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Scaling in a Gluon Model in Three Dimensions

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We study the scaling behavior of the structure functions of both deep-inelastic electron-proton scattering and inclusive electron-positron pair annihilation into a proton in a simple gluon model in three dimensions. We find, firstly, that both sets of structure functions have similar scaling properties, as expected on the basis of light-cone dominance, and that they are simply related to one another. Secondly, we obtain nonvanishing longitudinal structure functions in spite of the underlying bilinear fermion structure of the current. This confirms a positivity constraint discussed earlier and implies that interactions alter the free light-cone commutator structure. Thirdly, the transverse functions increase linearly with large ω, again confirming the results of our earlier paper.

I. INTRODUCTION

In spite of the widespread interest in scaling phenomena and light-cone physics, some ideas connecting the two have in general not been amenable to discussion in perturbation theory, for the simple reason that the latter violates scaling at all orders. In view of this difficulty, we propose to study in this paper some of these ideas in a model where scaling is indeed obeyed to (at least) the first nontrivial order of the perturbation series. This is a gluon model in a space of three dimensions: one time and two space dimensions. We consider a fermion field interacting via a Yukawa-type interaction with a scalar field. The current is bilinear in the fermion field and satisfies free-field commutation relations analogous to its counterpart in four dimensions.

General belief has it that these commutation relations (in particular the internal symmetry and singularity structure on the light cone) survive all interaction effects. This, of course, has many consequences, the most direct of which is the identical vanishing of the longitudinal structure function $F_L(\omega)$ of deep-inelastic electron-hadron (proton) scattering $[e+\Pi(p)\to e+X]$ in the scaling limit. Fairly general assumptions would then also lead to a similar vanishing of the corresponding longitudinal structure function $\bar{F}_L(\omega)$ measured in the inclusive annihilation process of an electron-positron pair into a hadron (proton) $[e^+ + e^- \to \Pi(p)+X]$.

Now whereas it is generally accepted that the vanishing of $F_L(\omega)$ is consistent with experimental observation, little is known about $\bar{F}_L(\omega)$.

On the other hand, we have recently shown[1] that positivity and gauge invariance imply that $\bar{F}_L(\omega)$ may *not* vanish identically, and that if it does the transverse structure function $\bar{F}_T(\omega)$ must increase at least linearly with ω. We have also shown in Ref. 1 that, under the fairly general assumptions of light-cone dominance, the corresponding structure functions in the processes $e+\Pi(p)\to e+X$

and $e^+ + e^- \to \Pi(p) + X$ scale in a similar manner.

The nonvanishing of $\bar{F}_L(\omega)$ implies of course that the free light-cone singularity structure of the current commutator is modified. The linear increase in ω of $\bar{F}_T(\omega)$ implies that the bilocal operator, whose matrix elements it measures, has a short-distance singularity (as opposed to one for lightlike distances) in it, contrary to general assumptions.

In order to confirm the validity of the above remarks we calculate, in our three-dimensional gluon model, the relevant structure functions to the first nontrivial order of the perturbation series. This is done exactly. Upon taking the scaling limit we find the following results.

(1) The structure functions in both the annihilation and scattering processes scale as expected and in a similar way. This implies that the operator controlling this behavior has indeed only one leading light-cone singularity, as assumed in Ref. 1 (see Sec. IV below).

(2) The structure functions are related to each other simply:

$$\bar{F}_L(\omega) = -F_L(-\omega),$$
$$\bar{F}_T(\omega) = -F_T(-\omega).$$

(3) The longitudinal structure functions do not vanish identically, thus confirming the positivity constraint of Ref. 1.

(4) The transverse structure functions increase linearly with large ω, also confirming the positivity constraint of Ref. 1. This increase is not required if $\bar{F}_L(\omega)$ does not vanish but is only an alternative. In this case we see, however, that both alternatives are realized.

In order to perform our task, we have to study first the structure of scalar and spinor fields in three dimensions. This is done in Sec. II, with most of the details left to Appendix A.

In Sec. III, the canonical light-cone structure consistent with scale invariance is discussed. We show in Appendix B how the equal-time limits are to be calculated.[2,3]

In Sec. IV the scaling behavior expected on the basis of the singularity structure of Sec. III is presented.

The calculation of the structure functions is outlined in Sec. V, with some details left to Appendix C. In Sec. VI we give some conclusions and discussion.

II. FREE FIELDS IN THREE DIMENSIONS

The equations of motion for free fields in our three-dimensional space (one time and two space coordinates) may be constructed along analogous procedures already known for four dimensions. Quantization proceeds also along similar canonical commutation or anticommutation relations. The simplest cases are those of one- and two-component fields. These are analogous to the ordinary spin-0 and spin-$\frac{1}{2}$ Dirac fields, respectively. We present in this section some general results for these two cases, leaving more of the details to Appendix A.

