[15]It should be pointed out that Eqs. (A3) do not represent a units transformation (UT) [see R. H. Dicke, Phys. Rev. 125, 2163 (1962)]. Moreover, although the quantities appear to be dimensionless, only those involving the differential dt (such as $\lambda\!\!\!^{-}/H$, etc.) actually are dimensionless in the sense of the UT. This circumstance arises because the UT is defined on the differential dt and not on t itself [see R. E. Morganstern, Phys. Rev. D 4, 278 (1971), Sec. II and Ref. 4].

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Standing Pion Waves in Superdense Matter*

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Previous results for the ground-state energies of systems of nucleons interacting with a condensed running-wave mode of π^- mesons are extended to the case of standing-wave modes. The onset density of the condensation turns out to be the same as in the running-wave case; at higher densities the standing-wave case has a somewhat lower energy, at least in the case in which ordinary nuclear forces are not taken into account. There are reasons to believe that the effects of these nuclear forces on the condensation will be much greater for the standing-wave case than for the running-wave case. The possibilities of π^+ and π^0 condensation are discussed.

I. INTRODUCTION

Recently it has been suggested that the ground state of neutron star matter at densities greater than some critical density consists of neutrons, protons, and π^- mesons, the latter condensed in a single running-wave mode of momentum (per pion) greater than $m_\pi c$.[1-3] This condensed pion wave is sustained by its interaction with a particular coherent state of protons and neutrons, as calculated in the conventional theory of the interaction of pions with nucleons. It was suggested that the magnetic energy problem associated with the immense currents in this state be solved by a breakup of the medium into filaments, with opposed currents in neighboring filaments.

An alternative solution, suggested in Ref. 1, could be to use a standing-wave mode for the π^- field. However, the solution for the energy of the medium for the standing-wave case is not as simple as that given for the running-wave case in Refs. 2 and 3, and the variational estimate placed on this energy in Ref. 1 is only an upper bound.

In the present work the problem of the standing π^- mode is solved analytically for the case of small X, where X = (number of π^-/number of nucleons), and numerically for the case of larger X. We find that in some domains of the parameters involved, the standing-mode solution is significantly lower in energy than the running-mode solution. These results are obtained in the absence of nucleon-nucleon forces. In the running-wave problems it was found that all isospin-independent nucleon-nucleon forces can be neatly included without changing the energy difference between the ground states in the condensed-pion case and in the normal case. In the standing-wave case, however, we shall see that more nuclear physics enters the problem and that our results are therefore less conclusive.

In the present work we also investigate the possibility of π^+ waves, which we predict will develop in addition to the π^- ones. Finally we consider the case of π^0 waves, which turns out to be harder to evaluate because of a greater dependence on the nuclear physics involved.

II. FORMULATION OF THE PROBLEM

As in Refs. 1–3, we begin with a Hamiltonian describing the interaction of protons and neutrons with a single mode of the π^- field as given in the conventional theory for nonrelativistic nucleons. We shall consider two cases, a running π^- wave with wave vector $k\hat{z}$ and a standing wave in the z direction. We denote the single-pion annihilation and creation operators for the condensed mode by a_R and $a_R^\dagger$, respectively. In the running-wave case we had[3]

$$H = H_0 + \frac{kf}{m_\pi(\omega_k V)^{1/2}} \int d^3x [-ip^\dagger(\vec{x})\sigma_3 n(\vec{x}) e^{-ikz} a^* + in^\dagger(\vec{x})\sigma_3 p(\vec{x}) e^{ikz} a] . \tag{1}$$

For the standing-wave case we obtain instead

$$H = H_0 + \frac{kf\sqrt{2}}{m_\pi(\omega_k V)^{1/2}} \int d^3x [-ip^\dagger(\vec{x})\sigma_3 n(\vec{x})(\cos kz) a^* + in^\dagger(\vec{x})\sigma_3 p(\vec{x})(\cos kz) a] , \tag{2}$$

where H_0 contains the rest and kinetic energies of the nucleons and pions. The parameter f is equal to 1.1. V is the volume; n and p are the neutron and proton fields.

