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Harmonic Analysis on the Dalitz Plot and $\eta \to 3\pi$ Decays

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Three different types of two-variable expansions of three-body decay amplitudes and their square moduli are reviewed and compared – a recently suggested expansion based on the group O(4), the standard Dalitz-Fabri power series, and an SU(3) expansion suggested by B. W. Lee. Each of these expansions is then applied to analyze the recently obtained experimental Dalitz plot for the $\eta \to \pi^0\pi^+\pi^-$ decay.

I. INTRODUCTION

In two recent articles[1,2] (to be referred to as I and II) we have presented a general formalism for treating Dalitz-plot distributions in decays of the type

$$1 \to 2+3+4 \tag{1}$$

and have applied this formalism to analyze the results of recent experiments on the process $K^{\pm} \to \pi^{\pm}\pi^{\pm}\pi^{\mp}$.

The formalism of papers I and II applies to spinless particles and consists of a two-variable expansion of the decay amplitude for reaction (1) in terms of the basis functions of the irreducible representations of the rotation group O(4). Reaction (1) is considered in a center-of-mass-like frame of reference, in which the momenta satisfy

$$\vec{p}_1 = \vec{p}_2, \quad \vec{p}_3 = -\vec{p}_4$$

(and of course $p_1 = p_2 + p_3 + p_4$). Choosing the space axes in such a manner that all momenta $\vec{p}_i$ lie in the O_{xz} plane and $\vec{p}_3 = -\vec{p}_4$ is along the z axis, we can express the scattering amplitude as a function of the coordinates of momentum p_1. Since all momenta are on their mass shells, we can consider the decay amplitude to be a function of a point on the upper sheet of the hyperboloid $p^2 = p_0^2 - \vec{p}^2 = m^2$ (we put $p = p_1$). For scattering $1+2 \to 3+4$ the point p would indeed range over the entire sheet $p^2 = m^2$, $p_0 \geq m$ and our approach would lead to expansions in terms of the homogeneous Lorentz group O(3, 1), which is the group of motions of the corresponding manifold. Such two-variable expansions of scattering amplitudes have been investigated in a series of previous publications (see, e.g., Refs. 3–6). For decays, however, the physically available energy region is restricted, so that the momentum p only lies on a "cup" close to the vertex of the hyperboloid. This finiteness of the physical region for decays was used in I to map the physically accessible part of the hyperboloid onto a four-dimensional sphere. The group of motions of this manifold is O(4), hence we obtain O(4) expansions of decay amplitudes. For all details we refer to I and II. Let us mention that the "spinless expansions" of decay[1,2] and scattering amplitudes[3-6] have been generalized to O(4) and O(3, 1) expansions of helicity amplitudes for processes involving particles with arbitrary spins.[7,8]

The dominant feature of both the O(3, 1) and O(4) expansions is that they are "maximal" expansions, in that the entire dependence on the kinematic parameters (e.g., the Mandelstam variables s, t, and u) is exhibited explicitly in known functions [basis functions of representations in the spin-zero case, O(3, 1) or O(4) transformation matrices in the nonzero-spin case]. The "dynamics" of the reactions are represented by the expansion coefficients. In previous articles[1-8] we have shown that such a separation of "kinematics" and "dynamics" is meaningful, in that the special functions provided by the O(3, 1) and O(4) groups have reasonable "kinematic" behavior (e.g., behavior at thresholds and pseudothresholds, in asymptotic limits, etc.).

The most standard way of treating scattering amplitudes is to write single-variable expansions for them. Thus, in the usual partial-wave analysis the squared energy $s = (p_1 + p_2)^2$ is fixed and the dependence on the scattering angle is expanded in terms of Legendre polynomials [or O(3) D functions]. In Regge-pole theory the squared momentum transfer $t = (p_1 - p_3)^2$ is fixed and the energy dependence is expanded in terms of certain Legendre functions [or O(2, 1) D functions[9]]. Decay amplitudes, on the other hand, are customarily treated by writing two-variable expansions. Among them let us just mention the simple power expansion in terms of the Dalitz-Fabri variables,[10, 11] an equivalent trigonometric expansion first (to our knowledge) used by Weinberg[12] and an expansion in terms of functions, orthonormal over the Dalitz plot, suggested by Lee.[13]

In the future we plan to further develop the O(3, 1) two-variable expansions for scattering so as to be able to apply them to analyze data on two-body scattering (different two-variable expansions of scattering amplitudes also exist in the literature, see, e.g., Refs. 14, 15). The O(4) expansions for decays can however be applied immediately and compared to other expansions, so we are concentrating on them first.

