Bethe-Salpeter Equations with $s=0$ Solution as Input

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Bethe-Salpeter equations with energy-independent kernels are reexpressed in a form in which the kernel is eliminated and the input is the $s=0$ scattering solution. In a model of particle-antiparticle scattering in which each particle has mass M and spin $\frac{1}{2}$, the greater convergence of this form is exploited in the kinematic region $s \ll M^2$ to give analytic scattering solutions when certain approximations are made. A sufficiently strong $s=0$ input amplitude leads to bound states of mass $\ll M$ with explicit vertex functions. The decay amplitude of one bound state into two others is calculated explicitly and found to be of a strength characteristic of the strong interactions.

The Bethe-Salpeter (BS) equation is the only known way to find infinite sums of Feynman diagrams. These infinite sums are necessary to derive bound states from field theory. A great deal of work has been done on the BS equation and extensive reviews have been written on it.[1] However, it has always been extremely difficult to solve BS equations and obtain predictions from them. Let M be the mass of each of two particles whose interaction is described by a BS equation, and let s be the square of their c.m. energy. The intractability of the BS equation towards attempts at solving it is epitomized by the simple fact that (in spite of many people's efforts) bound-state solutions have never been obtained by analytic methods, however approximate, except in the low-energy, nonrelativistic kinematic region $|s-4M^2| \ll M^2$. This is precisely the region for which the Schrödinger equation is usually adequate and field theory is not really necessary.

In this article we would like to show that in certain ways, some BS equations are also surprisingly tractable in the kinematic region $s \ll M^2$. Specifically, we will point out that BS equations with energy-independent kernels (which include most cases treated in practice) can be reexpressed in a form with the following properties: (Sec. I) The kernel is eliminated in favor of the $s=0$ scattering solution. (Sec. II) The resultant equation is more convergent. (Sec. III) In the region $s \ll M^2$, with a suitable input and some approximations, analytic scattering solutions can be obtained. (Sec. IV) If the input $s=0$ amplitude is sufficiently strong, bound states lie in this region $s \ll M^2$ and their vertex functions are explicit. (Sec. V) These bound-state vertex functions can be used in a triangle diagram to calculate the decay amplitude of one bound state into two others. We calculate one such amplitude explicitly and find its magnitude to be characteristic of the strong interactions.

For definiteness, and because the problem is considered to be difficult, we will consider the BS equation for the interaction of a spin-$\frac{1}{2}$ particle and antiparticle. We denote their scattering Green's function by $T_K = T_{\alpha\rho;\sigma\beta}(p,q;K)$, in which the Dirac indices and four-momenta of the fermion lines are (final) $\alpha, p+\frac{1}{2}K$ and $\rho, p-\frac{1}{2}K$; (initial) $\beta, q+\frac{1}{2}K$ and $\sigma, q-\frac{1}{2}K$. K is the total four-momentum; $s=-K^2$. The BS equation can be written in symbolic form as $T_K = I + IS_K T_K$, where S_K stands for the pair of free-particle propagators. Most of the explicit BS equations which are considered use energy-independent kernels I. We will treat this case.

I. REFORMULATION IN TERMS OF $s=0$ AMPLITUDE

When $K=0$ (so $s=0$) the BS equation is $T_0 = I + IS_0T_0$. Using this to eliminate I from the $K\neq 0$ equation gives the equation $T_K = T_0 + T_0(S_K - S_0)T_K$. Written out in full, this equation reads

$$T_{\alpha\rho;\sigma\beta}(p,q;K) = T_{\alpha\rho;\sigma\beta}(p,q;0) + i\int \frac{d^4k}{(2\pi)^4}\, T_{\alpha\rho;\theta\delta}(p,k;0) \times [G_{\delta\epsilon}(k+\tfrac{1}{2}K)G_{\lambda\theta}(k-\tfrac{1}{2}K) - G_{\delta\epsilon}(k)G_{\lambda\theta}(k)]T_{\epsilon\lambda;\sigma\beta}(k,q;K)\,. \quad (1)$$

Wick rotation has been used: $p^0 = ip_4$, $d^4k = d\vec{k}\,dk_4$. In the infinite-series solution of the BS equation, Wick rotation is known to be justified term by term below threshold, the region to which we confine the application. Also, Wick rotation of the solutions to BS equations in renormalizable theories has been stated by Domokos to be justified.[2]

Only for the electromagnetic interaction is the kernel of the BS equation known well. In most other physical examples just as much – if not more – is known about the $s=0$ scattering amplitude, since it is the forward scattering amplitude in the crossed channel. Accordingly, it seems reasonable to examine the form (1) of the BS equation, regarding the amplitude T_0 as the input.