In either case we shall consider a system of N nucleons, with NX protons, NX π^- mesons, and $N(1-X)$ neutrons. The actual state of the meson field will be taken as a superposition of states, each containing a number of pions approximately equal to NX such that in the limit of infinite volume we can make the c-number replacement,

$$a, a^* \to \sqrt{NX} = \sqrt{X\rho V} , \tag{3}$$

where ρ is the average density of nucleons. We make a redefinition of the proton field $p(x) \to i\sigma_3 p(x)$ in (1) and (2) and perform the substitution (3), obtaining for the running-wave case

$$H = H_0 + \sqrt{X}\, M_k \int d^3x [p^\dagger(\vec{x}) n(\vec{x}) e^{-ikz} + n^\dagger(\vec{x}) p(\vec{x}) e^{ikz}] \tag{4}$$

and for the standing-wave case

$$H = H_0 + \sqrt{2X}\, M_k \int d^3x [p^\dagger(\vec{x}) n(\vec{x}) + n^\dagger(\vec{x}) p(\vec{x})] \cos kz , \tag{5}$$

where

$$M_k = kf m_\pi^{-1} \rho^{1/2} \omega_k^{-1/2} . \tag{6}$$

Now we want to find the state, $|\Psi_0\rangle$, which minimizes $\langle\Psi_0|H|\Psi_0\rangle$ subject to the constraint of electrical neutrality. To incorporate the constraint we construct an $\mathcal{K}$ defined by

$$\mathcal{K} = \mathcal{K}_0 + H_I , \tag{7}$$

where H_I is the same interaction term as in (4) or (5) and

$$\mathcal{K}_0 = H_0 + \mu \int d^3x\, p^\dagger(\vec{x}) p(\vec{x}) . \tag{8}$$

The solution to our problem will be the state $|\Psi_0(\mu)\rangle$ which minimizes $\langle\mathcal{K}\rangle$, with the relation between the (proton-neutron) chemical potential, μ, and the parameter X determined from

$$\left\langle \Psi_0(\mu) \middle| \int d^3x\, \bar{p}(\vec{x}) p(\vec{x}) \middle| \Psi_0(\mu) \right\rangle = XN . \tag{9}$$

III. SMALL-X SOLUTION

From (4) and (5) we see that the case of small fractional proton occupancy (small X) is effectively that of weak coupling. We now solve the standing-wave problem, and for comparison the running-wave problem, in perturbation theory, retaining terms in the energy per baryon to order X^2. The unperturbed state of the nucleons will be taken as that of the free Fermi gas,

$$|\Psi_0^{(0)}\rangle = \prod_{q,\,\text{spin}}^{p_F} n^\dagger(q)_{\text{spin}} \Big| 0 \Big\rangle . \tag{10}$$

The vacuum state, $|0\rangle$, in this case denotes a state with no nucleons present but with a number XN of condensed pions. Thus we have

$$\mathcal{K}_0 |\Psi_0^{(0)}\rangle = H_0 |\Psi_0^{(0)}\rangle = \left(\tfrac{3}{5} N \frac{p_F^2}{2m} + XN\omega_k \right) \Big| \Psi_0^{(0)} \Big\rangle . \tag{11}$$

Although this unperturbed state is not electrically neutral, we shall achieve neutrality for the perturbed state by fixing the proton-neutron chemical potential difference, μ, at that value which leads to XN protons in the perturbed eigenstate of $\mathcal{K}$.

Since the operators $\mathcal{K}$ given by (7), (4), and (5) are quadratic in the nucleon fields, the perturbed ground-state wave function Ψ_0 will be given by

$$|\Psi_0\rangle = \prod_{q}^{p_F} b^\dagger(q) \Big| 0 \Big\rangle , \tag{12}$$

where $b^\dagger(q)$ is the creation operator of that linear combination of single-proton and single-neutron states which arises from the perturbation of the single-neutron state of momentum $\vec{q}$. To determine this combination and the associated energy shift we begin with the matrix elements of $\mathcal{K}_0$ and $\mathcal{K}_I$, in which spin has been suppressed entirely.

Running wave:

$$\langle \text{proton}, \vec{p} | \mathcal{K}_I | \text{neutron}, \vec{q} \rangle = \delta_{\vec{p},\vec{q}-\vec{k}} \sqrt{X}\, M_k , \tag{13}$$

Standing wave:

$$\langle \text{proton}, \vec{p} | \mathcal{K}_I | \text{neutron}, \vec{q} \rangle = \delta_{\vec{p},\vec{q}-\vec{k}} \sqrt{X}\, M_k 2^{-1/2} + \delta_{\vec{p},\vec{q}+\vec{k}} \sqrt{X}\, M_k 2^{-1/2} , \tag{14}$$

$$\langle \text{neutron}, \vec{p}|\mathcal{H}_0|\text{neutron}, \vec{q}\rangle = \frac{\vec{q}^2}{2m}\delta_{\vec{p},\vec{q}}, \tag{15}$$

$$\langle \text{proton}, \vec{p}|\mathcal{H}_0|\text{proton}, \vec{q}\rangle = \left(\frac{\vec{q}^2}{2m}+\mu\right)\delta_{\vec{p},\vec{q}}, \tag{16}$$

where m stands for the nucleon mass.

