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Relativistic Motion in a Uniform Magnetic Field*

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The Lorentz-Dirac equation of motion for a particle moving in a uniform field is analyzed in detail. The perturbation solution obtained is exact to sixth order in the interaction constant $\lambda = 2e^3H/3m^2c^4$. We study the convergence of this solution and show explicitly how the radiated energy depends on the transverse energy and on the Schott energy.

I. INTRODUCTION

Recently there has been renewed interest in the classical motion of a radiating particle in a uniform magnetic field. Though there has been some disagreement as to the roles played by the transverse and longitudinal motion in contributing to the radiated energy,[1-4] the approximate solutions of the Lorentz-Dirac[5] equation obtained have been applied to the study of synchrotron emission from pulsars.[1,6] From a more conceptual viewpoint, the significance of the Schott[5] energy for the relativistic and nonrelativistic motion of a particle in a plane normal to a constant magnetic field has been clarified.[7-9] It is our purpose in this paper to present a detailed analysis of the Lorentz-Dirac equation for general, as compared with planar,[9] motion in a uniform magnetic field. The solution is given to sixth order in the interaction parameter, $\lambda = 2e^3H/3m^2c^4$. We study the validity of this expansion and consider the limiting cases when the initial energy of the particle is equal to, or very much greater than, the longitudinal energy. Using this solution we show explicitly the manner in which the radiated energy depends on the transverse energy and on the Schott energy.

II. EQUATIONS OF MOTION

The Lorentz-Dirac equation for the motion of a particle of charge e and rest mass m in an externally applied field, described by the antisymmetric field tensor $F_{\mu\nu}$, is

$$\dot v_\mu(\tau) - \frac{2}{3}\frac{e^2}{mc^3}\ddot v_\mu(\tau) + \frac{2}{3}\frac{e^2}{mc^3}v_\mu(\tau)\frac{\dot v_\nu \dot v_\nu}{c^2} = \frac{e}{mc}F_{\mu\nu}v_\nu(\tau). \tag{1}$$

The units are Gaussian, and we use a dot over a variable to denote differentiation with respect to the particle proper time τ. The four-velocity has the components

$$v_\mu(\tau) = (v_k = \dot x_k(\tau), ic\gamma(\tau)), \tag{2}$$

where the Latin subscript k takes on values 1 to 3. For a uniform magnetic field, $H_k = (0, 0, H)$, the nonvanishing components of the electromagnetic field tensor are

$$F_{12} = -F_{21} = H. \tag{3}$$

When we introduce the complex four-velocity variable, $w = v_1 + iv_2$, of the particle in a plane normal to the direction of the magnetic field, Eq. (1) reduces to the three equations

$$\omega\dot w - \lambda\ddot w + \lambda w\left(\frac{\dot w\dot w^* + \dot v_3\dot v_3}{c^2} - \dot\gamma^2\right) = -i\omega^2 w, \tag{4a}$$

$$\omega\dot v_3 - \lambda\ddot v_3 + \lambda v_3\left(\frac{\dot w\dot w^* + \dot v_3\dot v_3}{c^2} - \dot\gamma^2\right) = 0, \tag{4b}$$

$$\omega\dot\gamma - \lambda\ddot\gamma + \lambda\gamma\left(\frac{\dot w\dot w^* + \dot v_3\dot v_3}{c^2} - \dot\gamma^2\right) = 0, \tag{4c}$$

with the cyclotron frequency $\omega = eH/mc$ and the interaction parameter $\lambda = 2e^3H/3m^2c^4$. We represent the solution to these equations in terms of the initial values of the variables and two functions of the proper time, $\phi(\tau)$ and $\theta(\tau)$, which vanish for $\tau = 0$. Thus we write

$$\gamma(\tau) = \gamma(0)\left[\frac{\gamma^2(0)}{\gamma_L^2} - \left(\frac{\gamma^2(0)}{\gamma_L^2} - 1\right) \times \exp[-2\lambda\phi(\tau)]\right]^{-1/2}, \tag{5a}$$

$$w(\tau) = w(0)\frac{\gamma(\tau)}{\gamma(0)}\exp[-\lambda\phi(\tau) - i\theta(\tau)], \tag{5b}$$

$$\frac{v_3(\tau)}{\gamma(\tau)} = \frac{v_3(0)}{\gamma(0)} = u_3(0). \tag{5c}$$

Here $\gamma(0)$ equals the initial total energy (in units of mc^2), and $w(0)$ the initial transverse complex four-velocity, while γ_L is the energy associated with the constant longitudinal velocity[2] $u_3(0)$, that is,

$$\gamma_L^2 = \frac{1}{1 - u_3^2(0)/c^2}. \tag{6}$$

Corresponding to this assumed form of solution, we find that the two functions, ϕ and θ, must satisfy the coupled differential equations

$$\omega\dot{\phi} - \dot{\theta}^2 - \lambda\ddot{\phi} - \lambda^2\dot{\phi}^2 + 2\lambda^2\dot{\phi}^2\frac{\gamma^2}{\gamma_L^2} = 0 \tag{7}$$

and

$$\omega\dot{\theta} - \omega^2 - \lambda\ddot{\theta} + 2\lambda^2\dot{\phi}\dot{\theta}\frac{\gamma^2}{\gamma_L^2} = 0. \tag{8}$$

