Bandyopadhyay's work concerns the annihilation process $e^- + e^+ \to \nu_e + \bar{\nu}_e$ and also allows the process $\mu^- + \mu^+ \to \nu_\mu + \bar{\nu}_\mu$. Correspondingly, from his prediction for the coupling strength we would find $g_\mu{}^2 = g_e{}^2(m_\mu/m_e)^4 \approx 2\times10^{-13}$ which is a factor of hundred larger than our result.

We mention that the photon-neutrino-antineutrino coupling has some interesting features and in the future maybe the decay $\rho^0 \to \nu_\mu \bar{\nu}_\mu$ is possible to be observed since it is more likely to occur than $\rho^0 \to \nu_e \bar{\nu}_e$ although the branching ratio for this process would be very small due to the very weak coupling strength.

---

†Work supported in part by an Arizona State University Faculty research grant-in-aid.

[1]P. Bandyopadhyay and P. Chaudhuri, Phys. Rev. D 3, 1378 (1971).

[2]P. Bandyopadhyay, Phys. Rev. 173, 1481 (1968); Nuovo Cimento 55A, 367 (1968).

[3]J. Bernstein, M. Ruderman, and G. Feinberg, Phys. Rev. 132, 1227 (1963).

[4]R. E. Marshak, Riazuddin, and C. P. Ryan, *Theory of Weak Interactions in Particle Physics* (Wiley-Interscience, New York, 1969), pp. 233, 706.

[5]M. G. Hare, Lett. Nuovo Cimento 4, 693 (1972).

[6]T. D. Lee, Phys. Rev. Letters 26, 801 (1971).

PHYSICAL REVIEW D VOLUME 7, NUMBER 5 1 MARCH 1973

Extension of a Bootstrap Model of Inclusive Reactions*

Richard Kronenfeld† and R. D. Peccei

Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California 94305

(Received 25 August 1972)

We extend the bootstrap model of inclusive reactions proposed by Finkelstein and Peccei and by Krzywicki and Petersson to include quantum numbers. In this extended model we obtain matrix integral equations for the inclusive distribution functions in terms of leading-particle distribution functions. The single-particle inclusive distributions in the central region are found, in the model, to be independent of the process. Furthermore, if the target (projectile) leading-particle distributions are independent of the projectile (target), then all inclusive distributions in the target (projectile) fragmentation region are independent of the projectile (target), and multiparticle inclusive distributions exhibit factorization properties among particles traveling in opposite directions. An example involving the scattering of mesons and baryons is discussed.

Recently a bootstrap model of inclusive reactions was proposed[1] in which one envisaged particle production processes as occurring via the formation of a leading particle and a fireball. The bootstrap hypothesis entered in assuming that the fireball decay distribution, in its own rest system, was the same as the over-all distribution in the c.m. system save for the fact that what played the role of s, the square of the c.m. energy, for the fireball decay was the fireball invariant mass squared, M^2. In the scaling limit, this bootstrap hypothesis leads to integral equations for the inclusive distribution functions in terms of an (unknown) leading-particle distribution. Knowledge of the single-particle inclusive distribution, however, permits in principle the elimination of the leading-particle distribution from the multiparticle distribution equations, thus effectively determining these functions in terms of the single-particle spectrum.

We should comment briefly on the physical motivation for the bootstrap hypothesis. The assumption made is that the distribution of produced particles is so chaotic that if one subtracts out the leading-particle contribution, what remains – the fireball – has a distribution which, apart from a scaling down in energy, is just like the over-all distribution. We should also note that the leading-particle distribution extends down to $x=0$, although it vanishes there, and thus what we call the leading particle may not be in actuality "leading" in all events. This precludes an experimental definition of this quantity, but a precise meaning for this distribution can be given in models.[2]

For simplicity, in Ref. 1 a world of only one kind of particle was considered. This, of course, permitted only a qualitative comparison of the model with experiment. In this paper we would like to partly remedy this situation by extending the model to a more realistic case in which particles of different quantum numbers are present. This extension not only makes the model more amenable to direct experimental test, but also yields several interesting predictions which could not be abstracted from the simple model of Ref. 1.

