive divergence of the theory is quartic. Even if the theory is renormalizable, due to profuse miraculous cancellation, one should still worry if such singular behavior may cause ambiguities due to different assignments of momenta in a diagram.[8] One can track down the origin of this problem. It has to do with the fact that the quantization of the vector particle here is not manifestly Lorentz-covariant, i.e., the spatial components and the temporal components are treated differently.

Inasmuch as the transformation from rectangular coordinates to cylindrical coordinates is not a linear one, the connection between the physical amplitudes in those two gauges must be complicated. In fact, they are rather remote.

There is a different choice of gauge which is, in our opinion, a happy compromise between these two extremes. It is the purpose of this paper to show this.

One can understand why the gauge excitation (denoted by the field Φ from now on) is massless. This is in part due to the choice $\partial_\mu A^\mu = 0$. Now, in a theory where we have complete symmetry between χ and Φ, where χ is the other component of the scalar field in a model defined below, this is a reasonable choice. We will, therefore, call this gauge the symmetrical gauge.

In a theory where the symmetry is eventually broken, albeit spontaneously, there is less compelling reason to make a symmetrical choice. In fact, a better alternative as we will show, is the asymmetrical gauge $\partial_\mu A^\mu = (\kappa^2/m)\Phi$,[12] where m is the mass of the vector particle A_μ and κ^2 has the dimension of (mass)^2. It will be further shown that κ^2 is in fact the mass parameter of the gauge excitation. As a corollary, we therefore state that the gauge excitation has nothing whatsoever to do with the Goldstone boson.

We will show that by taking the formal limit of $\kappa^2 \to \infty$, one can reach the unitarity gauge. However, this limit must be taken with extreme care, since it may interfere with the high-momentum limits of internal-loop integrations. It is precisely for this reason that the effective Hamiltonian in the unitarity gauge has quartic divergent compensating terms. Besides, there can exist ambiguous quadratic and lower-order divergent terms, if unheeded limiting procedure is taken. The situation here is quite like that discussed by Lee and Yang[13] in their ξ-limiting procedure.

The emphasis of this paper is more on using the gauge freedom to perform a calculation. Thus, the formal problems of gauge transformation properties of the Heisenberg fields, the connection of Green's functions in different gauges, etc. are not given here. We hope to return to these problems in the future. Instead, we will give several examples to show that (a) physical quantities are gauge-independent, (b) the calculations simplify drastically at the very beginning if a proper gauge is chosen, and (c) renormalization constants depend on gauge.

The plan of this paper is as follows: In Sec. I we introduce a simplified Abelian-gauge model, which was used by Appelquist and Quinn.[8] We briefly discuss the spontaneous-symmetry-breaking mechanism,[14, 15] There is nothing new here; its inclusion merely serves to introduce our notations.

In Sec. II we will quantize the system in the asymmetrical gauge. This will be done immediately in the interaction picture. We will not follow the functional integration technique[16] which is thus far quite popular in this area. Instead, we will employ the Lagrange-multiplier method, which is perhaps more familiar. In a certain sense our work here is an extension of some consideration we gave before to a related problem.[12]

In Sec. III we construct the energy tensor of the free fields and show that the scalar-gauge excitation and the scalar ghost of A_μ do not appear. Plane-wave expansions are also given.

In Sec. IV we use the commutation relation derived in Sec. II to construct the free propagators. These quantities are all manifestly Lorentz-covariant and therefore the perturbation expansion is immediate. We will discuss somewhat various choices of gauge parameters.

In Sec. V the formal transition to the unitarity gauge is given and discussed by taking the limit $\kappa^2 \to \infty$.

In Sec. VI examples are provided to illustrate the cancellation of gauge-dependent terms in some simple physical processes: (a) $\Psi + \Psi \to \Psi + \Psi$, (b) $\chi + \Psi \to A_\mu + \Psi$, and (c) the second-order anomalous magnetic moment of the "electron" Ψ. These also help to demonstrate that a clever choice of gauge simplifies the calculation right at the start. We then give several examples to illustrate that renormalization constants are gauge-dependent. These examples also indicate that such renormalization constants, for whatever they are worth, can be defined only for certain sets of gauge parameters. We also show the pitfall one may encounter in the limit $\kappa^2 \to \infty$.

A short conclusion and discussion is given in Sec. VII.

In the Appendix, we will discuss the compensating term due to ghost loops.

I. A MODEL

The model we consider here is an Abelian-gauge model used by Appelquist and Quinn.[8] We do not consider the non-Abelian extension at this stage, since nothing we want to do is gained or lost from its complexity. The Lagrangian is[17]

$$\mathcal{L} = + \tfrac{1}{4}G_{\mu\nu}G^{\mu\nu} - \tfrac{1}{2}G^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \bar{\Phi}^*_\mu \bar{\Phi}^\mu - \bar{\Phi}^*_\mu(\partial^\mu - ieA^\mu)\bar{\Phi} - (\partial^\mu + ieA^\mu)\bar{\Phi}^* \bar{\Phi}_\mu - \bar{\mu}^2 \bar{\Phi}^* \bar{\Phi}$$
$$- h(\bar{\Phi}^*\bar{\Phi})^2 - \bar{\Psi}[\gamma_\mu (1/i)\partial_\mu - g\gamma_\mu A^\mu \tfrac{1}{2}(1 + i\gamma_5)]\Psi - f\bar{\Psi}[\tfrac{1}{2}(1 - i\gamma_5)\bar{\Phi} + \tfrac{1}{2}(1 + i\gamma_5)\bar{\Phi}^*]\Psi \qquad (h > 0), \tag{1}$$

which is invariant under the transformation

$$A_\mu \to A_\mu + \partial_\mu \Lambda\,, \tag{2a}$$
$$\bar{\Phi} \to e^{ig\Lambda}\bar{\Phi}\,, \tag{2b}$$
$$\Psi \to e^{ig(1+i\gamma_5)\Lambda/2}\Psi\,, \tag{2c}$$

where Λ is an arbitrary continuous space-time-dependent gauge parameter. Clearly, independent variations of $G_{\mu\nu}$ and Φ_μ and stationary action yield

$$G_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,, \tag{3a}$$

$$\overline{\Phi}_\mu=(\partial_\mu-ieA_\mu)\overline{\Phi}\,, \tag{3b}$$

respectively.

We now turn to briefly discuss the spontaneous-symmetry-breaking mechanism.[14,15] Let us imagine that Ψ and A_μ are temporarily turned off in Eq. (1). Thus, the Hamiltonian of this complex-scalar-field system is

$$\mathcal{H}=\partial^0\overline{\Phi}^*\partial^0\overline{\Phi}+\partial_k\overline{\Phi}^*\partial_k\overline{\Phi}+\overline{\mu}^2\overline{\Phi}^*\overline{\Phi}+h(\overline{\Phi}^*\overline{\Phi})^2 \qquad (h>0)\,. \tag{4}$$

In the extremely long-wavelength and low-frequency limit, i.e.,

$$\partial_0\overline{\Phi}\to 0 \quad\text{and}\quad \vec{\partial}\,\overline{\Phi}\to 0\,,$$

denoted as $\overline{\Phi}_{k=0}$, the state with the lowest energy is $\overline{\Phi}_{k=0}=0$ if $\overline{\mu}^2>0$. On the other hand if $\overline{\mu}^2<0$, then the extremum solutions are $\overline{\Phi}^*\overline{\Phi}_{k^2=0}=0$ or

$$\overline{\Phi}^*\overline{\Phi}_{k=0}=-\tfrac{1}{2}\overline{\mu}^2/h\,. \tag{5}$$