1. *The Scalar Field $\phi(x)$.*

The equation of motion for the scalar field is the analog of the Klein-Gordon equation in three dimensions, namely ($\hbar = c = 1$)

$$(\partial^\mu \partial_\mu + \mu^2)\phi(x) = 0, \tag{1}$$

where

$$\partial^\mu \partial_\mu = \left(\frac{\partial}{\partial x^0}\right)^2 - \left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2. \tag{2}$$

The canonical commutation relation is given by

$$[\phi(x), \phi(y)] \equiv i\Delta(x-y). \tag{3}$$

$\Delta(x-y)$ is an invariant singular function given by

$$\Delta(x-y) = -i\int \frac{d^2k}{(2\pi)^2 2\omega_k}\left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right). \tag{4}$$

In particular we have

$$[\dot{\phi}(x), \phi(y)]_{x^0=y^0} = i\partial_0 \Delta(x-y)|_{x^0=y^0}$$
$$= -i\delta^2(\vec{x}-\vec{y}), \tag{5}$$

a two-dimensional δ function. The Feynman propagator is found to be

$$i\Delta_F(q) = \frac{i}{q^2 - \mu^2 + i\epsilon}. \tag{6}$$

Now Eq. (5) determines the physical dimension (in units of mass or inverse length) of $\phi(x)$ to be $\frac{1}{2}$. We shall show shortly that its canonical dimension under scale transformations is also $\frac{1}{2}$.

2. *The Two-Component (Dirac) Field.*

As shown in Appendix A, the simplest multicomponent field is a two-component spinor $\psi(x)$ which satisfies an equation of motion analogous to the Dirac equation. It reads

$$(\gamma^\mu \partial_\mu + M)\psi(x) = 0, \tag{7}$$

where

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

and

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}1, \tag{8}$$

with a metric

$$g^{\mu\nu} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}.$$

The solutions at rest are of the form

$$\psi^{(r)}(x) = \omega^{(r)}(0)e^{-i\epsilon_r Mt}, \qquad r = 1, 2$$

$$\epsilon_r = \begin{cases} +1, & r=1 \\ -1, & r=2 \end{cases}$$

and

$$\omega^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \omega^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The solution with $r=1$ is the positive-energy solution, and that with $r=2$ the negative-energy solution. These also correspond to eigenvalues of $\gamma^0 = S_3$ of $+1$ and -1, respectively. Thus, only one "spin" state occurs at rest for each solution in contrast to the ordinary Dirac particle. The single-particle wave functions and their properties are given in Appendix A. Canonical quantization is achieved by the anticommutation relation

$$\{\psi_\alpha(x), \overline{\psi}_\beta(y)\} = i(i\not\partial_x + M)_{\alpha\beta}\Delta(x-y) = iS_{\alpha\beta}(x-y). \tag{9}$$

The Feynman propagator is then

$$iS_F(p) = (\not p + M)/(p^2 - M^2 + i\epsilon). \tag{10}$$

At equal times we have

$$\{\psi_\alpha(\vec{x}, t), \psi_\beta^\dagger(\vec{y}, t)\} = \delta_{\alpha\beta}\delta^2(\vec{x} - \vec{y}). \tag{11}$$

This then fixes the physical dimension of $\psi(x)$ at unity. Let us show now that this agrees with its canonical dimension under scale transformations.

The free Lagrangian density incorporating these two fields takes the form

$$\mathcal{L} = \tfrac{1}{2}[\partial_\mu\phi(x)]^2 - \tfrac{1}{2}\mu^2\phi^2(x) + \overline{\psi}(x)(i\not\partial - M)\psi(x). \tag{12}$$

Let D be the generator of scale transformations. Then the dimensions of the fields $\phi(x)$ and $\psi(x)$ are defined by

$$\begin{aligned}[\phi(x), D] &= i(d_\phi + x\cdot\partial)\phi(x), \\ [\psi(x), D] &= i(d_\psi + x\cdot\partial)\psi(x).\end{aligned} \tag{13}$$

Consider the Lagrangian density of Eq. (12) in the massless limit. We have then

$$\begin{aligned}[D, \mathcal{L}_0(0)] &= -id\mathcal{L}_0(0) \\ &= -i\{(2d_\psi + 1)\overline{\psi}i\not\partial\psi(0) + (2d_\phi + 2)[\partial_\mu\phi(0)]^2\}.\end{aligned} \tag{14}$$

Now, in general, under infinitesimal transformations $\delta\lambda$ we have

$$\delta\mathcal{L}_0(x) = \delta\lambda(d + x\cdot\partial)\mathcal{L}_0(x). \tag{15}$$

If the Lagrangian, which is the integral over all (three-dimensional) space of $\mathcal{L}_0$, is to be invariant then $\delta\mathcal{L}_0(x)$ must be a total derivative. From Eq. (15) we see that this is possible only if $d=3$. For then we have

$$\delta\mathcal{L}_0(x) = \partial^\mu[\delta\lambda\, x_\mu \mathcal{L}_0(x)].$$

Using $d=3$ in Eq. (14) we find that

$$d_\psi = 1, \quad d_\phi = \tfrac{1}{2}, \tag{16}$$

which agree with the physical values determined by the equal-time commutation relations of Eq. (5) and Eq. (11).

