Is a Subtraction Necessary for the Pion Mechanical Form Factor?

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We investigate the consistency of ϵ and S^* dominance of the pion mechanical form factor with unsubtractedness, Ward identities, and s-wave $\pi\pi$ phenomenology. We conclude that the ϵ is not dominant as a result of the strong coupling of the S^* to both $\pi\pi$ and $K\bar{K}$ channels and an unsubtracted form factor is not ruled out in contrast to the conclusions of previous authors.

In recent years considerable attention has been paid to the concept of approximate scale invariance.[1] These ideas have both high- and low-energy consequences; it is the latter on which we wish to concentrate in this paper. The essence of the method for obtaining low-energy predictions from broken-scale invariance is to assume that the symmetry is realized through the Goldstone mechanism where the appropriate Goldstone boson is the scalar isoscalar σ (or ϵ) meson around 700 MeV. It emphasizes analogy with the corresponding approach in chiral symmetry where the pseudoscalar mesons are the Goldstone particles. Using such methods several authors[2-4] have derived Ward identities (WI) for the gravitational form factors of the pion in an attempt to gain some understanding of their dynamical behavior and the precise role of the ϵ.

In this paper we will only be interested in one of these form factors involving the nonvanishing trace of the energy-momentum tensor:

$$\langle\pi(p_1)|\Theta^\mu_\mu(0)|\pi(p_2)\rangle(4\omega_1\omega_2)^{1/2} \equiv f_\pi^{\,2}m_\pi^{\,4}\Gamma(q^2, m_\pi^{\,2}, m_\pi^{\,2}) = 2m_\pi^{\,2}G_S(q^2)\,, \tag{1}$$

where $q=p_1-p_2$ and $G_S(0)=1$. Assuming only ϵ dominance of the mechanical form factor $G_S(q^2)$, Kleinert and Weisz[2] (KW) have obtained a minimal form satisfying the Ward identities:

$$f_\pi^{\,2}m_\pi^{\,4}\Gamma(q^2, p_1^{\,2}, p_2^{\,2}) = \frac{aq^2+bm_\epsilon^{\,2}}{q^2-m_\epsilon^{\,2}}\,, \tag{2a}$$

$$\begin{aligned} a &= -m_\epsilon^{\,2}+(4-d)m_\pi^{\,2}\,,\\ b &= -2m_\pi^{\,2}+(1-d)(p_1^{\,2}+p_2^{\,2}-2m_\pi^{\,2})\,, \end{aligned} \tag{2b}$$

where the energy-momentum density has been assumed to have the form

$$\begin{aligned} \Theta_{00} &= \bar{\Theta}_{00}+\delta+u\,,\\ u &= u_0+cu_8\,. \end{aligned} \tag{3}$$

$\bar{\Theta}_{00}$ is scale- and SU(3)$\otimes$SU(3)-invariant; δ only breaks scale invariance; and u, which breaks both symmetries, has scale dimension d. The structure of $\Gamma(q^2)$ given in Eq. (2a) has two interesting aspects:

(a) Since the form factor does not vanish as $q^2 \to \infty$, there may either be a genuine subtraction in the dispersion relations for $\Gamma(q^2)$, or since the approximations involved in its derivation are low-energy ones, there may be other important contributions besides the ϵ which have been neglected.

(b) Equation (2a) indicates a large slope at zero momentum transfer,

$$2m_\pi^{\,2}G_S'(0) = 1+(d-2)m_\pi^{\,2}/m_\epsilon^{\,2}\,, \tag{4}$$

suggesting a large scalar mass radius,[3]

$$\begin{aligned} \tfrac{1}{6}(r_S^\pi)^2 &= G_S'(0)\\ &\approx 1/2m_\pi^{\,2}\,, \end{aligned} \tag{5}$$

approximately 15 times the corresponding expression for the pion's charge radius $\frac{1}{6}(r_{\rm ch}^\pi)^2 \approx 1/m_\rho^{\,2}$.

In this paper we adopt the point of view that neither of these properties is satisfactory. We impose a requirement that $\Gamma(q^2)\to 0$ as $q^2\to\infty$, and achieve this by taking account of the S^* pole, which recent data[5] suggest lies near the $K\bar{K}$ threshold around 1100 MeV. We show how dominance by the ϵ and S^* leads to a smooth and unsubtracted form for $\Gamma(q^2)$, which satisfies the WI and allows almost equal mass and charge radii for the pion.

