New Class of Solutions of the Einstein-Maxwell Fields

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Maxwell fields in static axially symmetric space-time are discussed. A new method for generating the Weyl class as well as a more general class of electromagnetic fields is outlined. It has been shown that by the same procedure a new class of Maxwell fields can be generated which are not of Weyl type. A particular solution of this class is derived which represents an asymptotically flat gravitational field of a body possessing an electric or magnetic dipole moment. At large distances and in the case of a vanishing dipole parameter, the gravitational field goes over to the Schwarzschild field.

I. INTRODUCTION

We consider solutions of the combined Einstein-Maxwell fields which depend on at most two spatial variables, the metric for which may be taken as

$$ds^2 = e^{2u} dt^2 - e^{2k-2u}[(dx^1)^2 + (dx^2)^2] - h^2 e^{-2u} d\phi^2 ,$$

where $u(x^1, x^2)$ and $k(x^1, x^2)$ are functions to be determined. h is a harmonic function and its choice determines the coordinate system. In particular, we choose $h = x^1 = \rho$ and $x^2 = z$, the coordinates ρ, z being known as Weyl canonical coordinates.[1] The Einstein-Maxwell equations are[2]

$$G_{ij} = -8\pi E_{ij} , \tag{1}$$

$$F^{ij}{}_{;j} = 0 , \tag{2}$$

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 , \tag{3}$$

$$E_{ij} = \frac{1}{4\pi}(g^{kl}F_{ik}F_{jl} - \tfrac{1}{4}g_{ij}F_{kl}F^{kl}) . \tag{4}$$

The electromagnetic potential vector has only two nonvanishing components for this problem. There are various ways to define the electromagnetic field in terms of the components of the four-potential, the most common being $F_{ij} = A_{i,j} - A_{j,i}$. However, another definition[3,4] given below has the advantage of reducing the nontrivial Maxwell equations as well as the components of the stress-tensor (4) in a symmetrical form with respect to the components of the potential vector. Thus, we define

$$F^{31} = \frac{1}{\rho} e^{2u-2k} A_{,2}, \quad F^{23} = \frac{1}{\rho} e^{2u-2k} A_{,1}, \tag{5a}$$

$$F_{01} = B_{,1}, \qquad F_{02} = B_{,2}, \tag{5b}$$

where obviously $A(\rho, z)$ and $B(\rho, z)$ are the magnetic and electric potentials, respectively. With the above stipulation, the Einstein-Maxwell field equations are

$$u_{,11} + u_{,22} + u_{,1}/\rho = -e^{-2u}(A_{,1}{}^2 + A_{,2}{}^2 + B_{,1}{}^2 + B_{,2}{}^2) , \tag{6}$$

$$k_{,1}/\rho = (u_{,1}{}^2 - u_{,2}{}^2) + e^{-2u}(A_{,1}{}^2 - A_{,2}{}^2 + B_{,1}{}^2 - B_{,2}{}^2) , \tag{7a}$$

$$k_{,2}/\rho = 2u_{,1}u_{,2} + 2e^{-2u}(A_{,1}A_{,2} + B_{,1}B_{,2}) , \tag{7b}$$

$$A_{,11} + A_{,22} + A_{,1}/\rho - 2(u_{,1}A_{,1} + u_{,2}A_{,2}) = 0 , \tag{8}$$

$$B_{,11} + B_{,22} + B_{,1}/\rho - 2(u_{,1}B_{,1} + u_{,2}B_{,2}) = 0 , \tag{9}$$

$$A_{,1}B_{,2} = A_{,2}B_{,1} . \tag{10}$$

The symmetrical occurrence of the potentials A and B in Eqs. (6)–(9) as well as Eq. (10) suggests a duality rotation,[5] $A = C\cos\beta$, $B = C\sin\beta$, where C is a new potential and β is a constant.

One thus obtains

$$u_{,11} + u_{,22} + u_{,1}/\rho = -e^{-2u}(C_{,1}{}^2 + C_{,2}{}^2), \tag{11}$$

$$k_{,1}/\rho = (u_{,1}{}^2 - u_{,2}{}^2) + e^{-2u}(C_{,1}{}^2 - C_{,2}{}^2), \tag{12a}$$

$$k_{,2}/\rho = 2u_{,1}u_{,2} + 2e^{-2u}C_{,1}C_{,2}, \tag{12b}$$

$$C_{,11} + C_{,22} + C_{,1}/\rho = 2(C_{,1}u_{,1} + C_{,2}u_{,2}) . \tag{13}$$

As suggested by Bonnor[6] Eqs. (11)–(13) constitute a completely determinate system of differential equations.