Let us consider an initial process in the c.m. system which is determined by s and the (additive) quantum number of the initial particles. We shall use Q_L (Q_R) as a shorthand notation for the collection of quantum numbers belonging to the particle which is incident on the left (right), and shall find it convenient to denote the process itself by $[Q_L, Q_R]_s$. All events are taken to proceed via the formation of a leading particle and a fireball, with the leading particle carrying quantum numbers q_l to the left (or, alternatively, q_r to the right). Again here q_l (q_r) are to be understood as shorthand for the collection of quantum numbers of the left (right) leading particles. The most natural way to extend the bootstrap hypothesis is to consider that the fireball decay distribution, in its rest system, is the same as the over-all distribution in the c.m. system of a process, at energy squared M^2, in which the quantum numbers of the left and right impinging particles are respectively $Q_L - q_l$ and Q_R (or alternatively Q_L and $Q_R - q_r$). Thus we may write, symbolically,

$$[Q_L, Q_R]_s = \sum_l (q_l + [Q_L - q_l, Q_R]_{M^2}) + \sum_r ([Q_L, Q_R - q_r]_{M^2} + q_r). \tag{1}$$

Generally speaking, it will be necessary to impose certain restrictions on what kind of leading particles are allowed to be produced in a given process so that the problem remains manageable. An example of this will be illustrated with a simple model below.

Let us denote by $N_i^{ab}(p, s)$ the single-particle inclusive distribution of particle i in the ab process:

$$N_i^{ab}(p, s) = \frac{1}{\sigma_{\text{tot}}^{ab}} \frac{d\sigma_i^{ab}}{dp}; \quad dp \equiv \frac{d^3p}{E}. \tag{2}$$

Let $L_i^{ab}(p, s)$ and $R_i^{ab}(p, s)$ be, respectively, the left and right leading-particle distributions of a particle of type i. Because there is only one leading particle per event we have

$$\sum_i \int dp [L_i^{ab}(p, s) + R_i^{ab}(p, s)] = 1. \tag{3}$$

From Eq. (1) and the bootstrap hypothesis, it follows, just as in Ref. 1, that

$$N_i^{ab}(p, s) = L_i^{ab}(p, s) + R_i^{ab}(p, s) + \sum_j \int dp' \, L_j^{ab}(p', s) N_i^{a-j\, b}(\Lambda_{p'} p, M^2) + \sum_k \int dp' \, N_i^{a\, b-k}(\Lambda_{p'} p, M^2) R_k^{ab}(p', s). \tag{4}$$

Here $\Lambda_{p'}$ is the Lorentz transformation that takes one from the c.m. frame to the fireball frame, and the notation $N_i^{a\, b-k}$ signifies that this is the inclusive distribution for the production of particles of type i in a reaction where the initial quantum numbers are those of a impinging on the left and those of $b-k$ impinging on the right.

Equation (4) simplifies considerably in the scaling limit and if we integrate over the (limited) transverse momentum. Since particle masses are unimportant in this limit for the kinematics, one obtains precisely the same kind of equation as in Ref. 1 save for the quantum-number indices. Let us define

$$f_i^{ab}(x) = \int d^2p_\perp N_i^{ab}(p_\perp, x),$$
$$l_i^{ab}(x) = \int d^2p_\perp L_i^{ab}(p_\perp, x), \tag{5}$$
$$r_i^{ab}(x) = \int d^2p_\perp R_i^{ab}(p_\perp, x),$$

where by hypothesis $l_i^{ab}(x \geq 0) = r_i^{ab}(x \leq 0) = 0$. Then we can write Eq. (4) in the scaling limit and integrated over transverse momentum as

$$f_i^{ab}(x) = l_i^{ab}(x) + r_i^{ab}(x) + \sum_j \int_{-1}^{0} \frac{dy}{|y|} l_j^{ab}(y) f_i^{a-j\, b}(z(x, y)) + \sum_k \int_0^1 \frac{dy}{y} f_i^{a\, b-k}(z(x, y)) r_k^{ab}(y), \tag{6}$$

where

$$z(x,y)=\begin{cases} x & \text{if } xy<0 \\ \dfrac{x}{1-|y|} & \text{if } xy>0 . \end{cases} \qquad (7)$$

Equation (6) provides the desired generalization of the bootstrap model to the case in which quantum numbers are included. As can be seen the net effect of the introduction of quantum numbers has been to give a set of coupled integral equations for the inclusive distributions $f_i^{ab}(x)$ without altering the analytic structure of the model.