We begin by giving modification of the WI analysis of KW assuming ϵ and S^* pole dominance, starting from their expressions[2]

$$f_\pi^{\,2}m_\pi^{\,4}\Gamma(0,p^2,p^2) = 2m_\pi^{\,2}+2(d-1)(p^2-m_\pi^{\,2})\,, \tag{6}$$

$$\left.\frac{\partial\Gamma}{\partial q^2}(q^2,0,0)\right|_{q^2=0} = \Delta^{-2}(0)[\dot{\Delta}_{\Theta\sigma}(0)-\dot{\Delta}(0)]\,, \tag{7}$$

$$\left.\frac{\partial\Gamma}{\partial p_1^{\,2}}(0,p_1^{\,2},0)\right|_{p_1^2=0} = (1-d)\Delta^{-2}(0)\dot{\Delta}(0)\,, \tag{8}$$

$$\Gamma(q^2,0,q^2) = \Delta^{-1}(0)[\Delta_{\Theta\sigma}(q^2)\Delta^{-1}(q^2)-(4-d)]\,, \tag{9}$$

$$\Delta_{\Theta\sigma}(0) = d\Delta(0)\,, \tag{10}$$

where the generalized propagators are

$$\Delta(q^2) \equiv -i \int d^4x\, e^{iq\cdot x} \langle T\{\partial_\mu A^\mu(x) \partial_\nu A^\nu(0)\}\rangle_0 ,$$
$$\Delta_{\Theta\sigma}(q^2) \equiv -i \int d^4x\, e^{iq\cdot x} \langle T\{\Theta^\mu_\mu(x) \sigma(0)\}\rangle_0 . \tag{11}$$

σ is the SU(2)$\otimes$SU(2)-violating part of the SU(3) $\otimes$SU(3)-breaking term given in Eq. (3) for the energy-momentum density.

$$u=\sigma+\tau ,$$
$$\sigma=\tfrac{1}{3}(\sqrt{2}+c)(\sqrt{2}u_0+u_8) , \tag{12}$$
$$\tau=\tfrac{1}{3}(1-\sqrt{2}c)(u_0-\sqrt{2}u_8) .$$

We now assume pion pole dominance of $\Delta(q^2)$ and ϵ, and S^* dominance of $\Delta_{\Theta\sigma}(q^2)$:

$$\Delta(q^2) \approx \frac{f_\pi^2 m_\pi^4}{q^2-m_\pi^2} , \tag{13}$$

$$\Delta_{\Theta\sigma}(q^2) \approx \frac{f_\pi^2 m_\pi^4}{q^2-m_\epsilon^2} g_1 + \frac{f_\pi^2 m_\pi^4}{q^2-m_{S^*}^2} g_2 , \tag{14}$$

where g_1 and g_2 are unknown. They are related by Eq. (10):

$$g_1 = d\frac{m_\epsilon^2}{m_\pi^2} - g_2 \frac{m_\epsilon^2}{m_{S^*}^2} . \tag{15}$$

Finally, Eqs. (13)–(15) are inserted into Eqs. (6)–(9) to give a minimal unsubtracted off-shell form factor

$$f_\pi^2 m_\pi^4 \Gamma(q^2, p_1^2, p_2^2) = \frac{A+B(p_1^2+p_2^2)}{q^2-m_\epsilon^2} + \frac{C+D(p_1^2+p_2^2)}{q^2-m_{S^*}^2} , \tag{16}$$