In Sec. II of this article we reproduce the O(4) expansions obtained in I both for amplitudes and square moduli of amplitudes and compare them with other expansions. We also obtain some relevant formulas for the Lee expansions.[13] In Sec. III we briefly discuss the data on $\eta \to 3\pi$ decays[16, 17] and our fitting procedures, using the O(4), Lee, and Dalitz-Fabri expansions. We also present and discuss the results of the various numerical fits to the data. Finally, in Sec. IV we summarize our conclusions and discuss the future outlook.

II. EXPANSION FORMULAS FOR DECAY AMPLITUDES

A. Expansions of Amplitudes

Let us consider the decay process (1) and introduce the Mandelstam variables

$$s=(p_1-p_2)^2,\quad t=(p_1-p_3)^2,\quad u=(p_1-p_4)^2,$$
$$s+t+u=m_1^{\,2}+m_2^{\,2}+m_3^{\,2}+m_4^{\,2}, \tag{2}$$

where p_i and m_i are the particle momenta and masses. Let us now consider three different types of two-variable expansions.

The O(4) variables of papers I and II are

$$\cos\alpha = 1-\frac{[(m_1+m_2)^2-s][(m_1-m_2)^2-s]}{2m_1^{\,2}R^2 s}, \tag{3}$$

$$\cos\theta=\frac{2s(t-m_1^{\,2}-m_3^{\,2})+(s+m_1^{\,2}-m_2^{\,2})(s+m_3^{\,2}-m_4^{\,2})}{\{[-s+(m_1+m_2)^2][-s+(m_1-m_2)^2][s-(m_3+m_4)^2][s-(m_3-m_4)^2]\}^{1/2}}, \tag{4}$$

where

$$R^2=\frac{[(m_1+m_2)^2-(m_3+m_4)^2][(m_1-m_2)^2-(m_3+m_4)^2]}{4m_1^{\,2}(m_3+m_4)^2}.$$

The O(4) expansion of the decay amplitude can be written as

$$F(s,t)=\sum_{n=0}^{\infty}\sum_{l=0}^{n} a_{nl}\,\phi_{nl}(\alpha,\theta), \tag{5}$$

where a_{nl} are the O(4) partial-wave amplitudes and

$$\phi_{nl}(\alpha,\theta)=e^{il\pi/2}\,\frac{2^{l+1/2}l\,!}{2\pi}\left((2l+1)\frac{(n+1)(n-l)!}{(n+l+1)!}\right)^{1/2}(\sin\alpha)^l\, C_{n-l}^{l+1}(\cos\alpha)P_l(\cos\theta). \tag{6}$$

[$C_{n-l}^{l+1}(\cos\alpha)$ and $P_l(\cos\theta)$ are Gegenbauer and Legendre polynomials, respectively.]

The Dalitz-Fabri variables are[10]

$$x=\sqrt{3}\;\frac{T_3-T_4}{Q},\quad y=\frac{3T_2-Q}{Q}, \tag{7}$$

where T_2, T_3, and T_4 are the kinetic energies of the final particles in the rest frame of the decaying particle 1. We have

$$Q=T_2+T_3+T_4=m_1-m_2-m_3-m_4$$

and

$$x=\frac{\sqrt{3}}{2m_1Q}\,[(m_3-m_4)(m_3+m_4-2m_1)-t+u],$$

$$y=\frac{1}{2m_1Q}\,[3(m_1-m_2)^2-2m_1Q-3s].$$

The corresponding "xy expansion" is simply

$$F(s,t)=\sum_{k,m} R_{km}\,x^k y^m. \tag{8}$$

Finally we shall make use of the Lee expansion[13] in which the variables are a and β (α and β in Ref. 13). It is difficult to give an explicit expression for a and β, but they can be related to more usual variables as follows. Let us put[13, 18]

$$T_2=\tfrac{1}{3}Q\,[1+\rho\cos\phi]\,,\qquad x=\rho\sin\phi\,,$$
$$T_3=\tfrac{1}{3}Q\,[1+\rho\cos(\phi-\tfrac{2}{3}\pi)]\,,\quad y=\rho\cos\phi\,,\tag{9}$$
$$T_4=\tfrac{1}{3}Q\,[1+\rho\cos(\phi+\tfrac{2}{3}\pi)]\,.$$

