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Covariant Perturbation Theory for Chiral Lagrangians

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The formalism "partial differential equations with respect to coupling constants" is developed involving the total Lagrangian density. By means of this formalism the covariant perturbation theory is developed using the example of chiral Lagrangian for massless pions. Because our formalism involves the total Lagrangian density, the covariant perturbation theory easily follows by integrating the partial differential equation with respect to coupling constant for the S matrix. Thus our approach is more direct and simpler than the other approaches involving the interaction Hamiltonian density and which either use the functional techniques or the generating functionals of the time-ordered Green's functions.

I. INTRODUCTION

There has been some confusion recently with the formulation of the covariant perturbation theory for Lagrangians which depend quadratically on derivatives of field operators. Namely, it has been known for quite some time that if $\mathcal{L}_{\text{int}}$ contains one derivative of a field operator,[1] the covariant perturbation theory is achieved by simply replacing in the Dyson expression for the S matrix, $\mathcal{H}_{\text{int}}^{\text{in}}$ by $-\mathcal{L}_{\text{int}}^{\text{in}}$ and the T product by its invariant modification, the T^* product. [The T^* product will be defined later in the text; its main feature is that in the contractions of operators all the noncovariant (normal-dependent) parts are omitted.[2]] When this rule is extended also to $\mathcal{L}_{\text{int}}$, which depends quadratically on derivative of field operators, some anomalies were encountered. The particular case where this happened was the chiral-invariant Lagrangian for zero-mass pions.[3] The anomalies soon disappeared after the correct Feynman rules were implemented. These rules were established essentially in two different ways: one which uses the functional-derivative technique starting directly from the Dyson expression for the S matrix,[4] and the other which employs the generating functional of the time-ordered Green's functions.[5] Let us also point out that similar Feynman rules were developed in the context of Yang-Mills theories.[6]

We wish to show in the example of a chiral-invariant Lagrangian for zero-mass pions, how one can develop a covariant perturbation theory by means of the formalism of "partial differential equations with respect to coupling constants (PDECC)".[7] Although we shall develop a covariant perturbation theory in this particular example, the method itself will be general enough as to be easily generalized to any other example.

Since our formalism PDECC involves the total Lagrangian density, our approach to the derivation of a covariant perturbation theory is more direct and simpler when compared to the other approaches which involve the interaction Hamiltonian density and use either the functional techniques[4] or the generating functionals of the time-ordered Green's functions.[5,6] The method itself may also be possible to develop for a case in which one has to reckon with two or more independent interactions (described with two or more independent coupling constants). An example of this is the system of particles interacting strongly and electromagnetically. Thus we may even be able to approach the question of seagull terms[8] from a new angle.

Section II is devoted to some preliminary remarks concerning the formalism of PDECC in which the total Hamiltonian density operator appears. The observable masses of interacting particles are to be considered as input parameters.

In Sec. III the total Lagrangian density is introduced into the formalism of PDECC. This will prove to be important for the formulation of the covariant perturbation theory for the case of chiral dynamics involving massless pions.

A simple "classical" model of derivative coupling between neutral scalar mesons with fermions[1] is considered in Sec. IV for the demonstration of the formalism developed in Secs. II and III. This model is historically the first one for which the covariant perturbation rules were derived. However, these rules generally are not true for other models. Besides deriving "classically" the covariant perturbation theory for this model by means of our formalism, we also introduce the T_n product in Sec. IV. The T_n product is numerically equal to the ordinary T product; however, the rules that go with them are different.

In Sec. V, by means of our formalism, the covariant perturbation theory is developed for the case of chiral-invariant Lagrangian for massless pions. In our approach one starts with the partial differential equation with respect to coupling constant for the S matrix which involves the total Lagrangian density. Since, in general, the Lagrangian densities are simpler than the Hamiltonian densities, our approach is not only simple but also rather direct. With such a covariant perturbation theory, Adler theorems (vanishing of amplitudes in the soft-pion limit), masslessness of the pion, and the equivalence of all pion fields are maintained beyond the tree approximation.[4,5]

Finally, in Sec. VI we summarize the results and discuss the unitarity of the derived expression for the S matrix.

II. PRELIMINARIES

For the sake of completeness, we shall briefly outline in this section the formalism of PDECC in which the total Hamiltonian density operator appears.[7] At least in principle we can assume that the system of particles interact through many independent interactions. These independent interactions we can characterize with the set of coupling constants g_i $(i=1,2,\ldots)$, which we can vary independently between zero and their physical values.

Let the system of interacting particles be described by a set of independent Heisenberg field operators $\phi(x)$ and their canonical conjugates $\pi(x)$. For the corresponding "in" field operators $\phi_{\rm in}(x)$ and their canonical conjugates $\pi_{\rm in}(x)$ we assume the following macrocausality relations to hold:

$$\frac{\partial}{\partial g_i}\phi_{\rm in}(x)=0\,,\qquad \frac{\partial}{\partial g_i}\pi_{\rm in}(x)=0\,. \tag{1a}$$

As a consequence of (1a) we have for the observable mass m associated with some "in" field operator $\phi_{\rm in}(x)$ that

$$\frac{\partial}{\partial g_i}m=0\,. \tag{1b}$$

Therefore, in our formalism the observable masses are to be considered as input parameters. This still allows us to treat a great variety of cases, examples being quantum electrodynamics and the theories which employ effective Lagrangians.[9] Of course, we can also have a composite particle in our formalism as long as, with respect to interactions characterized with coupling constants g_i $(i=1,2,\ldots)$, its mass is allowed to be considered as an input parameter.[10]

Next, we wish to introduce the "prime" partial derivatives $\partial'/\partial g_i$ which by definition have the property

$$\frac{\partial'}{\partial g_i}\phi(x)=0\,,\qquad \frac{\partial'}{\partial g_i}\pi(x)=0\,. \tag{2}$$

Otherwise, $\partial'/\partial g_i$ acts as the ordinary derivatives $\partial/\partial g_i$ on the coupling constants' dependent coefficients that may multiply ϕ's, π's, etc., as in this example:

$$\frac{\partial'}{\partial g}[g\phi^4(x)]=\left(\frac{\partial}{\partial g}g\right)\phi^4(x)=\phi^4(x)\,.$$

The total Hamiltonian density $\mathcal{H}(x)$ can be written as $\mathcal{H}(x)=\mathcal{H}_f(x)+\mathcal{H}_{\rm int}(x)$, where $\mathcal{H}_f(x)$ is the free part of the Hamiltonian density which we assume contains the observable masses. Then according to (1b) and (2) we have

$$\frac{\partial'}{\partial g_i}\mathcal{H}_f(x)=0\,. \tag{3}$$

In view of the fact that the Dyson expression for the S matrix is

$$S=T\exp\left[-i\int d^4x\,\mathcal{H}_{\rm int}^{\rm in}(x)\right]\,, \tag{4}$$

and that $\phi(x)$ and $\pi(x)$ are connected to $\phi_{\rm in}(x)$ and $\pi_{\rm in}(x)$ by the relations

$$\phi(x)=S^\dagger T(\phi_{\rm in}(x)S)\,,\qquad \pi(x)=S^\dagger T(\pi_{\rm in}(x)S)\,, \tag{5}$$

we get, by the help of (1a) and (5), the following differential equations for the S-matrix:

$$\frac{1}{i}\frac{\partial}{\partial g_i}S=-S\int d^4x\,\frac{\partial'}{\partial g_i}\mathcal{H}(x)\,. \tag{6}$$