Applying Brillouin-Wigner perturbation theory to second order, we obtain for the single-particle states, $|\Psi(q)\rangle = b^\dagger(q)|0\rangle$,

$$|\Psi(q)\rangle = \mathfrak{N}(|\text{neutron}, \vec{q}\rangle + a_1|\text{proton}, \vec{q}+\vec{k}\rangle + a_{-1}|\text{proton}, \vec{q}-\vec{k}\rangle + a_2|\text{neutron}, \vec{q}+2\vec{k}\rangle + a_{-2}|\text{neutron}, \vec{q}-2\vec{k}\rangle), \tag{17}$$

where in the standing-wave case we find

$$\begin{aligned} a_{\pm1} &= M_k\sqrt{X}\,2^{-1/2}[E-(2m)^{-1}(\vec{q}\pm\vec{k})^2-\mu]^{-1}, \\ a_{\pm2} &= M_k^2 X 2^{-1}[E-(2m)^{-1}(\vec{q}\pm2\vec{k})^2]^{-1}[E-(2m)^{-1}(\vec{q}\pm\vec{k})^2-\mu]^{-1}. \end{aligned} \tag{18}$$

In the running-wave case we find

$$\begin{aligned} a_1 &= a_2 = a_{-2} = 0 \\ a_{-1} &= M_k\sqrt{X}\,[E-(2m)^{-1}(\vec{q}-\vec{k})^2-\mu]^{-1}. \end{aligned} \tag{19}$$

We define the term "single-particle energy," $E(q)$, from the relation

$$[b^\dagger(q), \mathcal{H}] = -E(q)b^\dagger(q), \tag{20}$$

where $b^\dagger(q)$ is that of (12) and $\mathcal{H}$ is related to the real Hamiltonian by (7). We calculate the single-particle energies in fourth-order Brillouin-Wigner perturbation theory and obtain the following.

Standing wave:

$$\begin{aligned} E(q) = \vec{q}^2(2m)^{-1} &+ 2^{-1}M_k^2X[E(q)-(2m)^{-1}(\vec{q}+\vec{k})^2-\mu]^{-1} + 2^{-1}M_k^2X[E(q)-(2m)^{-1}(\vec{q}-\vec{k})^2-\mu]^{-1} \\ &+ 4^{-1}M_k^4X^2[E(q)-(2m)^{-1}(\vec{q}+\vec{k})^2-\mu]^{-2}[E(q)-(2m)^{-1}(\vec{q}+2\vec{k})^2]^{-1} \\ &+ 4^{-1}M_k^4X^2[E(q)-(2m)^{-1}(\vec{q}-\vec{k})^2-\mu]^{-2}[E(q)-(2m)^{-1}(\vec{q}-2\vec{k})^2]^{-1}, \end{aligned} \tag{21}$$

Running wave:

$$E(q) = \vec{q}^2(2m)^{-1} + M_k^2X[E(q)-(2m)^{-1}(\vec{q}-\vec{k})^2-\mu]^{-1}, \tag{22}$$

where we note the absence of fourth-order terms in the running-wave case.

We want to solve (21) for the energies to order X^2, and determine the expansion coefficients, a_m, of (18) well enough to solve for the parameter μ to order X from the neutralization condition, (9).

First we determine the zeroth-order value of μ. From (18) and (9) we find

$$X = \langle M_k^2 2^{-1}X\{[E(q)-(2m)^{-1}(\vec{q}+\vec{k})^2-\mu]^{-2} + [E(q)-(2m)^{-1}(\vec{q}-\vec{k})^2-\mu]^{-2}\}\rangle + O(X^2), \tag{23}$$

where $\langle\ \rangle$ means the average over a Fermi sphere in the variable $\vec{q}$. Now we expand the $(\vec{q}\cdot\vec{k})$ dependence of the denominator functions, to obtain

$$\begin{aligned} X = \langle M_k^2X[E(q)-(2m)^{-1}\vec{q}^2-(2m)^{-1}\vec{k}^2-\mu]^{-2}\rangle \\ + 3M_k^2Xm^{-2}\mu^{-4}\langle(\vec{q}\cdot\vec{k})^2\rangle + O(\langle(\vec{q}\cdot\vec{k})^4\rangle), \end{aligned} \tag{24}$$

where in the $\langle(\vec{q}\cdot\vec{k})^2\rangle$ term we have already substituted $E(q)\approx(2m)^{-1}\vec{q}^2$ and used $\vec{k}^2(2m)^{-1}\ll\mu$. The $\langle(\vec{q}\cdot\vec{k})^2\rangle$ term will contribute less than 10% to the value of μ determined from (24), and for simplicity we neglect it. The higher-order terms in $(\vec{q}\cdot\vec{k})$ will give negligible contributions.