In these variables the boundary of the decay region is

$$1=(1+\kappa)\rho^2+\kappa\rho^3\cos\phi\,,\tag{10}$$

where $\kappa=2QM(2M-Q)^{-2}$, M is the mass of the decaying particle and the masses of the three final particles are assumed to be equal. Finding the real root of (10) we obtain the boundary of the Dalitz plot as

$$\rho_B(\phi)=R(\cos3\phi,\,\kappa)\,.$$

The Lee variables then are

$$a=\frac{\rho}{R(\cos\phi,\,\kappa)}\,,\tag{11}$$

$$\beta=\frac{\pi}{3}\,\frac{\int_0^{\phi}d\phi' R^2(\cos3\phi',\,\kappa)}{\int_0^{\pi/3}d\phi' R^2(\cos3\phi',\,\kappa)}\,,\tag{12}$$

and we have

$$0\leqslant a\leqslant 1\,,\quad 0\leqslant\beta\leqslant 2\pi\,.$$

The corresponding expansion in terms of functions, orthogonal over the Dalitz plot can be written as

$$F(s,t)=\sum_{\kappa=\pm1}\sum_{m,n=0}^{\infty}A_{mn}^{\kappa}\lambda_m^{n\kappa}(a,\beta)\,,\tag{13}$$

with

$$\int_0^1 ada\int_0^{2\pi}d\beta\,\lambda_m^{n\kappa}(a,\beta)\lambda_{m'}^{n'\kappa'}(a,\beta)=\delta_{mm'}\delta_{nn'}\,\delta_{\kappa\kappa'}\,.\tag{14}$$

Following Frankel and Van Dyck,[19] we have separated the decay amplitude into an even ($\kappa=+1$) and odd ($\kappa=-1$) part with respect to $\beta\to-\beta$, i.e., even and odd with respect to the interchange of particles 3 and 4. In (13) we have

$$\lambda_m^{n\kappa}(a,\beta)=[2(2n+m+1)]^{1/2}a^m P_n^{(0,\,m)}(2a^2-1)B_m^{\kappa}(\beta)\,,\tag{15}$$

where $P_n^{(0,m)}(x)$ is a Jacobi polynomial and

$$B_m^{1}(\beta)=N_m\cos m\beta\,,$$
$$B_m^{-1}(\beta)=N_m\sin m\beta\,,\tag{16}$$

with

$$N_m=\begin{cases}\dfrac{1}{\sqrt{\pi}}\,, & m\geqslant 1\\[2ex] \dfrac{1}{\sqrt{2\pi}}\,, & m=0\,.\end{cases}\tag{17}$$

B. Expansions of Square Moduli of Amplitudes

Using the above expansions it is a simple matter to calculate the square modulus of the decay amplitude $F(s,t)$ and thus the density of points on the Dalitz plot.

For the O(4) expansion (5) this has been done in paper I and the result is

$$|F(s,t)|^2=\sum_{N=0}^{\infty}\sum_{L=0}^{N}b_{NL}\phi_{NL}(\alpha,\theta)\,,\tag{18}$$

with

$$b_{NL}=\frac{(N+1)^{1/2}}{\sqrt{2}\,\pi}\sum_{nln'l'}[(n+1)(n'+1)(2l+1)(2l'+1)]^{1/2}(l0l'0|L0)\begin{Bmatrix}\frac{1}{2}n & \frac{1}{2}n & l\\ \frac{1}{2}n' & \frac{1}{2}n' & l'\\ \frac{1}{2}N & \frac{1}{2}N & L\end{Bmatrix}e^{i\pi l'}a_{nl}\,a_{n'l'}^{*}\,,\tag{19}$$

where $(l0l'0|L0)$ is an O(3) group Clebsch-Gordan coefficient and the curly brackets represent an O(3) $9J$ symbol.

For the xy expansion we have

$$|F(s,t)|^2=\sum_{K,M=0}^{\infty}S_{KM}\,x^Ky^M\,,\tag{20}$$

with

$$S_{KM}=\sum_{k=0}^{K}\sum_{m=0}^{M}R_{km}R_{K-k,\,M-m}^{*}\,.\tag{21}$$

Finally, let us calculate the square of the amplitude in terms of the Lee expansion coefficients. Squaring (13) and making use of reality of $\lambda_m^{n\kappa}(a,\beta)$, we have

$$|F(s,t)|^2=\sum_{K=-1}^{+1}\sum_{M,N=0}^{\infty}B_{MN}^{K}\lambda_M^{NK}(a,\beta)\,,\tag{22}$$

where

$$B^K_{MN} = \sum_{\kappa,\kappa'} \sum_{mm'nn'} \{mn\kappa, m'n'\kappa' | M, N, K\} a^{\kappa}_{mn} a^{\kappa' *}_{m'n'} . \tag{23}$$

The "addition coefficients" $\{mn\kappa, m'n'\kappa' | MNK\}$ were introduced using somewhat different notations, by Frankel and Van Dyck[19] who also studied some of their properties. Since the functions $\lambda^{n\kappa}_m(a,\beta)$ are related to certain SU(3) harmonic functions,[13] the "addition coefficients" can be obviously related to the SU(3) Clebsch-Gordan coefficients. In this article we shall however simply calculate these coefficients directly, using only the O(3) group properties of the functions $\lambda^{n\kappa}_m(a,\beta)$.