Let us point out now that we may even decompose $\mathcal{H}(x)=\mathcal{H}_f'(x)+\mathcal{H}_{\rm int}'(x)$ where $\mathcal{H}_f'(x)$ contains the bare masses m_0's. However, now $(\partial'/\partial g_i)\mathcal{H}_f'\neq 0$, since in view of relation $m=m_0+\Delta m$, we have $(\partial/\partial g_i)m_0=-(\partial/\partial g_i)\Delta m\neq 0$, Δm being the mass shift. In other words, once having formulated the theory in terms of the total Hamiltonian density, we have the most general formulation. Now using (6) we get from (5)

$$\begin{aligned}\frac{1}{i}\frac{\partial}{\partial g_i}\phi(x)&=\int d^4y\left[-T\left(\phi(x)\frac{\partial'}{\partial g_i}\mathcal{H}(y)\right)+\left(\frac{\partial'}{\partial g_i}\mathcal{H}(y)\right)\phi(x)\right]\,,\\ \frac{1}{i}\frac{\partial}{\partial g_i}\pi(x)&=\int d^4y\left[-T\left(\pi(x)\frac{\partial'}{\partial g_i}\mathcal{H}(y)\right)+\left(\frac{\partial'}{\partial g_i}\mathcal{H}(y)\right)\pi(x)\right]\,.\end{aligned} \tag{7}$$

Now let the operator $F(x)$ be a functional of $\phi(x)$'s, $\pi(x)$'s, and their space derivatives. Also let $F(x)$ depend explicitly on the coupling constants g_i. Then from (7) we have at once[11]

$$\frac{1}{i}\frac{\partial}{\partial g_i}F(x)=\int d^4y\left[-T\left(F(x)\frac{\partial'}{\partial g_i}\mathcal{H}(y)\right)+\left(\frac{\partial'}{\partial g_i}\mathcal{H}(y)\right)F(x)\right]+\frac{1}{i}\frac{\partial'}{\partial g_i}F(x). \tag{8}$$

Equation (8) holds for any $F(x)$; thus it should also hold for $\dot{F}(x)$:

$$\frac{1}{i}\frac{\partial}{\partial g_i}\dot{F}(x)=\int d^4y\left[-T\left(\dot{F}(x)\frac{\partial'}{\partial g_i}\mathcal{H}(y)\right)+\left(\frac{\partial'}{\partial g_i}\mathcal{H}(y)\right)\dot{F}(x)\right]+\frac{1}{i}\frac{\partial'}{\partial g_i}\dot{F}(x). \tag{9}$$

Assuming that in general

$$\left[\frac{\partial}{\partial g_i},\frac{\partial}{\partial x_\mu}\right]=0,$$

after taking the time derivative of (8) and comparing with (9), we get

$$\frac{1}{i}\left[\frac{\partial'}{\partial g_i},\frac{\partial}{\partial x^\mu}\right]F(x)=-g_\mu^4\int d^4y\,\delta(x^4-y^4)\left[F(x),\frac{\partial'}{\partial g_i}\mathcal{H}(y)\right], \tag{10}$$

i.e., $\partial'/\partial g_i$ do not generally commute with $\partial/\partial x^4$. Let us point out that (10) is fully consistent with the Heisenberg equations of motion.

III. FORMULATION OF THE THEORY IN TERMS OF THE TOTAL LAGRANGIAN DENSITY

In order to proceed any further we have to introduce yet another partial derivative with respect to coupling constants g_i, a "star" partial derivative $\partial^*/\partial g_i$ with the property

$$\frac{\partial^*}{\partial g_i}\phi(x)=0,\qquad \left[\frac{\partial^*}{\partial g_i},\frac{\partial}{\partial x_\mu}\right]=0. \tag{11}$$

The "star" partial derivative otherwise acts as the ordinary derivative on the coupling constants' dependent coefficients that may multiply $\phi(x)$, $\partial_\mu\phi(x)$, etc. As we know, any quantity $F(x)$ can be expressed in terms of ϕ's, $\partial_r\phi$'s, and π's. However, since in general we assume that the Lagrangian density depends on first-order derivatives of field operators, we see that the π's, and consequently any $F(x)$, can be expressed in terms of ϕ's, $\partial_r\phi$'s, and $\dot\phi$'s. Therefore, when some π is expressed in terms of ϕ's, $\partial_r\phi$'s, and $\dot\phi$'s, an explicit dependence on coupling constants g_i ($i=1,2,\ldots$) may also develop. So generally we shall have

$$\frac{\partial^*}{\partial g_i}\pi(x)\neq 0. \tag{12}$$

The next thing that we would like to do is to express $(\partial'/\partial g_i)\mathcal{H}(x)$ in terms of $\mathcal{L}(x)$. Let us for the moment ignore the fact that we have quantized fields, so that we do not have to worry about the ordering of the operators. Then using the relation $\mathcal{H}=\sum_\phi \pi\dot\phi-\mathcal{L}$, we get

$$\frac{\partial'}{\partial g_i}\mathcal{H}=\sum_\phi \pi\frac{\partial'\dot\phi}{\partial g_i}-\frac{\partial'}{\partial g_i}\mathcal{L}, \tag{13}$$

where, in the sum, π is canonically conjugate to ϕ. We can evaluate $(\partial'/\partial g_i)\mathcal{L}$ easily if we note that for c-number fields we have generally for any F

$$\frac{\partial'}{\partial g_i}F=\frac{\partial^*}{\partial g_i}F+\sum_\phi\frac{\partial F}{\partial\dot\phi}\frac{\partial'\dot\phi}{\partial g_i}. \tag{14}$$

Since $\partial\mathcal{L}/\partial\dot\phi=\pi$, from (13) we get at once

$$\frac{\partial'}{\partial g_i}\mathcal{H}(x)=-\frac{\partial^*}{\partial g_i}\mathcal{L}(x). \tag{15}$$

We accept, of course, relation (15) also for the case of quantized field theory, and it can be verified on specific models. Relation (15) resembles greatly the case of no derivative coupling when $\mathcal{H}_{\rm int}=-\mathcal{L}_{\rm int}$. Of course, (15) is valid for cases of derivative and nonderivative couplings. Relation (15) tells us that although $\mathcal{H}$ may be dependent on a normal of the spacelike surface, $(\partial'/\partial g_i)\mathcal{H}$ ceases to be so. This is due to the property (10) of $\partial'/\partial g_i$. Incidentally, from (6) and (15) we see that the S matrix is a Lorentz-invariant quantity, which is due to the fact that if $\mathcal{L}(x)$ is a Lorentz scalar, then because of (11), $(\partial^*/\partial g_i)\mathcal{L}(x)$ is a Lorentz scalar too.