II. GREATER CONVERGENCE OF INTEGRAL

The fact that at high loop momentum k, S_K has the comparatively slow dropoff $S_K \sim 1/k^2$ [we suppose $G(k) = 1/(\gamma\cdot k + M)$] causes the integral in spin-$\frac{1}{2}$ BS equations to converge rather slowly, which has led to well-known difficulties (inapplicability of Fredholm theory, undefinable norms).[3] We will show that Eq. (1) has better convergence properties, by choosing a simple specific model as an example.

Assumption 1. The input amplitude T_0 has the form

$$T_{\alpha\rho;\sigma\beta}(p,q;0) = -iT(p,q;0)(i\gamma_5)_{\alpha\rho}(i\gamma_5)_{\sigma\beta} \tag{2}$$

(Dirac matrix factorization in the s channel). Substitution of this input into Eq. (1) shows that the solution has the same form:

$$T_{\alpha\rho;\sigma\beta}(p,q;K) = -iT(p,q;K)(i\gamma_5)_{\alpha\rho}(i\gamma_5)_{\sigma\beta}\,, \tag{3}$$

where the scalar $T(p,q;K)$ satisfies the scalar equation

$$T(p,q;K) = T(p,q;0) + \int \frac{d^4k}{(2\pi)^4}\,T(p,k;0)\frac{-3(k^2+M^2)^2K^2 + 4(k^2+M^2)(k\cdot K)^2 - \frac{1}{4}(k^2+M^2)K^4}{(k^2+M^2)^2[(k^2+M^2+\frac{1}{4}K^2)^2 - (k\cdot K)^2]}\,T(k,q;K)\,. \tag{4}$$

It is clear that in this case the term $S_K - S_0$ behaves as $1/k^4$ at large k. (In any model it behaves as $1/k^n$, $n \geq 3$.) The asymptotic behavior of this term now resembles that of the conventional scalar particle BS equation, which is known to be more amenable to conventional discussions (Fredholm theory, definable norms).[4] We will exploit this property in the case $s \ll M^2$.

III. SOLUTIONS WHEN $s \ll M^2$

Assumption 2. We stay in the kinematic region $s \ll M^2$. In this case Eq. (4) to lowest order in s/M^2 is explicitly

$$T(p,q;K) = T(p,q;0) + 3s\int \frac{d^4k}{(2\pi)^4}T(p,k;0)\frac{1}{(k^2+M^2)^2}\left(1 - \frac{4}{3}\,\frac{k_4{}^2}{k^2+M^2}\right)T(k,q;K) \tag{5}$$

in the c.m. frame.

Remark 1. The error in (5) is of order $(s/M^2)^2$, which we can make as small as we like.

Remark 2. Equation (5) has the form of an O(4)-invariant part, plus a non-O(4)-invariant part involving the term $R \equiv -\frac{4}{3}k_4{}^2/(k^2+M^2)$.

Remark 3. If the term R is neglected, it is well known that choosing the specific input $T(p,q;0) = 1/[\epsilon(p-q)^2]$ allows equations of the form of (5) to be solved analytically by a method applied by Cutkosky to the scalar-particle BS equation.

Remark 3 suggests the possibility of choosing inputs to Eq. (5) for which Cutkosky's analytic method of solution can be used; Remark 1 suggests the possibility of treating the term R by first-order perturbation theory. That is how we will proceed.

Cutkosky used the following transformations[5]:

$$u_i = \frac{2Mp_i}{p^2+M^2}\ \ (i=1,2,3,4),\quad u_5 = \frac{p^2-M^2}{p^2+M^2}\,, \tag{6}$$

(abbreviated as $p \to u$) and

$$T(p,q;K) = \frac{2M}{p^2+M^2}\,t(u,v;s)\,\frac{2M}{q^2+M^2} \tag{7}$$

(where $q \to v$). Then Eq. (5) transforms into the following equation on the compact unit 5-sphere $u^2=1$:

$$t(u,v;s) = t(u,v;0) + \frac{3s}{4M^2(2\pi)^4}\int d\Omega_w\, t(u,w;0)(1+R)t(w,v;s)\,, \tag{8}$$

where $k \to w$ and

$$R \equiv -\frac{4}{3}\frac{k_4^{\,2}}{k^2+M^2} = -\frac{2}{3}\frac{w_4^{\,2}}{1-w_5}\,.$$