II. DERIVATION OF THE NEW FIELDS

Let us now introduce a new complex function E in the following manner:

$$E = e^u + iC . \tag{14}$$

Equations (11) and (13) are satisfied identically if E satisfies

$$E_{,11} + E_{,22} + E_{,1}/\rho = e^{-u}(E_{,1}{}^2 + E_{,2}{}^2) . \tag{15}$$

Introduction of another complex function X as

$$E=\frac{X-1}{X+1} \tag{16}$$

transforms (15) into

$$X_{,11}+X_{,22}+\frac{X_{,1}}{\rho}=\frac{2\bar{X}}{X\bar{X}-1}(X_{,1}{}^2+X_{,2}{}^2). \tag{17}$$

It is easy to see that

$$X=-e^{i\alpha}\coth\tfrac{1}{2}\psi \tag{18}$$

is a solution of Eq. (17) with α an arbitrary real constant and ψ given by

$$\psi_{,11}+\psi_{,22}+\psi_{,1}/\rho=0. \tag{19}$$

Further, one obtains from Eq. (12)

$$k_{,1}/\rho=\psi_{,1}{}^2-\psi_{,2}{}^2, \tag{20}$$

$$k_{,2}/\rho=2\psi_{,1}\psi_{,2}. \tag{21}$$

Equations (19)–(21) are the field equations of the vacuum Weyl fields.

Now if a solution of Eq. (19) is given, we may construct the functions X and E with the help of Eqs. (18) and (16), respectively, and thus a class of solutions of the combined Einstein-Maxwell field equations may be obtained with a nontrivial Maxwell field. In particular, we have the following cases:

(a) If $\alpha=0$, the electromagnetic field vanishes and one obtains the class of vacuum Weyl fields.

(b) If $\alpha=\frac{1}{2}\pi$, one easily obtains

$$e^u=\operatorname{sech}\psi,\quad C=\tanh\psi. \tag{22}$$

The above equations are the well-known transformation equations for generating the Weyl class of electromagnetic fields. Thus, our investigations lead to a new derivation of the theorem enabling one to construct electromagnetic fields from vacuum fields.[7-9]

(c) If α is arbitrary one obtains a new class of fields which is more general than the Weyl class.

Thus we have now the following interesting results: Given a solution (ψ, k) of the vacuum Weyl fields, one can generate a solution of the combined Einstein-Maxwell fields by a method discussed in this note.

Attention is also drawn to the exact similarity of Eq. (17) with the gravitational field equations of the stationary axially symmetric problem.[10] But one defines the complex function E in the stationary axially symmetric problem as $E=e^{2u}+i\phi$, where ϕ is the twist potential. Thus, there is an interesting formal analogy between the roles of the potential ϕ of the stationary axially symmetric problem and the electromagnetic potential C. However, the analogy ends there and the properties of the two metrics will be altogether different.

III. AN EXAMPLE

Equation (17) is interesting in another way also. This is due to the fact that with the help of this equation we can obtain another class of electromagnetic fields which do not come under the categories discussed earlier. This class of fields is interesting in the sense that such fields are possibly due to the bodies possessing an electric or magnetic dipole moment or both.[5] We give below one such solution to illustrate our point.

If Eqs. (19) and (17) are expressed in spheroidal coordinates (λ, μ) which are defined as[11, 12]

$$\rho=(\lambda^2-1)^{1/2}(1-\mu^2)^{1/2},\quad z=\lambda\mu, \tag{23}$$

one obtains

$$\frac{\partial}{\partial\lambda}\left[(\lambda^2-1)\frac{\partial\psi}{\partial\lambda}\right]+\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu}\right]=0, \tag{24}$$

and

$$\frac{\partial}{\partial\lambda}\left[(\lambda^2-1)\frac{\partial X}{\partial\lambda}\right]+\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial X}{\partial\mu}\right]=\frac{2\bar{X}}{(X\bar{X}-1)}\left[(\lambda^2-1)\left(\frac{\partial X}{\partial\lambda}\right)^2+(1-\mu^2)\left(\frac{\partial X}{\partial\mu}\right)^2\right]. \tag{25}$$

It is interesting to note that Eqs. (24) and (25) are symmetric with respect to the interchange of λ and μ. Consequently if $X(\lambda, \mu)$ is a solution of (17), then so is $X(\mu, \lambda)$. The above remark is also true for Eq. (19).