To find the zeroth-order value of μ from (24) we insert the zeroth-order energy $E(q)=(2m)^{-1}q^2$, obtaining

$$X = M_k^2X[\mu+(2m)^{-1}\vec{k}^2]^{-2}$$

or

$$\mu = M_k - \frac{\vec{k}^2}{2m} + O(X). \tag{25}$$

The term $\vec{k}^2/2m$ will be a small correction to μ for those values of M_k at which a condensation will occur.

Next we determine the energies to order X from (25) and (21):

$$\begin{aligned} E(q)-(2m)^{-1}\vec{q}^2 = -M_kX - \frac{(\vec{q}\cdot\vec{k})^2}{M_km^2}X \\ + O((\vec{q}\cdot\vec{k})^4X) + O(X^2). \end{aligned} \tag{26}$$

Once again we shall simplify by dropping the $(\vec{q}\cdot\vec{k})^2$ term, which, on the average, will not change the coefficient of X by more than 10%. With this simplification the standing- and running-mode solutions for $E(q)$ are identical to order X. [The expressions for the total energy of the system would be the same in the two cases even if the $(\vec{q}\cdot\vec{k})^2$ terms had been retained.] From the energy to first order we can now obtain the chemical potential μ to first order by adding terms of order X^2 to the right-hand side of the neutralization condition (23), which becomes [after dropping the $(\vec{k}\cdot\vec{q})^2$ terms as usual and using (26)]

$$X=\left\langle M_k^2 X\left(M_k X+\frac{k^2}{2m}+\mu\right)^{-2}(1+X)^{-1}\right\rangle . \tag{27}$$

Here we have used the expression (18) for $a_{\pm1}$ and computed the normalization $\mathfrak{N}$ of (17) to first order in X. Note that to obtain μ to order X the coefficients $a_{\pm2}$ of (18) are not needed, since in the equation

$$X=\mathfrak{N}^2(a_1^2+a_{-1}^2) \tag{28}$$

they enter only in the computation of $\mathfrak{N}$, which we need only to order X.

Since nothing in (27) depends on $\vec{q}$ we can remove the bracket and solve for μ:

$$\mu=M_k(1-\tfrac{3}{2}X)-\frac{k^2}{2M}+O(X^2), \tag{29}$$

which is valid for both the running-wave and standing-wave cases. Now we return to (21) for the computation of the terms of order X^2 in $E(q)$ [neglecting the $(\vec{q}\cdot\vec{k})^2$ terms *except* in the fourth-order part, in which we make no expansion], obtaining

$$E(q)=(2m)^{-1}\vec{q}^{\,2}-XM_k-\tfrac{1}{2}M_kX^2 - 2^{-1}M_k^2X^2m[(\vec{k}^2-\vec{q}\cdot\vec{k})^{-1}+(\vec{k}^2+\vec{q}\cdot\vec{k})^{-1}] \tag{30}$$

for the standing-wave case, and the same result without the last term on the right in the running-wave case.

The lowest eigenvalue of $\mathcal{H}$ is now found by integrating $E(q)$ over the entire Fermi sphere and adding the free-meson energy,

$$\langle\mathcal{H}\rangle=\tfrac{3}{5}N\frac{p_F^2}{2m}+XN\omega_k-XNM_k-\tfrac{1}{2}X^2NM_k-mM_k^2X^2N\rho^{-1}\int^{p_F}\frac{d^3q}{(2\pi)^3}[(\vec{k}^2+\vec{q}\cdot\vec{k})^{-1}+(\vec{k}^2-\vec{q}\cdot\vec{k})^{-1}]. \tag{31}$$

The energy per nucleon of the system is given by[4]

$$\frac{E}{N}=\frac{\langle\mathcal{H}\rangle}{N}-\mu X$$

$$=\frac{3}{5}\frac{p_F^2}{2m}+X\omega_k-2XM_k+X^2M_k+\frac{k^2}{2m}X-(4\pi^2\rho)^{-1}mM_k^2X^2\left[(p_F^2-k^2)k^{-1}\ln\left(\frac{p_F+k}{p_F-k}\right)+2p_F\right]. \tag{32}$$

In the running-wave case we obtain

$$\frac{E}{N}=\frac{3}{5}\frac{p_F^2}{2m}+X\omega_k-2XM_k+X^2M_k+\frac{k^2}{2m}X, \tag{33}$$

which agrees with the result of Ref. 2, Eq. (15), when it is expanded in powers of X.