Indeed, from (13), (14), (22), and (23) we have

$$\{mn\kappa, m'n'\kappa' | M, N, K\} = \int_0^1 a\,da \int_0^{2\pi} d\beta\, \lambda^{n\kappa}_m(a,\beta)\lambda^{n'\kappa'}_{m'}(a,\beta)\lambda^{NK}_M(a,\beta) . \tag{24}$$

Let us use (15)–(17) and calculate the two integrals separately. Put

$$I^{\kappa,\kappa',K}_{m,m',M} = \int_0^{2\pi} B^{\kappa}_m(\beta) B^{\kappa'}_{m'}(\beta) B^K_M(\beta) d\beta . \tag{25}$$

It can readily be checked that the only nonzero coefficients of the type (25) are

$$\begin{aligned}
&I^{\kappa,\kappa',\kappa\kappa'}_{m,m',m+m'} = \frac{1}{2\sqrt{\pi}}\, e^{i(\kappa+\kappa'-\kappa\kappa'-1)\pi/4}, \quad m \neq 0, \quad m' \neq 0 \\
&I^{\kappa,\kappa',\kappa\kappa'}_{m,m',|m-m'|} = \frac{1}{2\sqrt{\pi}} \frac{m+\kappa\kappa' m'}{|m+\kappa\kappa' m'|}\, e^{i4(\kappa-\kappa'-\kappa\kappa'+1)\pi/4}, \quad 0 \neq m \neq m' \neq 0 \\
&I^{111}_{mmo} = I^{111}_{mom} = I^{111}_{omm} = I^{111}_{ooo} = \frac{1}{\sqrt{2\pi}} , \\
&I^{-1-11}_{mmo} = I^{1-1-1}_{omm} = I^{-11-1}_{mom} = \frac{1}{\sqrt{2\pi}} , \quad m \neq 0 .
\end{aligned} \tag{26}$$

The integration over a in (24) can also be performed and we have:

$$Y^{nn'N}_{mm'M} \equiv [8(2n+m+1)(2n'+m'+1)(2N+M+1)]^{1/2} \int_0^1 a^{m+m'+M+1} P^{(0,m)}_n(2a^2-1) P^{(0,m')}_{n'}(2a^2-1) P^{(0,M)}_N(2a^2-1) da . \tag{27}$$

Let us now make the substitution

$$2a^2 - 1 = \cos x$$

in (27) and express the Jacobi polynomials in terms of the Wigner d functions[20] for the group O(3)

$$P^{(0,M)}_N(\cos x) = (\cos\tfrac{1}{2}x)^{-M} d^{N+M/2}_{M/2,M/2}(x) .$$

The resulting integral can be calculated using standard angular momentum theory and we obtain

$$Y^{nn'N}_{mm'm\pm m'} = [2(2n+m+1)(2n'+m'+1)(2N+m\pm m'+1)]^{1/2} \begin{pmatrix} n+\frac{1}{2}m & n'+\frac{1}{2}m' & N+\frac{1}{2}(m\pm m') \\ \frac{1}{2}m & \pm\frac{1}{2}m' & -\frac{1}{2}(m\pm m') \end{pmatrix}^2 , \tag{28}$$

where the last entry is a Wigner $3j$ symbol.

Finally, the nonzero addition coefficients can be written as

$$\{mn\kappa, m'n'\kappa' | \, |m\pm m'|, N, \kappa\kappa'\} = I^{\kappa,\kappa',\kappa\kappa'}_{mm'm\pm m'} Y^{nn'N}_{mm'm\pm m'} . \tag{29}$$

In particular, if we are only interested in expanding functions that are symmetric under the interchange of particles 3 and 4, then we have $\kappa = \kappa' = K = 1$ and

$$\begin{aligned}
\{mn1, m'n'1 | m\pm m', N, 1\} = R^{m\pm m'}_{m,m'} \frac{1}{2\sqrt{\pi}} & [2(2n+m+1)(2n'+m'+1)(2N+m\pm m'+1)]^{1/2} \\
& \times \begin{pmatrix} n+\frac{1}{2}m & n'+\frac{1}{2}m' & N+\frac{1}{2}(m\pm m') \\ \frac{1}{2}m & \pm\frac{1}{2}m' & -\frac{1}{2}(m\pm m') \end{pmatrix}^2 ,
\end{aligned} \tag{30}$$

with

$$R_{m,m'}^{m\pm m'}=\begin{cases}1 \text{ for } 0\neq m\neq m'\neq 0\\ \sqrt{2} \text{ for } m, m', \text{ or } m\pm m'=0\,.\end{cases} \tag{31}$$

C. Comments on the Expansion Formulas

Formulas (5), (8), and (13) [or (18), (20), and (22)] thus represent O(4), xy, and Lee expansions of the decay amplitude (or the square modulus of the decay amplitude) for a three-body decay involving spinless particles. The O(4) and xy expansions are written for arbitrary final-state masses, the Lee expansions are written only for the case of equal masses.