where

$$A = -\frac{m_\epsilon^2}{(m_{S^*}^2-m_\epsilon^2)}\left[m_\epsilon^2 m_{S^*}^2 + 2m_\pi^2 m_\epsilon^2(d-2) - dm_\pi^2 m_{S^*}^2 + \frac{g_2 m_\pi^4(m_{S^*}^2-m_\epsilon^2)}{m_{S^*}^2}\right] ,$$
$$B = \frac{1}{(m_{S^*}^2-m_\epsilon^2)}\left[m_\epsilon^2 m_{S^*}^2 + 2m_\pi^2 m_\epsilon^2(d-2) - dm_\pi^2 m_{S^*}^2 + (m_{S^*}^2-m_\epsilon^2)\left(\frac{g_2 m_\pi^2 m_\epsilon^2}{m_{S^*}^2} - d(m_\epsilon^2-m_\pi^2)\right)\right] , \tag{17}$$
$$C = \frac{m_{S^*}^2}{(m_{S^*}^2-m_\epsilon^2)}\left[m_\epsilon^2 m_{S^*}^2 + 2m_\pi^2 m_{S^*}^2(d-2) - dm_\pi^2 m_{S^*}^2 + \frac{g_2 m_\pi^4(m_{S^*}^2-m_\epsilon^2)}{m_{S^*}^2}\right] ,$$
$$D = -\frac{1}{(m_{S^*}^2-m_\epsilon^2)}[m_\epsilon^2 m_{S^*}^2 + 2m_\pi^2 m_{S^*}^2(d-2) - dm_\pi^2 m_{S^*}^2 + g_2 m_\pi^2(m_{S^*}^2-m_\epsilon^2)] .$$

This expression is consistent with our requirement that $\Gamma(q^2)\to 0$ as $q^2\to\infty$ while at the same time satisfying all the Ward identities. We have gained this flexibility by inserting an extra pole suggested by experiment[5]; but we have paid for this by having an extra free parameter g_2. Of course, because the approximation is essentially a low-energy one we cannot claim to have proved that the S^* implies absence of subtractions; other effects could always alter this. This is why we insist that unsubtractedness is an assumption,[6] and the form (16) is a minimal one consistent with the WI.

One point which should be made at this stage is that the results of KW cannot be obtained from ours by simply taking $g_2=0$, but rather by taking $m_{S^*}\to\infty$. This is because S^* is playing the role of the "distant" singularities that led KW to a subtracted form factor. In particular we obtain from Eqs. (16) and (23) their low-energy relation

$$f_\epsilon G_{\epsilon\pi\pi} = m_\epsilon^2 + (d-2)m_\pi^2 .$$

We can now compute the on-shell slope at $q^2=0$ from Eq. (16):

$$2m_\pi^2 \frac{\partial G_S(q^2)}{\partial q^2}\bigg|_{q^2=0} = f_\pi^2 m_\pi^4 \Gamma'(0, m_\pi^2, m_\pi^2)$$
$$\approx 1 + (d-2)\frac{m_\pi^2}{m_\epsilon^2} - 2\frac{m_\pi^2}{m_{S^*}^2}\left(1-(4-d)\frac{m_\pi^2}{m_\epsilon^2}\right) - \frac{g_2 m_\pi^4(m_{S^*}^2-m_\epsilon^2)}{m_{S^*}^4 m_\epsilon^2} , \tag{18}$$

where we have neglected higher powers of m_π^2 but not terms involving g_2. The fact that g_2 enters this formula gives the additional freedom to alter the slope from the value 1 given by single-pole model.

Okubo[7] has actually derived a bound for the slope when $\Gamma(q^2, m_\pi^2, m_\pi^2)$ does not require a subtraction. He obtains

$$f_\pi^2 m_\pi^4 \Gamma'(0, m_\pi^2, m_\pi^2) \lesssim 0.5 , \tag{19}$$

which is one way of setting a bound on g_2. However, rather than doing this we shall connect our work with that of Renner and Staunton[8] (RS) by estimating $\Gamma'(0, m_\pi^2, m_\pi^2)$ from the Omnès formula:

$$f_\pi^2 m_\pi^4 \Gamma(q^2, m_\pi^2, m_\pi^2) \approx 2m_\pi^2 P(q^2) \exp\left(\frac{q^2}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\phi(s)ds}{s(s-q^2)}\right) \tag{20}$$

