Noncausal Propagation of Classical Rarita-Schwinger Waves*

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The Rarita-Schwinger field coupled to scalar and Dirac fields is considered at the classical level. It is shown that the fermion fields satisfy a system of hyperbolic equations supplemented by initial conditions whose solutions propagate faster than light, thereby violating causality.

Recently Nath, Etemadi, and Kimel[1] have attempted to quantize the Rarita-Schwinger field interacting with a spin-$\frac{1}{2}$ field and the derivative of a spin-zero field. Hagen[2] has shown, however, that a consistent quantization of such a theory is not possible within the framework of the action principle, and that the inconsistencies which one encounters are in fact quite similar to those which occur in the case of the electromagnetic interaction of the spin-$\frac{3}{2}$ field.[3]

Velo and Zwanziger[4] have demonstrated that in the case of the electromagnetic interaction there are difficulties even at the classical level inasmuch as the equations of motion admit solutions that propagate faster than light, in violation of causality. In the present work we examine the field equations satisfied by the fermion fields ψ^μ and ψ, interacting with the derivative of an external scalar field ϕ in the manner proposed by Nath *et al.* It is found, as in Ref. 4, that noncausal solutions are present.

The field equations are[5,6]

$$\left(\frac{1}{i}\gamma\cdot\partial+m\right)\psi-ig\sigma_{\mu\nu}\psi^\nu\partial^\mu\phi=0\,, \tag{1}$$

$$[\gamma^\mu g_{\alpha\beta}-(\delta^\mu_\alpha\gamma_\beta+\delta^\mu_\beta\gamma_\alpha)-\gamma_\alpha\gamma^\mu\gamma_\beta]\frac{1}{i}\partial_\mu\psi^\beta+m(g_{\alpha\beta}+\gamma_\alpha\gamma_\beta)\psi^\beta-ig\sigma_{\alpha\nu}\psi\partial^\nu\phi=0\,, \tag{2}$$

where

$$\sigma_{\mu\nu}=\tfrac{1}{2}i[\gamma_\mu,\gamma_\nu]\,. \tag{3}$$

Not all of Eqs. (2) are true equations of motion. Considering the equation with $\alpha=0$ one gets the primary constraint

$$(i\vec{\gamma}\cdot\vec{\nabla}+m)(\vec{\gamma}\cdot\vec{\psi})+i(\vec{\nabla}\cdot\vec{\psi})+ig\beta\sigma_{0\nu}\psi\partial^\nu\phi=0\,. \tag{4}$$

Upon contraction of Eq. (2) with γ^α and $i\partial^\alpha$ one obtains

$$2i[(\gamma\cdot\partial)(\gamma\cdot\psi)+(\partial\cdot\psi)]+3m(\gamma\cdot\psi)+ig\gamma^\mu\sigma_{\mu\nu}\psi\partial^\nu\phi=0\,, \tag{5}$$

$$im[(\gamma\cdot\partial)(\gamma\cdot\psi)+(\partial\cdot\psi)]+g\partial^\mu(\sigma_{\mu\nu}\psi\partial^\nu\phi)=0\,, \tag{6}$$

which in turn lead to the secondary constraint

$$(\gamma\cdot\psi)-\tfrac{2}{3}gm^{-2}\partial^\mu(\sigma_{\mu\nu}\psi\partial^\nu\phi)+\tfrac{1}{3}igm^{-1}\gamma^\mu\sigma_{\mu\nu}\psi\partial^\nu\phi=0\,. \tag{7}$$

Substituting for $(\gamma\cdot\psi)$ from this into Eq. (5) or (6) there follows the useful relation

$$(\partial\cdot\psi)-\tfrac{1}{3}igm^{-2}[(3m+2i\gamma\cdot\partial)\partial^\mu+m(\gamma\cdot\partial)\gamma^\mu]\sigma_{\mu\nu}\psi\partial^\nu\phi=0\,. \tag{8}$$

Using these, one obtains equations of motion for all field components, which, when supplemented by certain initial conditions, are equivalent to the field equations (1) and (2). Eliminating $(\gamma\cdot\psi)$ and $(\partial\cdot\psi)$ from Eq. (2) with the help of Eqs. (7) and (8) one gets

$$(-i\gamma\cdot\partial+m)\psi^\alpha+\tfrac{2}{3}igm^{-2}\partial^\alpha\partial^\mu(\sigma_{\mu\nu}\psi\partial^\nu\phi)-\tfrac{1}{3}igm^{-1}[3mg^{\alpha\mu}+(m\gamma^\alpha+i\partial^\alpha)\gamma^\mu-i\gamma^\alpha\partial^\mu]\sigma_{\mu\nu}\psi\partial^\nu\phi=0\,, \tag{9}$$

which is, now, a true equation of motion for ψ^α. Since we want the equations to be of the first order while Eq. (9) contains second derivatives of the spinor field, it is necessary to introduce new field variables χ^k, $k=1,2,3$ by

$$\chi^k-\partial^k\psi=0\,, \tag{10}$$

for which one has the equation

$$(-i\gamma\cdot\partial+m)\chi^k-ig\sigma_{\mu\nu}\partial^k(\psi^\nu\partial^\mu\phi)=0\,. \tag{11}$$

Since this definition is noncovariant, manifest covariance is lost from this point on.