All the expansions can be directly applied to analyze Dalitz plots, i.e., we can keep a finite number of terms in the corresponding expansion and obtain values of the expansion coefficients from best fits to the data. The most natural cutoff procedures that we apply in fitting $\eta\to 3\pi$ data, using the expansions for amplitudes (or square-moduli) are the following:

(a) O(4) expansions. Fix n_0 and take $0\leq n\leq n_0$, $0\leq l\leq n$.

(b) xy expansions. Fix $\tilde{n}_0$ and take all $k\geq 0$, $m\geq 0$ such that $k+m\leq\tilde{n}_0$.

(c) Lee expansions. Fix $\tilde{\tilde{n}}_0$ and take all $m\geq 0$, $n\geq 0$ such that $2n+m\leq\tilde{\tilde{n}}_0$. Sum over $\kappa=\pm 1$.

Various selection rules may of course restrict the allowed values of some of the above summation indices. Thus, if particles 3 and 4 are identical or if they are antiparticles of each other and C invariance holds, then l must be even in (5) [L even in (18)], k must be even in (8) [K even in (20)] and we must have $\kappa=1$ only in (13) [$K=1$ in (22)].

Let us mention several features of the above expansions.

The O(4) expansions are written in terms of variables α and θ that have a simple physical meaning, they are written for arbitrary masses and have been generalized to arbitrary spins.[8] The boundary of the physical region is given by the equation $\cos\theta=\pm 1$, the presence of the factor $(\sin\alpha)^l$ in (6) ensures the correct behavior of the amplitude at the physical threshold $s=(m_3+m_4)^2$ and pseudothreshold $s=(m_1-m_2)^2$. The angular momentum l of particles 3 and 4 in the center-of-mass-like frame of reference (see papers I and II) is displayed explicitly and indeed the θ dependence in (5) [see (6)] is given by a standard O(3) partial-wave expansion. The "four-dimensional angular momentum" n also has a physical meaning, which however only becomes meaningful when the particles have spin.[8] The O(4) expansion is in terms of functions, orthogonal over the Dalitz plot. However, the expression for an element of phase space is rather complicated (see paper II). Most important, the O(4) expansions are a modification of O(3, 1) expansions for scattering amplitudes, so that similar two-variable techniques should be applicable for analyzing scattering.

The xy power-series expansion (8) has the advantage of extreme simplicity and for any reasonably smooth distribution it will provide a good fit with a few parameters simply because it is an expansion around the point $x=y=0$, i.e., $T_2=T_3=T_4=Q/3$ (the center of the Dalitz plot). The functions are not orthogonal, the expansion has no helpful group-theoretical interpretation, none of the standard kinematical features (boundary of physical region, thresholds, etc.) are incorporated. The expression for an element of phase space is extremely simple, namely, $dxdy$.

The Lee expansion (13) uses the rather complicated variables a and β of (11) and (12), the boundary of the physical region is simply $a=1$. A major advantage is that the functions (15) are orthogonal over the Dalitz plot with a measure, proportional to an element of phase space $ada d\beta$. Assuming that the decay matrix element were uniform over the Dalitz plot, then this orthogonality would ensure the statistical independence of the coefficients in (13) and thus the stability of this expansion with respect to truncation[13] (see also paper II). The expansion has a group theoretical interpretation in terms of an SU(3) group, relevant to the classification of three-particle states.[21] No generalizations to arbitrary masses and spins or to the treatment of scattering have been presented as yet.

III. ANALYSIS OF THE $\eta\to 3\pi$ DALITZ PLOT

We apply the three types of expansions, discussed in the previous section, to analyze the data on $\eta\to\pi^0\pi^+\pi^-$ decays, obtained at the Princeton-Pennsylvania Accelerator by the Columbia University group.[16,17] Our main purpose is to test and discuss the relative merits of the different expansions, when applied to this problem. The analyzed Dalitz plot was reconstructed from the data consisting of over 220 000 $\eta\to\pi^0\pi^+\pi^-$ events and is divided into 54 bins. The square of the matrix element $|F_i|^2$, corresponding to each bin ($i=1,\ldots,54$), as well as its statistical uncertainty Δ_i are obtained as

$$|F_i|^2=\frac{N_i}{E_i}\,,\quad \Delta_i=\frac{\sqrt{N_i}}{E_i}\,,$$

where N_i is the number of experimental events in each bin and E_i is the experimental efficiency.