and

$$\tan\phi = \frac{\eta \sin 2\delta_0^0}{1+\eta\cos 2\delta_0^0}, \tag{21}$$

where δ_0^0 is the $I=0$ $\pi\pi$ s-wave phase shift and η is the inelasticity. This formula is approximate: It contains only two-particle unitarity, although coupling to other channels than $\pi\pi$ is approximately accounted for by η. There is also an ambiguity in that $P(q^2)$ is in general an arbitrary polynomial in q^2. Our assumption that $\Gamma(q^2)\to 0$ as $q^2\to\infty$ implies that $P(q^2)=1$ and that $\delta_0^0(\infty)>0$; following RS we will take $\delta_0^0(\infty)=\pi$. Orginally these authors took $\eta=1$, but in a recent analysis Pennington[9] has used the more recent data around the $K\bar{K}$ threshold region for which $\eta\neq 1$. This enhances the importance of the S^* region enormously because of its strong coupling both to $\pi\pi$ and to $K\bar{K}$. The result is that $\Gamma(q^2, m_\pi^2, m_\pi^2)$ appears to be dominated by S^* rather than ϵ (Ref. 10); the role of the latter is to broaden the effective S^* pole. This provides a justification for our neglect of the ϵ and S^* widths in evaluating the left-hand side of (20) from (16); all our tests are made near $q^2=0$, which is at least about 500 MeV away from the dominant effective S^* peak.

These assumptions then lead to a value for the slope at $q^2=0$, using the most recent experimental data,[5,10] of

$$f_\pi^2 m_\pi^4 \Gamma'(0, m_\pi^2, m_\pi^2) \approx \frac{2m_\pi^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\phi(s)ds}{s^2} \approx 0.15\,. \tag{22}$$

It is satisfying to note that this value is consistent with Okubo's bound (19) and therefore consistent with our assumption that $\Gamma(q^2)$ requires no subtraction. This then leads to a numerical estimate of g_2 by comparing Eqs. (18) and (22), which is given in Table I for a range of d between 1 and 3. Of course, the absolute values of g_1 and g_2 are not important because they depend on the definitions in Eq. (14), but their ratio is significant because it characterizes the relative importance of the two poles. The negative sign of g_2/g_1 is responsible for the mutual cancellation needed to give a small slope. This then leads to a scalar mass radius

$$\tfrac{1}{6}(r_S^\pi)^2 \approx \frac{0.15}{2m_\pi^2} \approx \frac{2.3}{m_\rho^2},$$

which is more like the charge radius, as one might naively expect.

These values for g_2 are also consistent with an independent analysis by Crewther,[11] who obtained a value for the $\epsilon\pi\pi$ coupling $G_{\epsilon\pi\pi}$ from collinear dispersion relations in broken-scale invariance. In our notation this coupling[12] occurs in

$$f_\pi^2 m_\pi^4 \Gamma(q^2, m_\pi^2, m_\pi^2) = -\frac{f_\epsilon m_\epsilon^2 G_{\epsilon\pi\pi}}{q^2-m_\epsilon^2} - \frac{f_{S^*} m_{S^*}^2 G_{S^*\pi\pi}}{q^2-m_{S^*}^2} \tag{23}$$

and therefore has the value (at $q^2=m_\epsilon^2$) given in Eq. (16):

$$f_\epsilon G_{\epsilon\pi\pi} \approx m_\epsilon^2\left(\frac{m_{S^*}^2}{m_{S^*}^2-m_\epsilon^2} - g_2\frac{m_\pi^4}{m_{S^*}^2 m_\epsilon^2}\right).$$

From Table I this is around $0.44m_\epsilon^2$ (for $d=1$), which compares favorably with Crewther's value[11]

$$f_\epsilon G_{\epsilon\pi\pi} \approx m_\epsilon^2(1-m_\epsilon^2/m_A^2) \approx 0.6m_\epsilon^2\,.$$

Before proceeding with our main argument we would like to note an interesting sum rule that emerges from Eqs. (16) and (23) at $q^2=0$:

$$f_\epsilon G_{\epsilon\pi\pi} + f_{S^*}G_{S^*\pi\pi} = 2m_\pi^2\,, \tag{24}$$

which was first obtained by Carruthers[1] from a simple unsubtracted form for $G_S(q^2)$.