An immediate consequence of Eq. (6) can be obtained when $x=0$. In this case the integral equations reduce to algebraic ones for the $f_i^{ab}(0)$:

$$f_i^{ab}(0)=\sum_j\left[\int_{-1}^{0}\frac{dy}{|y|}\,l_j^{ab}(y)\right]f_i^{a-j\,b}(0)+\sum_k f_i^{a\,b-k}\left[\int_0^1\frac{dy}{y}\,r_k^{ab}(y)\right], \qquad (8)$$

or in matrix notation

$$A(l,r)f_i(0)=0. \qquad (9)$$

The matrix $A(l, r)$ has determinant zero since the sum of each of its rows vanishes due to the normalization condition

$$\sum_j\int_{-1}^{0}\frac{dy}{|y|}\,l_j^{ab}(y)+\sum_k\int_0^1\frac{dy}{y}\,r_k^{ab}(y)=1. \qquad (10)$$

Furthermore, if $A(l, r)$ is an $n\times n$ matrix, in general its rank will be $n-1$. Hence Eq. (9) has a unique solution in terms of an arbitrary parameter. Because the sum of the rows of $A(l, r)$ vanishes it follows that this solution is just

$$f_i^{ab}(0)=c_i. \qquad (11)$$

Thus the value of the single-particle distributions at $x=0$ is independent of the process in this model. Such a result is typical of an analysis of inclusive reactions along the lines of Mueller with a factorizable Pomeranchuk singularity.[3]

A related factorization property of $f_i^{ab}(x)$ can be proven provided one makes a very plausible assumption for the leading-particle distribution functions. If $l_i^{ab}(x)=l_i^a(x)$ and $r_i^{ab}(x)=r_i^b(x)$ then it follows that $f_i^{ab}(x)$ is independent of b for $x\le 0$ and independent of a for $x\ge 0$. That is, provided that the target (projectile) leading-particle distributions are independent of the nature of the projectile (target), the inclusive distribution in the target (projectile) fragmentation region will also be independent of the nature of the projectile (target). To prove this assertion it is convenient to recast Eq. (6) into a set of algebraic equations by introducing the transform[4]

$$\hat f_i^{ab}(\lambda)=\int_0^1\frac{dx}{x}\,x^\lambda f_i^{ab}(x). \qquad (12)$$

Defining

$$h_{r\,i}^{b}(x)=\frac{x r_i^b(1-x)}{1-x}, \qquad (13)$$

we have

$$\hat f_i^{ab}(\lambda)=\hat r_i^b(\lambda)+\sum_j\left[\int_{-1}^{0}\frac{dy}{|y|}\,l_j^a(y)\right]\hat f_i^{a-j\,b}(\lambda)+\sum_k \hat f_i^{a\,b-k}(\lambda)\,\hat h_{r\,i}^{b}(\lambda). \qquad (14)$$

Since

$$\sum_j\int_{-1}^{0}\frac{dy}{|y|}\,l_j^a(y)=\tfrac{1}{2}$$

is independent of a, it is clear that a solution of the above equations is

$$\hat f_i^{ab}(\lambda)=\hat f_i^b(\lambda). \qquad (15)$$

However, since the above equations have a unique solution, Eq. (15) is the *only* solution proving the desired factorization property.