When we express these equations in terms of the variables $x = \dot{\phi}/\omega$, $y = \dot{\theta}/\omega$, and $\zeta = \gamma^2/\gamma_L^2$, we obtain the pair of first-order differential equations

$$x - y^2 + \lambda^2(2\zeta - 1)x^2 + \lambda^2 2\zeta(\zeta - 1)x\frac{dx}{d\zeta} = 0 \tag{9}$$

and

$$y - 1 + \lambda^2 2\zeta xy + \lambda^2 2\zeta(\zeta - 1)x\frac{dy}{d\zeta} = 0. \tag{10}$$

In making ζ the independent variable in these last equations, we have used the relationship

$$\frac{d\zeta}{d\tau} = -\lambda 2\zeta(\zeta - 1)\dot{\phi}, \tag{11}$$

derivable from Eq. (5a). At this point we treat λ as a small parameter and derive the expansions

$$x = 1 + \lambda^2(1 - 6\zeta) + \lambda^4 2(40\zeta^2 - 20\zeta + 1) - \lambda^6(1568\zeta^3 - 1368\zeta^2 + 234\zeta - 5) + \cdots \tag{12}$$

and

$$y = 1 - \lambda^2 2\zeta + \lambda^4 2\zeta(10\zeta - 3) - \lambda^6 4\zeta(88\zeta^2 - 60\zeta + 5) + \cdots. \tag{13}$$

We now expand $\zeta = \gamma^2/\gamma_L^2$, as given by Eq. (5a), into a Taylor series in the variable $(\lambda\phi)$. The result is

$$\zeta/\zeta_0 = 1 - 2(\zeta_0 - 1)\lambda\phi + 2(\zeta_0 - 1)(2\zeta_0 - 1)(\lambda\phi)^2 - \tfrac{4}{3}(\zeta_0 - 1)(6\zeta_0^2 - 6\zeta_0 + 1)(\lambda\phi)^3 + \tfrac{2}{3}(\zeta_0 - 1)(24\zeta_0^3 - 36\zeta_0^2 + 14\zeta_0 - 1)(\lambda\phi)^4 + \cdots, \tag{14}$$

wherein $\zeta_0 = \gamma^2(0)/\gamma_L^2$. Inserting this expansion into Eqs. (12) and (13) yields, after integration, the desired ϕ and θ functions:

$$\begin{aligned}\phi(\tau) = {}& \omega\tau + \lambda^2(1 - 6\zeta_0)\omega\tau + \lambda^3 6\zeta_0(\zeta_0 - 1)\omega^2\tau^2 + \lambda^4 2(40\zeta_0^2 - 20\zeta_0 + 1)\omega\tau - \lambda^4 4\zeta_0(\zeta_0 - 1)(2\zeta_0 - 1)\omega^3\tau^3 \\ & - \lambda^5 2\zeta_0(\zeta_0 - 1)(98\zeta_0 - 23)\omega^2\tau^2 + \lambda^5 2\zeta_0(\zeta_0 - 1)(6\zeta_0^2 - 6\zeta_0 + 1)\omega^4\tau^4 - \lambda^6(1568\zeta_0^3 - 1368\zeta_0^2 + 234\zeta_0 - 5)\omega\tau \\ & + \lambda^6 \tfrac{8}{3}\zeta_0(\zeta_0 - 1)(165\zeta_0^2 - 133\zeta_0 + 13)\omega^3\tau^3 - \lambda^6 \tfrac{4}{5}\zeta_0(\zeta_0 - 1)(24\zeta_0^3 - 36\zeta_0^2 + 14\zeta_0 - 1)\omega^5\tau^5 + \cdots\end{aligned} \tag{15}$$

