Helicity Conservation for Pion-Nucleon Scattering in the Brick-Wall Coordinate System

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It is shown that present data on πN elastic scattering are consistent with asymptotic (with energy) helicity conservation in the nucleon brick-wall coordinate system.

A great deal of interest has been generated recently in the question of helicity conservation in elastic and diffractive processes.[1] For mesonic reactions, helicity seems to be conserved in the direct (s) channel for γ-ρ^0 transitions,[2] but for[3] π-A_1 and[4] K-Q transitions the conservation seems to be in the t-channel frame. High-energy model amplitudes for πN scattering are consistent with nucleon helicity conservation in the direct channel.[5] Low-energy phase-shift and resonance data are also consistent with this via finite-energy sum rules[5] (FESR) and unsubtracted dispersion relations.[6] A recent model-independent amplitude analysis of πN scattering at 6 GeV/c has revealed that the ratio of s-channel helicity-flip to helicity-nonflip amplitudes is small but nonzero.[7] This brings up the question of energy dependence, i.e., is s-channel helicity conservation valid asymptotically? Of course, this can only be answered definitely with measurements of polarization parameters at higher energies. Here we seek answers to two possible alternatives. (a) Is there a coordinate system in which helicity is conserved at finite energies? (b) Is there a coordinate system other than the s-channel center-of-mass system in which helicity is conserved asymptotically, and is consistent with the data at finite energy? The answer to (a) is yes, but the resulting coordinate system lacks any simple physical interpretation and is probably energy-dependent. The answer to (b) is also yes, and is the subject of this paper.

We use the notation of Barger and Halzen.[8] The quantity of interest is

$$\alpha \equiv \frac{2M}{\sqrt{-t}}\,\frac{F^0_{+-}}{F^0_{++}}\,, \tag{1}$$

where the F's are s-channel center-of-mass helicity amplitudes, M the nucleon mass, t the invariant momentum transfer, and the superscript 0 indicates isospin zero in the crossed (t) channel. The amplitude analysis at[7] 6 GeV/c reveals that $\mathrm{Re}\,\alpha \approx -0.4 \pm 0.2$ for $|t| \leqslant 0.625$ (GeV/c)^2, and $\mathrm{Im}\,\alpha$ smaller by a factor of 5 to 7. Barger and Halzen[5] have noted that polarization data alone give some information on helicity conservation, and hence some energy dependence can be obtained. They consider the combination

$$\Sigma' P \equiv P^+\frac{d\sigma^+}{dt} + P^-\frac{d\sigma^-}{dt} - P^0\frac{d\sigma^0}{dt} = 4|F^0_{+-}|\,|F^0_{++}|\sin(\phi^0_{++}-\phi^0_{+-})\,, \tag{2}$$

where the $\pm, 0$ refer to elastic $\pi^\pm p$ and $\pi^- p \to \pi^0 n$ reactions. This is directly related to the ratio of interest

$$\Sigma' P = -4\left(\frac{-t}{4M^2}\right)^{1/2} |F^0_{++}|^2\,\mathrm{Im}\,\alpha\,. \tag{3}$$

The data for laboratory momentum between 2 and 11 GeV/c show a sharp decrease of $\Sigma' P$ at fixed t, like P_{lab}^{-2}. This is evidence for either decrease of $|F^0_{+-}|$ (s-channel helicity conservation) or decrease of $\sin(\phi^0_{++}-\phi^0_{+-})$ (equal phases for both helicity amplitudes). FESR determinations of the phase angles[8] indicate that $\sin(\phi^0_{++}-\phi^0_{+-})$ does not decrease as fast as P_{lab}^{-2} in the energy range considered, so that this could be evidence for asymptotic s-channel helicity conservation. However, in the energy range from 5 to 11 GeV/c the same quantity is consistent with P_{lab}^{-2} energy dependence, so that $|F^0_{+-}|$ need not decrease in this energy region. In any event, the large contribution of F^0_{+-} is in $\mathrm{Re}\,\alpha$, and the energy dependence of this quantity is only available from a complete amplitude analysis, requiring R- or A-parameter measurements at high energy.