The absence of a C-violating asymmetry was established directly from the raw experimental data and was then assumed in the reconstruction of the Dalitz plot, corrected for the efficiency. The data available to us were thus essentially half of a Dalitz plot, in which events with $T_3 > T_4$ were plotted on top of those with $T_3 < T_4$ (where, e.g., $T_3 = T_\pi^+$ and $T_4 = T_\pi^-$). The distribution must thus automatically be symmetric under the interchange of the two charged pions and we have no way at all of considering a possible C violation.

Since two of the final pions are charged, long-range final-state electromagnetic interactions will definitely be present (besides the final-state strong interactions). In order to estimate the influence of these electromagnetic final-state interactions we expanded decay amplitudes representing three different sets of data. The first set was simply the actual data, corrected for the experimental efficiencies.[16,17] The second and third sets were obtained from the first by dividing out a Coulomb correction factor:

$$|F_i|^2_{\text{Coul. corr.}} = \frac{|F_i|^2_{\text{exp}}}{\int_{\text{bin}} C(x, y)dxdy}\,. \qquad (32)$$

The integral in the denominator of (32) is over the area of the ith bin and $C(x, y)$ is a Coulomb correction factor. The simplest form of $C(x, y)$ corresponds to a nonrelativistic Coulomb term[22]

$$C_{\text{Dalitz}} = \frac{2\pi\alpha}{\nu}\,\frac{1}{1 - e^{-2\pi\alpha/\nu}}\,,$$

where α is the fine-structure constant and ν is the relative velocity of the two charged pions. A somewhat more sophisticated calculation of the radiative corrections to $\eta \to \pi^+\pi^-\pi^0$ can be obtained from an article by Neveu and Scherk.[23] The corrections are calculated to first order in the fine-structure constant, the influence of strong interactions (and of uncalculable diagrams) is neglected. We shall not reproduce the expression[23] for C_{NS} here.

For $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$ decays, in which all three final pions are charged. The inclusion of the (Dalitz) Coulomb corrections in paper II improved the fits to data noticeably. For $\eta \to 3\pi$ decays on the other hand the modification of the data by formula (32), using C_{Dalitz} and C_{NS} turned out to be completely insignificant, in that it did not change the best-fit values of any of the expansion coefficients, nor the values of the corresponding χ^2 functions. The reason for this difference between $K^\pm$ and η decays seems to be that the Coulomb correction function C (in both its versions) varies smoothly over the Dalitz plot when only two of the final pions are charged, but has a more complicated behavior when all three of them are.

Having the above comments on Coulomb corrections in mind we shall only present a treatment of the uncorrected data.

The squared matrix elements $|F(s,t)|^2$ obtained from the Dalitz plot were fitted in the usual manner by minimizing the χ^2 function. Fits directly to $|F(s,t)|^2$ were obtained by minimizing

$$\chi^2 = \sum_i \left(\frac{|F_i|^2 - \sum c_{jk}\psi_{jk}}{\Delta_i}\right)^2, \qquad (33)$$

where ψ_{jk} were successively taken to be O(4) functions as in (18), the xy powers as in (20) or the $\lambda_M^{N_1}(\alpha, \beta)$ functions of (22), evaluated at the center of each bin.

Fits to the matrix element itself, rather than to the square modulus, were obtained by minimizing

$$\chi^2 = \sum_i \left(\frac{|F_i|^2 - |\sum c_{jk}\psi_{jk}|^2}{\Delta_i}\right). \qquad (34)$$

$\sum C_{jk}\psi_{jk}$ represents the O(4) expansion (5) or the Lee expansion (13).

Note that the coefficients C_{jk} in (33) are real, whereas c_{jk} in (34) are complex. As mentioned above, the amplitude is by assumption symmetric

TABLE I. Best-fit parameter values (and statistical errors) for the expansions of $\eta \to \pi^0\pi^+\pi^-$ decay amplitudes. There are 54 bins. The number of degrees of freedom (NDF) is the number of bins less the number of free parameters. The coefficients in the O(4) expansion were multiplied by the normalization constant N.

	O(4) expansion (5)			Lee expansion (13)	
NDF	51	47	NDF	51	47
χ^2/NDF	1.265	1.128	χ^2/NDF	1.373	1.136
χ^2	64.49	53.03	χ^2	70.00	53.41
a_{00}	2.57 ± 0.02	2.50 ± 0.03	A_{00}^1	1.00	1.00
Rea_{10}	0.88 ± 0.03	0.91 ± 0.04	ReA_{10}^1	-0.254 ± 0.004	-0.268 ± 0.006
Ima_{10}	0.95 ± 0.07	1.02 ± 0.03	ImA_{10}^1	0.00 ± 0.033	0.016 ± 0.107
Rea_{20}		-0.13 ± 0.03	ReA_{01}^1		-0.012 ± 0.007
Ima_{20}		$+0.25 \pm 0.06$	ImA_{01}^1		-0.083 ± 0.065
Rea_{22}		$+0.02 \pm 0.02$	ReA_{20}^1		-0.012 ± 0.005
Ima_{22}		$+0.02 \pm 0.08$	ImA_{20}^1		0.008 ± 0.059

under the interchange of particles 3 and 4.