The Lagrangian density of Eq. (12) leads to a conserved vector current $J_\mu(x)$ given by

$$J_\mu(x) = \phi(x)\overleftrightarrow{\partial}_\mu\phi(x) + [\overline{\psi}(x), \gamma_\mu\psi(x)]. \tag{17}$$

This current has then a dimension of two. Alternately, if we assume that the constant "charge" it defines is scale-invariant, we arrive at the values in Eq. (16) for the field dimensions.

Using the free-field canonical commutation-anticommutation rules of Eq. (3) and Eq. (9) we obtain

$$\begin{aligned}[J_\mu(x), J_\nu(y)] = {} & i\partial_\mu^x\partial_\nu^y[\Delta(x-y)\phi(x)\phi(y)] - [\partial_\lambda^x\Delta(x-y)][\overline{\psi}(x)\gamma_\mu\gamma^\lambda\gamma_\nu\psi(y) + \overline{\psi}(y)\gamma_\nu\gamma^\lambda\gamma_\mu\psi(x)] \\ & + iM\Delta(x-y)[\overline{\psi}(x)\gamma_\mu\gamma_\nu\psi(y) + \overline{\psi}(y)\gamma_\nu\gamma_\mu\psi(x)].\end{aligned} \tag{18}$$

The first term is due to the scalar-field part of the current, and the last two to the fermion-field part. Equation (18) specifies the behavior of the free commutator everywhere, and in particular on the light cone $(x-y)^2 = 0$. We shall investigate this behavior and its consequences in the following sections.

III. LIGHT-CONE EXPANSIONS OF OPERATOR PRODUCTS

We shall be interested mainly in the conserved vector current $J_\mu(x)$ presented earlier. The free-field expression in Eq. (18) above suggests that the current commutator on the light cone may be expressed as follows[3]:

$$[J_\mu(x), J_\nu(0)] = (\partial_\mu\partial_\nu - \Box g_{\mu\nu})C_1(x^2)R_1(x,0) + (g_{\mu\nu}\partial_\alpha\partial_\beta - g_{\alpha\nu}\partial_\beta\partial_\mu - g_{\alpha\mu}\partial_\beta\partial_\nu + g_{\alpha\mu}g_{\beta\nu}\Box)C_2(x^2)R_2^{\alpha\beta}(x,0), \tag{19}$$

where $C_{1,2}(x^2)$ are c-number singular functions and $R_1(x, 0)$ and $R_2^{\alpha\beta}(x, 0)$ are, respectively, scalar and symmetric second-rank tensor bilocal operators. The singularity of the commutator is assumed to be defined completely by $C_1(x^2)$ and $C_2(x^2)$. The bilocal operators may then be expanded in terms of an infinite set of local operators as follows:

$$R_1(x,0) = \sum_{n=0}^{\infty} x^{\alpha_1}\cdots x^{\alpha_n} R_{1,\alpha_1\cdots\alpha_n}(0),$$
$$R_2^{\alpha\beta}(x,0) = \sum_{n=0}^{\infty} x^{\alpha_1}\cdots x^{\alpha_n} R^{\alpha\beta}_{2,\alpha_1\cdots\alpha_n}(0). \tag{20}$$

Current conservation is then explicit in Eq. (19). The current $J_\mu(x)$ has a dimension of two; hence if Eq. (19) is scale-invariant, as we assume, one must have

$$d_{C_1} + d_{R_1} = d_{C_2} + d_{R_2} = 2. \tag{21}$$

In a theory with only scalar fields the lowest-dimensional scalar bilocal operator is $\phi(x)\phi(0)$ of dimension one. The lowest-dimensional symmetric tensor is $\phi(x)\partial^\alpha\partial^\beta\phi(0)$ with dimension three. Thus, in scalar field theory,

$$d_{C_1} = 1, \quad d_{C_2} = -1. \tag{22}$$

In a theory of Dirac fields ψ the lowest-dimensional scalar bilocal is $\bar\psi(x)\psi(0)$ of dimension two, and the lowest-dimensional symmetric tensor is

$$\bar\psi(x)\gamma_\mu\partial_\nu\psi(y) + \bar\psi(y)\gamma_\nu\partial_\mu\psi(x)$$

of dimension three, leading to

$$d_{C_1} = 0, \quad d_{C_2} = -1. \tag{23}$$

Therefore in a mixed free theory the leading singularity in $C_1(x^2)$ comes from scalar fields, and that in $C_2(x^2)$ comes from both fields. We have in general

$$C_1(x^2) = (x^2 - i\epsilon x_0)^{-d_{C_1}/2} - (x^2 + i\epsilon x_0)^{-d_{C_1}/2},$$
$$C_2(x^2) = (x^2 - i\epsilon x_0)^{-d_{C_2}/2} - (x^2 + i\epsilon x_0)^{-d_{C_2}/2},$$

with $d_{C_1} = 1$ and $d_{C_2} = -1$.

The expansion in Eq. (19) determines the short-distance behavior and in particular the equal-time limit. We show in Appendix B how the equal-time limit should be calculated.