If nature chooses the second solution, then we have a symmetry which is spontaneously broken. We can readjust the energy scale, so that the lowest energy again is zero. This state is the vacuum. Sandwiching Eq. (5) between vacuum states and using translational invariance, we can see that

$$\langle\overline{\Phi}\rangle_{k=0}\equiv\langle 0|\overline{\Phi}_{k=0}|0\rangle\neq 0\,, \tag{6}$$

if we assume that $|0\rangle$ is the only state with vanishing energy.[18] We choose as our convention that it is the real (Hermitian) part in $\overline{\Phi}$ which has nonvanishing vacuum expectation value $(1/\sqrt{2})\,v$ and write

$$\overline{\Phi}=\frac{1}{\sqrt{2}}(v+\chi+i\Phi)\,. \tag{7}$$

From this definition, we have implicitly imposed the requirements

$$\langle\chi\rangle=\langle\Phi\rangle=0$$

and

$$v^2=-\overline{\mu}^2/h\,. \tag{8}$$

We now assume that the same situation persists with the presence of Ψ and A_μ. Then we reexpress the Lagrangian of Eq. (1) as

$$\mathcal{L}=\mathcal{L}_{\text{free}}+\mathcal{L}_{\text{int}}\,, \tag{9}$$

where

$$\begin{aligned}\mathcal{L}_{\text{free}}={}&+\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu}-\tfrac{1}{2}G^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)-\tfrac{1}{2}m^2A_\mu A^\mu-\Phi_\mu\partial^\mu\Phi+\tfrac{1}{2}\Phi_\mu\Phi^\mu\\&+mA^\mu\partial_\mu\Phi-\chi_\mu\partial^\mu\chi+\tfrac{1}{2}\chi_\mu\chi^\mu-\tfrac{1}{2}\mu^2\chi\chi-\overline{\Psi}[\gamma_\mu(1/i)\partial^\mu+m_e]\Psi\end{aligned} \tag{10}$$

and

$$\begin{aligned}\mathcal{L}_{\text{int}}={}&gA^\mu(\chi\partial_\mu\Phi-\Phi\partial_\mu\chi)-g\,mA_\mu A^\mu\chi-\tfrac{1}{2}g^2A_\mu A^\mu(\chi^2+\Phi^2)-\mu(\tfrac{1}{2}h)^{1/2}\chi(\chi^2+\Phi^2)-\tfrac{1}{4}h(\chi^2+\Phi^2)^2\\&+g\overline{\Psi}\gamma_\mu\tfrac{1}{2}(1+i\gamma_5)\Psi A^\mu-(m_e/m)g\overline{\Psi}\Psi\chi-(m_e/m)g\overline{\Psi}\gamma_5\Psi\Phi\,,\end{aligned} \tag{11}$$

where

$$m=gv\,, \tag{12a}$$

$$\mu=\sqrt{2h}\;v\,, \tag{12b}$$

$$m_e=fv/\sqrt{2}\,. \tag{12c}$$

It is seen that the vacuon $v/\sqrt{2}=\langle\overline{\Phi}\rangle$ gives masses to Ψ and A_μ. It may also appear from the free Lagrangian that Φ is massless. However, this need not be the case, as we will show in the next section.

II. QUANTIZATION OF A_μ and Φ

The parts which describe χ and Ψ in the free Lagrangian Eq. (10) are familiar ones. We need not consider them any further. Let us look at those parts which describe A_μ and Φ:

$$\mathcal{L}' = +\tfrac{1}{4}G^{\mu\nu}G_{\mu\nu} - \tfrac{1}{2}G^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) - \tfrac{1}{2}m^2 A_\mu A^\mu - \Phi_\mu \partial^\mu \Phi + \tfrac{1}{2}\Phi_\mu \Phi^\mu + mA^\mu \partial_\mu \Phi . \tag{13}$$

This is invariant under the transformation

$$G^{\mu\nu} \to G^{\mu\nu}, \tag{14a}$$

$$A^\mu \to A^\mu + \partial^\mu \lambda, \tag{14b}$$

$$\Phi \to \Phi + m\lambda, \tag{14c}$$

$$\Phi_\mu \to \Phi_\mu + m\partial_\mu \lambda . \tag{14d}$$

Note that the last term in Eq. (13) is a derivative bilinear coupling between A_μ and Φ. Because of it, we cannot quantize it in the usual fashion, otherwise the canonical relations will be inconsistent with the equations of motion. This has been pointed out in our previous work.[12] The reason is that in the canonical scheme of quantization, $-\partial^0\Phi + mA^0$ is at the same time both the momentum conjugate to Φ and a dependent variable, expressible in terms of the momentum-conjugate variables of A_k (i.e., G_{0k}). It fails to play these two roles simultaneously. Thus, the proper way to quantize Eq. (13) is to introduce a Lagrange multiplier G. We modify Eq. (13) to read

$$\mathcal{L}_0 = -\tfrac{1}{2}G^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \tfrac{1}{4}G^{\mu\nu}G_{\mu\nu} - \tfrac{1}{2}m^2 A_\mu A^\mu - \Phi^\mu \partial_\mu \Phi + \tfrac{1}{2}\Phi^\mu \Phi_\mu + mA_\mu \partial^\mu \Phi - G[\partial_\mu A^\mu - (\kappa^2/m)\Phi] + \tfrac{1}{2}\xi^{-1}G^2 , \tag{15}$$

where, relying on our experience in electrodynamics, we demand that G be an operator which will help to generate the gauge transformations given by Eqs. (14a)–(14d). It follows that G must annihilate all *right* physical states,[19] since they are gauge-invariant. That is,

$$G|\Psi\rangle = \partial G|\Psi\rangle = 0 . \tag{16}$$

Now, under an infinitesimal gauge transformation $\delta\lambda$

$$G \to G \tag{17}$$

and we have

$$\mathcal{L}_0 \to \mathcal{L}_0 - G(\partial^2 - \kappa^2)\delta\lambda .$$

By the action principle, the generator which effects the transformation (14a)–(14d) is

$$G_{\delta\lambda} = \int d^3x\,(G\,\partial_0\delta\lambda - \delta\lambda\,\partial_0 G) \tag{18}$$

and the stationary requirement gives

$$(\partial^2 - \kappa^2)G = 0 . \tag{19}$$

The Heisenberg equations

$$\frac{1}{i}[A_\mu, G_{\delta\lambda}] = \partial_\mu \delta\lambda , \qquad \frac{1}{i}[\Phi, G_{\delta\lambda}] = m\delta\lambda , \quad \text{etc.} \tag{20}$$

give the following nonvanishing commutation relations at equal time:

$$\frac{1}{i}[A_0(x), G(x')] = \delta(\vec{x} - \vec{x}'), \tag{21a}$$

$$\frac{1}{i}[A_k(x), \partial_0 G(x')] = -\partial_k \delta(\vec{x} - \vec{x}'), \tag{21b}$$

$$\frac{1}{i}[\Phi(x), \partial_0 G(x')] = -m\delta(\vec{x} - \vec{x}'), \tag{21c}$$

$$\frac{1}{i}[\Phi_0(x), G(x')] = m\delta(\vec{x} - \vec{x}') . \tag{21d}$$

Since the transformation is Abelian, we must have

$$[G_{\delta\lambda_1}, G_{\delta\lambda_2}] = 0$$

or

$$[G(x), G(x')] = [G(x), \partial_0 G(x')] = [\partial_0 G(x), \partial_0 G(x')] = 0 . \tag{22}$$

Finally, the usual equal-time commutation relations

$$\frac{1}{i}[\Phi(x), \partial_0 \Phi(x')] = \delta(\vec{x} - \vec{x}')$$

and

$$\frac{1}{i}[G^{0k}(x), A_l(x')] = \delta^k_l \delta(\vec{x} - \vec{x}') . \tag{23}$$

are true. Note that

$$[A^0, G^{0k}] = [A^0, A^k] = [A^0, \Phi] = [A^0, \Phi^0] = 0 \tag{24}$$

which means that A^0 is an independent dynamical variable. We have elevated it to equal status with the spatial components A^k. This, as we will see, makes the propagators for A_μ manifestly Lorentz-covariant.