From (32) and (33) we see that in the present models the onset of the condensation (small X) can occur at a density for which

$$M_k=\tfrac{1}{2}\omega_k+\frac{k^2}{4m} \tag{34}$$

for either the standing-wave or the running-wave case. As in Refs. 1 and 2, the lowest-density onset will be for a value of $k\approx1.2m_\pi$. Since the sign of the fourth-order term in (32) is negative, the standing wave would seem to give lower energies at densities significantly above the onset density. However, we shall comment at the end of the next section on how the nuclear forces may disrupt this result.

IV. SOLUTION FOR LARGE X

Let a single-particle state which is an eigenstate of $\mathcal{H}$ in the running-wave case be created by the operator $b^\dagger$,

$$b^\dagger=\int n^\dagger(x)f(x)+\int p^\dagger(x)g(x). \tag{35}$$

Then the equations obeyed by f and g follow from (7) and (8):

$$(-2m)^{-1}\nabla^2\begin{pmatrix}f\\g\end{pmatrix}+\begin{pmatrix}0&\sqrt{2}\,M_k\cos kz\\ \sqrt{2}\,M_k\cos kz&\mu\end{pmatrix}\begin{pmatrix}f\\g\end{pmatrix}=E\begin{pmatrix}f\\g\end{pmatrix}. \tag{36}$$

Because of the periodicity of the potential a band structure will arise in this case. In the perturbation solution of the last section the evidence for the

band structure was the vanishing denominator in the integral in (31). However, to order X^2 it was correct to use the principal-part definition, rather than using degenerate perturbation theory, for those states satisfying $q \cdot k \approx -k^2$.

To solve for the eigenstates and eigenvalues it is convenient to return to momentum space. We find an infinite, discrete matrix, $A(q)$, to be diagonalized, the rows and columns of which we label with the positive and negative integers, $n = 0, \pm 1, \pm 2, \ldots$,

$$\begin{aligned} &A(q)_{2n,2n} = (2m)^{-1}(\vec{q} + 2n\vec{k})^2, \\ &A(q)_{2n+1,2n+1} = 2m^{-1}[\vec{q} + (2n+1)\vec{k}]^2 + \mu, \\ &A(q)_{n,n\pm1} = (\tfrac{1}{2}X)^{1/2} M_k, \\ &\text{and all other } A(q)_{n,n'} = 0. \end{aligned} \tag{37}$$

The equation to be solved is $A(q)\xi(q) = E(q)\xi(q)$, where the even components of ξ, ξ_{2n}, refer to neutrons, and the odd ones, ξ_{2n+1}, to protons. The z component of q will be restricted to the domain $-k < q_z < k$ in the reduced-zone scheme.

We truncated the eigenvalue problem at $\xi_{\pm 4}$, that is, by approximating A by a 9×9 matrix, and numerically determined the eigenvalues and eigenvectors. By comparing with the 7×7 case we have verified that the truncation errors are negligible.

The task is now to pick a value of μ, to fill the nucleon states up to the desired average density ρ in the lowest-energy configuration, and then to use the eigenfunctions $\xi(q)$ of the filled states to determine the fractional proton occupancy, X, of the whole system. Since X is an input parameter in the matrix (37) this process must be made self-consistent; that is, the output X must equal the input X. In general, for any allowed chemical potential difference, μ, this can happen for only one value of X. After self-consistency is achieved we calculate the energy from

$$E/N = N^{-1} \sum_{\text{occupied states}} E_i(q) - \mu X + \omega_k X, \tag{38}$$

where μ has been determined in terms of X by the process just described. In (38) q specifies the position of the state within the reduced zone and i specifies the energy band.

In practice two facts which we have verified computationally make the task easier:

(a) For a well-developed condensation ($\rho > 0.4$ F^{-3}) the single-particle energies minus the transverse kinetic energies, $E(q) - (2m)^{-1}(q_x^2 + q_y^2)$, are almost constant as we go through the zone from $q_z = -k$ to $q_z = k$.

(b) The minimum-energy configuration will involve occupancy only of the first two bands for those values of k and ρ which we shall consider.