IV. SCALING AND CURRENT-COMMUTATOR SINGULARITIES ON THE LIGHT CONE

It is well known that the behavior near the light cone of the electromagnetic current commutator controls the structure functions of deep-inelastic electron-hadron scattering in the Bjorken scaling limit. We have also shown[1] that under the assumption of light-cone dominance it also controls the scaling behavior of the structure functions measured in the annihilation of an electron-position pair into a hadron plus anything. In this section we wish to outline this relationship in three-dimensional space. The notation is the same as in Ref. 1, and we shall have occasion to refer to some equations in this reference.

1. Deep-Inelastic Electron-Hadron Scattering.

It is well known that in the process $e + \Pi(p) \to e + X$ [$\Pi(p)$ is a hadron of momentum p and mass M; X denotes "anything"] one is probing the structure of the absorptive part of the forward virtual Compton scattering amplitude. We study (in three dimensions)

$$W_{\mu\nu}(q,p) = \int d^3x\, e^{iq\cdot x}\langle \Pi(p)|J_\mu(x)|N\rangle\langle N|J_\nu(0)|\Pi(p)\rangle$$
$$= \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right)W_1(q^2,\nu) + \left(p_\mu - \nu\frac{q_\mu}{q^2}\right)\left(p_\nu - \nu\frac{q_\nu}{q^2}\right)W_2(q^2,\nu). \tag{25}$$

q^2 is the mass of the virtual photon, and $\nu = q\cdot p$ its energy in the laboratory system. In the above process, $q^2 < 0$ spacelike and $\nu > 0$. Thus $W_{\mu\nu}(q,p)$ may be written as in Ref. 1 in the form

$$W_{\mu\nu}(q,p) = \int d^3x\, e^{iq\cdot x} A^{(1)}_{\mu\nu}(x,p), \tag{26}$$

where

$$A^{(1)}_{\mu\nu}(x,p) = \int d^3y\, d^3z\, e^{ip\cdot y} e^{-ip\cdot z} K_y K_z \langle 0|R(J_\mu(x);\phi(y))R(J_\nu(0);\phi^*(z))|0\rangle. \tag{27}$$

R denotes the retarded product, and $K_y = \partial_\mu \partial^\mu + M^2$ in three dimensions. In Eq. (27) the hadron was assumed spinless.

$A^{(1)}_{\mu\nu}(x, p)$ is identical with

$$\langle \Pi(p) | [J_\mu(x), J_\nu(0)] | \Pi(p) \rangle$$

for $q^2 < 0$, $\nu > 0$, and hence must have at least the same singularity structure on the light cone. More singular terms may exist only for $q^2 > 0$. However, in the spirit of light-cone dominance, one expects $A^{(1)}_{\mu\nu}(x, p)$ to have a single leading singularity which dominates at all q^2.

Let us assume that the operator product $A^{(1)}_{\mu\nu}(x, p)$ has the following decomposition in position space:

$$A^{(1)}_{\mu\nu}(x, p) = (\partial_\mu \partial_\nu - g_{\mu\nu}\Box) A^{(1)}_1(x^2, x\cdot p) + [\Box p_\mu p_\nu - p\cdot\partial(p_\mu \partial_\nu + p_\nu \partial_\mu) + (p\cdot\partial)^2 g_{\mu\nu}] A^{(1)}_2(x^2, x\cdot p). \tag{28}$$

Let us also assume, in view of the light-cone expansion of Eq. (19), that as $x^2 \to 0$ we have

$$\begin{aligned} A^{(1)}_1(x^2, x\cdot p) &\sim \left(\frac{1}{-x^2 + i\epsilon x_0}\right)^a f_1(x^2, x\cdot p) + \text{less-singular terms}, \\ A^{(1)}_2(x^2, x\cdot p) &\sim \left(\frac{1}{-x^2 + i\epsilon x_0}\right)^b f_2(x^2, x\cdot p) + \text{less-singular terms}, \end{aligned} \tag{29}$$

where $a = \frac{1}{2}$ and $b = -\frac{1}{2}$.

Here $f_{1,2}(x^2, x\cdot p)$ are assumed to be nonsingular as x^2 (but not necessarily x) approaches zero. As is well known, the expressions of Eq. (19) determine the behavior of the structure functions $W_1(q^2, \nu)$ and $W_2(q^2, \nu)$ in the Bjorken scaling limit (B): $\nu \to \infty$, $q^2 \to -\infty$, and $\omega = -q^2/2\nu$ fixed. We wish to exhibit this relationship here, as it is slightly different than in four-dimensional space.