The minimization of χ^2 was performed in exactly the same manner as in paper II for $K^{\pm} \to 3\pi$ decays, using the MINUIT computer program,[24] starting from random points. The results of the numerical fits that we have performed are presented in Tables I and II. The statistical errors in the parameters correspond to an increase of one unit in the value of χ^2. The fits presented in the tables are essentially unique and thus correspond to a global minimum χ_0^2. All the expansions were written in the form $N(1 + a_1\psi_1 + a_2\psi_2 + \cdots)$. The coefficients a_i were minimized numerically, the normalization constant ($N > 0$) was simultaneously

TABLE II. Best-fit parameter values (and statistical errors) for expansions of the square moduli of $\eta \to \pi^0\pi^+\pi^-$ decay amplitudes. Data and notations are the same as in Table I. The coefficients in the O(4) and xy expansions were multiplied by the normalization constant N.

	O(4) expansion (18)		
NDF	52	50	48
χ^2/NDF	5.9	1.280	1.142
χ^2	308.7	64.01	54.82
b_{00}	14.4 ± 0.2	16.0 ± 0.2	15.6 ± 0.3
b_{10}	11.8 ± 0.2	8.7 ± 0.3	9.3 ± 0.3
b_{20}		3.2 ± 0.2	2.4 ± 0.3
b_{22}		-0.2 ± 0.2	-0.2 ± 0.3
b_{30}			0.7 ± 0.2
b_{32}			0.0 ± 0.2
	xy expansion (20)		
NDF	52	50	48
χ^2/NDF	1.122	1.119	1.118
χ^2	58.32	55.95	53.68
S_{00}	48.91 ± 0.3	48.06 ± 0.7	47.9 ± 0.7
S_{01}	-52.2 ± 0.8	-51.8 ± 0.8	-53.3 ± 2.2
S_{02}		2.3 ± 1.8	3.7 ± 2.1
S_{20}		2.5 ± 2.1	2.3 ± 2.1
S_{03}			4.8 ± 4.2
S_{21}			-3.5 ± 6.6
	Lee expansion (22)		
NDF	52	50	48
χ^2/NDF	1.340	1.071	1.090
χ^2	69.67	53.54	52.32
B^1_{00}	1.00	1.00	1.00
B^1_{10}	-0.5 ± 0.008	-0.494 ± 0.009	-0.493 ± 0.01
B^1_{01}		0.014 ± 0.008	0.014 ± 0.008
B^1_{20}		0.029 ± 0.008	0.032 ± 0.009
B^1_{11}			0.004 ± 0.008
B^1_{30}			0.009 ± 0.010

fitted analytically. We also tried to apply different cutoff procedures, but the results were always worse than the ones presented in the tables. An incorporation of still higher-order terms in the expansions was not justified, since it leads to an increase in χ^2/NDF and also the solutions were no longer unique (NDF = number of degrees of freedom).

When comparing the different expansions, representing the same data, important questions are the following: Which gives the best fit (lowest χ^2) for the least number of parameters? What is the stability with respect to truncation of the series? How unique are the solutions? How sensitive are the expansion coefficients to interesting dynamical features?

We have already mentioned that all presented solutions are essentially unique. As far as stability and the χ^2 value goes, Table I indicates that the O(4) expansion seems to be somewhat better than the Lee expansion for the amplitude itself. For the square modulus of the amplitude, however, the Lee expansion gives the best χ^2 fit, the xy expansion is second, the O(4) one the worst. All three expansions are essentially stable within statistical errors. The results, however, are so close that we do not attach any deep significance to the difference and only conclude that all the fits with three or more parameters are perfectly adequate. No definite conclusions can as yet be drawn as to the relative merits of the individual expansions.

IV. CONCLUSIONS

In this article we have written out three different expansions of a three-body decay amplitude $F(s, t)$ – the O(4) expansion, the "xy expansion," and the "Lee expansion." We also gave expressions for the square of the decay matrix element in terms of the coefficients in each of the above expansions.

The expansions were then applied to analyze the recently obtained $\eta \to \pi^0\pi^+\pi^-$ Dalitz plot[16,17] and the results are presented in Tables I and II. The numerical results do not indicate that any one of the expansions has any particular advantages over the other ones for this specific analysis.