We now obtain the Euler's equations by independent variations of $\mathcal{L}_0$ in Eq. (15) with respect to

$$\delta G^{\mu\nu}: \quad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{25a}$$

$$\delta A_\mu: \quad -\partial_\nu G^{\mu\nu} - m^2 A^\mu + m\,\partial^\mu \Phi + \partial^\mu G = 0\,, \tag{25b}$$

$$\delta \Phi_\mu: \quad \Phi_\mu = \partial_\mu \Phi \,, \tag{25c}$$

$$\delta \Phi: \quad \partial_\mu \Phi^\mu - m\,\partial_\mu A^\mu + (\kappa^2/m) G = 0\,, \tag{25d}$$

$$\delta G: \quad G = \xi[\partial_\mu A^\mu - (\kappa^2/m)\Phi]\,. \tag{25e}$$

It is not difficult to show that the equations of motion are now consistent with the commutation relations.

III. ENERGY-MOMENTUM TENSOR DENSITIES AND PLANE-WAVE EXPANSION OF A_μ and Φ

By standard procedure one can deduce the energy-momentum tensor densities from the Lagrangian of Eq. (15). They are

$$\Theta^{00} = T^{00} - A^l \partial_l G - A^0(\partial_l G^{0l} + m^2 A^0 - m\,\partial^0 \Phi) - (\kappa^2/m) G\Phi - \tfrac{1}{2}\xi^{-1} G^2 \tag{26a}$$

and

$$\Theta^{0k} = T^{0k} - A^0 \partial^k G - A^k(\partial_l G^{0l} + m^2 A_0 - m\,\partial^0 \Phi)\,, \tag{26b}$$

where

$$T^{00} = \tfrac{1}{4}(G^{kl})^2 + \tfrac{1}{2}(G^{0k})^2 + \tfrac{1}{2} m^2 (A^k - \Phi^k/m)^2 + \tfrac{1}{2} m^2 (A^0 - \Phi^0/m)^2 \tag{27a}$$

and

$$T^{0k} = G^{0l} G^k{}_l + (\Phi^0 - m A^0)(\partial^k \Phi - m A^k)\,. \tag{27b}$$

When acting on physical states, we have

$$\Theta^{00} | \Psi \rangle = T^{00} | \Psi \rangle \tag{28a}$$

and

$$\Theta^{0k} | \Psi \rangle = T^{0k} | \Psi \rangle \,. \tag{28b}$$

$\Theta^{0\mu}$ are the extended energy-momentum tensor densities, derivable directly from the Lagrangian, whereas $T^{0\mu}$ are true energy-momentum tensor densities as measured by the physical states. $T^{0\mu}$ are gauge-invariant.

We now want to write down plane-wave expansions for A_μ and Φ. Since creation and annihilation operators have meaning only when acting on states, in the remaining of this section all equations are to be understood with $|\Psi\rangle$ standing to the right.[19] Thus, Eq. (21e) becomes

$$\partial_\mu A^\mu = (\kappa^2/m)\Phi\,. \tag{29}$$

Together with Eqs. (25c) and (25d), we have

$$\partial^2 \Phi = \kappa^2 \Phi\,. \tag{30}$$

This suggests that κ is the "mass" of the gauge excitation. We see that it has no connection with the boson in the Goldstone mechanism at all.

Similarly, Eq. (29) and Eqs. (25a) and (25b) lead to

$$\partial^2 A^\mu - m^2 A^\mu + \frac{m^2 - \kappa^2}{\kappa^2} \partial^\mu \partial_\nu A^\nu = 0\,. \tag{31}$$

These last three equations define the plane-wave solutions uniquely. They are

$$\vec{A} = \sum_{\vec{q},\,t} \left\{ \left(\frac{1}{2\pi}\right)^{3/2} \left(\frac{1}{2q^0}\right)^{1/2} \left[(a_q^t e^{iq\cdot x} + a_q^{t\dagger} e^{-iq\cdot x}) \vec{\epsilon}_q^{\,t} + (a_q^l e^{iq\cdot x} + a_q^{l\dagger} e^{-iq\cdot x}) q^0 \frac{\hat{q}}{m} \right] + \left(\frac{1}{2\pi}\right)^{3/2} \left(\frac{1}{2q'^0}\right)^{1/2} (a_{q'}^s e^{iq'\cdot x} + a_{q'}^{s\dagger} e^{-iq'\cdot x}) \frac{\vec{q}}{m} \right\}, \tag{32a}$$

$$A^0 = \sum_{\vec{q}} \left[\left(\frac{1}{2\pi}\right)^{3/2} \left(\frac{1}{2q^0}\right)^{1/2} (a_q^l e^{iq\cdot x} + a_q^{l\dagger} e^{-iq\cdot x}) \frac{|\vec{q}|}{m} + \left(\frac{1}{2\pi}\right)^{3/2} \left(\frac{1}{2q'^0}\right)^{1/2} (a_{q'}^s e^{iq'\cdot x} + a_{q'}^{s\dagger} e^{-iq'\cdot x}) \frac{q'^0}{m} \right], \tag{32b}$$

and

$$\Phi=-i\sum_{\vec{q}}\left(\frac{1}{2\pi}\right)^{3/2}\left(\frac{1}{2q'^0}\right)^{1/2}(a^s_{q'}e^{iq'\cdot x}-a^{s\dagger}_{q'}e^{-iq'\cdot x}), \tag{32c}$$

where

$$q^0=(\vec{q}^{\,2}+m^2)^{1/2}, \qquad q'^0=(\vec{q}^{\,2}+\kappa^2)^{1/2}, \qquad \text{and} \quad q'^\mu=(q'^0,\vec{q}).$$

ϵ^t_q $(t=1,2)$ are the transverse polarization vectors, and together with the unit vector $\hat{q}=\vec{q}/|\vec{q}|$ they form a right-handed orthonormal triad.

It is easy to verify that the gauge-invariant combinations $A_\mu-\partial_\mu\Phi/m$ and $G_{\mu\nu}$ do not contain the scalar modes $a^s_{q'}$ and $a^{s\dagger}_{q'}$. It follows from Eqs. (27a) and (27b) that we need to concern ourselves with the transverse and the (3-dimensional) longitudinal modes only. The Fock space so constructed is identical to the usual one for a neutral massive vector meson. The gauge excitation never appears in the asymptotic states.

IV. PROPAGATORS

We derive in the following the propagators which result from Eqs. (25a)–(25e).

We define

$$G_{\mu\nu}(x-y)=\langle(A_\mu(x)A_\nu(y))_+\rangle, \tag{33a}$$

$$G_\mu(x-y)=\langle(\Phi(x)A_\mu(y))_+\rangle, \tag{33b}$$

and

$$G(x-y)=\langle(\Phi(x)\Phi(y))_+\rangle. \tag{33c}$$

From Eqs. (25a)–(25c) we have the equation

$$\partial^2A^\mu-m^2A^\mu-(1-\xi)\partial^\mu\partial_\nu A^\nu+m[1-\xi(\kappa^2/m^2)]\partial^\mu\Phi=0. \tag{34a}$$

Likewise Eqs. (25c)–(25e) give

$$[\partial^2-\xi(\kappa^4/m^2)]\Phi+(1/m)(\xi\kappa^2-m^2)\partial_\mu A^\mu=0. \tag{34b}$$

Now, it is not difficult to use the commutation relations of Eqs. (21)–(23) to obtain

$$\partial_0{}^2G_{\mu\nu}(x-y)=\langle(\partial_0{}^2A_\mu(x)A_\nu(y))_+\rangle+(i/\xi)g_{\mu0}g_{\nu0}\delta(x-y)-i(g_{\mu\nu}-g_{\mu0}g_{\nu0})\delta(x-y)$$

and

$$\partial_\mu\partial^\lambda G_{\lambda\nu}(x-y)=\langle(\partial_\mu\partial_\lambda A^\lambda(x)A_\nu(y))_+\rangle-(i/\xi)g_{\mu0}g_{\nu0}\delta(x-y)$$

which will lead to the equation

$$(-\partial^2+m^2)G_{\mu\nu}(x-y)=-(1-\xi)\partial_\mu\partial^\lambda G_{\lambda\nu}(x-y)+m[1-\xi(\kappa^2/m^2)]\partial_\mu G_\nu(x-y)-ig_{\mu\nu}\delta(x-y). \tag{35}$$