Define

$$g_i(\alpha) = \frac{1}{2\pi} \int d(x\cdot p) e^{-i(x\cdot p)\alpha} f_i(0, x\cdot p), \qquad i = 1, 2. \tag{30}$$

From Eqs. (15) and (16) we then have

$$\begin{aligned} W_1(q^2, \nu) + \frac{\nu^2}{q^2} W_2(\nu, q^2) &= -q^2 \int d\alpha\, g_1(\alpha) K(a) \frac{-i}{[-(q+\alpha p)^2 + i\epsilon(q+\alpha p)^0]^{(3-2a)/2}}, \\ W_2 &= -q^2 \int d\alpha g_2(\alpha) K(b) \frac{-i}{[-(q+\alpha p)^2 + i\epsilon(q+\alpha p)^0]^{(3-2b)/2}}, \end{aligned} \tag{31}$$

where

$$K(a) = 2^{3-2a} \pi^{3/2} \Gamma^{3/2}(\tfrac{1}{2}(3-2a)) \Gamma(a).$$

If we take the scaling limit (B) of the integrals in Eq. (21), we find

$$\begin{aligned} W_1(q^2, \nu) + \frac{\nu^2}{q^2} W_2(\nu, q^2) &\underset{B}{\sim} \frac{\omega}{(2\nu)^{1/2-a}} K(a) \int d\alpha\, g_1(\alpha) \frac{-i}{(\omega - \alpha + i\epsilon)^{3/2-a}} \underset{B}{\sim} (2\nu)^{a-1/2} F_L(\omega), \\ W_2(q^2, \nu) &\underset{B}{\sim} \frac{\omega}{(2\nu)^{1/2-b}} K(b) \int d\alpha\, g_2(\alpha) \frac{-i}{(\omega - \alpha + i\epsilon)^{3/2-b}} \underset{B}{\sim} (2\nu)^{b-1/2} F_2(\omega). \end{aligned} \tag{32}$$

A positivity requirement exists[4] such that $0 \le W_1 \le (1 - \nu^2/q^2) W_2$. From Eq. (32) we then conclude that $b \ge a - 1$.

Comparing Eq. (29) with Eqs. (18), (19), and (21), we find for free fields the leading singularity to be that of $\Delta(x)$ in three dimensions. $\Delta(x)$ has dimension unity so it behaves as

$$\left(\frac{1}{-x^2 + i\epsilon x_0}\right)^{1/2} - \text{c.c.}$$

as $x^2 \to 0$. Thus, $a = \frac{1}{2}$, $b = -\frac{1}{2}$, and it follows that

$$\begin{aligned} \left(W_1 + \frac{\nu^2}{q^2} W_2\right) &\underset{B}{\sim} F_L(\omega), \\ (2\nu W_2) &\underset{B}{\sim} F_2(\omega), \end{aligned} \tag{33}$$

and hence

$$W_1 \underset{B}{\sim} F_1(\omega)$$

just as in the four-dimensional case.

From Eq. (15), and using well-known techniques,[5] we notice that in the absence of scalar fields $F_L(\omega)\equiv 0$, while in the absence of Dirac fields $F_1(\omega)\equiv 0$. These results are similar to the four-dimensional case, and follow if the commutator structure near the light cone is not altered by interactions.

2. *The Annihilation Process* $e^+e^- \to \Pi(p)+X$.

To the order that this process is mediated by a single timelike photon one may summarize the hadron structure being probed by

$$\begin{aligned}\overline{W}_{\mu\nu}(q,p) &= \int d^3x\, e^{iq\cdot x}\langle 0|J_\mu(x)|\Pi(p)N\rangle\langle N\Pi(p)|J_\nu(0)|0\rangle \\ &= \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right)\overline{W}_1(q^2,\nu) + \left(p_\mu - \nu\frac{q_\mu}{q^2}\right)\left(p_\nu - \nu\frac{q_\nu}{q^2}\right)\overline{W}_2(q^2,\nu)\,, \end{aligned} \tag{34}$$

where the physical region is for $\nu>0$ and $q^2 \geq 2\nu$ timelike. As pointed out in Ref. 1, the behavior of $\overline{W}_{\mu\nu}(q,p)$ in the scaling limit is also controlled by the light-cone singularity of $A^{(1)}_{\mu\nu}(x,p)$, for we have

$$\overline{W}_{\mu\nu}(q,p) = \int d^3x\, e^{-iq\cdot x} A^{(1)}_{\mu\nu}(x,p)\,. \tag{35}$$

If $A^{(1)}_{\mu\nu}(x,p)$ has only one leading light-cone singularity as specified in Eq. (28) and Eq. (29), then we must have

$$\begin{aligned}&\left(\overline{W}_1(q^2,\nu) + \frac{\nu^2}{q^2}\overline{W}_2(\nu,q^2)\right) \underset{B}{\sim} (2\nu)^{a-1/2}\overline{F}_L(\omega)\,,\\ &\overline{W}_2(q^2,\nu) \underset{B}{\sim} (2\nu)^{b-1/2}\overline{F}_2(\omega)\,,\end{aligned} \tag{36}$$

and hence

$$\overline{W}_1(q^2,\nu) \underset{B}{\sim} \overline{F}_1(\omega)\,.$$

For $a=\frac{1}{2}$, $b=-\frac{1}{2}$ we get the expected scaling behavior familiar in four dimensions. We also have in general

$$\begin{aligned}\overline{F}_L(\omega) &= \omega K(a)\int d\alpha\, g_1(\alpha)\frac{-i}{(-\omega-\alpha+i\epsilon)^{3/2-a}}\,,\\ \overline{F}_2(\omega) &= \omega K(b)\int d\alpha\, g_2(\alpha)\frac{-i}{(-\omega-\alpha+i\epsilon)^{3/2-b}}\,.\end{aligned} \tag{37}$$