Thus to determine the minimal way of filling the states for particular values of μ and X we need to know only the two lowest eigenvalues, E_1 and E_2, of the matrix A. The Fermi surfaces are cylindrical, bounded by $(q_x^2 + q_y^2)_{\text{band 1}} < R_1^2$ and $(q_x^2 + q_y^2)_{\text{band 2}} < R_2^2$, where R_1 and R_2 are set by the conditions

$$\begin{aligned} &(2\pi^2)^{-1} k (R_1^2 + R_2^2) = \rho, \\ &(2m)^{-1} R_1^2 + E_1 = (2m)^{-1} R_2^2 + E_2. \end{aligned} \tag{39}$$

In order to find the output value of X we need to know the fractional proton occupancies, X_1 and X_2 for bands 1 and 2, respectively. These are determined from the eigenvectors ξ,

$$X_{1,2} = \sum_{n \text{ odd}} |\xi_n^{1,2}|^2 \left(\sum_{\text{all } n} |\xi_n^{1,2}|^2 \right)^{-1}, \tag{40}$$

of a state in band 1 or band 2, respectively. Once again the $\xi^{1,2}$ are nearly independent of the position q_z in the reduced zone for the case of a well-developed condensate. For the over-all proton fraction we obtain

$$X = (R_1^2 X_1 + R_2^2 X_2)(R_1^2 + R_2^2)^{-1}. \tag{41}$$

In practice we pick a value of μ, guess an input value of X (the known points are $X = \frac{1}{2}$ for $\mu = 0$ and the value of $\mu \approx M_k$ for small X, as determined in Sec. III), compute the numbers $E_1, E_2, X_1, X_2, R_1, R_2$, and then recalculate X. This procedure is iterated until consistency is achieved. The energy per baryon is then calculated. For given values of ρ and k this calculation is repeated for different X (and μ) until the minimum-energy state (and the minimizing X) is determined.

The whole process has been repeated for various values of ρ and k, and the results are shown in Fig. 1, where they are compared with the results of Ref. 3 for the running-wave case. As expected from the small-X calculation, the energies for the standing-wave condensation are considerably lower than for the running-wave case.

However, these results have been derived in the absence of nuclear forces. Some difficult nuclear physics may stand in the way of realistic comparison of the two cases in the presence of nuclear forces. In Ref. 3 it was shown for the running-mode case that the results for the energy difference between the condensed pion state and the normal ground state are essentially maintained in the presence of spin- and isospin-independent nuclear forces. The origin of this result was the fact that the nucleon wave function in the condensed-pion case is identical to that in the normal pure-neutron ground state, except that the neutrons are replaced by quasiparticles which are coherent mixtures of proton and neutron. The expectation value of the nucleon density operator, $\bar{n}n + \bar{p}p$, and the density-density correlation function are the same as in the

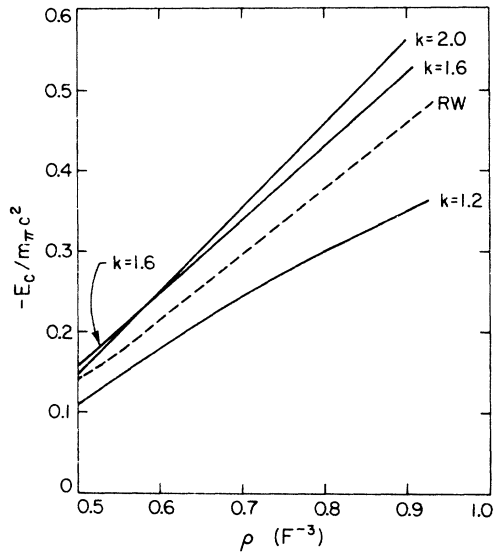

FIG. 1. The negative of the condensation energy versus density, for standing-wave solutions (solid lines) and for running-wave (RW) solutions (dashed line). In the running-wave case the value of k is chosen to minimize the condensation energy at each density. The standing-wave results are given for three different values of k (measured in units of $m_\pi c$). One can, however, see the fact that the value of k which gives the lowest energy increases with density. At onset ($\rho = 0.21$) according to our analytic calculation the $k = 1.2$ curve would pass through zero; the others will be below zero. However, our numerical calculation is unreliable below a density of about 0.5 F^{-3}.

normal ground state.

None of this will be true in the standing-mode case. The pion density wave will induce a similar wave in the nucleon density. We can calculate the magnitude of this density wave from the eigenvectors, ξ. For example, for an average density of 0.43 F^{-3} and a pion momentum of $1.6 m_\pi$, which are values for which we predict a well-developed condensation, we find a nucleon density wave,

$$\langle p^\dagger(\vec{x}) p(\vec{x}) + n^\dagger(\vec{x}) n(\vec{x}) \rangle \approx (1 + 0.8 \cos 2kz)\rho . \quad (42)$$

This will undoubtedly result in an increase in the nuclear interaction energy. The pion density wave inherent in the standing mode will itself lead to greater energies from $\pi^-\pi^-$ repulsions than in the constant-density running-wave case (for which the pion-pion effects were estimated in Ref. 3).