Similarly, since Φ and A_μ are independent, we have

$$\partial_0{}^2G_\nu(x-y)=\langle(\partial_0{}^2\Phi(x)A_\nu(y))_+\rangle,$$

which yields

$$[-\partial^2+\xi(\kappa^4/m^2)]G_\nu(x-y)=(1/m)(\xi\kappa^2-m^2)\partial^\lambda G_{\lambda\nu}(x-y). \tag{36}$$

We can solve the coupled Eqs. (35) and (36), which give us

$$G_{\mu\nu}(x-y)=-i\left(g_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{m^2}\frac{(m^2-\kappa^2)^2}{(-\partial^2+\kappa^2-i\epsilon)^2}\right)\frac{1}{-\partial^2+m^2-i\epsilon}\delta(x-y)-i\left(\frac{\xi-1}{\xi}\right)\frac{1}{(-\partial^2+\kappa^2-i\epsilon)^2}\partial_\mu\partial_\nu\delta(x-y) \tag{37}$$

and

$$G_\mu(x-y)=\frac{-i}{\xi m}(\xi\kappa^2-m^2)\frac{1}{(-\partial^2+\kappa^2-i\epsilon)^2}\partial_\mu\delta(x-y). \tag{38}$$

In like fashion, we use

$$\partial_0{}^2G(x-y)=\langle(\partial_0{}^2\Phi(x)\Phi(y))_+\rangle-i\delta^4(x-y)$$

to arrive at

$$[-\partial^2+\xi(\kappa^2/m^2)]G(x-y)=(1/m)(\xi\kappa^2-m^2)\langle(\partial_\mu A^\mu(x)\Phi(y))_+\rangle-i\delta^4(x-y)$$
$$=-(1/m)(\xi\kappa^2-m^2)\partial^\mu G_\mu(x-y)-i\delta^4(x-y) \tag{39}$$

or

$$G(x-y)=-i\frac{-\partial^2+m^2/\xi}{(-\partial^2+\kappa^2-i\epsilon)^2}\,\delta(x-y)\,. \tag{40}$$

(*Note added in proof.* The propagators of Eqs. (37), (38), and (40) were also considered by Y. Fujii, Phys. Rev. 138, B423 1965). See also the references therein.)

For completeness, we also write down the propagators for χ and Ψ. They are

$$\langle(\chi(x)\chi(y))_+\rangle=-i\frac{1}{-\partial^2+\mu^2-i\epsilon}\,\delta(x-y) \tag{41}$$

and

$$\langle(\Psi(x)\bar\Psi(y))_+\rangle=-i\frac{m_e-\gamma\cdot\frac{1}{2}\partial}{-\partial^2+m_e^2-i\epsilon}\,\delta(x-y)\,. \tag{42}$$

We notice that all these propagators are manifestly Lorentz-covariant.[20] The perturbation series can be generated by taking

$$\int d^4x\,\mathcal{H}_{\rm int}=\int d^4x\,(-\mathcal{L}_{\rm int})+\delta H\,, \tag{43a}$$

where $\mathcal{L}_{\rm int}$ is given by Eq. (11) and

$$\delta H=i\,{\rm Tr}\ln\left(1+\frac{\kappa^2}{-\partial^2+\kappa^2}\frac{g}{m}\chi\right)\,, \tag{43b}$$

the origin of which will be explained in the Appendix. (I would like to thank Professor H. Quinn for pointing out to me the necessity of this piece when loops with external χ's are present.) We hold m, μ, and m_e fixed and expand in powers of e and $\sqrt{\hbar}$.

The high-momentum behavior of all boson propagators Eqs. (37), (38), (40), and (41) goes like $1/-\partial^2$, and so a power counting argument shows that the theory is renormalizable.[21]

If we set $\kappa^2=0$, we obtain the results by other authors, where functional-integration techniques were used. However, it is our opinion that to avoid infrared problem in some of the diagrams, although they must cancel out at the end, other values are preferred. As we shall show by examples, there are two gauges which are particularly attractive.

The first gauge is to set $\xi=1$ and $\kappa^2=m^2$.[5] Then ($\xi=1$, $m^2=\kappa^2$)

$$G_{\mu\nu}(x-y)=-ig_{\mu\nu}\frac{1}{-\partial^2+m^2-i\epsilon}\,\delta(x-y)\,, \tag{44a}$$

$$G_\mu(x-y)=0\,, \tag{44b}$$

and

$$G(x-y)=-i\frac{1}{-\partial^2+m^2-i\epsilon}\,\delta(x-y)\,. \tag{44c}$$

In this gauge, all the poles are simple. At the same time, we do not have Φ, A_μ transition [Eq. (44b)], which simplifies the number of diagrams drastically.

The other gauge is to take the limit $\kappa^2\to\infty$ which we discuss in the next section.

V. TRANSITION TO THE UNITARITY GAUGE

In the limit $\kappa^2\to\infty$, we have

$$G_{\mu\nu}(x-y)\to-i\left(g_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{m^2}\right)\frac{1}{-\partial^2+m^2-i\epsilon}\,\delta(x-y)\,, \tag{45a}$$

$$G_\mu(x-y)\to0\,, \tag{45b}$$

and

$$G(x-y)\to 0\,. \tag{45c}$$

The interaction Lagrangian of Eq. (11) becomes

$$\mathfrak{L}_{\rm int}\to -gmA_\mu A^\mu\chi-\tfrac{1}{2}g^2A_\mu A^\mu\chi^2-\mu(\tfrac{1}{2}h)^{1/2}\chi^3-\tfrac{1}{4}h\chi^4+g\overline{\Psi}\gamma_\mu\tfrac{1}{2}(1+i\gamma_5)\Psi A^\mu-(m_e/m)g\overline{\Psi}\Psi\chi\,, \tag{46}$$

$$\delta H\to i\delta^4(0)\ln[1+(g/m)\chi]\,. \tag{47}$$

The propagator (45a) and the interaction above are what one has in the unitarity gauge. However, we must feel somewhat uneasy about the way this formal limit is taken. There are many internal loops in a general diagram. If the integration and the limit are finite, then the limiting procedure is justified. However, in other cases, there will be ambiguities.

For large $-\partial^2$, the vector propagator of Eq. (45a) behaves like a constant. This gives rise to quartic leading divergences in some amplitudes, which are canceled out by the quartic divergence of Eq. (47).

On the other hand, for any finite value of κ^2 we have the high-momentum behavior

$$G_{\mu\nu}(x-y)\sim 1/-\partial^2 \tag{48}$$

instead of constant as given by Eq. (45a). In order to avoid the difficulty which may result because of different assignments of momenta in a diagram when high-degree divergences occur, it seems to us that the proper way to do the unitarity gauge is to first find a convenient value for ξ, regularize the diagrams sufficiently, and then take $\kappa^2\to\infty$. This is similar to the spirit of Lee and Yang[13] in their formulation of a complex vector field coupled to an electromagnetic field.[22]

VI. EXAMPLES

We will present some simple examples here to show a proper choice of gauge at the start saves a lot of unnecessary cancellation. We see also that all physical quantities do not depend on ξ and κ^2, the gauge parameters. In other words, they are gauge-invariant. Finally, we calculate some renormalization constants and show that they are gauge-dependent and at the same time illustrate that the formal limit of $\kappa^2\to\infty$ must be taken with great care.

Before we carry on any calculation, we summarize our notations for diagrams in Fig. 1.