We now investigate question (a). Let the helicity amplitudes in the new coordinate system be $G_{\lambda\mu}$. The crossing relation between these amplitudes and the s-channel center-of-mass amplitudes is

$$G_{\lambda\mu} = \sum_{\lambda',\mu'=\pm\frac{1}{2}} d^{1/2}_{\lambda\lambda'}(X_1)\,d^{1/2}_{\mu\mu'}(X_2)\,F_{\lambda'\mu'}\,, \tag{4}$$

where X_1 and X_2 are the crossing angles to be determined later. If helicity is to be conserved in the new system, $G_{+-} \equiv 0$, which implies

$$\frac{F_{+-}}{F_{++}} = \tan\frac{X_1 - X_2}{2}\,. \tag{5}$$

We restrict ourselves to new coordinate systems which are reached by a Lorentz transformation characterized by β along a direction at an angle ψ

with respect to the incoming pion direction and in the plane of the reaction. Let θ be the s-channel scattering angle, and q_1, p_1, q_2, p_2 be the four-momenta of the particles $\pi N \to \pi N$, with the corresponding primed quantities in the new system. Then X_1 is the angle between $\vec{q}_1'$ and $\vec{p}_2'$ in the rest frame of p_1, and X_2 is the angle between $\vec{q}_2'$ and $\vec{p}_2'$ in the rest frame of p_2. Then one can make the required transformations and calculate

$$\cot X_1 = \frac{K}{M\beta \sin\psi} - \frac{E}{M}\cot\psi \tag{6}$$

and

$$\cot X_2 = \frac{-K}{M\beta\sin(\theta-\psi)} + \frac{E}{M}\cot(\theta-\psi), \tag{7}$$

where M is the nucleon mass and K and E are the momentum and energy of a nucleon in the s-channel center-of-mass system. It is obvious from (5), (6), and (7) that a given value of α does not give a unique coordinate system specified by a single pair of values (β, ψ) for which $G_{+-} \equiv 0$. (Note that G_{+-} cannot be identically zero unless F_{+-} and F_{++} have the same phase. We use the real part of α since it is much larger than the imaginary part at 6 GeV/c.) Conversely, for certain values of α, θ, t, and K there may be no solution. Unfortunately the complexity of the equations does not allow an analytic solution. Let us first try $\psi = 0$, i.e., transform along the incoming particle direction. One gets $X_1 = 0$, and from (5) and (7) the relation

$$\beta = \frac{K}{\sin\theta}\,\frac{1}{E\cot\theta - \frac{1}{2}M(F_{+-}/F_{++} - F_{++}/F_{+-})}$$

$$= \left\{\left(1+\frac{M^2}{K^2}\right)^{1/2}\left(1+\frac{t}{2K^2}\right) - \frac{M}{2K}\left(\frac{-t}{K^2}\right)^{1/2}\left[\left(\frac{4M^2}{-t}\right)^{1/2}\frac{1}{\alpha} - \alpha\left(\frac{-t}{4M^2}\right)^{1/2}\right]\right\}^{-1}. \tag{8}$$

If we use $\alpha = -0.4$ at $P_{\text{lab}} = 6$ GeV/c, we find

$$\beta \approx [1 + (t + 9M^2)/2K^2]^{-1}$$

which varies from 0.41 to 0.435 for $|t| < 0.6$ (GeV/c)^2 and corresponds to a ratio of incoming pion to nucleon momenta $|K_\pi/K_p|$ ranging from 2.73 to 2.98. Thus a helicity-conserving coordinate system can be found at this energy. However, there seems to be no other particular significance to this system.[9] In fact, one can make the same transformation along the outgoing pion direction $\psi = \theta$, to get another helicity-conservation frame. It is interesting, however, to consider the symmetric case where $\psi = \frac{1}{2}\theta$. This leads to $\cot X_1 = -\cot X_2$ and $\tan\frac{1}{2}(X_1 - X_2) = -\cot X_1$, or

$$\beta = \left[\frac{\alpha t}{4K^2} + \frac{E}{K}\left(1+\frac{t}{4K^2}\right)^{1/2}\right]^{-1}$$