The positivity requirements discussed in Ref. 1 lead to restrictions on the short-distance behavior of $A^{(1)}_{\mu\nu}(x,p)$. These, as shown in Ref. 1, lead to either or both of two results. The first is the nonvanishing of the longitudinal structure function $\overline{F}_L(\omega)$, and the second, in case the first does not hold, is that

$$\overline{F}_1(\omega) \underset{\omega\to\infty}{\geq} c\omega\,.$$

The latter follows from a necessary singularity in $f_2(0, x\cdot p)$ as $x\to 0$ of the form $\delta''(x\cdot p)$ in case a in Eq. (29) is less than $\frac{1}{2}$. For details of the derivation of the positivity constraint we refer the reader to Ref. 1. The same steps follow in three dimensions without any essential alterations.

V. SCALING IN A SIMPLE GLUON MODEL

We turn now to the study of the scaling behavior and various positivity constraints of Ref. 1, presented in the previous section, in a simple gluon model to the first nontrivial order in perturbation theory. We do this by calculating $W_{\mu\nu}(q,p)$ of Eq. (25) and $\overline{W}_{\mu\nu}(q,p)$ of Eq. (34) directly with an intermediate state of one nucleon and one meson and to lowest order in perturbation theory.

The current is taken to be bilocal in the field $\psi(x)$, namely,

$$J_\mu(x) = \overline{\psi}(x)\gamma_\mu\psi(x)\,. \tag{38}$$

The interaction Lagrangian density is of the form

$$\mathcal{L}_I = g\overline{\psi}(x)\psi(x)\phi(x)\,. \tag{39}$$

This interaction Lagrangian density has dimension $2\frac{1}{2}$ under scale transformations and hence violates scale invariance of the Lagrangian. A scale-invariant interaction is of the form $g'\overline{\psi}\psi\phi^2$ and contributes to the three-particle intermediate state. Needless to say, the calculation in this latter case is doable but considerably more involved than the one at hand. Since our aim is to demonstrate the validity of very general and model-independent results, we shall be satisfied with the simple model at hand. We shall refer to the particle described by the field ψ as the "fermion" and that by the field ϕ as the "meson."

The results of a similar calculation in four dimensions[6] show a nonvanishing longitudinal structure function $F_L(\omega)$ and a violation of scaling by logarithmic factors in $\nu W_2(\nu, q^2)$,

$$\nu W_2 \underset{B}{\sim} (\ln q^2) F_2(\omega) . \quad (40)$$

In view of the fermion field structure for the current, the nonvanishing of $F_L(\omega)$ is contrary to "folklore" and expectations based on the free-field light-cone structure for the commutator given in Eq. (18), namely (fermion part only),

$$[J_\mu(x), J_\nu(y)] = -[\partial_\lambda^x \Delta(x-y)][\overline{\psi}(x)\gamma_\mu\gamma^\lambda\gamma_\nu\psi(y) + \overline{\psi}(y)\gamma_\nu\gamma^\lambda\gamma_\mu\psi(x)] + iM\Delta(x-y)[\overline{\psi}(x)\gamma_\mu\gamma_\nu\psi(y) + \overline{\psi}(y)\gamma_\nu\gamma_\mu\psi(x)] . \quad (41)$$

It is argued, however, that the nonvanishing of $F_L(\omega)$ is not to be taken seriously in view of the fact that also scaling for νW_2 is violated; in other words that perturbation calculations are not a good laboratory for studying scaling phenomena.

Now the gluon model in three dimensions (at least to the first nontrivial order in perturbation theory) indeed exhibits *nonvanishing* longitudinal structure functions $F(\omega)$ and $\overline{F}(\omega)$, whereas W_1 and $\overline{W}_1$ (consequently then νW_2 and $\nu \overline{W}_2$) scale as expected, thus confirming the results of Ref. 1 and removing the objection to the four-dimensional calculation. Gauge invariance and positivity play an important role in the derivation of the results of Ref. 1, and we make sure here to satisfy them explicitly. We proceed now to present some aspects of the calculation, leaving more of the details to Appendix C.

1. Deep-Inelastic Scattering $e + \Pi(p) \to e + X$.

We calculate $W_{\mu\nu}(q,p)$ by calculating to lowest order the gauge-invariant expression for the matrix element

$$T_\mu(x) = \langle \Pi(p_1) | J_\mu(x) | \Pi(p_1) k \rangle , \quad (42)$$

where $\Pi(p_1)$ is a "nucleon" on the mass shell, with momentum p_1 and mass M and with k the momentum of an on-mass-shell meson with mass μ. We then have

$$W_{\mu\nu}(q,p) = \int d^3x e^{iq\cdot x} \int d^3k\, \delta(p_1{}^2 - M^2)\delta(k^2-\mu^2)\theta(p_1^0 - M)\theta(k^0-\mu)\delta(q+p-k-p_1) T_\mu(x) T_\nu^\dagger(0)] . \quad (43)$$

To lowest order T_μ receives contributions from the two graphs of Fig. 1:

$$T_\mu = B_\mu + C_\mu .$$

The current with momentum q couples with a vertex $-ie\gamma_\mu$ to the fermion line, and the meson is emitted off this line with strength g; the sum T_μ of the two graphs in Fig. 1 is explicitly gauge-invariant. The resulting expression for $W_{\mu\nu}$ is given in Appendix C.