A. $\Psi+\Psi\to\Psi+\Psi$

The lowest-order diagrams are shown in Fig. 2. Let us define the invariant transition amplitude by

$$S=1+i(2\pi)^4\delta(p_1+p_2-p_1'-p_2')NM\,, \tag{49}$$

where N is the usual normalization factor, and

$$M^{(a)}=g^2\bar{u}(p_1')\gamma^\mu\tfrac{1}{2}(1+i\gamma_5)u(p_1)\bar{u}(p_2')\gamma^\nu\tfrac{1}{2}(1+i\gamma_5)u(p_2)$$
$$\times\left\{\left[g_{\mu\nu}+\frac{k_\mu k_\nu}{m^2}\,\frac{(m^2-\kappa^2)^2}{(k^2+\kappa^2)^2}\right]\frac{1}{k^2+m^2}-\frac{\xi-1}{\xi}\,\frac{1}{(k^2+\kappa^2)^2}\,k_\mu k_\nu\right\}, \tag{50a}$$

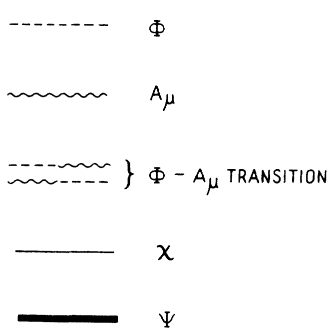


FIG. 1. Notations for lines.

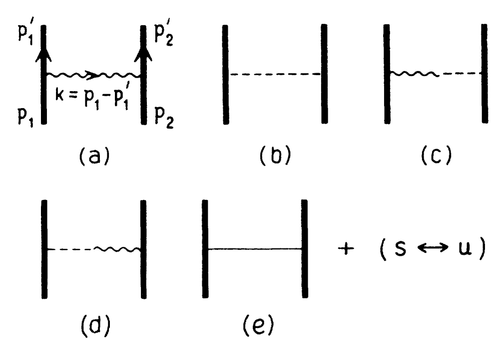


FIG. 2. Lowest-order diagrams for the process $\Psi+\Psi\to\Psi+\Psi$.

$$M^{(b)}=g^2\left(\frac{m_e}{m}\right)^2\bar{u}(p_1')\gamma_5 u(p_1)\bar{u}(p_2')\gamma_5 u(p_2)\frac{k^2+m^2/\xi}{(k^2+\kappa^2)^2}\ , \tag{50b}$$

$$M^{(c)}=-ig^2\left(\frac{m_e}{m}\right)\bar{u}(p_1')\gamma_\mu\tfrac{1}{2}(1+i\gamma_5)u(p_1)\bar{u}(p_2')\gamma_5 u(p_2)\frac{1}{\xi m}\,(\xi\kappa^2-m^2)\frac{1}{(k^2+\kappa^2)^2}\,k^\mu\ , \tag{50c}$$

$$M^{(d)}=ig^2\left(\frac{m_e}{m}\right)\bar{u}(p_1')\gamma_5 u(p_1')\bar{u}(p_2')\gamma_\mu\tfrac{1}{2}(1+i\gamma_5)u(p_2)\frac{1}{\xi m}\,(\xi\kappa^2-m^2)\frac{1}{(k^2+\kappa^2)^2}\,k^\mu\ , \tag{50d}$$

$$M^{(e)}=g^2\left(\frac{m_e}{m}\right)^2\bar{u}(p_1')u(p_1)\bar{u}(p_2')u(p_2)\frac{1}{k^2+\mu^2}\ . \tag{50e}$$

Using the identities, which are due to Dirac's equations

$$\bar{u}(p_1')\gamma\cdot k\,\tfrac{1}{2}(1+i\gamma_5)u(p_1)=m_e\bar{u}(p_1')i\gamma_5 u(p_1)$$

and (51)

$$\bar{u}(p_2')\gamma\cdot k\,\tfrac{1}{2}(1+i\gamma_5)u(p_2)=-m_e\bar{u}(p_2')i\gamma_5 u(p_2)\ ,$$

we have, after some algebra,

$$\begin{aligned}M^{(a)}+M^{(b)}+M^{(c)}+M^{(d)}+M^{(e)}\\ &=g^2\bar{u}(p_1')\gamma_\mu\tfrac{1}{2}(1+i\gamma_5)u(p_1)\bar{u}(p_2')\gamma^\mu\tfrac{1}{2}(1+i\gamma_5)u(p_2)\frac{1}{k^2+m^2}\\ &\quad+g^2\left(\frac{m_e}{m}\right)^2\bar{u}(p_1')\gamma_5 u(p_1)\bar{u}(p_2')\gamma_5 u(p_2)\,\frac{1}{k^2+m^2}+g^2\left(\frac{m_e}{m}\right)^2\bar{u}(p_1')u(p_1)\bar{u}(p_2')u(p_2)\frac{1}{k^2+\mu^2}\ .\end{aligned} \tag{52}$$

Clearly, had we used the gauge $\kappa^2=m^2$ and $\xi=1$,[5] Eqs. (44a)–(44c), we would have obtained this result right away. Note that (52) is gauge-independent.

B. $\chi+\Psi\to A_\mu+\Psi$

The lowest-order diagrams are depicted in Fig. 3. Again, using Eq. (51) and the transversality condition

$$p_1'\cdot\epsilon_1'=0\ ,$$

where ϵ_1' is the polarization vector of A_μ in the final state, we have

$$M^{(a)}+M^{(b)}+M^{(c)}+M^{(d)}=-2g^2 m\bar{u}(p_2')\gamma\cdot\epsilon_1'\,\tfrac{1}{2}(1+i\gamma_5)u(p_2)\frac{1}{k^2+m^2}+2ig^2\left(\frac{m_e}{m}\right)p_1\cdot\epsilon_1'\,\frac{\bar{u}(p_2')\gamma_5 u(p_2)}{k^2+m^2}\ . \tag{53}$$

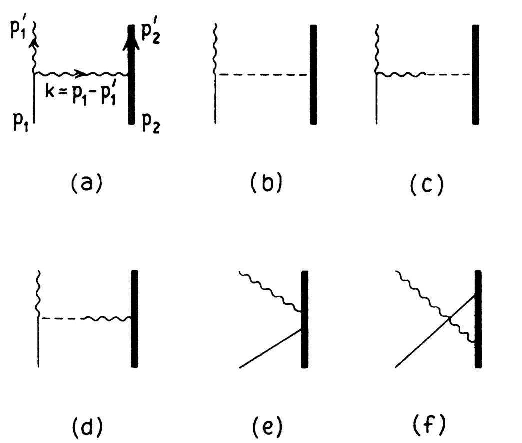


FIG. 3. Lowest-order diagrams for the process $\chi+\Psi\to A_\mu+\Psi$.

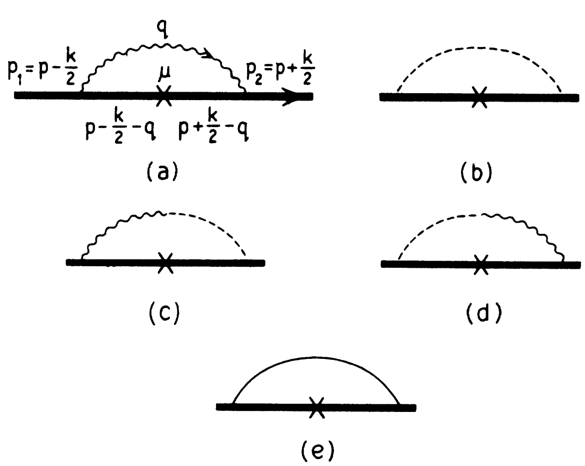


FIG. 4. Second-order nonelectromagnetic contribution to the electron anomalous magnetic moment.

Again, this result can be directly written down if we use the gauge $\xi = 1$ and $\kappa^2 = m^2$.