Multiparticle equations can be obtained in an analogous way as in Ref. 1, the introduction of quantum numbers merely making these equations matrix equations. As an example we write below the equation satisfied by the two-particle distribution function $f_{ij}^{ab}(x_i, x_j)$ in the case where $x_i<0$, $x_j>0$:

$$f_{ij}^{ab}(x_i,x_j)=l_i^{ab}(x_i)f_j^{ab}(x_j)+f_i^{ab}(x_i)r_j^{ab}(x_j)+\sum_k\int_{-1}^{0}\frac{dy}{|y|}\,l_k^{ab}(y)f_{ij}^{a-k\,b}\left(\frac{x_i}{1-|y|},x_j\right)+\sum_m\int_0^1\frac{dy}{y}f_{ij}^{a\,b-m}\left(x_i,\frac{x_j}{1-y}\right)r_m^{ab}(y). \qquad (16)$$

If one assumes that the leading-particle distributions factorize, then $f_{ij}^{ab}(x_i, x_j)$ does also:

$$f_{ij}^{ab}(x_i,x_j)=f_i^a(x_i)f_j^b(x_j) \qquad (x_i<0,\ x_j>0). \qquad (17)$$

To prove Eq. (17) it is again convenient to go to the transform space where Eq. (16) reads, in matrix notation,

$$A_{ij}^{ab,cd}(\lambda_i,\lambda_j)\hat{f}_{ij}^{cd}(\lambda_i,\lambda_j)=\hat{l}_i^a(\lambda_i)\hat{f}_j^b(\lambda_j)+\hat{f}_i^a(\lambda_i)\hat{r}_j^b(\lambda_j)\,, \tag{18}$$

with

$$\begin{aligned}A_{ij}^{ab,cd}(\lambda_i,\lambda_j)&=\delta^{ac}\delta^{bd}-\sum_k \hat{h}_{lk}^a(\lambda_i)\delta^{a-k\,c}\delta^{bd}-\sum_m \hat{h}_{rm}^b(\lambda_j)\delta^{ac}\delta^{b-m\,d}\\&=A_l^{ac}(\lambda_i)\delta^{bd}+\delta^{ac}A_r^{bd}(\lambda_j)\,.\end{aligned} \tag{19}$$

Here the matrices $A_l^{ac}(\lambda_i)$ and $A_r^{bd}(\lambda_j)$ are precisely the ones that enter into the equations for the single-particle distributions;

$$\begin{aligned}&A_l^{ac}(\lambda_i)\hat{f}_i^c(\lambda_i)=\hat{l}_i^a(\lambda_i)\,,\\&A_r^{bd}(\lambda_j)\hat{f}_i^d(\lambda_j)=\hat{r}_j^b(\lambda_j)\,.\end{aligned} \tag{20}$$

From the above it follows immediately that

$$\hat{f}_{ij}^{cd}(\lambda_i,\lambda_j)=\hat{f}_i^c(\lambda_i)\hat{f}_j^d(\lambda_j)\,, \tag{21}$$

which, on transforming back to x space, proves the stated factorization property, Eq. (17). Similar factorization properties can be established for multiparticle inclusive distributions among particles traveling in opposite directions.

To conclude, we shall illustrate the model by considering meson-baryon scattering. We shall consider a simplified problem where there is only one kind of meson, one kind of baryon, and one kind of antibaryon; they are denoted by M, B, and $\bar{B}$, respectively. It should be clear that this is not a necessary restriction since in principle the model can handle an arbitrary number of particles. To make the problem tractable we shall impose as a reasonable physical constraint that no leading antibaryons (baryons) can be produced by an incident baryon (antibaryon).[5] In this case the only nonvanishing right leading-particle distributions are r_M^M, r_B^M, $r_{\bar{B}}^M$; r_M^B, r_B^B; $r_M^{\bar{B}}$, $r_{\bar{B}}^{\bar{B}}$. These are not all independent, since by baryon-antibaryon conjugation one has

$$r_{\bar{B}}^M=r_B^M,\quad r_B^B=r_{\bar{B}}^{\bar{B}};\quad r_M^B=r_M^{\bar{B}}\,. \tag{22}$$