Remark 4. The well-known "solvable" input referred to in Remark 3, $T(p,q;0)=1/[\epsilon(p-q)^2]$, transforms by (6), (7) into the form

$$t(u,v;0)=\frac{1}{\epsilon(u-v)^2}$$
$$=\sum_{Pnlm}\frac{4\pi^2}{\epsilon(P+1)(P+2)}Y(Pnlm;u)Y^*(Pnlm;v)\,,$$

where the $Y(Pnlm;u)$ are orthonormal surface harmonics on the 5-sphere,[6] with $P \geq n \geq l \geq |m| \geq 0$.

Now we can implement the suggestions after Remark 3 by supposing the following:

Assumption 3. The input $T(p,q;0)$ of Eq. (5) transforms into the same type of diagonal form as the above:

$$t(u,v;0)=\sum_{Pnlm} t_{Pn}Y(Pnlm;u)Y^*(Pnlm;v)\,. \tag{9}$$

Assumption 4. The O(4)-breaking term R in Eq. (8) can be treated as a first-order perturbation. (The accuracy of this assumption will be examined below.)

With these assumptions it is elementary to solve Eq. (8). The solution, transformed by (6) and (7) back into the original amplitude (3), is

$$T_{\alpha\rho;\sigma\beta}(p,q;K)=i\sum_{Pnlm}\frac{\Gamma^{Pnlm}_{\alpha\rho}(p,K)\bar{\Gamma}^{Pnlm}_{\sigma\beta}(q,K)}{s-s_{Pnl}}\,, \tag{10}$$

where

$$\Gamma^{Pnlm}(p,K)=\frac{16\pi^2M^2}{p^2+M^2}\frac{1}{[3(1+\langle R\rangle_{Pnl})]^{1/2}}Y(Pnlm;u)i\gamma_5\,, \tag{11}$$

$$\bar{\Gamma}^{Pnlm}(q,K)=\frac{16\pi^2M^2}{q^2+M^2}\frac{1}{[3(1+\langle R\rangle_{Pnl})]^{1/2}}Y^*(Pnlm;v)i\gamma_5\,,$$

and

$$s_{Pnl}=\frac{4M^2(2\pi)^4}{3t_{Pn}(1+\langle R\rangle_{Pnl})}\,, \tag{12}$$

in which

$$\langle R\rangle_{Pnl}\equiv\int d\Omega_u Y^*(Pnlm;u)RY(Pnlm;u)$$
$$=-\frac{1}{6}\left[\frac{(n+\frac{3}{2})^2-(l+\frac{1}{2})^2}{(n+\frac{3}{2})^2-\frac{1}{4}}\right]-\frac{1}{6}\left[\frac{(n+\frac{1}{2})^2-(l+\frac{1}{2})^2}{(n+\frac{1}{2})^2-\frac{1}{4}}\right]. \tag{13}$$

Remark 5. Equation (13) shows that $\langle R\rangle_{Pnl}$ is in general small compared to unity, providing a verification of Assumption 4.

IV. BOUND STATES

Although the solution (10) ostensibly has the form of a sum of pole terms, it is valid only in the region $s \ll M^2$. Therefore it cannot be said to have bound state poles unless $s_{Pnl} \ll M^2$ also. It is not known yet whether nature contains bound states whose mass is small compared to that of their constituent particles. Since the possibility of such bound states remains open, we will take this opportunity of considering them.

Assumption 5. The input amplitude T_0 is strong enough to cause $s_{Pnl} \ll M^2$ for some P, n, l.

Remark 6. Assumption 5 implies that the solution (10) contains bound states of mass $m_{Pnl}=\sqrt{s_{Pnl}}$ which have explicit vertex functions (11).

Remark 7. It can be shown that the bound-state vertex functions (11) are correctly normalized.[7]

Remark 8. An explicit form for t_{Pn} leads, by (12) and (13), to an explicit form for the bound-state poles, which therefore lie on explicit Regge trajectories. For example, suppose that we choose $t_{Pn}=\delta_{Pn}/[\epsilon(P-\alpha_0)]$. Then it is easy to see that the leading trajectory will have the form

$$s=\frac{4(2\pi)^2\epsilon M^2}{3}(\alpha-\alpha_0)\frac{1}{1-1/3(\alpha+2)}\,,$$

which is almost linear in the physical region. (Dynamical models give s in terms of α.) To satisfy Assumption 5, ϵ must be small: $\epsilon << 1/(2\pi)^4$.