Near the onset density, when X is small, neither of these effects should be important, since they both give energies per baryon of order X^2. For larger X it will require a theory with more nuclear physics in it to decide which possibility gives the lower energy.

V. π^+ CONDENSATION

We now consider a running wave of π^- in the $+\hat{z}$ direction and one of π^+ in the $-\hat{z}$ direction, each with the same wavelength. Defining XN as the number of π^- minus the number of π^+, and YN as the number of π^+, we obtain, in place of (4),

$$H = H_0 + [(X+Y)^{1/2} + Y^{1/2}] M_k \int d^3x [\, p^\dagger(\vec{x}) n(\vec{x}) e^{-ikz} + n^\dagger(\vec{x}) p(\vec{x}) e^{ikz}] , \quad (43)$$

where the meson kinetic energy in H_0 is now given as $(X+2Y)N\omega_k$. The neutrality constraint is

$$XN = \left\langle \int p^\dagger(\vec{x}) p(\vec{x})\, d^3x \right\rangle . \quad (44)$$

We solve for the energy exactly as in Refs. 2 and 3, obtaining

$$\frac{E}{N} = \frac{3}{5}\left(\frac{p_F^2}{2m}\right) + X\left(\frac{k^2}{2m}\right) + \omega_k(X+2Y) - 2M_k[(X+Y)^{1/2} + Y^{1/2}][X(1-X)]^{1/2} . \quad (45)$$

Minimizing this expression with respect to X and Y reveals that it will always be advantageous to add a small number of π^+'s, with a corresponding increase in the number of π^-'s. The onset is now predicted to be at $\rho = 0.19$ F^{-3} instead of at 0.25 F^{-3}, with the π^+/π^- ratio at onset equal to 0.071. As the density is increased it is predicted that somewhat higher fractions of π^+ are present, and the energy is of course somewhat lower than in the pure π^- case. Numerical results are shown in Table I. We can expect a similar admixture of π^+

TABLE I. The minimum condensation energy per nucleon and the values of the parameters which minimize the condensation energy for different values of the density, for the case of a running π^- wave superimposed on a running π^+ wave (in the opposed direction). E_c is given by E/N (condensed pion phase) $-E/N$ (pure neutron phase). The condensation energies are more than 50% larger than in the pure π^- case discussed in Ref. 3, even though the fraction of π^+ mesons remains small. The parameter Y is the ratio of the number of π^+ to the number of neutrons; $X+Y$ is $N_\pi - /N$; $k*$ is the best value of pion momentum.

ρ (F^{-3})	E_c ($m_\pi c^2$)	X	Y	$k*$ ($m_\pi c$)
0.2	-5.29×10^{-3}	0.07	0.005	1.2
0.3	-7.81×10^{-2}	0.22	0.02	1.4
0.4	−0.184	0.27	0.04	1.5
0.5	−0.305	0.32	0.05	1.6
0.6	−0.430	0.35	0.07	1.7
0.7	−0.560	0.37	0.08	1.8
0.8	−0.687	0.40	0.09	1.9
0.9	−0.822	0.40	0.11	2.0

mesons in the lowest-energy standing-wave configuration.

VI. NEUTRAL-PION CONDENSATION

A Hamiltonian for a condensed π^0 mode in the running-wave case is

$$H = H_0 + \sqrt{2X}\, M_k \int d^3x [\, p^\dagger(x)\sigma_3 p(x) - n^\dagger(x)\sigma_3 n(x)] \cos kz\,, \tag{46}$$

while in the standing-wave case it is

$$H = H_0 + 2\sqrt{X}\, M_k \int d^3x [\, p^\dagger(x)\sigma_3 p(x) - n^\dagger(x)\sigma_3 n(x)] \cos kz\,, \tag{47}$$

where XN equals the number of π^0 mesons. Thus in this case the only difference between the standing- and running-wave cases is in the effective coupling constant, and we shall consider the standing-wave case only since it is the more strongly coupled.

We shall solve for the energy of a state consisting of π^0 mesons and neutrons only and calculate to order X only. The single-particle energy in second-order perturbation theory becomes

$$E(q) = (2m)^{-1}\vec{q}^{\,2} + XM_k^{\,2}\{[E(q) - (2m)^{-1}(\vec{q} - \vec{k})^2]^{-1} + [E(q) - (2m)^{-1}(\vec{q} + \vec{k})^2]^{-1}\}\,. \tag{48}$$

The differences between this result and that for the π^- standing wave, Eq. (20), are the following:

(a) The coupling is larger by a factor $\sqrt{2}$ here.