The aim of the calculation is an expression for the structure functions in Eq. (33). These may be projected from $W_{\mu\nu}$ as follows. Consider the longitudinal spacelike vector η_L^μ with the properties

$$\eta_L^\mu \eta_{L\mu} = -1, \quad q_\mu \eta_L^\mu = 0 . \quad (44)$$

η_L^μ may be represented by

$$\eta_L^\mu = \frac{1}{\sqrt{q^2}} (|\vec{q}|, q^0 \hat{q}) , \quad (45)$$

where $\hat{q}$ is a unit vector in the direction of the two-vector $\vec{q}$. We then have

$$\eta_L^\mu \eta_L^\nu W_{\mu\nu} = W_1 + \left(\frac{\nu^2}{q^2} - 4M^2\right) W_2 = W_L . \quad (46)$$

Comparing to Eq. (33), we see that in the scaling limit

$$W_L \underset{B}{\sim} F_L(\omega) . \quad (47)$$

Similarly, consider the transverse timelike vector η_T^μ with the properties

$$\eta_T^\mu \eta_{L\mu} = 0 ,$$
$$\eta_T^\mu \eta_{T\mu} = +1, \quad (48)$$
$$\eta_T^\mu q_\mu = 0 .$$

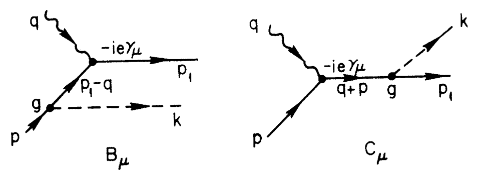

FIG. 1. $T_\mu = B_\mu + C_\mu$. Graphs contributing to the structure functions W_L and W_1 of deep-inelastic electron-proton scattering.

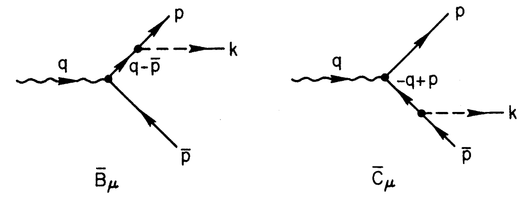

FIG. 2. $\overline{T}_\mu = \overline{B}_\mu + \overline{C}$. Graphs contributing to the structure functions $\overline{W}_L$ and $\overline{W}_1$ of the inclusive annihilation of an electron-positron pair into a proton.

Then

$$\eta_T^\mu = i(0, \vec{a}), \quad \vec{a}\cdot\vec{q} = 0 .$$

Using η_T^μ we find

$$-\eta_T^\mu \eta_T^\nu W_{\mu\nu} = W_T = W_1 , \tag{49}$$

which with Eq. (46) gives an expression for νW_2. In Appendix C we describe the steps in the calculation of W_L and W_T. In the scaling limit we then obtain

$$F_L(\omega) = -\frac{g^2e^2}{4M}\left[\left(\frac{32\pi(1-\omega)^2}{(2\zeta)^{3/2}}\right)\left(\frac{1}{8(1-\omega)}\,[M^2(1-2\omega)+3\omega\mu^2-\mu^2]-\tfrac{1}{4}\omega M^2\right)+\left(\tfrac{1}{4}\pi\,\frac{1-\omega}{(2\zeta)^{1/2}}\right)\right], \tag{50}$$

$$F_T(\omega) = \frac{g^2e^2}{4M}\left[\frac{8\pi}{(2\zeta)^{3/2}}\left\{-M^2\omega + 2M^2\,[(1-\omega^2)-4\omega^2(1-\omega)] + \mu^2[(8\omega^2-4\omega-1)(1-\omega)^2-8\omega(1-\omega)]\right\}\right], \tag{51}$$

where

$$\zeta = M^2 - (1-\omega)(M^2-\mu^2) + 2\mu^2\omega^2 , \tag{52}$$

and $0<\omega=-q^2/2\nu<1$ in the physical region.

These expressions are the result of an exact calculation of $W_L(\nu, q^2)$ and $W_T(\nu, q^2)$ followed by the scaling limit $\nu\to\infty$, $q^2\to-\infty$, with $\omega=-q^2/2\nu$ fixed. Note that for large ω (outside the physical region) $F_T(\omega)\sim C\omega$, as pointed out in Ref. 1.

2. *The Annihilation Process* $e^+ + e^- \to \Pi(p) + X$.