Taking into account (15), we now rewrite Eq. (6), the first equation of (7), Eq. (8), and Eq. (10) as

$$\frac{1}{i}\frac{\partial}{\partial g_i}S=S\int d^4x\,\frac{\partial^*}{\partial g_i}\mathcal{L}(x), \tag{6$'$}$$

$$\frac{1}{i}\frac{\partial}{\partial g_i}\phi(x)=\int d^4y\left[T\left(\phi(x)\frac{\partial^*}{\partial g_i}\mathcal{L}(y)\right)-\left(\frac{\partial^*}{\partial g_i}\mathcal{L}(y)\right)\phi(x)\right], \tag{7$'$}$$

$$\frac{1}{i}\frac{\partial}{\partial g_i}F(x)=\int d^4y\left[T\left(F(x)\frac{\partial^*}{\partial g_i}\mathcal{L}(y)\right)-\left(\frac{\partial^*}{\partial g_i}\mathcal{L}(y)\right)F(x)\right]+\frac{1}{i}\frac{\partial'}{\partial g_i}F(x)\,, \tag{8'}$$

$$\frac{1}{i}\left[\frac{\partial'}{\partial g_i},\frac{\partial}{\partial x^\mu}\right]F(x)=g_\mu^4\int d^4y\,\delta(x^4-y^4)\times\left[F(x),\frac{\partial^*}{\partial g_i}\mathcal{L}(y)\right]\,. \tag{10'}$$

$F(x)$ is now to be considered as being expressed in terms of ϕ's and $\partial_\mu\phi$'s. A canonical conjugate π, on the other hand, is simply a special case of F with the conditions $(\partial'/\partial g_i)\pi=0$ $(i=1,2,\ldots)$. Combining (6′) and (8′) we also get

$$\frac{1}{i}\frac{\partial}{\partial g_i}SF(x)=\int d^4y\,ST\left(F(x)\frac{\partial^*}{\partial g_i}\mathcal{L}(y)\right)+S\frac{1}{i}\frac{\partial'}{\partial g_i}F(x)\,. \tag{16}$$

It is not difficult to generalize (16) for the time-ordered product $T(F(x)G(y)\cdots Q(z))$:

$$\frac{1}{i}\frac{\partial}{\partial g_i}ST(F(x)G(y)\cdots Q(z))=\int d^4w\,ST\left(F(x)G(y)\cdots Q(z)\frac{\partial^*}{\partial g_i}\mathcal{L}(w)\right)+S\frac{1}{i}\frac{\partial'}{\partial g_i}T(F(x)G(y)\cdots Q(z))\,, \tag{17}$$

where the last term is to be evaluated as

$$\frac{1}{i}\frac{\partial'}{\partial g_i}T(F(x)G(y)\cdots Q(z))=T\left(\left(\frac{1}{i}\frac{\partial'}{\partial g_i}F(x)\right)G(y)\cdots Q(z)\right)+T\left(F(x)\left(\frac{1}{i}\frac{\partial'}{\partial g_i}G(y)\right)\cdots Q(z)\right)+\cdots+T\left(F(x)G(y)\cdots\frac{1}{i}\frac{\partial'}{\partial g_i}Q(z)\right)\,. \tag{18}$$

IV. A SIMPLE "CLASSICAL" MODEL

It may be worthwhile to illustrate the formalism from the preceding sections on a simple model of a neutral scalar field $\sigma(x)$ interacting with a spinor field $\psi(x)$. The interaction Lagrangian density is[1]

$$\mathcal{L}_{\rm int}(x)=g\bar\psi(x)\gamma_\mu\psi(x)\partial^\mu\sigma(x)\,. \tag{19}$$

The interaction Hamiltonian density is easy to obtain:

$$\mathcal{H}_{\rm int}(x)=-g\bar\psi(x)\vec\gamma\psi(x)\nabla\sigma(x)-g\bar\psi(x)\gamma^4\psi(x)\pi_\sigma(x)+\tfrac{1}{2}g^2(\bar\psi(x)\gamma^4\psi(x))^2\,, \tag{20}$$

and it clearly is normal-dependent. $\pi_\sigma(x)$ is canonically conjugate to $\sigma(x)$:

$$\pi_\sigma(x)=\frac{\partial\mathcal{L}(x)}{\partial\dot\sigma(x)}=\dot\sigma(x)+g\bar\psi(x)\gamma^4\psi(x)\,. \tag{21}$$

Let us first verify relation (15) in this example. First of all it is clear that $\mathcal{H}_{\rm int}\neq-\mathcal{L}_{\rm int}$. Secondly, since we are ignoring the mass renormalization,

$$\frac{\partial'}{\partial g}\mathcal{H}_f'=0\,,\quad\frac{\partial^*}{\partial g}\mathcal{L}_f'=0$$

($\mathcal{H}_f'$ and $\mathcal{L}_f'$ by definition depend on coupling-constant-dependent bare masses, in contrast to $\mathcal{H}_f$ and $\mathcal{L}_f$, which by definition depend on observable masses),[12] and all we have to show is that

$$\frac{\partial'}{\partial g}\mathcal{H}_{\rm int}=-\frac{\partial^*\mathcal{L}_{\rm int}}{\partial g}\,.$$

The proof is simple. From (20) we have

$$\begin{aligned}\frac{\partial'}{\partial g}\mathcal{H}_{\rm int}&=-\bar\psi\vec\gamma\psi\nabla\sigma-\bar\psi\gamma^4\psi\pi_\sigma+g(\bar\psi\gamma^4\psi)^2\\&=-\bar\psi\vec\gamma\psi\nabla\sigma-\bar\psi\gamma^4\psi(\pi_\sigma-g\bar\psi\gamma^4\psi)\\&=-\bar\psi\gamma_\mu\psi\partial^\mu\sigma\,,\end{aligned} \tag{22}$$

where (21) was taken into account. On the other hand, from (19) we have trivially

$$\frac{\partial^*}{\partial g}\mathcal{L}_{\rm int}=\bar\psi\gamma_\mu\psi\partial^\mu\sigma\,,$$

and the proof is completed. We can also illustrate relation (10) or (10′) by choosing $F=\sigma$ $[(\partial'/\partial g)\sigma=(\partial^*/\partial g)\sigma=0]$:

$$\begin{aligned}\frac{1}{i}\frac{\partial'}{\partial g}\dot\sigma(x)&=\int d^4y\,\delta(t_x-t_y)\times[\sigma(x),\bar\psi(y)\gamma^4\psi(y)\pi_\sigma(y)]\\&=+i\bar\psi(x)\gamma^4\psi(x)\,.\end{aligned} \tag{23}$$

On the other hand, from (21) we have

$$\frac{1}{i}\frac{\partial'}{\partial g}\pi_\sigma(x)=0=\frac{1}{i}\frac{\partial'}{\partial g}\dot\sigma(x)+\frac{1}{i}\bar\psi(x)\gamma^4\psi(x)\,,$$

from which we obtain the same thing.