(b) There is no parameter μ and no equation of constraint in the π^0 case.

(c) Since there is no μ in the denominators in (48) they will vanish for some q even in the second-order terms. The band splittings develop in first order in this model.

(d) Were we to perform a complete calculation for all X we should use a range of q_z in each band equal to one half that used in the π^- case; i.e., $-\frac{1}{2}k < q_z < \frac{1}{2}k$. However, to obtain the first-order result in X we can ignore the band structure and integrate straight through the singularity in (48), using the principal-value definition, when we calculate the total energy of the system.

In this way we obtain the result

$$E/N = \tfrac{3}{5}P_F^{\,2}(2m)^{-1} + \omega_k X - \tfrac{3}{2}M_k^{\,2}XmP_F^{-3}\left[P_F + k^{-1}(P_F^{\,2} - \tfrac{1}{4}k^2)\ln\left(\frac{2P_F + k}{2P_F - k}\right)\right]. \tag{49}$$

The negative term here is embarrassingly large. If we consider a pion momentum of $k = 1.2m_\pi$ then (49) would predict a π^0 condensation at nuclear densities. If a higher k were chosen the situation would be even worse.

However, we have reason to think that the result (49) will be altered much more by the nuclear forces than the results for the π^- cases. One evidence for this is the very different way the nucleon mass enters the formulas for the condensation energy in the two cases. In the π^- case it came in an insignificant correction term of magnitude $k^2/2m$ per proton. In the π^0 case the entire negative term was proportional to m. This reflects the different nature of the intermediate states in the two cases. In the π^0 case the only intermediate states which actually contributed consisted of neutrons near the Fermi surface excited to outside of the Fermi surface by addition of the momentum $\pm k\hat{z}$. In the π^- case the intermediate states replaced a neutron from anywhere in the sea by a proton with momentum differing by $\pm k\hat{z}$.

The large second-order effect in the π^0 case came from the fact that the kinetic energy differences between the unperturbed ground state and excited states were so small owing to the largeness of the nucleon mass. However, in real nuclear matter it is known that a large price in potential energy must be paid also; the effective mass at the Fermi surface is about one half of the free-nucleon mass.

If, for example, we use an effective mass of $0.6m$ in (49) the situation changes dramatically. Now the onset for π^0 condensation will be at densities above $\rho = 0.2\ \mathrm{F}^{-3}$ for any pion momentum up to $k = 2.4m_\pi$; at $k = 1.2m_\pi$ the onset would be raised to $0.490\ \mathrm{F}^{-3}$. Thus our conclusion is that the π^- (and π^+) condensation probably would have an onset at a lower density than would a π^0 condensation. The question of whether a π^0 mode could then be established in the presence of the π^- mode is then a very complicated one. Only one thing is clear: We cannot superpose the theories of the separate condensations. The demands made on the nucleon wave functions are at least partially in conflict.

VII. DISCUSSION

We have found a multitude of possibilities for pion condensation in superdense nuclear matter. Unfortunately we have not been able to analyze any

other case as thoroughly as that of the running π^- wave, which was discussed previously. A more complete discussion of the cases of a standing π^- wave or of a π^0 wave would seem to involve some hard nuclear physics. If any of the other possibilities were realized the effects on the equation of state could only be a further softening over that predicted in the running π^- case. Thus perhaps the most important immediate problem is the further development of the running-wave π^- case, through a more complete analysis of the effects of spin-dependent nuclear forces, through a more complete treatment of non-P-wave pion-nucleon forces, perhaps through inclusion of the multipion effects from chiral models, and especially through a careful understanding of electromagnetic effects and filament structure.

There is one further modification which would almost certainly lower the energy in all cases but which we have not yet examined. This is the addition of harmonics to the basic meson wave. The choice of a simple sinusoidal wave was completely arbitrary. For example, in the standing-wave case, which already is a variety of a crystal since translational invariance has been broken, it may be advantageous to localize the pions even more through addition of harmonics to the wave form.

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[1]R. F. Sawyer, Phys. Rev. Letters 29, 382 (1972).

[2]D. J. Scalapino, Phys. Rev. Letters 29, 386 (1972).

[3]R. F. Sawyer and D. J. Scalapino, Phys. Rev. D 7, 953 (1973).

[4]This result is in disagreement, both in the X and X^2 terms, with that of J. Kogut and J. T. Manassah [Phys. Letters 41A, 129 (1972)], who seem to have posed exactly the same mathematical problem but have arrived at a completely different answer.