Using Eqs. (3) and (10) we can replace first derivatives of ψ by terms involving ψ, χ^k, ψ^α. The second derivatives of ψ can be expressed in terms of the fields and their first derivatives. Carrying out this program in Eqs. (7) and (9) one gets, in that order,

$$(\gamma\cdot\psi)+\tfrac{1}{3}igm^{-1}\gamma^{\mu}\sigma_{\mu\nu}\psi\partial^{\nu}\phi+\tfrac{2}{3}gm^{-2}\{(\vec{\gamma}\cdot\vec{\nabla}\phi)[m\psi-ig\sigma_{\mu\nu}\psi^{\nu}\partial^{\mu}\phi]+i(\vec{\nabla}\phi+\beta\vec{\gamma}\partial_0\phi)\cdot\vec{\chi}\}=0\,, \tag{12}$$

$$(-i\gamma\cdot\partial+m)\psi^{\alpha}-\tfrac{2}{3}g^2m^{-2}(\vec{\gamma}\cdot\vec{\nabla}\phi)\sigma_{\mu\nu}(\partial^{\alpha}\psi^{\nu})\partial^{\mu}\phi+\tfrac{2}{3}gm^{-2}(\vec{\nabla}\phi+\beta\vec{\gamma}\partial_0\phi)\cdot\partial^{\alpha}\vec{\chi}+B^{\alpha}=0\,, \tag{13}$$

where B^{α} contains no field derivatives.

Consider Eqs. (1), (11), and (13). They are satisfied whenever the field equations are satisfied, and conversely, these equations supplemented by initial conditions (4), (10), and (12) imply the field equations (1) and (2). This can be shown using the same arguments as used in Appendix A of Ref. 4. Thus Eqs. (1), (11), and (13) with the initial conditions are completely equivalent to the field equations, but have the advantage of being true equations of motion.

We now proceed to examine the characteristic surfaces of our equations, as these provide the propagation properties of the solutions. We are interested in establishing that the characteristic surfaces are spacelike for some values of the external field ϕ. Since the maximum velocity of propagation of disturbances is given by the slope of the characteristic surface, we would have then established the existence of noncausal solutions.

The normals n_{μ} to the characteristic surface for a linear system of equations of the form

$$[(\Gamma^{\mu}\partial_{\mu})^{\alpha}{}_{\beta}+B^{\alpha}{}_{\beta}]\Psi^{\beta}=0$$

are given by[7]

$$D(n)=\det|(n_{\mu}\Gamma^{\mu})^{\alpha}{}_{\beta}|=0\,. \tag{14}$$

We want to see if a timelike vector n_{μ} satisfies (14). We try $n_{\mu}=(0,0,0;n)$. Taking the coefficient of ∂_0 in Eqs. (1), (11), and (13), and evaluating the resulting 32×32 determinant, one gets

$$D(n)=n^{32}[1-\tfrac{2}{3}g^2m^{-2}(\vec{\nabla}\phi)^2]^4\,. \tag{15}$$

If the external field is such that for each spacetime point x there exists a Lorentz frame such that

$$1-\tfrac{2}{3}g^2m^{-2}(\vec{\nabla}\phi)^2\geq 0,\quad \partial_{\mu}\phi\neq 0$$

then in some frame one has

$$1-\tfrac{2}{3}g^2m^{-2}(\vec{\nabla}\phi)^2=0 \tag{16}$$

and thus a normal in the time direction. Consequently the characteristic surface is spacelike and we have noncausal solutions.

One might hope that these solutions are eliminated by the initial conditions (4), (10), and (12). Such, however, is not the case. The discontinuities in the field derivatives propagate along the characteristic surfaces.[8] We shall show that those traveling along the bad characteristics are compatible with the constraints.

Let the discontinuities in the field derivatives be

$$\begin{aligned}[][\partial_{\mu}\psi]&=n_{\mu}u,\\ [\partial_{\mu}\psi^{\alpha}]&=n_{\mu}\omega^{\alpha},\\ [\partial_{\mu}\chi^{k}]&=n_{\mu}v^{k},\end{aligned} \tag{17}$$

where $[f]$ denotes the discontinuity in the function f. If these travel along the bad characteristics, n_{μ} should be $(0,0,0;n)$.