and

$$\begin{aligned}\theta(\tau) = {}& \omega\tau - \lambda^2 2\zeta_0\omega\tau + \lambda^3 2\zeta_0(\zeta_0 - 1)\omega^2\tau^2 + \lambda^4 2\zeta_0(10\zeta_0 - 3)\omega\tau - \lambda^4 \tfrac{4}{3}\zeta_0(\zeta_0 - 1)(2\zeta_0 - 1)\omega^3\tau^3 \\ & - \lambda^5 4\zeta_0(\zeta_0 - 1)(13\zeta_0 - 2)\omega^2\tau^2 + \lambda^5 \tfrac{2}{3}\zeta_0(\zeta_0 - 1)(6\zeta_0^2 - 6\zeta_0 + 1)\omega^4\tau^4 - \lambda^6 4\zeta_0(88\zeta_0^2 - 60\zeta_0 + 5)\omega\tau \\ & + \lambda^6 \tfrac{4}{3}\zeta_0(\zeta_0 - 1)(90\zeta_0^2 - 68\zeta_0 + 5)\omega^3\tau^3 - \lambda^6 \tfrac{4}{15}\zeta_0(\zeta_0 - 1)(24\zeta_0^3 - 36\zeta_0^2 + 14\zeta_0 - 1)\omega^5\tau^5 + \cdots.\end{aligned} \tag{16}$$

It is clear that the solution presented in this section is exact to sixth order in the interaction constant and can without fundamental difficulty be extended to higher order. It is to be compared with previous solutions.[1,6,9-12]

III. LIMITS OF VALIDITY

Bhabha,[13] some years ago, put forth the idea that the physical solutions of the Lorentz-Dirac equation should be continuous functions of the interaction constant at the point where the value of this constant is zero. Insofar as the solution given in the previous section satisfies this criterion as λ tends toward zero, it is a physical solution.[14] However, the Taylor series for ζ/ζ_0, Eq. (14), used to obtain the expansions for the functions ϕ and θ in terms of the proper time will be valid only within the circle of convergence. If we write Eq. (5a) as

$$\zeta/\zeta_0=[\zeta_0-(\zeta_0-1)e^{-\rho}]^{-1}, \tag{17}$$

we find that this radius is

$$|\rho|=\ln[\zeta_0/(\zeta_0-1)] \tag{18}$$

and corresponds to a limitation on the variable

$$|\phi|<\frac{1}{2|\lambda|}\ln[\zeta_0/(\zeta_0-1)]. \tag{19}$$

When the initial total energy is very much greater than the longitudinal energy, that is, $\gamma(0)\gg\gamma_L$, this allowable range of the variable ϕ is approximately ($|\lambda|\approx 10^{-16}H$ for an electron) given by

$$|\phi|<5\times10^{15}\gamma_L{}^2/H\gamma^2(0). \tag{20}$$

In contrast, when $\gamma(0)\approx\gamma_L$, we have

$$|\phi|<2.5\times10^{14}/H([\gamma(0)/\gamma_L]-1). \tag{21}$$

IV. MOTION WITH LOW AND HIGH TRANSVERSE ENERGY

When the motion of the particle is primarily in the direction of the magnetic field, we see from Eq. (21) that the range of the variable ϕ is large, at least for physically achievable magnitudes of the magnetic field, and the motion can then be followed for many revolutions (large values of θ). It is interesting to point out that, in the special case of $\gamma(0)\approx 1$ and $\gamma_L=1$, our solution corresponds to the nonrelativistic solution given by Plass.[10]

In the alternative case of a particle having an initial energy very much larger than its longitudinal energy, that is, a large transverse energy, the situation is quite different. Equation (20) shows that the range of ϕ becomes small as $\gamma(0)$ increases in magnitude. Under this condition the particle motion can be followed by considering successive intervals no larger than this limit. However, within each of these intervals, the quantity $(2\lambda\zeta_0\phi)$ is less than one. Therefore, since $|\lambda|\ll 1$, the values of both ϕ and θ as given by Eqs. (15) and (16) become very nearly equal to $\omega\tau$. Under these circumstances, Eqs. (5a) and (5b) simplify to

$$\zeta/\zeta_0=[\zeta_0-(\zeta_0-1)\exp(-2\lambda\omega\tau)]^{-1} \tag{22}$$

and

$$\frac{w}{w_0}=\frac{\gamma(\tau)}{\gamma(0)}\exp[-(\lambda+i)\omega\tau]. \tag{23}$$

For a succession of intervals defined by the proper times $(\tau_1-\tau_0)$, $(\tau_2-\tau_1),\ldots,$ these last relationships can be expressed as

$$\left(1-\frac{\gamma_L{}^2}{\gamma(\tau_0)^2}\right)\exp(2\lambda\omega\tau_0)=\left(1-\frac{\gamma_L{}^2}{\gamma(\tau_1)^2}\right)\exp(2\lambda\omega\tau_1)=\cdots \tag{24}$$

and

$$\frac{w(\tau_0)}{\gamma(\tau_0)}\exp[-(\lambda+i)\omega\tau_0]=\frac{w(\tau_1)}{\gamma(\tau_1)}\exp[-(\lambda+i)\omega\tau_1]=\cdots. \tag{25}$$

It consequently follows that Eqs. (22) and (23) describe the motion over all the intervals with τ as the total elapsed proper time.