which varies from 0.855 to 0.862 over the t range considered before at 6 GeV/c, with $\alpha = -0.4$. This particular frame is interesting because it is close to the so-called brick-wall frame, in which the initial and final nucleon momenta are equal and opposite. The transformation to this frame from the s-channel center-of-mass frame requires $\psi = \frac{1}{2}\theta$ and

$$\beta = \frac{K}{E}\left(1+\frac{t}{4K^2}\right)^{1/2}.$$

This β varies from 0.855 to 0.833 over the same t range and energy as before, so it does not coincide exactly with a helicity-conservation frame. Conversely, if one *assumes* helicity conservation in the brick-wall frame, this implies

$$\alpha = -\left(1 + \frac{4M^2 - t}{8K^2}\right),$$

which of course disagrees with the measured value of -0.4 at 6 GeV/c. Here we are faced with the situation of being able to find helicity-conserving frames with no apparent physical significance, and physically simple frames (s-channel center-of-mass, brick-wall) in which helicity is not conserved at finite energy. If one believes that s-channel helicity is conserved asymptotically,[10] then the deviation of α from zero at finite energy must be due to nonasymptotic terms. Hence a Regge model may be postulated in which the asymptotic term (Pomeranchukon exchange) conserves s-channel helicity with the nonasymptotic term (P' or f exchange) producing the deviation.[11] Similarly, one could assume asymptotic helicity conservation in the brick-wall frame, and the deviation of α (from -1, in this case) due to nonasymptotic terms. Certainly the data on Reα at 6 GeV/c does not seem compelling for one case or the other. To this end we parametrize the s-channel center-of-mass helicity amplitudes in the form

$$F^0_{++} = f_P + f_f, \tag{9}$$

$$F^0_{+-} = \alpha_P f_P e^{i\phi_P} + \alpha_f f_f e^{i\phi_f}, \tag{10}$$

where the two terms are the asymptotic and nonasymptotic terms for isospin zero in the s-channel. We have not assumed a pure pole model, but have allowed phase variations ϕ_P and ϕ_f between the different helicity amplitudes. We parametrize the ratio

$$\frac{f_f}{f_P} = \epsilon e^{i\theta} , \tag{11}$$

where ϵ is fixed by the energy dependence of the total cross-section sum $\sigma(\pi^+p)+\sigma(\pi^-p)$. Using data from 3 to 30 GeV/c, we find that [using $\theta = \frac{1}{2}\pi(\alpha_f - \alpha_P) \approx \frac{1}{4}\pi$] $\epsilon \approx 0.4$. We use this value with an energy dependence of $s^{\alpha_f - \alpha_P} \approx s^{-1/2}$ in all further calculations. The quantities of interest are

$$\mathrm{Re}\,\alpha = \alpha_P \frac{\cos\phi_P + \epsilon[\cos(\theta-\phi_P) + (\alpha_f/\alpha_P)\cos(\theta+\phi_f)] + \epsilon^2(\alpha_f/\alpha_P)\cos\phi_f}{1+2\epsilon\cos\theta+\epsilon^2} , \tag{12}$$

$$\mathrm{Im}\,\alpha = \alpha_P \frac{\sin\phi_P + \epsilon[(\alpha_f/\alpha_P)\sin(\theta+\phi_f) - \sin(\theta-\phi_P)] + \epsilon^2(\alpha_f/\alpha_P)\sin\phi_f}{1+2\epsilon\cos\theta+\epsilon^2} . \tag{13}$$

We then fit the data on Reα at 6 GeV/c and Imα from 2.5 to 11 GeV/c, using the value of α_P appropriate for asymptotic helicity conservation in the brick-wall frame. The adjustable parameters are α_f, ϕ_P, ϕ_f, and θ. Although the energy dependence of Imα constrains the parameters considerably, it is not enough to give unique values. We impose the additional constraints that the phase angles ϕ_P, ϕ_f, and θ are as close as possible to their asymptotic value (0, 0, and $\frac{1}{4}\pi$), and that they have at most logarithmic energy dependence. The values so obtained are