It should be clear that if we had more than one baryon, say p and n, we would have to consider leading-particle distributions like r_p^p and r_n^p. These in general would not be the same and would lead to distinct bootstrap equations. The relevant equations for the problem, written in the notation of Eq. (1), are

$$\begin{aligned}&[B,B]_s=(B+[M,B]_{M^2})+(M+[B,B]_{M^2})+([B,M]_{M^2}+B)+([B,B]_{M^2}+M)\,,\\&[B,M]_s=(B+[M,M]_{M^2})+(M+[B,M]_{M^2})+([B,B]_{M^2}+\bar{B})+([B,M]_{M^2}+M)+([B,\bar{B}]_{M^2}+B)\,,\\&[B,\bar{B}]_s=(B+[M,\bar{B}]_{M^2})+(M+[B,\bar{B}]_{M^2})+([B,M]_{M^2}+\bar{B})+([B,\bar{B}]_{M^2}+M)\,,\\&[M,M]_s=(\bar{B}+[B,M]_{M^2})+(M+[M,M]_{M^2})+(B+[\bar{B},M]_{M^2})+([M,B]_{M^2}+\bar{B})+([M,M]_{M^2}+M)+([M,\bar{B}]_{M^2}+B)\end{aligned} \tag{23}$$

plus five other equations which can be obtained from the above by the substitution $B\leftrightarrow\bar{B}$ and/or the interchange of the right and left incident particles.

It is straightforward to obtain a solution of Eqs. (23) in the transform space. One finds

$$\begin{aligned}&\hat{f}_i^M(\lambda)=\{M_i(\lambda)[\tfrac{1}{2}-\hat{h}_{rM}^B(\lambda)]+[B_i(\lambda)+\bar{B}_i(\lambda)]\hat{h}_{rB}^M(\lambda)\}\Delta^{-1}(\lambda)\,,\\&\hat{f}_i^B(\lambda)=(B_i(\lambda)\{[\tfrac{1}{2}-\hat{h}_{rM}^M(\lambda)][\tfrac{1}{2}-\hat{h}_{rM}^B(\lambda)]-\hat{h}_{rB}^M(\lambda)\hat{h}_{rB}^B(\lambda)\}\\&\qquad+M_i(\lambda)[\tfrac{1}{2}-\hat{h}_{rM}^B(\lambda)]h_{rB}^B(\lambda)+\bar{B}_i(\lambda)\hat{h}_{rB}^M(\lambda)\hat{h}_{rB}^B(\lambda))\Delta^{-1}(\lambda)[\tfrac{1}{2}-\hat{h}_{rM}^B(\lambda)]^{-1}\,,\\&\hat{f}_i^{\bar{B}}(\lambda)\leftrightarrow\hat{f}_i^B(\lambda)\quad\text{with } B_i(\lambda)\leftrightarrow\bar{B}_i(\lambda)\,,\end{aligned} \tag{24}$$

where

$$\Delta(\lambda)=\{[\tfrac{1}{2}-\hat{h}^{M}_{rM}(\lambda)][\tfrac{1}{2}-\hat{h}^{B}_{rM}(\lambda)]-2\hat{h}^{M}_{rB}(\lambda)\hat{h}^{B}_{rB}(\lambda)\}, \quad (25)$$

and

$$B_i(\lambda)=\begin{cases}\hat{r}^B_B(\lambda), & i=B\\ \hat{r}^B_M(\lambda), & i=M\\ 0, & i=\bar{B},\end{cases}$$

$$\bar{B}_i(\lambda)=\begin{cases}0, & i=B\\ \hat{r}^B_M(\lambda), & i=M\\ \hat{r}^B_B(\lambda), & i=\bar{B},\end{cases} \quad (26)$$

$$M_i(\lambda)=\begin{cases}\hat{r}^M_B(\lambda), & i=B\\ \hat{r}^M_M(\lambda), & i=M\\ \hat{r}^M_B(\lambda), & i=\bar{B}.\end{cases}$$