Taking the discontinuities of Eqs. (10) and (4), and of the four-gradient of Eq. (12), one gets the conditions imposed by the constraints:

$$n^{k}u=0\,, \tag{18}$$

$$\vec{\sigma}\cdot(\vec{n}\times\vec{\omega})=0\,, \tag{19}$$

$$(\gamma\cdot\omega)+\tfrac{1}{3}igm^{-1}\gamma^{\mu}\sigma_{\mu\nu}u\partial^{\nu}\phi+\tfrac{2}{3}gm^{-2}\{(\vec{\gamma}\cdot\vec{\nabla}\phi)[mu-ig\sigma_{\mu\nu}\omega^{\nu}\partial^{\mu}\phi]+i(\vec{\nabla}\phi+\beta\vec{\gamma}\partial_0\phi)\cdot\vec{v}\}=0\,, \tag{20}$$

where we have assumed that the external field is sufficiently differentiable.

Taking the discontinuities of the equations of motion one gets

$$(\gamma\cdot n)u=0\,, \tag{21}$$

$$(\gamma\cdot n)v^{k}+g\sigma_{\mu\nu}n^{k}\omega^{\nu}\partial^{\mu}\phi=0\,, \tag{22}$$

$$(\gamma\cdot n)\omega^{\alpha}-\tfrac{2}{3}ig^2m^{-2}(\vec{\gamma}\cdot\vec{\nabla}\phi)\sigma_{\mu\nu}n^{\alpha}\omega^{\nu}\partial^{\mu}\phi+\tfrac{2}{3}igm^{-2}(\vec{\nabla}\phi+\beta\vec{\gamma}\partial_0\phi)\cdot\vec{v}\,n^{\alpha}=0\,. \tag{23}$$

We now show that there are nontrivial solutions u, v^k, ω^{α} (i.e., not all of them identically vanish) satisfying Eqs. (18)–(23) that are compatible with $n_{\mu}\equiv(0,0,0;n)$ and Eq. (16).

With $n_{\mu}=(0,0,0;n)$, Eqs. (18), (19), and (22) are satisfied identically. If we further take

$$u=0,\quad v^k=0=\omega^k,\quad \text{and } \omega^0=f, \tag{24}$$

(21) is also identically satisfied. Equations (20) and (23) coincide, becoming

$$[\gamma^0+\tfrac{2}{3}ig^2m^{-2}(\vec{\gamma}\cdot\vec{\nabla}\phi)\sigma_{k0}\partial^{k}\phi]f=0\,. \tag{25}$$

Multiplying by γ^0 and substituting for σ_{0k}, this becomes

$$[1-\tfrac{2}{3}g^2m^{-2}(\vec{\nabla}\phi)^2]f=0\,,$$

which is identically satisfied by virtue of Eq. (16). This establishes the compatibility of the noncausal solutions with the constraints.

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Gauge Model of Vector-Meson Masses*

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We construct a model of the strong interactions with spontaneously broken U(3)×U(3) gauge symmetry based on the ideas of Bardakci and Halpern and interpret the gauge bosons as the observed strongly interacting vector mesons. We assume that global SU(3) is broken only by terms of dimensions less than four in the Lagrangian. The model that arises is a trivial generalization of the "strong" part of the Bars-Halpern-Yoshimura model. This leads to a well-satisfied relation among the ρ, K^*, A_1, K_A, ω, and ϕ masses.

In this paper we describe a model of strong interactions with spontaneously broken gauge symmetry, based on the ideas of Bardakci and Halpern.[1] We interpret the gauge particles in the model as the observed strongly interacting vector mesons and find a relation among the vector-meson masses. The Lagrangian which emerges is of little value as a phenomenological description of the strong interactions because of the large coupling constants involved; however, it may not be unreasonable to hope that the predicted relation among the vector-meson masses is more reliable, since it follows simply from the gauge structure of the theory.

A further justification for studying such a model can be found in the work of Bars, Halpern, and Yoshimura[2] on unifying strong, weak, and electromagnetic interactions. In fact, the model we construct is just a simple generalization of the "strong" part of their model, with weak and electromagnetic interactions turned off in an appropriate sense.

In constructing the model, we make three kinds of assumptions.

(i) Gauge symmetry properties. We assume that the Lagrangian which describes the strong interactions is invariant under gauge transformations in the chiral group U(3)×U(3). There are 18 gauge vector mesons R_i^μ and L_i^μ for $i=1$ to 8 and R_9^μ and L_9^μ, where $(R^\mu \pm L^\mu)$ are vector and axial-vector, respectively. The fermions in the model are the fractionally charged quarks

$$q=\begin{pmatrix}\mathcal{P}\\ \mathfrak{N}\\ \lambda\end{pmatrix}.$$

Under an infinitesimal gauge transformation, they transform like

$$\delta q_L=i\omega_L^i\tfrac{1}{2}\lambda_i q_L+i\beta\omega_L^9\tfrac{1}{2}\lambda_9 q_L\,,$$

$$\delta q_R=i\omega_R^i\tfrac{1}{2}\lambda_i q_R+i\beta\omega_R^9\tfrac{1}{2}\lambda_9 q_R\,,$$

where the λ's are the usual SU(3) matrices. The