In order to derive the covariant perturbation theory for this model, let us first note what is the difference between T and T^* products for "in" fields. According to Ref. 2, we have, for example, that

$$T^*\left(\frac{\partial}{\partial x_\mu}\phi_{\rm in}(x)\frac{\partial}{\partial y_\nu}\phi_{\rm in}(y)\cdots\right)=\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu}\cdots T(\phi_{\rm in}(x)\phi_{\rm in}(y)\cdots). \tag{24}$$

This definition makes T and T^* contractions between "in" field operators different, for while T contractions may not be covariant objects in general, the T^* contractions always will be. In particular for this model, here are some examples of T contractions:

$$\sigma_{\rm in}(x)^\cdot\sigma_{\rm in}(y)^\cdot=\langle 0|T(\sigma_{\rm in}(x)\sigma_{\rm in}(y))|0\rangle=-i\Delta_F(x-y), \tag{25a}$$

$$\frac{\partial}{\partial x^\mu}\sigma_{\rm in}(x)^\cdot\sigma_{\rm in}(y)^\cdot=\left\langle 0\left|T\left(\frac{\partial}{\partial x^\mu}\sigma_{\rm in}(x)\sigma_{\rm in}(y)\right)\right|0\right\rangle=-i\frac{\partial}{\partial x^\mu}\Delta_F(x-y), \tag{25b}$$

$$\frac{\partial}{\partial x^\mu}\sigma_{\rm in}(x)^\cdot\frac{\partial}{\partial y^\nu}\sigma_{\rm in}(y)^\cdot=\left\langle 0\left|T\left(\frac{\partial}{\partial x^\mu}\sigma_{\rm in}(x)\frac{\partial}{\partial y^\nu}\sigma_{\rm in}(y)\right)\right|0\right\rangle=-i\left[\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\Delta_F(x-y)+g^4_\mu g^4_\nu\delta^{(4)}(x-y)\right]. \tag{25c}$$

T^* contractions of the same operators, according to (23), are

$$[\sigma_{\rm in}(x)^\cdot\sigma_{\rm in}(y)^\cdot]^*=\langle 0|T^*(\sigma_{\rm in}(x)\sigma_{\rm in}(y))|0\rangle=-i\Delta_F(x-y), \tag{26a}$$

$$\left[\frac{\partial}{\partial x^\mu}\sigma_{\rm in}(x)^\cdot\sigma_{\rm in}(y)^\cdot\right]^*=\left\langle 0\left|T^*\left(\frac{\partial}{\partial x^\mu}\sigma_{\rm in}(x)\sigma_{\rm in}(y)\right)\right|0\right\rangle=-i\frac{\partial}{\partial x^\mu}\Delta_F(x-y), \tag{26b}$$

$$\left[\frac{\partial}{\partial x^\mu}\sigma_{\rm in}(x)^\cdot\frac{\partial}{\partial y^\nu}\sigma_{\rm in}(y)^\cdot\right]^*=\left\langle 0\left|T^*\left(\frac{\partial}{\partial x^\mu}\sigma_{\rm in}(x)\frac{\partial}{\partial y^\nu}\sigma_{\rm in}(y)\right)\right|0\right\rangle=-i\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\Delta_F(x-y). \tag{26c}$$

We see that only (26c) and (25c) differ by the noncovariant (normal-dependent) term $g^4_\mu g^4_\nu\delta^{(4)}(x-y)$. Equations (25) and (26) suggest the definition of pure noncovariant contractions as

$$\begin{aligned}&[\sigma_{\rm in}(x)^\cdot\sigma_{\rm in}(y)^\cdot]^n=0,\\&[\partial_\mu\sigma_{\rm in}(x)^\cdot\sigma_{\rm in}(y)^\cdot]^n=0,\\&[\partial_\mu\sigma^{\rm in}(x)^\cdot\partial_\nu\sigma^{\rm in}(y)^\cdot]^n=-ig^4_\mu g^4_\nu\delta^{(4)}(x-y),\quad\text{etc.}\end{aligned} \tag{27}$$

With these definitions we see that we can formally write the T product as

$$T=T^*+T^*(T_n-1), \tag{28}$$

where the T_n product has the following meaning: When T_n acts on a product of "in" operators, one sums up over all possible pure noncovariant contractions according to (27). (−1) in (28) subtracts the zeroth pure noncovariant contraction, and the T^* which is in front of (T_n-1) means that one still has to sum up over all possible pure covariant contractions, according to (26), of what is left over. The proof of (28) is straightforward by noticing that a T product means summing up over all possible contractions according to (25). Obviously, the result will contain pure covariant contractions defined by (26) and pure noncovariant contractions as defined by (27). It is clear that the result can be expanded in terms of pieces containing pure noncovariant contractions. The zeroth term in the expansion represents the sum over all possible covariant contractions, and the rest is represented by $T^*(T_n-1)$. We know that the T and the T^* products of "in" operators are reduced to a sum of normal products by means of corresponding contractions. As we see from (28), because T^* is in front of T_n-1, the T_n product of in operators is reduced to a sum of ordinary products of in operators by means of pure noncovariant contractions. However, we would like to have similar rules for the T_n product as we have for the T (or the T^*) product.[13] Thus, consistent with (28), we define the following rules (where $1,2,\ldots,$ is a shorthand notation for "in" field operators):

$$\begin{aligned}T_n(12\cdots r)=T^*(12\cdots r)&+[1^\cdot 2^\cdot]^n T^*(3\cdots r)\\&+[1^\cdot 3^\cdot]^n T^*(2\cdots r)\\&+\cdots+[1^\cdot 2^\cdot]^n[3^\cdot 4^\cdot]^n T^*(5\cdots r)+\cdots.\end{aligned} \tag{R1}$$

Since the T^* product is evaluated only by means of pure covariant contractions, we obviously have

$$T_nT^*(12\cdots s)=T^*(12\cdots s). \tag{R2}$$

Finally, from (R1) and (R2), we see that we also have

$$T_n(123\cdots s)=T_n(T_n(123\cdots s))=T_n(123T_n(4\cdots s))=\text{etc.} \tag{R3}$$

As we see, formally the T_n product denotes a reduction into the T^* products by means of pure noncovariant contractions. There is still one more important rule that holds for the T_n product:

$$T_n(T^*(12\cdots r)T^*(1'2'\cdots s'))=T^*(12\cdots r1'2'\cdots s')$$

$$+[1^{\bullet}1'^{\bullet}]^n T^*(2\cdots r\cdot 2'\cdots s')$$
$$+[1^{\bullet}2'^{\bullet}]^n T^*(2\cdots r1'3'\cdots s') + \text{etc.} \tag{R4}$$

Clearly the T product and the T_n product are numerically equal; however, the rules that go with them are *different*.