At this stage one might argue that we should compare our expression for $\Delta_{\Theta\sigma}(0)$ in Eq. (14) with that obtained for $\Delta_{\Theta\Theta}(0)$ by RS, as well as by Pennington using experimental data. We shall not do this because we believe that the connection between $\Delta_{\Theta\sigma}$ and $\Delta_{\Theta\Theta}$ is not as simple as the above authors assumed. The general connection between these propagators is given by the virial theorem[1]

$$\Theta_\mu^\mu = (4-d)(\sigma+\tau)\,,$$

which implies

$$\Delta_{\Theta\Theta}(0) = (4-d)[\Delta_{\Theta\sigma}(0)+\Delta_{\Theta\tau}(0)]\,.$$

In order to compare this expression with experiment, RS and Pennington use low-energy relations for $\Delta_{\Theta\sigma}$ and neglect $\Delta_{\Theta\tau}$. This approximation, combined with the low-energy relation[2]

$$\Delta_{\Theta\Theta}(0) = d(4-d)\langle 0|u_0 + cu_8|0\rangle\,,$$

TABLE I. Phenomenological values of g_1 and g_2 for a range of d.

d	g_1	g_2	g_2/g_1
1	−830	1945	−2.3
2	−850	2040	−2.4
3	−865	2135	−2.5

implies from the Ward-identity equation (10) that

$$\Delta(0) \approx \langle 0 | u_0 + c u_8 | 0 \rangle .$$

However, this equation is badly violated: Pion pole dominance gives (in units $m_\pi = 1$) $\Delta(0) = -f_\pi^2 m_\pi^2 \approx -0.5$, while RS (Ref. 8) use -12.6 for the right-hand side. The discrepancy probably lies in the neglect of $\Delta_{\Theta T}(0)$.

We then conclude that at this time no reasonable estimate can be made of $\Delta_{\Theta T}$. It is not possible to go beyond the point we have reached in this paper, and we therefore believe that there is no convincing evidence for a subtraction, provided due account is taken of the S^* contribution. Moreover, although it might be true that ϵ is the Goldstone boson of scale invariance, its role is complicated by an equally broad nearby S^* which couples strongly both to $\pi\pi$ and $K\bar{K}$. We should perhaps add a note of caution that these calculations lean heavily on experimental results which are obtained not directly but through an extrapolation.

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[1]For a review see the following: M. Gell-Mann, in *Proceedings of the Third Hawaii Topical Conference on Particle Physics*, edited by S. F. Tuan (Western Periodicals, North Hollywood, Calif., 1970); P. Carruthers, Phys. Reports 1C, 1 (1971); H. Kleinert, Berlin University Report, 1972 (unpublished).

[2]H. Kleinert and P. H. Weisz, Nucl. Phys. B27, 23 (1971); Nuovo Cimento 3A, 479 (1971).

[3]K. Raman, Phys. Rev. Letters 26, 1069 (1971); Phys. Rev. D 3, 2900 (1971).

[4]S. P. de Alwis, Nucl. Phys. B28, 594 (1971); Sh. Eliezer and R. Dutt, Phys. Letters 40B, 250 (1972).

[5]J. P. Baton, G. Laurens, and J. Reignier, Phys. Letters 33B, 525 (1970); 33B, 528 (1970); M. Alston-Garnjost *et al.*, *ibid.* 36B, 152 (1971); S. M. Flatté *et al.*, *ibid.* 38B, 232 (1972).

[6]D. N. Levin and D. R. Palmer, Phys. Rev. D 4, 3716 (1971). In this paper, it is shown that the pion mechanical form factor is smooth and unsubtracted in a particular model in which the entire SU(2) ⊗ SU(2)-violating term and the entire SU(3) ⊗ SU(3)-violating term have different dimensions (zero and two, respectively). However, we doubt whether it is consistently possible to assign different scale dimensions to operators within the same $(3,\bar{3}) \oplus (\bar{3},3)$ multiplet while keeping current algebra and dimensionless charges.

[7]S. Okubo, Phys. Rev. D 3, 2807 (1971); 4, 725 (1971).

[8]B. Renner and L. P. Staunton, Phys. Rev. D 5, 1676 (1972). This paper will be referred to as RS.

[9]M. R. Pennington, Phys. Rev. D 6, 1458 (1972).

[10]See the graph of $|G_S(q^2)|$ for $\beta = 0$ in Ref. 9.

[11]R. J. Crewther, Phys. Rev. D 3, 3152 (1971); Phys. Letters 33B, 305 (1970).

[12]We have neglected the effect of SU(3) mixing, and so the physical fields of ϵ and S^* correspond to the SU(3) singlet and octet states σ_0 and σ_8, respectively.