$$\theta = \tfrac{1}{4}\pi - \frac{1.0}{(\ln s)^{3/2}} , \tag{14a}$$

$$\phi_P = \frac{1.3}{\ln s + 1.5} , \tag{14b}$$

$$\phi_f = -\frac{0.52}{\ln s + 1.1} . \tag{14c}$$

Note that the values are typically less than 25°

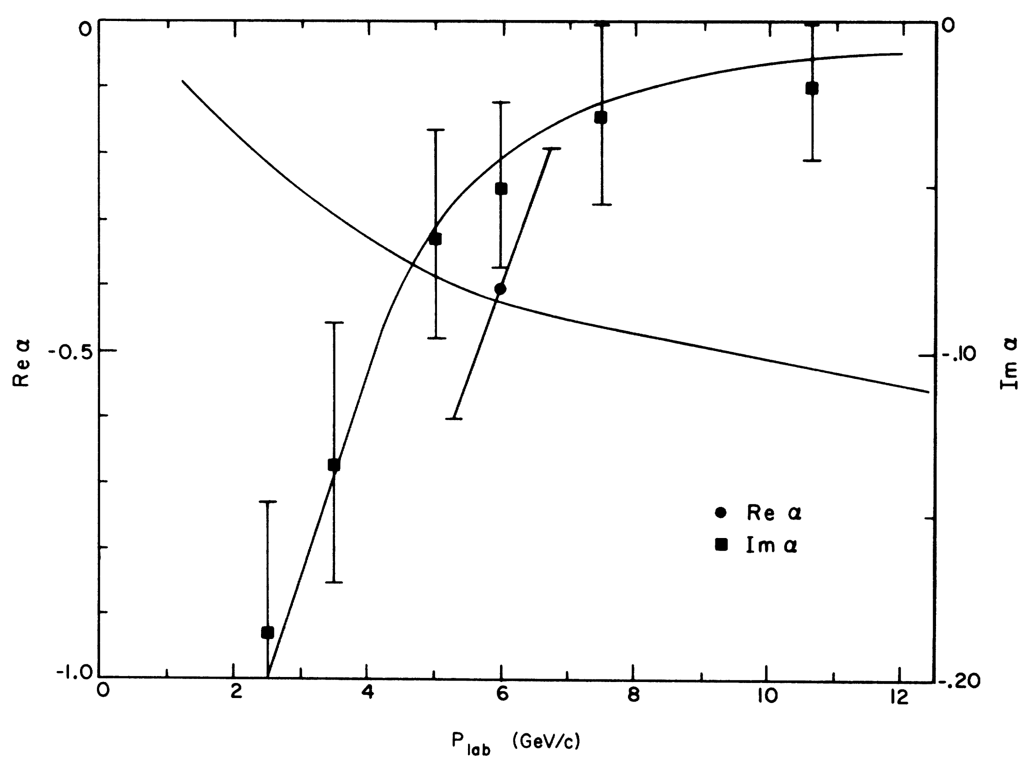

FIG. 1. Real and imaginary parts of α as a function of energy predicted from brick-wall helicity conservation. Data on Reα at 6 GeV/c from Ref. 7 and data on Imα from curves given in Ref. 5.