In order to deduce the covariant perturbation theory for our model, let us write down the differential equation for the S matrix:

$$\frac{1}{i}\frac{\partial}{\partial g}S = -S\int d^4x \frac{\partial'}{\partial g}\mathcal{H}_{\text{int}}(x) = S\int d^4x \frac{\partial^*}{\partial g}\mathcal{L}_{\text{int}}(x). \tag{29}$$

Using (16) and (17), from (29) we can get the S matrix as a power series in g (see Ref. 7). The result is given by two equivalent expressions:

$$S = T\exp\left[-i\int d^4x \sum_{n=1}^{\infty}\frac{g^n}{n!}\left(\frac{\partial'^n}{\partial g^n}\mathcal{H}_{\text{int}}(x)\right)_{g=0}\right], \tag{30a}$$

$$S = T\exp\left[i\int d^4x \sum_{n=1}^{\infty}\frac{g^n}{n!}\left(\frac{\partial'^{n-1}}{\partial g^{n-1}}\frac{\partial^*}{\partial g}\mathcal{L}_{\text{int}}(x)\right)_{g=0}\right]. \tag{30b}$$

Equation (30b) also follows from (30a) by taking into account (15) for our model [see the text after relation (21)]. We can easily verify (30a) by noticing that

$$\sum_{n=1}^{\infty}\frac{g^n}{n!}\left(\frac{\partial'^n}{\partial g^n}\mathcal{H}_{\text{int}}(x)\right)_{g=0} = \sum_{n=1}^{\infty}\frac{g^n}{n!}\left(\frac{\partial^n}{\partial g^n}\mathcal{H}^{\text{in}}_{\text{int}}(x)\right)_{g=0} = \mathcal{H}^{\text{in}}_{\text{int}}(x), \tag{31}$$

where in view of (1a) we used the fact that

$$\sigma(x)|_{g=0} = \sigma_{\text{in}}(x),$$
$$\pi_\sigma(x)|_{g=0} = \pi_\sigma^{\text{in}}(x), \quad \text{etc.}$$

In (30b) the sum actually terminates with $n=2$, because in view of (23)

$$\frac{1}{i}\frac{\partial'}{\partial g}\frac{\partial^*}{\partial g}\mathcal{L}_{\text{int}}(x) = i[\bar\psi(x)\gamma^4\psi(x)]^2, \tag{32}$$

and obviously

$$\frac{\partial'^r}{\partial g^r}\frac{\partial^*}{\partial g}\mathcal{L}_{\text{int}} = 0 \text{ for } r \geq 2.$$

In what follows we shall find relation (28) to be more useful in this equivalent form:

$$T = T^*T_n. \tag{33}$$

For this particular model we wish to derive the covariant perturbation theory as "classically" as possible; i.e., showing how the noncovariant terms arising from $\mathcal{H}^{\text{in}}_{\text{int}}$ [compare (31) and (30)] and the noncovariant terms arising from the pure noncovariant contractions cancel each other. Thus taking into account (33), we rewrite (30b) as

$$S = T^*\left\{\exp\left[i\frac{g^2}{2}\int d^4x\left(\frac{\partial'}{\partial g}\frac{\partial^*}{\partial g}\mathcal{L}_{\text{int}}(x)\right)_{g=0}\right]T_n\exp\left[ig\int d^4x\left(\frac{\partial^*}{\partial g}\mathcal{L}_{\text{int}}(x)\right)_{g=0}\right]\right\}, \tag{34}$$

where, in view of (32), the term

$$\left(\frac{\partial'}{\partial g}\frac{\partial^*}{\partial g}\mathcal{L}_{\text{int}}(x)\right)_{g=0}$$

obviously does not contribute to pure noncovariant contractions and has been pulled to the left of T_n. Noticing that

$$\left(\frac{\partial^*}{\partial g}\mathcal{L}_{\text{int}}\right)_{g=0} = \bar\psi_{\text{in}}\gamma^\mu\psi_{\text{in}}\partial_\mu\sigma_{\text{in}}$$

we wish to evaluate

$$S_n = T_n\exp\left[ig\int d^4x\,\bar\psi_{\text{in}}(x)\gamma^\mu\psi_{\text{in}}(x)\partial_\mu\sigma_{\text{in}}(x)\right]. \tag{35}$$

With the notation

$$\delta_{\mu\nu}(x-y) = g^4_\mu g^4_\nu\delta^{(4)}(x-y),$$

(35) can be also written as[14]

$$S_n = T^*\exp\left\{ig\int d^4x\,\bar\psi_{\text{in}}(x)\gamma^\mu\psi_{\text{in}}(x)\left[\partial_\mu\sigma_{\text{in}}(x) - i\int d^4y\,\delta_{\mu\nu}(x-y)\frac{\delta}{\delta\partial_\nu\sigma_{\text{in}}(y)}\right]\right\},$$

from which we deduce the differential equation

$$\frac{1}{i}\frac{\partial}{\partial g}S_n = T^*\int d^4x\,\overline{\psi}_{\rm in}(x)\gamma^\mu\psi_{\rm in}(x)\left[\partial_\mu\sigma_{\rm in}(x) - i\int d^4y\,\delta_{\mu\nu}(x-y)\frac{\delta}{\delta\partial_\nu\sigma_{\rm in}(y)}\right]S_n$$

$$= T^*\int d^4x\,\overline{\psi}_{\rm in}(x)\gamma^\mu\psi_{\rm in}(x)\left[\partial_\mu\sigma_{\rm in}(x) + g^4_\mu g^4_\nu g\overline{\psi}_{\rm in}(x)\gamma^\nu\psi_{\rm in}(x)\right]S_n\,,$$

which has the solution

$$S_n = T^*\exp\left\{ig\int d^4x\,\overline{\psi}_{\rm in}(x)\gamma_\mu\psi_{\rm in}(x)\partial^\mu\sigma_{\rm in}(x) + i\frac{g^2}{2}\int d^4x[\overline{\psi}_{\rm in}(x)\gamma^4\psi_{\rm in}(x)]^2\right\}\;. \tag{36}$$

We see that in view of (32), the last term in (36) cancels completely the first term in (34), and the result is

$$S = T^*\exp\left\{i\int d^4x\,g\overline{\psi}_{\rm in}(x)\gamma_\mu\psi_{\rm in}(x)\partial^\mu\sigma_{\rm in}(x)\right\}\;. \tag{37}$$

In other words, in this model the noncovariant terms resulting from the partial derivative $\partial'/\partial g$ cancel completely the contribution from pure noncovariant contractions. However, let us note that the contributions from pure noncovariant contractions give a pure noncovariant result in this model, and it must be canceled somehow. Furthermore, there is no *a priori* reason in general why pure noncovariant contractions should not, besides noncovariant, also give covariant contributions, which need not be canceled from the contributions from the partial prime derivatives $\partial'/\partial g$, $\partial'^2/\partial g^2, \ldots,$ etc. We shall see that the model in the next section has, in fact, this property.

