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Some Absolute Bounds for K_{l3} Decay Parameters

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Assuming that the scalar [or vector] K_{l3} form factor $D(t)$ [or $f_+(t)$] is either univalent or a function satisfying at most a once-subtracted dispersion relation with a definite sign for its imaginary part, we can derive several exact bounds for these form factors without introducing any arbitrary parameters.

There are many theoretical calculations on K_{l3} decay parameters. However, most of these calculations are based upon varieties of approximations whose validity is not obvious at all. Recently, exact inequalities which are relatively free of theoretical uncertainties were derived by several authors.[1-7] In particular, the exact bound for the scalar slope parameter Λ_0 defined by

$$\Lambda_0 = \lambda_+ + \frac{m_\pi^2}{m_K^2 - m_\pi^2}\xi \tag{1}$$

was found[3,5] to be very small and positive, with

$$0.008 \le \lambda_0 \le 0.019\,. \tag{2}$$

This must be compared with the world-averaged experimental value[8] of

$$\Lambda_0 = -0.11 \pm 0.03\,. \tag{3}$$

However, in view of mutually contradicting experimental data,[8,9] this discrepancy should not be regarded as final. The main assumptions needed in deriving the bound in Eq. (2) are the following: (i) the chiral SW(3) Hamiltonian[10] of Gell-Mann, Oakes, and Renner and of Glashow and Weinberg, (ii) the K_{l3} soft-pion theorem,[11] (iii) a weak form[12] of the Ademollo-Gatto theorem, i.e., $f_+(0) \lesssim 1$, and finally (iv) some technical assumptions[2,3] concerning two-point Green's functions for divergences of weak currents.

In this note, we shall present an entirely different approach with fewer assumptions and show that we can derive similar exact bounds for K_{l3} decay parameters. With this objective, we shall consider two different classes of analytic functions. Hereafter, $f(t)$ will always represent a real analytic function of t [i.e., $f^*(t^*) = f(t)$] with a right-hand cut on the real axis at $t_0 \le t < \infty$. Without loss of generality, we can assume that the threshold t_0 is positive. First, a univalent function is a function $f(t)$ which never assumes the same value twice, i.e., it satisfies $f(t_1) \ne f(t_2)$ in the entire cut plane whenever $t_1 \ne t_2$. Second, let us introduce another class of analytic functions, which we shall call semimonotonic for lack of a better terminology. This is a real analytic function $f(t)$ which satisfies (i) a standard dispersion relation with a finite number of subtractions, and (ii) a condition that $\mathrm{Im} f(t + i\epsilon)$ does not change its sign on the entire cut at $\infty > t \ge t_0$. Then, the main conclusions of this paper are as follows: We assume either that $f(t)$ is a univalent function of t in the cut plane or that $f(t)$ is semimonotonic and satisfies an at most once-subtracted dispersion relation. Under either of these conditions, we can prove the following inequalities:

$$\frac{t_0-t}{t_0}\left|\frac{f(t)-f(0)}{t}\right| \leqslant |f'(0)| \leqslant \left|\frac{f(t)-f(0)}{t}\right| \quad \text{for } t_0 > t \geqslant 0, \quad (4)$$

$$\frac{t_0-t}{t_0}\left|\frac{f(t)-f(0)}{t}\right| \geqslant |f'(0)| \geqslant \left|\frac{f(t)-f(0)}{t}\right| \quad \text{for } 0 \geqslant t, \quad (5)$$

$$0 \leqslant \frac{t_0 f''(0)}{f'(0)} \leqslant 2. \quad (6)$$

In addition, if we assume that $f(t)$ satisfies the unsubtracted dispersion relation or that $f(t)$ is univalent with no zero point in the entire cut plane, then we have

$$t_0|f'(0)| \leqslant |f(0)| \quad (7)$$

as well as

$$\frac{t_0-t}{t_0} \leqslant \frac{f(t)}{f(0)} \leqslant \frac{t_0}{t_0-t} \qquad (t_0 > t \geqslant 0). \quad (8)$$

Actually, we can derive a few other inequalities, but these are sufficient for the moment.

Before proving these bounds, we remark that for the K_{l3} decay problem we are mainly concerned with we have

$$t_0 = (m_K + m_\pi)^2. \quad (9)$$

Next, let us identify $f(t)$ first with the K_{l3} scalar form factor $D(t)$ defined by

$$D(t) = (m_K^2 - m_\pi^2) f_+(t) + t f_-(t), \quad (10)$$

where $f_\pm(t)$ are the standard form factors.[13] Then the inequality (7) immediately gives us

$$|\Lambda_0| \leqslant \frac{m_\pi^2}{t_0} = 0.046, \quad (11)$$

which contradicts the present experimental value Eq. (3). This may imply that the extra assumption of either an unsubtracted dispersion relation or a no-zero-point hypothesis for the univalent case may be experimentally ruled out for $D(t)$, and hence we shall not assume this extra ansatz for a while.