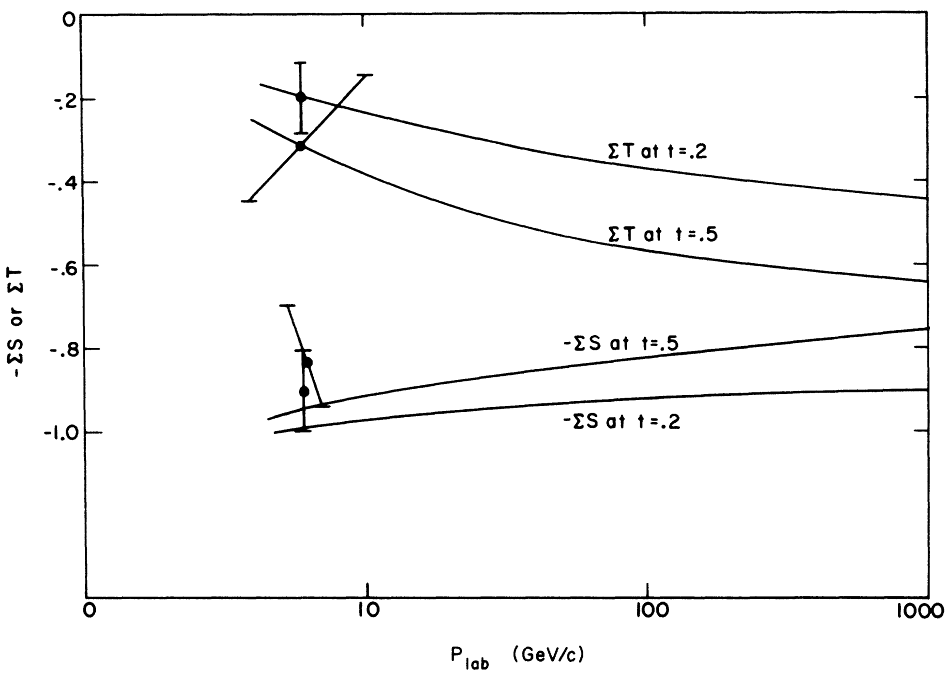

FIG. 2. Predictions for ΣT and ΣS polarization at high energy for $t=0.2$ and 0.5 $(\mathrm{GeV}/c)^2$. Data at 6 GeV/c from Ref. 8.

from their asymptotic values, but definitely not compatible with a pure pole model with asymptotic phases. This is consistent with FESR amplitude analysis and dispersion-relation results.[12] The energy dependence is not uniquely determined in the fit, but merely a reasonable interpolating formula for values determined separately at each energy. The remaining parameter α_f/α_P was found to be fairly well restricted to -1.3 ± 0.1 in order to fit Reα at 6 GeV/c, independent of the phase-angle values.

Figure 1 shows the fitted Reα and Imα along with the data. Note that as energy increases, Im$\alpha \to 0$ due to the phases ϕ_{++}^0 and ϕ_{+-}^0 becoming equal, and not due to $|F_{+-}^0| \to 0$, as would be the case for s-channel helicity conservation. This model predicts that Re$\alpha \to -1$ as energy increases, as contrasted with Re$\alpha \to 0$ for s-channel helicity conservation. Amplitude analysis at higher energies will be required to check this. An alternative expression of brick-wall helicity conservation results at high energy is given in Fig. 2. Here we plot the polarization parameter combinations ΣS and ΣT (see Ref. 8 for definitions) for fixed t as a function of energy. Contrast this behavior with that for s-channel conservation where $\Sigma T\to 0$, $\Sigma S \to 1$ as $s\to\infty$. We of course do not claim that the ability to get a fit proves that helicity is conserved in the brick-wall frame, but merely that there is no contradiction with present data. Note that nonconservation of nucleon helicity in the s channel will have virtually no effect on the helicity-conservation results for ρ^0 photoproduction, since the helicity structures are effectively decoupled at the energies considered.[13,14]

There is one additional consequence of s-channel helicity conservation which must be examined. It is obvious that in this case the asymptotic contribution to the invariant amplitude A must vanish.[1] If one makes the additional assumption that the nonasymptotic (f) contribution also vanishes, then A will satisfy an unsubtracted dispersion relation. Pfeffer, Cheng, Dutta-Roy, and Renninger[6] have shown that this is quite consistent with the low-energy data in πN and KN scattering. If, however, the helicity conservation is in the brick-wall frame, there will be an asymptotic contribution to A and the dispersion relation will need a subtraction. To examine this possibility, we write a subtracted dispersion relation for A at $t=0$, using the parameters from the preceding fit to α.

$$\mathrm{Re}A(\nu)=\mathrm{Re}A(\nu_1)+\frac{2(\nu^2-\nu_1^2)}{\pi}\int_{\nu_0}^{\infty}\frac{\nu' d\nu' \,\mathrm{Im}A(\nu')}{(\nu'^2-\nu^2)(\nu'^2-\nu_1^2)}\,. \tag{15}$$