V. ENERGY CONSERVATION AND THE SCHOTT ENERGY

In this section we discuss briefly the conservation of energy characterizing the motion of a particle in a uniform magnetic field. To do this we first integrate Eq. (4c) from the initial proper time to a later time τ. Thereby we obtain

$$W_R(0,\tau)=\gamma(0)-\gamma(\tau)+W_S(0)-W_S(\tau), \tag{26}$$

where the radiated or Larmor energy is

$$W_R(0,\tau)=\frac{\lambda}{\omega}\int_0^\tau d\sigma\,\gamma(\sigma)\left(\frac{\dot w\dot w^*+\dot v_3\dot v_3}{c^2}-\dot\gamma^2\right), \tag{27}$$

while the acceleration or Schott energy at time τ is

$$W_S(\tau)=-\frac{\lambda}{\omega}\dot\gamma(\tau). \tag{28}$$

With the help of Eqs. (5a) and (12), this expression for the acceleration energy can be written in the form

$$W_S(\tau)=\lambda^2\gamma(\tau)\left(\frac{\gamma^2(\tau)}{\gamma_L{}^2}-1\right)\times\left[1+\lambda^2\left(1-6\frac{\gamma^2(\tau)}{\gamma_L{}^2}\right)+\cdots\right]. \tag{29}$$

It is evident that at any given time the Schott energy associated with a particle in a uniform field depends only on the total and longitudinal energies of the particle.

For the energy radiated over the entire motion, that is, until the particle has lost all its transverse energy, we write the relationship

$$W_R(0,\infty)\simeq[\gamma(0)-\gamma_L]\left(1+\lambda^2\frac{\gamma(0)[\gamma(0)+\gamma_L]}{\gamma_L{}^2}\right). \tag{30}$$

This expression is of interest since it shows that a general solution of the Lorentz-Dirac equation requires that a particle moving in a uniform magnetic field radiate an energy equal to its transverse kinetic energy, and in addition, the initial Schott energy. Though for realistic magnetic fields the additional energy is small, it is essential for the conservation of energy.[8,9,15]

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Observational Constraints Imposed by Brans-Dicke Cosmologies*

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The flat-space Brans-Dicke (BD) Friedmann cosmologies previously found are analyzed in more detail. Further exact relations among the observable quantities ρ, H, $\dot{\Lambda}$ (fractional time variation of G), t, and q (deceleration parameter) are found and subsequently used to discuss the over-all consistency of the cosmological solution with the observed values of these quantities and their associated uncertainties. It is found that consistency with observables is possible over almost the entire range of the solution parameter z from 1 to ∞. The greatest upper bound on $\dot{\Lambda}_0$ which is marginally consistent with the other observables ($t_{\text{galac}} = 7.3\times10^9$ yr and $t_H = 19.5\times10^7$ yr) is found to be $\dot{\Lambda} = 3.85\times10^{-11}$/yr (for $z\simeq1.4$) and corresponds to a density of $\rho = 1.95\times10^{-30}$ g/cm^3. Since in the $z\to1$ limit the curved- and flat-space solutions are identical, the above bound on $\dot{\Lambda}$ is also a reasonable one for curved space. In any case, the limiting values of $\dot{\Lambda}/H$ as $z\to1$ give a greatest upper bound on $\dot{\Lambda}$ for curved spaces, and one finds $\dot{\Lambda}_{\max} = 9.7\times10^{-11}$/yr for $t_H = 13\times10^9$ yr. The previous "upper bound" $\dot{\Lambda}\simeq10^{-11}$/yr (for flat space) found by using the $z\to\infty$ solution is therefore actually an upper bound in terms of ϵ only. Finally the values for the deceleration parameter are found to range from $q = 0.538$, 1.0 (for $\epsilon = 0, \frac{1}{3}$) to $q = 2.0$ (all ϵ) as z goes from ∞ to 1. For the case of marginal consistency with the other observables ($z\simeq1.4$) one finds $q = 1.42$, 1.52 for $\epsilon = 0, \frac{1}{3}$. This range of (flat-space) values for the deceleration parameter, consistent with other observables, indicates the nonexistence of a unique relation between the deceleration parameter and the sign of spatial curvature in the BD theory.

I. INTRODUCTION

In a previous paper[1] exact solutions to flat Friedmann-type Brans-Dicke[2] (BD) cosmologies were found and an upper bound[3] on $\dot{\Lambda}_0$, the fractional time variation of the gravitational constant at the present epoch, was obtained. The bound consisted merely in establishing a relation between $\dot{\Lambda}_0$ and t_0 (or H_0) at the present epoch. The analysis of the cosmological solution in Ref. 3 was incomplete in several respects. We wish to present here a more detailed analysis which will bring to light certain features of the solution not apparent from the previous work.

First of all, we find some new exact relations which exist among the observables ρ_0, H_0, $\dot{\Lambda}_0$,