Since at $\lambda=0$

$$\begin{aligned}\hat{h}^B_{rM}(0)&=\hat{r}^B_M(0)=\omega^B_M,\\ \hat{h}^B_{rB}(0)&=\hat{r}^B_B(0)=\omega^B_B,\\ \hat{h}^M_{rM}(0)&=\hat{r}^M_M(0)=\omega^M_M,\\ \hat{h}^M_{rB}(0)&=\hat{r}^M_B(0)=\omega^M_B,\end{aligned} \quad (27)$$

with the normalization condition, Eq. (10), implying

$$\begin{aligned}\omega^B_M+\omega^B_B&=\tfrac{1}{2},\\ \omega^M_M+2\omega^M_B&=\tfrac{1}{2},\end{aligned} \quad (28)$$

it follows that all $\hat{f}(\lambda)$ have a pole at $\lambda=0$ since $\Delta(\lambda)$ vanishes. As can be checked, the residue of this pole is independent of the process, confirming that indeed $f^{ab}_i(x=0)$ is independent of a and b. A straightforward calculation gives

$$\begin{aligned}f_M(x=0)&=(2\omega^B_M\omega^M_B+\omega^M_M\omega^B_B)R^{-1},\\ f_B(x=0)&=f_{\bar{B}}(x=0)\\ &=(2\omega^B_B\omega^M_B)R^{-1},\end{aligned} \quad (29)$$

where

$$R=-\left\{\omega^B_B\int_0^1\frac{dx}{x}[r^M_M(x)+2r^M_B(x)]\ln(1-x)+2\omega^M_B\int_0^1\frac{dx}{x}[r^B_M(x)+r^B_B(x)]\ln(1-x)\right\}. \quad (30)$$

The fact that $f_B(0)=f_{\bar{B}}(0)$ is not surprising since it merely reflects baryon conservation.[6]

The ratio of meson production to baryon production at $x=0$ in the model is

$$R\left(\frac{M}{B}\right)=\frac{\omega^M_M\omega^B_B+2\omega^B_M\omega^M_B}{2\omega^B_B\omega^M_B}. \quad (31)$$

Experimentally it is well known that this ratio is large. This fact can be easily obtained in the model by requiring, rather naturally, that the probability of a baryon being a leading particle in a meson-induced reaction and the probability of a meson being a leading particle in a baryon-induced reaction be small.[7] In this case

$$R\left(\frac{M}{B}\right)\simeq\frac{\omega^M_M}{2\omega^M_B}=\frac{\omega^M_M}{\frac{1}{2}-\omega^M_M}\gg 1. \quad (32)$$

We would like to thank J. Finkelstein for much useful advice and encouragement and for reading the manuscript. Part of this work was developed by one of us (R. D. P.) at the Aspen Center for Physics and he would like to thank the Center for its hospitality.

---

*Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under AFOSR Contract No. F44620-71-C-0044.

†National Science Foundation Predoctoral Fellow.

[1]J. Finkelstein and R. D. Peccei, Phys. Rev. D 6, 2606 (1972); A. Krzywicki and B. Petersson, *ibid.* 6, 924 (1972); B. Petersson, in *Proceedings of the Seventh Rencontre de Moriond,* edited by J. Tran Thanh Van (CNRS, Paris, 1972).

[2]In a multiperipheral model, what we call the leading-particle distribution would correspond to the contribution of end diagrams. See, for example, N. F. Bali, A. Pignotti, and D. Steele, Phys. Rev. D 3, 1167 (1971).

[3]A. H. Mueller, Phys. Rev. D 2, 2963 (1970).

[4]We consider for definiteness $x\geq 0$.

[5]This restriction is necessary if we want to deal only with processes involving particles which do not have exotic quantum numbers, and is one which we would impose in general.

[6]The condition $f_B(0)=f_{\bar{B}}(0)$ follows in general and does not depend on our having made use of baryon-antibaryon conjugation for the leading-particle distributions.

[7]This ratio can also be made large if $\omega^B_M\gg\omega^B_B$ and $\omega^M_B\gg\omega^M_M$, but this is a rather unnatural solution.