We divide the integral into two parts at $\nu'=N$ (where the Regge asymptotic behavior is valid for $\nu'>N$), set $\nu=\nu_0$, and take the limit $\nu_1\to\infty$, to get

$$\mathrm{Re}A(\nu_0)-\frac{2}{\pi}\int_{\nu_0}^{N}\frac{\nu' d\nu' \,\mathrm{Im}A(\nu')}{\nu'^2-\nu_0^2}=\lim_{\nu_1\to\infty}\left(\mathrm{Re}A(\nu_1)+\frac{2(\nu_0^2-\nu_1^2)}{\pi}\int_N^{\infty}\frac{\nu' d\nu'\,\mathrm{Im}A(\nu')}{(\nu'^2-\nu_0^2)(\nu'^2-\nu_1^2)}\right). \tag{16}$$

Note that the left-hand side is just the difference between the threshold value of A and its evaluation via an unsubtracted dispersion relation cut off at the upper limit $\nu'=N$. In Ref. 6, numerical evaluation for $N=(2.42\text{ GeV})^2$ gave the integral a value of 95 ± 9.5 mb GeV, while the threshold amplitude is 93 ± 2.5 mb GeV. The right-hand side gives the deviation from the unsubtracted dispersion relation result if the A amplitude needs a subtraction. We evaluate it by using the asymptotic form $\mathrm{Im}A(\nu')=\beta(\nu'^2-\nu_0^2)^{\alpha/2}$ for $\nu'>N$, and evaluating a subtracted (at ν_0) dispersion relation for this amplitude. The result for the right-hand side of (16) is

$$\Delta A(\nu_0)=\lim_{\nu_1\to\infty}\frac{2(\nu_1^2-\nu_0^2)}{\pi}\int_{\nu_0}^{N}\frac{\nu' d\nu' \beta(\nu'^2-\nu_0^2)^{\alpha/2}}{\nu'^2-\nu_0^2}$$
$$\approx -\frac{2\beta N^{\alpha}}{\pi\alpha}\,. \tag{17}$$

Note that this α is the Regge trajectory value at $t=0$, not the quantity defined in (1). We evaluate this term using the P and f contributions with parameters determined in the amplitude analysis fit, and with over-all normalization from total cross-section sums. We use

$$\sigma_T^0=\tfrac{1}{2}[\sigma(\pi^+p)+\sigma(\pi-p)]$$
$$=\frac{\mathrm{Im}F_{++}^0}{P_{\mathrm{lab}}}\xrightarrow[s\to\infty]{}\frac{\mathrm{Im}A}{\alpha P_{\mathrm{lab}}},$$

where α is defined in (1) and we have used

$$\lim_{s\to\infty}F_{+-}^0=\left(\frac{-t}{4m^2}\right)^{1/2}A\,.$$

The result is

$$\Delta A(\nu_0)=\frac{-2\beta_P N}{\pi}-\frac{2\beta_f N^{1/2}}{\frac{1}{2}\pi}$$
$$\approx -\frac{20W}{\pi M}(W-2\alpha_f)\text{ mb GeV}, \tag{18}$$

where W (center-of-mass energy) and M (nucleon mass) are in GeV. Using $W=2.42$ GeV and $\alpha_f=+1.3\pm 0.1$, we get $\Delta A(\nu_0)=-3\pm 3$ mb GeV. This is smaller than the error in the evaluation of the low-energy integral of the unsubtracted dispersion relation. Hence cutting off the integral at 2.42 GeV is not a sensitive test of unsubtracted dispersion relation vs brick-wall helicity conservation. If the cutoff were increased to $W=3.5$ GeV, the expected deviation goes up to 20 mb GeV, so that low-energy data up to this region are necessary for a decisive test. Note, however, that the value of α_f as determined from the amplitude-analysis fit was crucial in maintaining consistency of brick-wall helicity conservation with the unsubtracted-dispersion-relation results. Either the P or f contribution alone predicts $\Delta A(\nu_0)\approx 40$ mb GeV for $W=2.42$ GeV – well above the uncertainty in the low-energy integral.

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[9]One possibility would seem to be the quark-quark center-of-mass system. However, this requires $|K_\pi/K_p|=\frac{2}{3}$.

[10]This has been a tacit assumption for high-energy elastic scattering in most absorption models. If it proves to be untrue, then all of the results of these models must be reexamined. Work is in progress along these lines for charge- and hypercharge-exchange reactions.

[11]Note, however, the statement in Ref. 7 that the non-helicity-conserving term seems to be in the Pomeranchukon rather than f in a Regge pole model.

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