Consider now the following solution of (24):

$$\psi=\ln\frac{\lambda-1}{\lambda+1}. \tag{26}$$

In view of (18) the solution (26) is equivalent to $X=\lambda$. Hence, the above-mentioned symmetry leads to the solution $X=\mu$. A linear combination of these two solutions will also be a solution of (24). If this linear combination of the above-mentioned solutions also satisfies (17), i.e., (25), one has obviously obtained a new solution which will be of a different class than that discussed earlier.[8] Now, in the case of (26), the linear combination may clearly be taken as

$$X = \lambda \cos x + i\mu \sin x, \tag{27}$$

which is a solution of Eq. (25). x is a real parameter which fixes the scale of length in transforming canonical coordinates to spheroidal coordinates.

We define

$$\tan x = e, \quad \sec x = m, \tag{28}$$

where m and e are constants and we are measuring the length in units of $(m^2 - e^2)^{1/2}$. When the entire metric is constructed, one obtains

$$ds^2 = \left[\frac{r^2 + e^2\cos^2\theta - 2mr}{r^2 + e^2\cos^2\theta}\right]^2 dt^2 - \frac{(r^2 - 2mr + a^2\cos^2\theta)^2 (r^2 + e^2\cos^2\theta)^2}{[r^2 - 2mr + e^2\cos^2\theta + m^2\sin^2\theta]^3}\left[\frac{dr^2}{r^2 - 2mr + e^2} + d^2\theta\right] - \frac{(r^2 + e^2\cos^2\theta)^2 (r^2 - 2mr + e^2)}{(r^2 - 2mr + e^2\cos^2\theta)^2}\sin^2\theta\, d\phi^2, \tag{29}$$

where the coordinates r and θ are defined as

$$r = \lambda(m^2 - e^2)^{1/2} + m, \quad \cos\theta = \mu. \tag{30}$$

For canonical coordinates the transformation is given by Eq. (23) as

$$\rho = (r^2 + e^2 - 2mr)^{1/2}\sin\theta, \quad z = (r - m)\cos\theta. \tag{31}$$

The electromagnetic potential is given by

$$C = \frac{2me\cos\theta}{r^2 + e^2\cos^2\theta}. \tag{32}$$

This represents the static field of an electric or magnetic dipole; its asymptotic form is

$$C \approx \frac{2me\cos\theta}{r^2} \quad (r \to \infty). \tag{33}$$

Thus, the constant $2me$ may be interpreted as the electric or magnetic dipole moment of the system. Similarly the metric given by (29) is the gravitational field of a body possessing an electric or magnetic dipole moment. It may be easily seen that this metric is asymptotically flat. This metric has an essential singularity given by

$$S \equiv r^2 + e^2\cos^2\theta = 0 \tag{34}$$

which is a ring singularity. Besides, this metric has nonessential singularities also. The form of these singularities reminds one of the singularities of the Kerr metric.[13] If the parameter e vanishes the asymptotic form of the metric is given by

$$ds^2 = (1 - 4m/r)dt^2 - \frac{dr^2}{(1 - 4m/r)} - \frac{r^2}{(1 - 2m/r)}(d\theta^2 + \sin^2\theta d\phi^2) \quad (r \to \infty), \tag{35}$$

which may be identified as the Schwarzschild metric by a redefinition of the r coordinate. This enables one to interpret the parameter m as the mass of the source. This appears to us an interesting result and the reason for this may be seen in the choice of the solution (26).

Note added in proof. After submitting this paper for publication it was found that Bonnor had discovered a solution for a magnetic dipole similar to our Eq. (29) [W. B. Bonnor, Z. Physik 190, 444 (1966)]. But our solution is more general in the sense that it refers to an electromagnetic field.

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