The inequality (6) leads to

$$0 \leqslant \frac{m_\pi^2 D''(0)}{D'(0)} \leqslant \frac{2m_\pi^2}{t_0} = 0.092. \quad (12)$$

So far the accurate experimental value for $m_\pi^2 D''(0)/D'(0)$ is not available. However, if we assume that the slope of $f_-(t)$ is negligible, then the present data appear to indicate[8] a relatively large and negative value for $m_\pi^2 D''(0)/D'(0)$, thus contradicting our bound Eq. (12).

Up to now, we did not make use of the soft-pion theorem[11]

$$\frac{D(\delta)}{D(0)} \simeq \frac{f_K}{f_\pi f_+(0)} \simeq 1.28, \qquad \delta = m_K^2 - m_\pi^2. \quad (13)$$

If we assume this extra information, then Eq. (4) with $t = \delta = m_K^2 - m_\pi^2$ leads to

$$0.010 \leqslant |\Lambda_0| \leqslant 0.023. \quad (14)$$

Again, this contradicts the present world-averaged value. Therefore, we conclude (i) that $D(t)$ cannot be a univalent function of t, and (ii) that $\operatorname{Im} D(t+i\epsilon)$ must assume both positive and negative values on the cut, if $D(t)$ satisfies an at most once-subtracted dispersion relation. In particular, this last statement explains the reason why the so-called κ-dominance model does not work. Of course, we must keep in mind the fact that the present experimental situation on the K_{l3} decay is very confused and far from being final. Hence, it is still possible that our bounds may turn out to be still compatible with the experiment. Then our results will be of some interest since our theory does not contain any adjustable free parameters at all. However, on the other hand, it must be emphasized that there is *a priori* no theoretical basis for believing in the validity of our assumptions, apart from their simplicity.

Next, we can apply our theorems to the vector form factor $f_+(t)$. In that case, Eq. (7) gives us

$$|\lambda_+| \leqslant \frac{m_\pi^2}{t_0} = 0.046, \quad (15)$$

which is consistent with the experimental value[8] of $\lambda_+ = 0.012 \pm 0.005$. Also, the inequality (6) leads to

$$0 \leqslant \frac{m_\pi^2 f_+''(0)}{f_+'(0)} \leqslant \frac{2m_\pi^2}{t_0} = 0.092, \quad (16)$$

which should be compared with the experimental value[8] of

$$\frac{m_\pi^4 f_+''(0)}{f_+(0)} = 0.0104 \pm 0.0026,$$

$$\frac{m_\pi^2 f_+'(0)}{f_+(0)} = 0.012 \pm 0.005.$$

Now, let us proceed to prove our inequalities. First, let us consider the case when $f(t)$ is univalent in the cut t plane. It is then convenient to map our cut t plane inside the interior of the unit disk $|z| < 1$ by the conformal mapping

$$(t_0 - t)^{1/2} = (t_0)^{1/2} \frac{1+z}{1-z} \quad (17)$$

and set

$$f(t) \equiv F(z). \quad (18)$$

Then, $F(z)$ is a real univalent function in $|z|<1$. Moreover, if we define

$$g(z)=\frac{1}{F'(0)}[F(z)-F(0)], \tag{19}$$

then $g(z)$ is also a univalent function and satisfies the constraints

$$g(0)=0, \quad g'(0)=1, \quad g^*(z^*)=g(z). \tag{20}$$

It is well known[14] that such a function must satisfy

$$\frac{|z|}{(1+|z|)^2} \leq |g(z)| \leq \frac{|z|}{(1-|z|)^2} \quad (|z|\leq 1), \tag{21}$$

$$\left|\frac{1}{n!}g^{(n)}(0)\right| \leq n \qquad (n=2,3,4,\ldots). \tag{22}$$

Rewriting these inequalities in terms of $f(t)$, we find Eqs. (4)–(6) immediately.

If $f(t)$ satisfies an unsubtracted dispersion relation, then we must impose an additional constraint

$$\lim_{t\to\infty} f(t)=0.$$

In particular, letting $t\to-\infty$ in Eq. (5), this reproduces Eq. (7). Alternatively we can also prove Eq. (7), or more generally Eq. (8), if we instead assume that $f(t)$ has no zero point at all in the entire cut plane. In that case, the inverse function $1/F(z)$ is a real univalent analytic function, and this fact leads to an inequality similar to Eq. (21). From these, we can easily find that $F(z)$ must then satisfy

$$\left(\frac{1-|z|}{1+|z|}\right)^2 \leq \left|\frac{F(z)}{F(0)}\right| \leq \left(\frac{1+|z|}{1-|z|}\right)^2, \tag{23}$$

which is rewritten as Eq. (8), while Eq. (7) is obtained from Eq. (8) by letting $t\to 0$.