C. Second-Order Anomalous Magnetic Moment of Ψ

Let us assume that only the Ψ field carries electric charge. Upon the introduction of an electromagnetic field $[A_\mu(\gamma)]$, an extra term appears in the interaction Lagrangian of Eq. (11):

$$e\bar{\Psi}\gamma_\mu \Psi A(\gamma)^\mu . \tag{54}$$

The second-order nonelectromagnetic corrections are shown in Fig. 4. The corresponding proper vertex functions can be written down

$$\bar{u}(p_2)\Lambda_\mu^{(a)} u(p_1) = \tfrac{1}{4}g^2 \int \frac{d^4q}{(2\pi)^4 i}\, \bar{u}(p_2)\left[\gamma_\lambda \frac{1}{m_e + \gamma\cdot(p_2-q)}\gamma_\mu \frac{1}{m_e+\gamma\cdot(p_1-q)}\gamma_\kappa + \gamma_\lambda \frac{1}{m_e - \gamma\cdot(p_2-q)}\gamma_\mu \frac{1}{m_e-\gamma\cdot(p_1-q)}\gamma_\kappa\right] \times u(p_1)\left\{\left[g^{\lambda\kappa} + \frac{q^\lambda q^\kappa}{m^2}\frac{(m^2-\kappa^2)^2}{(q^2+\kappa^2)^2}\right]\frac{1}{q^2+m^2} - \frac{\xi-1}{\xi}\frac{q^\lambda q^\kappa}{(q^2+\kappa^2)^2}\right\}, \text{ etc.} \tag{55}$$

Now, we write,

$$\bar{u}(p_2)\Lambda_\mu u(p_1) = \bar{u}(p_2)(a\gamma_\mu + bi\sigma_{\mu\nu}k^\nu)u(p_1), \tag{56}$$

then it can be shown easily that to order linear in k,[23]

$$16\, b m_e^2 k^\lambda = \mathrm{Tr}(m_e - \gamma\cdot p_2)\Lambda_\mu(m_e-\gamma\cdot p_1)\sigma^{\mu\lambda}/i - \tfrac{3}{2}(k^\lambda/m_e)\,\mathrm{Tr}(m_e-\gamma\cdot p_2)\Lambda_\mu(m_e-\gamma\cdot p_1)\gamma^\mu . \tag{57}$$

After some tedious trace calculation, one finds that Figs. (4a)–(4d) give

$$16\, b^{(a-d)} m_e^2 k^\nu = \tfrac{1}{2}g^2 \int \frac{d^4q}{(2\pi)^4 i}\,\frac{64 m_e^3 k^\nu + 96 m_e p\cdot q k^\nu - 16 m_e k\cdot q q^\nu + 16 m_e q^2 k^\nu + 48[(p\cdot q)^2/m_e]k^\nu}{[m_e^2 + (p-q)^2]^2(m^2+q^2)} + \left(\frac{m_e}{m}\right)^2 g^2 \int \frac{d^4q}{(2\pi)^4 i}\,\frac{8m_e q^2 k^\nu + 24[(p\cdot q)^2/m_e]k^\nu - 8m_e q\cdot k q^\nu}{[m_e^2+(p-q)^2]^2(m^2+q^2)} + R^\nu , \tag{58}$$

where R^ν is a gauge-dependent term:

$$R^\nu = \tfrac{1}{2}g^2\,\frac{m_e}{m}(\kappa^2 - m^2)\int \frac{d^4q}{(2\pi)^4 i}\,\frac{N^\nu}{[(p-q)^2+m_e^2]^2(q^2+m^2)(q^2+\kappa^2)}, \tag{59}$$

with

$$N^\nu = -32 m_e (p\cdot q)^2 k^\nu + 16 m_e p\cdot q\, q^2 k^\nu + 32 m_e^3 q\cdot k q^\nu . \tag{60}$$

The first term in Eq. (58) is due to the $g_{\mu\nu}$ part of the vector-meson propagator in Eq. (55). The second term together with R^ν is due to the rest of Figs. 4(a)–4(d). In fact, the second term is the same object as we would obtain from Fig. 4(b) alone if we set $\xi = 1$ and $\kappa^2 = m^2$.

We proceed to show that R^ν actually gives a vanishing contribution. This is done by writing

$$\kappa^2 - m^2 = (q^2+\kappa^2) - (q^2+m^2),$$

to split Eq. (59) into two terms:

$$R^\nu = \tfrac{1}{2}g^2(m_e/m)^2(R^\nu_{m^2} - R^\nu_{\kappa^2}), \tag{61}$$

where

$$R^\nu_{\kappa^2} = \int \frac{d^4q}{(2\pi)^4 i}\,\frac{N^\nu}{[(p-q)^2+m_e^2]^2(q^2+\kappa^2)}, \tag{62}$$

and $R^\nu_{m^2}$ is obtained from $R^\nu_{\kappa^2}$ by the substitution $\kappa^2 \to m^2$.

We introduce Feynman's parameters and make a shift of origin in q integration which results in

$$R^\nu_{\kappa^2} = 32\, k^\nu m_e^3 \int \alpha_1 d\alpha_1 d\alpha_2 \delta(1-\alpha_1-\alpha_2)\int \frac{d^4q'}{(2\pi)^4 i}\,\frac{(1-\tfrac{3}{2}\alpha_1)q'^2 - m_e^2\alpha_1^2(2-\alpha_1)}{(q'^2+\alpha_1^2 m_e^2 + \alpha_2\kappa^2)^3} \quad (q' = q - \alpha_1 p). \tag{63}$$

This integral is logarithmically divergent. We regularize it by a cutoff (Λ) after the Euclidean rotation. Then

$$R^{\nu}_{\kappa^2}=\frac{1}{16\pi^2}\,32\,k^{\nu}m_e^{3}\int \alpha_1 d\alpha_1 d\alpha_2\delta(1-\alpha_1-\alpha_2)$$
$$\times\left[(1-\tfrac{3}{2}\alpha_1)\ln\Lambda^2-(1-\tfrac{3}{2}\alpha_1)\ln(\alpha_1^{2}m_e^{2}+\alpha_2\kappa^2)+\frac{-\frac{1}{2}m_e^{2}\alpha_1^{2}(2-\alpha_1)}{\alpha_1^{2}m_e^{2}+\alpha_2\kappa^2}-\tfrac{3}{2}(1-\tfrac{3}{2}\alpha_1)\right]. \tag{64}$$

Now, we write the part with logarithmic integrand as

$$-\int \alpha_1 d\alpha_1 d\alpha_2\delta(1-\alpha_1-\alpha_2)(1-\tfrac{3}{2}\alpha_1)\ln(\alpha_1^{2}m_e^{2}+\alpha_2\kappa^2)=-\int_0^1[d(\tfrac{1}{2}\alpha_1^{2}-\tfrac{1}{2}\alpha_1^{3})]\ln[\alpha_1^{2}m_e^{2}+(1-\alpha_1)\kappa^2]$$
$$=\int_0^1 d\alpha_1\frac{\frac{1}{2}\alpha_1^{2}(1-\alpha_1)(2\alpha_1 m_e^{2}-\kappa^2)}{\alpha_1^{2}m_e^{2}+(1-\alpha_1)\kappa^2} \tag{65}$$

in which the last term is obtained by an integration by parts.

All in all, $R^{\nu}_{\kappa^2}$ can be cast into

$$R^{\nu}_{\kappa^2}=\frac{1}{16\pi^2}\,32\,k^{\nu}m_e^{3}\int_0^1 d\alpha_1[-\tfrac{1}{2}\alpha_1^{2}-\tfrac{3}{2}\alpha_1(1-\tfrac{3}{2}\alpha_1)+\alpha_1(1-\tfrac{3}{2}\alpha_1)\ln\Lambda^2], \tag{66}$$

which is independent of κ^2. Similarly, $R^{\nu}_{m^2}$ is independent of m^2 and consequently $R^{\nu}=0$.