Let us point out that the model in the present section serves here only as an example of the formalism since, with a suitable canonical transformation of fermion fields, $\mathcal{L}_{\rm int}$ can be made to vanish.

V. CHIRAL-INVARIANT LAGRANGIAN DENSITY FOR MASSLESS PIONS

The result from the preceding section suggests indeed that in general all that one should do in order to get the covariant perturbation theory is to change T by T^* and $\mathcal{H}_{\rm int}$ by $-\mathcal{L}_{\rm int}$ in the Dyson expression for the S matrix. If these kind of rules are applied to calculations with chiral-invariant pion Lagrangians,[3] it will be found that the worst divergences in such a perturbation theory violate the Adler condition for π-π scattering and that the pion acquires a mass. Since these anomalies disappear for a particular definition of a pion field, it also means that within the framework of such a perturbation theory, the S matrix is not invariant under the canonical transformations of Heisenberg fields.[15] Of course, as mentioned already,[4,5] after applying the correct covariant perturbation theory, the masslessness of the pion, the Adler zeros, and the general current-algebra theorems were established.

Let us now show in this example of a chiral-invariant Lagrangian for massless pions how our formalism of PDECC yields in a rather natural way the correct covariant perturbation theory. Denoting with ϕ_a the pion field, the Lagrangian density can be written as

$$\mathcal{L} = -\tfrac{1}{2}\phi_{\mu,a}G_{ab}(\phi)\phi^\mu_b\,,$$
$$\phi_{\mu,a} = \partial_\mu\phi_a\,, \tag{38}$$

where the dependence of G_{ab} on the pion field is determined by the requirement that $\mathcal{L}$ be SU(2) $\times$SU(2)-invariant.[16] Different SU(2)$\times$SU(2)-nonlinear-representation assignments for ϕ_a require different G_{ab}'s. We shall write $G_{ab}(\phi)$ as

$$G_{ab}(\phi) = \delta_{ab} - g\overline{G}_{ab}(\phi)\,, \tag{39}$$

where g is a "mathematical" coupling constant which we shall vary between 0 and 1. The physical coupling constant is "frozen" in $\overline{G}_{ab}$. This is done in order to simplify the derivation, and since the final result will be written in a closed form, it will be independent of whether we vary the physical coupling constant or the above defined "mathematical" one. The canonical momentum π_a conjugate to the pion field ϕ_a is

$$\pi_a = \frac{\partial\mathcal{L}}{\partial\phi_{4,a}} = G_{ab}(\phi)\phi_{4,b} = [\delta_{ab} - g\overline{G}_{ab}(\phi)]\,\phi_{4,b}\,. \tag{40}$$

Since we shall need the differential equation with respect to coupling constant g for the S matrix [see (6')], we need $(\partial^*/\partial g)\mathcal{L}$. From (38) we have

$$\frac{\partial^*}{\partial g}\mathcal{L} = \tfrac{1}{2}\phi_{\mu,a}\overline{G}_{ab}(\phi)\phi^\mu_a \equiv \mathcal{L}_{\rm int}\,. \tag{41}$$

Let us now write down the definitions of various contractions between "in" pion field operators. The T-contractions are

$$\phi^{\rm in}_a(x)^\cdot\phi^{\rm in}_b(y)^\cdot = \langle 0|T(\phi^{\rm in}_a(x)\phi^{\rm in}_b(y))|0\rangle$$
$$= -i\Delta_{F,ab}(x-y)\,, \tag{42a}$$

$$\phi^{\rm in}_{\mu,a}(x)^\cdot\phi^{\rm in}_b(y)^\cdot = \langle 0|T(\phi^{\rm in}_{\mu,a}(x)\phi^{\rm in}_b(y))|0\rangle$$
$$= -i\partial_\mu(x)\Delta_{F,ab}(x-y)\,, \tag{42b}$$

$$\phi^{\text{in}}_{\mu,a}(x)^{\boldsymbol\cdot}\phi^{\text{in}}_{\nu,b}(y)^{\boldsymbol\cdot}=\langle 0|T(\phi^{\text{in}}_{\mu,a}(x)\phi^{\text{in}}_{\nu,b}(y))|0\rangle$$
$$=-i\{\partial_\mu(x)\partial_\nu(y)\Delta_{F,ab}(x-y)+\delta_{\mu\nu,ab}(x-y)\},\qquad(42c)$$

where

$$\delta_{\mu\nu,ab}(x-y)=g^4_\mu g^4_\nu\delta_{ab}\delta^{(4)}(x-y).\qquad(42d)$$

The T^*, or what we call pure covariant, contractions are

$$[\phi^{\text{in}}_a(x)^{\boldsymbol\cdot}\phi^{\text{in}}_b(y)^{\boldsymbol\cdot}]^*=\langle 0|T^*(\phi^{\text{in}}_a(x)\phi^{\text{in}}_b(y))|0\rangle=-i\Delta_{F,ab}(x-y),\qquad(43a)$$

$$[\phi^{\text{in}}_{\mu,a}(x)^{\boldsymbol\cdot}\phi^{\text{in}}_b(y)^{\boldsymbol\cdot}]^*=\langle 0|T^*(\phi^{\text{in}}_{\mu,a}(x)\phi^{\text{in}}_b(y))|0\rangle=-i\partial_\mu(x)\Delta_{F,ab}(x-y),\qquad(43b)$$

$$[\phi^{\text{in}}_{\mu,a}(x)^{\boldsymbol\cdot}\phi^{\text{in}}_{\nu,b}(y)^{\boldsymbol\cdot}]^*=\langle 0|T^*(\phi^{\text{in}}_{\mu,a}(x)\phi^{\text{in}}_{\nu,b}(y))|0\rangle=-i\partial_\mu(x)\partial_\nu(y)\Delta_{F,ab}(x-y).\qquad(43c)$$

Therefore, for what we call the pure noncovariant contractions, we shall have

$$[\phi^{\text{in}}_a(x)^{\boldsymbol\cdot}\phi^{\text{in}}_b(y)^{\boldsymbol\cdot}]^n=0,\qquad(44a)$$

$$[\phi^{\text{in}}_{\mu,a}(x)^{\boldsymbol\cdot}\phi^{\text{in}}_b(y)^{\boldsymbol\cdot}]^n=0,\qquad(44b)$$

$$[\phi^{\text{in}}_{\mu,a}(x)^{\boldsymbol\cdot}\phi^{\text{in}}_{\nu,b}(y)^{\boldsymbol\cdot}]^n=-i\delta_{\mu\nu,ab}(x-y).\qquad(44c)$$