Next, let us discuss the case where $f(t)$ is now semimonotonic and satisfies a once-subtracted dispersion relation:

$$f(t)=f(0)+\frac{t}{\pi}\int_{t_0}^{\infty}dt'\,\frac{1}{t'(t'-t)}\,\mathrm{Im}f(t'+i\epsilon). \tag{24}$$

By assumption of semimonotonicity, we have either $\mathrm{Im}f(t'+i\epsilon)\geq 0$ or $\mathrm{Im}f(t'+i\epsilon)\leq 0$ on the entire cut so that Eq. (24) gives us

$$\left|\frac{f(t)-f(0)}{t}\right| = \frac{1}{\pi}\int_{t_0}^{\infty}dt'\,\frac{|\mathrm{Im}f(t'+i\epsilon)|}{t'(t'-t)} \quad (t\leq t_0),$$

$$\left|\frac{1}{n!}f^{(n)}(0)\right| = \frac{1}{\pi}\int_{t_0}^{\infty}dt'\,\frac{|\mathrm{Im}f(t'+i\epsilon)|}{(t')^{n+1}} \quad (n\geq 1).$$

Then it is an easy matter to check the validity of Eqs. (4)–(6) from these equations.

We can also prove

$$\frac{t_0}{t_0-t}\left|\frac{f(\delta)-f(0)}{\delta}\right| \geq \left|\frac{f(t)-f(\delta)}{t-\delta}\right| \geq \left|\frac{f(\delta)-f(0)}{\delta}\right| \tag{25}$$

for $t_0>\delta>t>0$. Identifying $f(t)=D(t)$ with $\delta=m_K^2-m_\pi^2$ provides a bound for physical values of $D(t)$ in terms of $D(0)$ and $D(\delta)$. This relation is analogous to that discussed by Bourrely.[7] Just as in that case, the present inequality (25) appears to be badly violated by the experimental data. Also, as we shall observe shortly, we can prove

$$\frac{d^2}{dt^2}\ln\left|\frac{f(t)-f(0)}{t}\right| \geq 0 \quad (t\leq t_0). \tag{26}$$

If a semimonotonic function $f(t)$ satisfies an unsubtracted dispersion relation, then we can derive better bounds, since we now have

$$f(t)=\frac{1}{\pi}\int_{t_0}^{\infty}dt'\,\frac{1}{t'-t}\,\mathrm{Im}f(t'+i\epsilon). \tag{27}$$

Then it is easy to derive

$$1\leq\frac{f(t)}{f(0)}\leq\frac{t_0}{t_0-t} \quad (t_0>t\geq 0), \tag{28}$$

which improves Eq. (8). Moreover, Eq. (27) together with the semimonotonicity of $f(t)$ implies

$$\frac{d^n}{dt^n}|f(t)|\geq 0 \tag{29}$$

for all $t\leq t_0$ and for all $n=0,1,2,\ldots$. Any real function satisfying such a condition is known as an absolutely monotonic function,[15] and must automatically obey an inequality[15]

$$\frac{d^2}{dt^2}\ln|f(t)|\geq 0 \quad (t\leq t_0), \tag{30}$$

i.e., $\ln|f(t)|$ is a convex function of t below the threshold. First, the inequality (30) gives us

$$\frac{f''(0)}{f'(0)}\geq\frac{f'(0)}{f(0)}\geq 0, \tag{31}$$

which improves the lower bound in Eq. (6). Also, the convexity of $\ln|f(t)|$ demands that we have

$$\lambda\ln|f(t_1)|+(1-\lambda)\ln|f(t_2)|\geq\ln|f(t)|,$$
$$t=\lambda t_1+(1-\lambda)t_2,$$
$$1>\lambda>0 \tag{32}$$

where t_1 and t_2 are two arbitrary real points less than t_0. This inequality may be tested for the vector form factor $f_+(t)$, if the experimental data become more accurate. Also, we can apply all of our inequalities to electromagnetic form factors of pions and nucleons. However, in view of the smallness of $t_0 = 4m_\pi^2$ for this case, our bounds are not strong enough to be of special interest, except possibly for the convex inequality (32).

If $f(t)$ satisfies the once-subtracted dispersion relation instead of the unsubtracted form, then a similar method proves Eq. (26). Moreover, if $f(t)$ requires two subtractions, we can prove only a weaker relation:

$$t^2|f''(0)| \leq 2|f(t) - f(0) - tf'(0)|$$

for $0 \leq t \leq t_0$.

Finally, it may be worthwhile to observe a possible connection between semimonotonicity and univalency. If $f(t)$ satisfies a once-subtracted dispersion relation and if it is semimonotonic, then a function $g(t)$ defined by

$$g(t) = \int_0^t dt' \frac{f(t') - f(0)}{t'}$$

is an univalent function in the cut t plane where the integral path is to be taken as a straight line. The proof for this statement is essentially the same as that given by Khuri and Kinoshita[16] for a different problem. However, this fact does not appear to give any better inequality in our case.

Last, any univalent function $F(z)$ which is analytic in the unit disk $|z| < 1$ [see Eq. (18)] is known[17] to belong automatically to the H^p class for all p with $0 < p < \frac{1}{2}$. Moreover, its singular part of its standard factorization[17] is an identity. In particular, these imply that the finite boundary function $f(t + i\epsilon)$ exists almost everywhere for $t \geq t_0$, and that $f(t)$ satisfies an at most once-subtracted ordinary dispersion relation, if the possible singularity of $f(t)$ on the cut is to be found only at infinity as is usually assumed.

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