Remark 9. Equations (9) and (12) show that there is a close relation between the positions of the bound-state poles and the input amplitude T_0. In fact the relationship between one and the other is so direct, under the assumptions made above, that it is hardly possible to say that one is more fundamental than the other. In studies of physical situations satisfying assumptions of the sort adopted here, one would try to correlate both together. (Note that $t_{Pn}=0$ implies the absence of poles s_{Pnl} from the spectrum.)

Remark 10. The situation is just the opposite for the bound-state vertex functions Γ, Eq. (11). They are independent of the coefficients t_{Pn} of the input amplitude. Thus in models of this general type, the vertex functions depend on the quantum numbers but not on the bound-state mass.

In view of Remark 10 it is clear that in possible physical applications, statements derived from the bound states' vertex functions would be less subject to arbitrary parametric variation than statements involving their masses. To conclude this paper we will ask a question answerable from the vertex functions alone. What is the strength of the amplitude for one of these bound states to decay into two others?

V. BOUND-STATE→TWO-BOUND-STATE DECAY AMPLITUDE

We will calculate one such amplitude from its triangle diagrams. The sides of the triangle are composed of the spin-$\frac{1}{2}$ constituents of the bound states. To fix ideas we will give the amplitude for a $J^PI^G=1^-1^+$ bound state V to decay into two $J^PI^G=0^-1^-$ bound states P.

The bound states whose vertex functions have been given above can be provided with unit isospin by multiplying their vertex functions by the factor $\vec{\tau}/\sqrt{2}$ (which preserves normalization). Using the $P=n=l=0$ vertex function in (11), the vertex function of the bound state P would be[6]

$$\Gamma^P(p,K)=\frac{16\,\pi^2M^2}{p^2+M^2}\left(\frac{3}{40\,\pi^2}\right)^{1/2} i\gamma_5\vec{\tau}\,.$$

None of the vertex functions (11) could be combined with $\vec{\tau}/\sqrt{2}$ to represent a V bound state. The quantum numbers are wrong. However, a second input containing scalar Dirac matrices 1 instead of $i\gamma_5$ can be added to the input (2). The equations decouple. The scalar solution has a similar form. The vertex function of the bound state V could be $\vec{\tau}/\sqrt{2}$ times the $P=n=l=1$ vertex function of this solution; it is

$$\Gamma^V(p,K)=\frac{16\,\pi^2M^2}{p^2+M^2}\left(\frac{63}{64\,\pi^2}\right)^{1/2}\frac{2M}{p^2+M^2}\vec{\epsilon}\cdot\vec{p}\,1\vec{\tau}\,.$$

The vertex functions Γ^P and Γ^V can now be used in the calculation of the $V\to PP$ decay amplitude (or vertex, if $2m_P>m_V$). The calculation of the triangle diagrams is simplified by Assumption 5. Furthermore, although the propagators along the sides of the triangle have the form $1/(\gamma\cdot k+M)$, the factors $1/(k^2+M^2)$ in each vertex function prevent the Feynman integral from diverging.

The $V\to PP$ amplitude given by this calculation is $(24\pi\sqrt{7}/25)\epsilon_{ijk}\vec{\epsilon}\cdot(\vec{p}_j-\vec{p}_k)$. (Here ϵ_{ijk} is the isospin factor and $\vec{p}_j, \vec{p}_k$ are the P momenta.) Denoting the numerical coefficient by f, we have $f^2/4\pi=5.07$. The physical ρ, π mesons have the same quantum numbers as our V, P bound states, and the $\rho\to\pi\pi$ decay amplitude has the same form, with $f^2/4\pi\approx 2.4$. This comparison shows that the decay amplitude of one bound state into two others has the magnitude of a strong interaction.

We recall that in this calculation the bound-state vertex functions used were obtained under the assumption (among others) that the masses of the bound states were small compared to the masses of their constituents. This assumption is the same as occurs in some current speculations on the composition of strongly interacting particles. Because of this, and because the decay amplitude we have just found is of the magnitude of a strong interaction, we are led to wonder whether BS equations in the kinematic region $s<<M^2$ might be relevant to the real world.

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