We could now try to solve "classically" (as in the preceding section) the question of covariant perturbation theory starting with (30a) or (30b). Although this task can be accomplished in this way, it is quite involved and lengthy. Instead we start directly with the differential equation for the S matrix (6′) which, in view of (41), is

$$\frac{1}{i}\frac{\partial}{\partial g}S=S\int d^4x\,\tfrac{1}{2}\phi_{\mu,a}(x)\overline{G}_{ab}(\phi(x))\phi^\mu_b(x).\qquad(45)$$

The S matrix must be a relativistic invariant quantity [(45) is consistent with this requirement]; therefore, we can seek its solution for any g in the form

$$S=T^*\exp\left\{i\int d^4x[g\tfrac{1}{2}\phi^{\text{in}}_{\mu,a}(x)\overline{G}_{ab}(\phi^{\text{in}}(x))\,\phi^{\text{in}\,\mu}_b(x)+C(\phi^{\text{in}}(x))]\right\},\qquad(46a)$$

from which we get

$$\frac{1}{i}\frac{\partial}{\partial g}S=T^*\left\{\int d^4x[\tfrac{1}{2}\phi^{\text{in}}_{\mu,a}(x)\overline{G}_{ab}(\phi^{\text{in}}(x))\,\phi^{\text{in}\,\mu}_b(x)+\frac{\partial}{\partial g}C(\phi^{\text{in}}(x))]S\right\}.\qquad(46b)$$

C in (46a) and (46b) does not depend on $\phi^{\text{in}}_{\mu,a}$. In fact, it is not difficult to see that (46a) is the most general form for the S matrix in our case. From (45) we have

$$\frac{1}{i}\frac{\partial}{\partial g}S=T\left\{\int d^4x\,\tfrac{1}{2}[\phi_{\mu,a}(x)]^{\text{in}}\overline{G}_{ab}(\phi^{\text{in}}(x))[\phi^\mu_b(x)]^{\text{in}}S\right\},\qquad(47)$$

where we acknowledged the fact that the asymptotic limit of a time derivative of an operator is generally different from the time derivative of the asymptotic limit of the same operator. However, utilizing (40) we get

$$\pi^{\text{in}}_a=\phi^{\text{in}}_{4,a}=G_{ab}(\phi^{\text{in}})[\phi_{4,b}]^{\text{in}}.\qquad(48)$$

Since $\phi^{\text{in}}_{r,a}=[\phi_{r,a}]^{\text{in}}$, we can write in general

$$[\phi_{\mu,a}]^{\text{in}}=\{g_{\mu\nu}\delta_{ab}-g^4_\mu g^4_\nu[G^{-1}(\phi^{\text{in}})-1]_{ab}\}\phi^{\text{in}\,\nu}_b.\qquad(49)$$

Taking into account that $T=T^*T_n$ [see (33)] and that the pure noncovariant contractions (44c) can be achieved by the following functional method (see also the preceding section),

$$T_n[\cdots\phi^{\text{in}}_{\mu,a}(x)\cdots]=T^*\left[\cdots\left(\phi^{\text{in}}_{\mu,a}(x)-i\int d^4y\,\delta_{\mu\nu,ab}(x-y)\frac{\delta}{\delta\phi^{\text{in}}_{\nu,b}(y)}\right)\cdots\right],$$

we find that (47) can be rewritten as

$$\frac{1}{i}\frac{\partial}{\partial g}S=T^*\left\{\int d^4x\,\frac{1}{2}\left[\phi^{\text{in}\,\nu}_c(x)-i\int d^4y\,\delta^{\kappa\nu}_{dc}(y-x)\frac{\delta}{\delta\phi^{\text{in}}_{\kappa,d}(y)}\right]\right.$$
$$\times[g_{\nu\mu}-g^4_\nu g^4_\mu(G^{-1}(\phi^{\text{in}}(x))-1)]_{ca}\overline{G}_{ab}(\phi^{\text{in}}(x))$$
$$\left.\times[g^{\mu\rho}-g^{\mu4}g^{\rho4}(G^{-1}(\phi^{\text{in}}(x))-1)]_{be}\left[\phi^{\text{in}}_{\rho,e}(x)-i\int d^4z\,\delta_{\rho\epsilon,ef}(x-z)\frac{\delta}{\delta\phi^{\text{in}}_{\epsilon,f}(z)}\right]S\right\}.\qquad(50)$$

Inserting the ansatz (46a) for S in (50), after carrying out straightforwardly the indicated functional derivatives, we obtain

$$\frac{1}{i}\frac{\partial}{\partial g}S=T^*\left(\int d^4x\{\tfrac{1}{2}\phi^{\text{in}}_{\mu,a}(x)\overline{G}_{ab}(\phi^{\text{in}}(x))\,\phi^{\text{in}\,\mu}_b(x)+\tfrac{1}{2}i\delta^{(4)}(0)\,\text{Tr}[G^{-1}(\phi^{\text{in}}(x))\overline{G}(\phi^{\text{in}}(x))]\}S\right).\qquad(51)$$

In deriving (51) we used the fact that

$$[g^{\mu\nu} - g^{\mu 4}g^{\nu 4}(G^{-1}(\phi^{\rm in}(x)) - 1)]_{ab}[g_{\nu\kappa} + gg^4_\nu g^4_\kappa \overline{G}(\phi^{\rm in}(x))]_{bc} = g^\mu_\kappa \delta_{ac}$$

and

$$\mathrm{Tr}[g_{\mu\nu} - g^4_\mu g^4_\nu (G^{-1}(\phi^{\rm in}(x)) - 1)]\overline{G}(\phi^{\rm in}(x))g^\mu_4 g^\nu_4 = -\mathrm{Tr}G^{-1}(\phi^{\rm in}(x))\overline{G}(\phi^{\rm in}(x)) .$$

Identifying (51) with (46b), we get

$$\frac{\partial}{\partial g} C(\phi^{\rm in}(x)) = \tfrac{1}{2} i \delta^{(4)}(0)\, \mathrm{Tr}[G^{-1}(\phi^{\rm in}(x))\overline{G}(\phi^{\rm in}(x))] . \tag{52}$$

Noticing that

$$\mathrm{Tr}[G^{-1}(\phi^{\rm in}(x))\overline{G}(\phi^{\rm in}(x))] = -\frac{\partial}{\partial g}\, \mathrm{Tr}\ln[1 - g\overline{G}(\phi^{\rm in}(x))] ,$$

we get

$$C(\phi^{\rm in}(x)) = -\mathrm{Tr}\ln[1 - g\overline{G}(\phi^{\rm in}(x))] , \tag{53}$$

where we imposed $C|_{g=0} = 0$. Therefore, the S matrix (for $g=1$) is given as

$$S = T^* \exp\left(i \int d^4x \{\mathcal{L}_{\rm int}(\phi^{\rm in}(x)) - \tfrac{1}{2} i \delta^{(4)}(0) \mathrm{Tr}\ln[1 - \overline{G}(\phi^{\rm in}(x))]\}\right) , \tag{54}$$

where it is important to notice that T^* is applied to the whole expression.