Elementary integration yields the gyromagnetic ratio $\frac{1}{2}(g_e-2)$ due to Figs. 4(a)–4(e) as

$$\tfrac{1}{2}(g_e-2)=-\frac{1}{2\pi}\left(\frac{g^2}{4\pi}\right)\left[\int_0^1 d\alpha_1\frac{\alpha_1(1-\alpha_1)(2-\alpha_1)}{\alpha_1^{2}+(1-\alpha_1)(m/m_e)^2}+\left(\frac{m_e}{m}\right)^2\int_0^1 d\alpha_1\frac{\alpha_1^{3}}{\alpha_1^{2}+(1-\alpha_1)(m/m_e)^2}\right.$$
$$\left.+\left(\frac{m_e}{m}\right)^2\int_0^1 d\alpha_1\frac{\alpha_1^{2}(\alpha_1-2)}{\alpha_1^{2}+(1-\alpha_1)(\mu/m_e)^2}\right], \tag{67}$$

where the first term is due to the vector meson in the gauge $\xi=1$ and $\kappa^2=m^2$. The second term is due to Φ in the same gauge and the third term is due to χ.

D. χ to Vacuum Transition

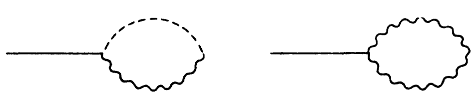
The last three examples demonstrated that the gauge $\xi=1$ and $\kappa^2=m^2$ simplified the calculations considerably. Here we want to use the χ to vacuum transition to illustrate the transition to the unitarity gauge by taking $\kappa^2\to\infty$ and point out the possible ambiguity.

The part of the Hamiltonian relevant to us now is

$$\int d^4x\,\mathcal{H}=\int d^4x\,[-gA^{\mu}(\chi\partial_\mu\Phi-\Phi\partial_\mu\chi)+gmA_\mu A^{\mu}\chi]$$
$$+\,i\mathrm{Tr}\frac{\kappa^2}{-\partial^2+\kappa^2}\frac{g}{m}\chi \tag{68}$$

and the diagrams are shown in Fig. 5. The amplitudes are, respectively,

$$A^{(a)}=-\frac{g}{\xi m}\int\frac{d^4q}{(2\pi)^4}(\xi\kappa^2-m^2)\frac{1}{(q^2+\kappa^2)^2}\,q^2,$$



(a) (b)

FIG. 5. χ to vacuum transition.

$$A^{(b)}=-gm\int\frac{d^4q}{(2\pi)^4}\left[\left(4+\frac{q^2}{m^2}\right)\frac{(m^2-\kappa^2)^2}{(q^2+\kappa^2)^2}\frac{1}{q^2+m^2}\right.$$
$$\left.-\frac{\xi-1}{\xi}\frac{1}{(q^2+\kappa^2)^2}\,q^2\right],$$

and

$$A^{\delta H}=\frac{g}{m}\int\frac{d^4q}{(2\pi)^4}\frac{\kappa^2}{q^2+\kappa^2}. \tag{69}$$

Some simple algebra allows us to write

$$A^{(a)}+A^{(b)}+A^{\delta H}=-gm\int\frac{d^4q}{(2\pi)^4}\frac{3}{q^2+m^2}. \tag{70}$$

The corresponding expressions in the unitarity gauge are

$$\int d^4x\,\mathcal{H}_u=\int d^4x\,gmA^{\mu}A_\mu\chi+i\delta^4(0)g\chi/m, \tag{71}$$

$$A_u^{(a)}=0,$$
$$A_u^{(b)}=-\frac{g}{m}\int\frac{d^4q}{(2\pi)^4}-gm\int\frac{d^4q}{(2\pi)^4}\frac{3}{q^2+m^2}, \tag{72}$$
$$A_u^{\delta H}=\frac{g}{m}\int\frac{d^4q}{(2\pi)^4},$$

and

$$A_u^{(a)}+A_u^{(b)}+A_u^{\delta H}=-gm\int\frac{d^4q}{(2\pi)^4}\frac{3}{q^2+m^2}. \tag{73}$$

Although the end results are the same in the two calculations, we notice that the degrees of divergence in Eq. (69) and Eq. (72) are different. In fact, Eq. (69) is quadratically divergent, which is less difficult to regularize than Eq. (72).

E. Radiative Correction to $\chi \to e^+ + e^-$ Due to $\chi\Phi^2$ Coupling

As the last example, we want to illustrate that the vertex correction of $\chi \to e^+ + e^-$ makes sense only if $\kappa^2 > \frac{1}{4}\mu^2$. This is easily understood, since if $\mu^2 > 4\kappa^2$, then χ can decay into two Φ's as a real process. The unfortunate circumstance here is that the Φ's are not physical particles, and therefore we really cannot claim that we understand this unphysical singularity.

In the course of this calculation, we will also prove that as a consistency requirement the $\chi\Phi^2$ vertex cannot give rise to an induced pseudoscalar coupling of χ to Ψ, up to the order we consider.

The diagrams we have in mind are drawn in Fig. 6. For example, the proper vertex function for Fig. 6(a) is

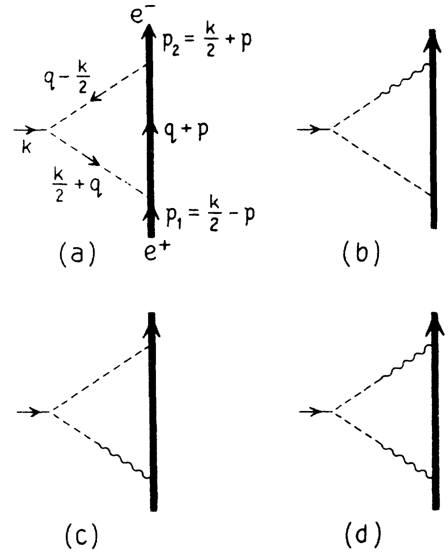


FIG. 6. Radiative corrections to $\chi\bar{\Psi}\Psi$.

$$\bar{v}(p_1)\Lambda u(p_2) = 2\mu(\tfrac{1}{2}h)^{1/2} g^2 \left(\frac{m_e}{m}\right)^2 \int \frac{d^4q}{(2\pi)^4 i}\, \bar{v}(p_1)\gamma_5 \frac{1}{m_e + \gamma\cdot(p+q)}\gamma_5 u(p_2) \frac{(q+\frac{1}{2}k)^2 + m^2/\xi}{[(q+\frac{1}{2}k)^2+\kappa^2]^2}\,\frac{(q-\frac{1}{2}k)^2+m^2/\xi}{[(q-\frac{1}{2}k)^2+\kappa^2]^2}\,. \tag{74}$$

Clearly, this quantity cannot give rise to a pseudoscalar quantity. Now in a gauge which satisfies $\xi\kappa^2 = m^2$, we do not have Figs. 6(b)–6(d). Consistency thereupon demands that Figs. 6(b)–6(d) should not give an induced pseudoscalar part in any gauge since it is a physical quantity. Indeed, if one does some algebra and introduces Feynman parameters, one sees that they do not. The scalar vertex correction can now be calculated, and we find that

$$\bar{v}(p_1)\Lambda_{\text{scalar}}\, u(p_2) \sim \int d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\alpha_1-\alpha_2-\alpha_3)\frac{N_{1,2,3}}{D_{1,2,3}}\,,$$

where

$$D = \alpha_1{}^2 m_e{}^2 + [(\alpha_2-\alpha_3)^2 + \alpha_1(1-\alpha_1)]\tfrac{1}{4}\mu^2 + (\alpha_2+\alpha_3)(\kappa^2 - \tfrac{1}{4}\mu^2) \tag{75}$$

and the N's are explicitly gauge-dependent functions. One can absorb the over-all effect by a vertex renormalization.

The points we want to make are that (a) examining Eq. (75), we see that the integral is well defined only if $\kappa^2 > \frac{1}{4}\mu^2$; (b) the renormalization constants are gauge-dependent, which is well known in QED.