No attempt at any consideration of the actual physics (or "dynamics") of the $\eta \to 3\pi$ decay was made in this paper. The presently available data (or rather the available experimental efficiencies) do not allow us to consider the question of a manifestation of a possible C violation, which would lead to the presence of additional terms in all of the expansions (antisymmetric under the exchange of the π^+ and π^- mesons). It is quite possible that one of the expansions would be more sensitive than the others to such an asymmetry. Indeed, we have mentioned in paper II that one of the coefficients in the O(4) expansion (18), namely b_{10}, turned out to be sensitive to a difference between the $K^+ \to 3\pi$ and $K^- \to 3\pi$ Dalitz plots. A further question of obvious dynamic importance is the extraction of information on final-state interactions between the pions, i.e., information on the $\pi\pi$ scattering amplitude, various vertices, the presence of resonances, etc. We plan to return to all these important physical problems, specifically in the context of the O(4) two-variable expansions, in a future publication.

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[1]H. R. Hicks and P. Winternitz, Phys. Rev. D 4, 2339 (1971).

[2]H. R. Hicks and P. Winternitz, Phys. Rev. D 5, 2877 (1972).

[3]N. Ya. Vilenkin and Ya. A. Smorodinskiĭ, Zh. Eksp. Teor. Fiz. 46, 1793 (1964) [Sov. Phys. JETP 19, 1209 (1964)].

[4]P. Winternitz and I. Friš, Yad. Fiz. 1, 889 (1965) [Sov. J. Nucl. Phys. 1, 636 (1965)].

[5]P. Winternitz, Ya. A. Smorodinskiĭ, and M. B. Sheftel, Yad. Fiz. 7, 1325 (1968) [Sov. J. Nucl. Phys. 7, 785 (1968)]; Yad. Fiz. 8, 833 (1968) [Sov. J. Nucl. Phys. 8, 485 (1968)].

[6]P. Winternitz, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Colorado Associated Univ. Press, Boulder, Colo., 1971), Vol. 13 (contains numerous further references).

[7]C. Shukre and P. Winternitz, Phys. Rev. D 6, 3592 (1972).

[8]C. Shukre and P. Winternitz, Phys. Rev. D 6, 3607 (1972).

[9]P. D. B. Collins and E. J. Squires, *Regge Poles in Particle Physics* (Springer, Berlin, 1968).

[10]R. H. Dalitz, Phil. Mag. 44, 1068 (1953); E. Fabri, Nuovo Cimento 11, 479 (1954).

[11]R. E. Marshak, Riazuddin, and C. P. Ryan, *Theory of Weak Interactions in Particle Physics* (Wiley, New York, 1969).

[12]S. Weinberg, Phys. Rev. Letters 4, 87 (1960).

[13]B. W. Lee, New Phys. Suppl. 7, 41 (1968).

[14]A. P. Balachandran and J. Nuyts, Phys. Rev. 172, 1821 (1968); A. P. Balachandran, W. T. Meggs, and P. Ramond, *ibid.* 187, 2080 (1969); A. P. Balachandran, W. Case, and M. Modjtehedzadeh, Phys. Rev. D 1, 1773 (1970).

[15]J. Charap and B. Minton, J. Math. Phys. 10, 1823 (1969).

[16]M. Gormley, E. Hyman, W. Lee, T. Nash, J. Peoples, C. Schultz, and S. Stein, Phys. Rev. Letters 21, 402 (1968); M. Gormley, E. Hyman, W. Lee, T. Nash, J. Peoples, C. Schultz, and S. Stein, Phys. Rev. D 2, 501 (1970); J. G. Layter, J. A. Appel, A. Kotlewski, W. Lee, S. Stein, and J. J. Thaler, Phys. Rev. Letters 29, 316 (1972).

[17]J. G. Layter, J. A. Appel, A. Kotlewski, W. Lee, S. Stein, and J. J. Thaler, Phys. Rev. Letters 29, 316 (1972).

[18]G. Källén, *Elementary Particle Physics* (Addison-Wesley, Reading, Mass., 1964).

[19]S. Frankel and O. Van Dyck, University of Pennsylvania report, 1968 (unpublished).

[20]D. M. Brink and G. R. Satchler, *Angular Momentum* (Clarendon, Oxford, 1968); A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton Univ. Press, Princeton, N. J., 1957).

[21]A. J. Dragt, J. Math. Phys. 6, 533 (1965).

[22]R. H. Dalitz, Proc. Phys. Soc. (London) A69, 527 (1956); L. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1955).

[23]A. Neveu and J. Scherk, Phys. Letters 27B, 384 (1968).

[24]F. James and M. Roos, report (unpublished).