Let us point out that applying the same procedure to the "classical" model from Sec. IV, we would arrive at result (37) for the S matrix in a few lines.

Now it can be shown that the disappearance of the "anomalies" found in Ref. 3 [violation of Adler's condition for π-π scattering (the nonvanishing of amplitudes in the soft-pion limit), pion acquiring the mass, dependence of amplitudes on the choice of pion field] is due to the term

$$-\tfrac{1}{2} i \delta^{(4)}(0) \mathrm{Tr}\ln[1 - \overline{G}(\phi^{\rm in}(x))] ,$$

in the expression for the S matrix, (54). This term was absent in Ref. 3 (for details of this analysis, see Refs. 4 and 5). As noted in Ref. 4, the definition of the unique pion field by

$$\mathrm{Tr}\ln[1 - \overline{G}(\phi^{\rm in}(x))] = 0 ,$$

although simplifying the expression for the S matrix, makes the pion field itself quite complicated in the sense that it obeys rather complicated nonlinear transformation properties under SU(2) ×SU(2). Incidentally, because of identity Tr ln = ln det, the above condition reduces to

$$\det[1 - \overline{G}(\phi^{\rm in}(x))] = 1 ,$$

a condition found in Ref. 3 for the disappearance of the "anomalies." Of course, now we see why this condition must be imposed if we do not have the term

$$(-\tfrac{1}{2} i) \delta^{(4)}(0)\, \mathrm{Tr}\ln[1 - \overline{G}(\phi^{\rm in}(x))]$$

in the S matrix.

VI. CONCLUSION AND DISCUSSION

One of the remarkable things from Sec. V is the fact that using our formalism of PDECC, one can formulate the covariant perturbation theory without explicit use of the Hamiltonian density [see relation (45)]. We feel that this is of considerable advantage for, on one hand, the Hamiltonian densities are usually more complicated objects than the Lagrangian densities, and, on the other hand, the symmetry requirements are usually expressed through Lagrangians.

The other advantage of our formalism, we feel, is the fact that we do not need to change the notion of the Lagrangian density from $\mathcal{L}$ to $\overline{\mathcal{L}}$ with

$$\overline{\mathcal{L}} = \mathcal{L} - \tfrac{1}{2} i \delta^{(4)}(0) \ln\det[1 - \overline{G}(\phi)] ,$$

which as we see becomes non-Hermitian and which is used in the method employing the generating functional of the time-ordered Green's functions.

For these reasons, we believe that our approach to the formulation of a covariant perturbation theory is more direct and simpler than the ones involving the interaction Hamiltonian density[4] or the generating functional of the time-ordered Green's functions.[5,6]

The question which naturally arises with the expression for the S matrix (54) is whether the S matrix is unitary, since an anti-Hermitian term is added to the $\mathcal{L}^{\rm in}_{\rm int}$. It is, of course, unitary since (54) is equivalent to

$$S = T \exp\left(-i \int d^4x\, \mathcal{H}^{\rm in}_{\rm int}(x)\right) ,$$

for which the unitarity is easily proven. The other way to see that expression (54) for the S matrix is unitary is to realize that it was derived from the partial differential equation with respect to coupling constant (45), which implies the unitarity of the S matrix. To see this, let us denote with α the following:

$$\alpha = \int d^4x \, \frac{\partial^* \mathcal{L}(x)}{\partial g} \, .$$

Then, according to (45), we have at once that

$$\frac{1}{i} \frac{\partial}{\partial g}(SS^\dagger) = S\alpha S^\dagger - S\alpha S^\dagger = 0 \, ,$$

which allows $SS^\dagger = 1$. With this result one shows also that $S^\dagger S = 1$.

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[8]For an excellent discussion of this subject see R. Jackiw, in BNL Report No. BNL15732, 1971 (unpublished).

[9]Of course, some symmetry requirement may relate various observable masses. Nevertheless, in effective Lagrangians they still can be considered as input parameters.

[10]Of course, we admit the existence of some other interactions [different from those described with coupling constants g_i $(i = 1, 2, \ldots)$] which are responsible for the mass of a composite particle. In numerical determination of a mass of a composite particle, the relevant coupling constants will enter as a consequence of boundary conditions on a wave function.

[11]Note that $T(A(x)B(y)) - B(y)A(x) = \theta(x^4 - y^4)[A(x), B(y)]$, a relation which can quite simplify the deduction of (8).

[12]According to (1b) and the discussion after relation (6), the bare mass and the corresponding mass shift are both coupling-constant-dependent and satisfy

$$\frac{\partial}{\partial g} M_0(g) = -\frac{\partial}{\partial g} \Delta M(g) .$$

As we know, mass renormalizations are usually achieved by adding counterterms with $\Delta M(g)$'s to $\mathcal{L}_{\rm int}$. However, this is now equivalent to having free Hamiltonian density or Lagrangian density dependent on bare masses, which we denote as $\mathcal{H}_f'$ and $\mathcal{L}_f'$. If we ignore the mass renormalization (by considering only tree graphs, for example), formally $\Delta M(g) = 0$ and consequently

$$\frac{\partial'}{\partial g} \mathcal{H}_f' = 0, \quad \frac{\partial^*}{\partial g} \mathcal{L}_f' = 0.$$

More on this question can be found in the first paper of Ref. 7.

[13]See, for example, S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson, Evanston, Ill., 1961).

[14]Here we adopt the functional technique usually used to reduce the T product into normal products [see, e.g., N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (State Publishing House, Moscow, 1957)]. However, we wish to point out that during this procedure, we can formally treat the field operators as c-number fields, since the whole thing will be time ordered by means of T^*.

[15]It has been shown that as long as the chiral-symmetry-[(SU(2) × SU(2) or SU(3) × SU(3)] breaking part in the Lagrangian density has fixed transformation properties, the physical results are independent of representation assignments to Heisenberg fields [S. Coleman, J. Wess, and B. Zumino, Phys. Rev. 177, 2239 (1969); C. G. Callan, S. Coleman, J. Wess, and B. Zumino, *ibid.* 177, 2247 (1969)]. In the framework of chiral U(3) × U(3) it has been verified that the renormalized meson-baryon coupling constants are invariant under the canonical transformations of baryon fields [J. Šoln, Phys. Rev. D 2, 2404 (1970)].

[16]See, for example, S. Weinberg, Phys. Rev. 166, 1568 (1968). See also Ref. 4.