CONCLUSION

We have quantized a theory with continuous gauge symmetry, which is subsequently broken by the ground state, by the Lagrange-multiplier method. We have shown that the gauge excitation field Φ can have mass κ, and therefore is not a Goldstone scalar. By direct examples, the gauge with parameters $\xi = 1$ and $\kappa^2 = m^2$ is shown to be particularly useful for finite calculation. The connection of a renormalizable gauge (κ^2 finite) to the unitarity gauge ($\kappa^2 \to \infty$) has been established and is particularly satisfying.

Clearly, nothing we discuss here cannot be extended to a theory with non-Abelian symmetry.[24] We intend to return to this extension and the other formal aspects, such as connection of Green's functions in different gauges, etc., in the future.

The reader may now, if not earlier, wonder what the point is to give a canonical formulation of gauge theory since we have already the elegant functional-integration approach. The canonical quantization is a differential characterization of the system. Because of the fundamental commutation relations, it allows us to study the local algebraic structure. For example, the current algebra can be constructed and subsequent sum rules can be derived. After all, this has been a very fruitful approach in the past when electromagnetic and weak interaction properties of hadrons are investigated.

ACKNOWLEDGMENTS

I would like to thank my colleagues, especially Professor Y. Tomozawa, for their interest, which prompted me to look into this area again.

APPENDIX

We discuss the origin of the extra piece δH in the canonical formulation.

In the Lagrange-multiplier method as presented, what we have done is to enlarge the right (or left) space to include the gauge parameter Λ. Let us denote these states by the symbol $|\Psi\rangle_\Lambda$.

The extended system, which is described by the Lagrangian of Eqs. (9)–(11), is invariant under the infinitesimal transformation

$$A^\mu \to A^\mu + \partial^\mu \delta\Lambda\,,$$
$$\phi \to \phi + g(v+\chi)\delta\Lambda = \phi + m\delta\Lambda + g\chi\delta\Lambda\,, \tag{A1}$$

and

$$\chi \to \chi - g\phi\delta\Lambda\,.$$

The physical states are projected out by restricting the gauge parameter Λ to

$$[\partial_\mu A^\mu(x) - (\kappa^2/m)\phi(x)]\,|\Psi\rangle_\Lambda^{\text{physical}} = 0\,. \tag{A2}$$

Thus, symbolically the solution is

$$\begin{aligned}|\Psi\rangle_\Lambda^{\text{physical}} &= \prod_{\text{all } x} \delta(\partial_\mu A^\mu(x) - (\kappa^2/m)\phi(x))\,|\Psi\rangle_{\Lambda=0}\\ &= \prod_{\text{all } x} \delta(\Lambda(x))\,\frac{1}{\left|\dfrac{d}{d\Lambda(x)}[\partial_\mu A^\mu(x) - (\kappa^2/m)\phi(x)]\right|}\,|\Psi\rangle_{\Lambda=0}\\ &= \prod_{\text{all } x} \delta(\Lambda(x))\frac{1}{|-\partial^2+\kappa^2+\kappa^2(g/m)\chi|}\,|\Psi\rangle_{\Lambda=0}\,.\end{aligned} \tag{A3}$$

Now we turn to the interaction description of Eq. (15). The group of the gauge parameter is restricted differently, since now the invariance is

$$A^\mu_{\text{int}} \to A^\mu_{\text{int}} + \partial^\mu\delta\Lambda$$

and (A4)

$$\phi_{\text{int}} \to \phi_{\text{int}} + m\delta\Lambda\,.$$

The realizable states here are projected out by the condition

$$[\partial_\mu A^\mu_{\text{int}}(x) - (\kappa^2/m)\phi_{\text{int}}(x)]\,|\Psi\rangle_\Lambda^{\text{int}} = 0\,. \tag{A5}$$

As before, we write

$$\begin{aligned}|\Psi\rangle_\Lambda^{\text{int}} &= \prod_{\text{all } x} \delta(\Lambda(x))\frac{1}{|-\partial^2+\kappa^2|}\,|\Psi\rangle_{\Lambda=0}\\ &= \prod_{\text{all } x} \frac{-\partial^2+\kappa^2+\kappa^2(g/m)\chi}{-\partial^2+\kappa^2}\,|\Psi\rangle_\Lambda^{\text{physical}}\\ &= \det\left(1+\frac{\kappa^2}{-\partial^2+\kappa^2}\,\frac{g}{m}\,\chi\right)|\Psi\rangle_\Lambda^{\text{physical}}\\ &= \exp\left[\operatorname{Tr}\ln\left(1+\frac{\kappa^2}{-\partial^2+\kappa^2}\,\frac{g}{m}\,\chi\right)\right]|\Psi\rangle_\Lambda^{\text{physical}}\,.\end{aligned} \tag{A6}$$

The S-matrix element of the process $a \to b$ is given by

$$\langle a|S|b\rangle_\Lambda^{\text{int}} = \left\langle a\,\middle|\,S\exp\left[\operatorname{Tr}\ln\left(1+\frac{\kappa^2}{-\partial^2+\kappa^2}\,\frac{g}{m}\,\chi\right)\right]\middle|\,b\right\rangle_\Lambda^{\text{physical}}\,, \tag{A7}$$

where S is of course generated by the Hamiltonian $(-\mathcal{L}_{\text{int}})$. The effective Hamiltonian is then

$$\int d^4x\, \mathcal{H}_{\text{eff}} = \int d^4x\,(-\mathcal{L}_{\text{int}}) + \delta H\,, \tag{A8}$$

with

$$\delta H = i\,\mathrm{Tr}\ln\left(1+\frac{\kappa^2}{-\partial^2+\kappa^2}\,\frac{g}{m}\,\chi\right). \tag{A9}$$

We see that the compensating term δH comes about because in going from the Heisenberg representation to the interaction representation, we have changed the group of gauge parameter admitted by the physical states.

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[17]Our notations are such that the metric tensor is $g=(-1,\,1,\,1,\,1)$. Latin indices run from 1 to 3, and Greek indices run from 0 to 3.

$$\{\gamma_\mu,\,\gamma_\nu\} = -2g_{\mu\nu}\,,\quad \gamma_k^\dagger = -\gamma_k\,,\quad \gamma_0 = \gamma_0^\dagger\,,$$

$$\gamma_5 = \gamma^0\gamma_1\gamma_2\gamma_3 = -\gamma_5^\dagger\,,\quad \sigma_{\mu\nu} = \tfrac{1}{2}i\,[\gamma^\mu,\,\gamma^\nu]\,.$$

The Dirac spinors satisfy $(m_e+\gamma\cdot p)\,u(p)=0$ and $\bar{v}(p)(m_e-\gamma\cdot p)=0$.

[18]As we will show, there is no massless particle in the physical states.

[19]We can, if we wish, demand this with respect to the left physical states. However, as we learned in QED, we cannot make this requirement on *both* the right *and* the left physical states *at the same time* without running into ill-defined quantities.

[20]To be more precise, we must use

$$\langle(\partial_\mu\Phi(x)\partial'_\nu\Phi(y))\rangle \to \partial_\mu\partial'_\nu\langle(\Phi(x)\Phi(y))_+\rangle$$

and a similar treatment for the χ's. This is the rule for scalar electrodynamics and, of course, can be proved.

[21]See the paper by Lee in Ref. 6. We neglect the complication due to the triangular anomaly.

[22]See, in particular, Appendix E of Ref. 13.

[23]There are other projection operators we can use. We find this combination to be the simplest for our present purpose.

[24]After this work was completed, we learned that J. Zinn-Justin, B. W. Lee, and their co-workers have also considered a generalized gauge of this sort for both the Abelian and the non-Abelian symmetries. *Note added in proof.* G.'t Hooft and M. Veltman considered the gauge invariance problem in a recent Utrecht report (unpublished).