We calculate $\overline{W}_{\mu\nu}(q, p)$ in a similar manner to the above by calculating to lowest order the gauge-invariant expression for the matrix element

$$\overline{T}_\mu(x) = \langle 0 | J_\mu(x) | \Pi(p)\Pi(\bar{p})k\rangle . \tag{53}$$

We then have

$$\overline{W}_{\mu\nu}(q,p) = \int d^3x\, e^{iq\cdot x}\int d^3\bar{p}\int d^3k\,\delta(\bar{p}^2-M^2)\delta(k^2-\mu^2)\theta(\bar{p}^0-M^2)\theta(k^0-\mu)\delta(q-p-k-\bar{p})\overline{T}_\mu(x)\overline{T}_\nu^\dagger(0) . \tag{54}$$

To lowest order $\overline{T}_\mu$ receives contributions from the two graphs of Fig. 2

$$\overline{T}_\mu = \overline{B}_\mu + \overline{C}_\mu .$$

The sum $\overline{T}_\mu$ of the two graphs in Fig. 2 is explicitly gauge-invariant. The resulting expression for $\overline{W}_{\mu\nu}$ is given in Appendix C. Using the vectors η_L^μ and η_T^μ we define

$$\overline{W}_L = \overline{W}_1 + \left(\frac{\nu^2}{q^2} - 4M^2\right)\overline{W}_2 = \eta_L^\mu\eta_L^\nu \overline{W}_{\mu\nu} ,$$
$$\overline{W}_T = \overline{W}_1 = -\eta_T^\mu\eta_T^\nu \overline{W}_{\mu\nu} . \tag{55}$$

Upon comparing the resulting expression for $W_{\mu\nu}(q,p)$ and $\overline{W}_{\mu\nu}(q,p)$ we find that they are related by the relation

$$\overline{W}_{\mu\nu}(q,p) = -W_{\mu\nu}(q,-p) . \tag{56}$$

Thus we obtain that the annihilation structure functions are the same as deep-inelastic ones with the replacement of $-\omega$ for ω. The physical region for ω in this process, however, is $-\infty<\omega<-1$. We thus have

$$\overline{F}_L(\omega)=+\frac{g^2e^2}{4M}\left[\frac{32\pi(1+\omega)^2}{(2\overline{\zeta})^{3/2}}\left(\frac{1}{8(1+\omega)}[M^2(1+2\omega)-3\omega\mu^2-\mu^2]+\tfrac{1}{4}\omega M^2\right)+\tfrac{1}{4}\pi\frac{1+\omega}{(2\overline{\zeta})^{1/2}}\right], \tag{57}$$

and

$$\overline{F}_T(\omega)=-\frac{g^2e^2}{4M}\left[\frac{8\pi}{(2\overline{\zeta})^{3/2}}\{M^2\omega+2M^2[(1-\omega^2)-4\omega^2(1+\omega)]+\mu^2[(8\omega^2+4\omega-1)(1+\omega)^2+8\omega(1+\omega)]\}\right], \tag{58}$$

where

$$\overline{\zeta}=M^2-(1+\omega)(M^2-\mu^2)+2\mu^2\omega^2 \tag{59}$$

and $-\infty\leqslant\omega=-q^2/2\nu\leqslant-1$ in the physical region. Note here again that for $\omega\to\infty$, $\overline{F}_T(\omega)\sim C\omega$ as expected from positivity and gauge invariance in Ref. 1.

VI. CONCLUSIONS AND DISCUSSION

The main purpose of the above discussion has been to investigate in a simple model the validity of the general results obtained in Ref. 1. The calculation confirms the result that W_1, W_L and $\overline{W}_1$, $\overline{W}_L$ scale in a similar fashion. To this order one finds also that they are simply related to each other as functions of ω.

More importantly one finds nonvanishing longitudinal structure functions and transverse functions that increase linearly with large ω. This confirms these results as obtained in Ref. 1 from positivity gauge invariance and light-cone dominance. The nonvanishing of $F_L(\omega)$ and $\overline{F}_L(\omega)$ implies that the free-field current commutator on the light cone has been altered by interaction. The linear increase with large ω of $F_T(\omega)$ and $\overline{F}_T(\omega)$ implies a short-distance singularity of the form $\delta''(\eta)$ in $f_2(0,\eta)$ of Eq. (29) and Eq. (28) as explained in Ref. 1. Thus the bilocal operator leading to $f_2(x^2, x\cdot p)$ seems to be singular at short distances, contrary to "canonical" expectations. It of course is *not* singular for lightlike separations, and hence the expected scaling with $b=-\frac{1}{2}$ is not affected.

Naturally many other connections between the light cone and scaling, such as a large number of sum rules, may be investigated in our model. This however is beyond the scope of the present paper and is deferred to a separate publication.

ACKNOWLEDGMENT

I wish to thank Professor K. Wilson for suggesting a look at gluon models in three dimensions, and Dr. Carl Kaysen and the faculty of the School of Natural Sciences for their hospitality.

APPENDIX A

1. Scalar Field $\phi(x)$

The real scalar field $\phi(x)$ satisfies the Klein-Gordon equation of motion in three dimensions

$$\left[\left(\frac{\partial}{\partial x^0}\right)^2-\left(\frac{\partial}{\partial x^1}\right)^2-\left(\frac{\partial}{\partial x^2}\right)^2+\mu^2\right]\phi(x)=0\,. \tag{A1}$$

The momentum-space decomposition of $\phi(x)$ is

$$\phi(\vec{x},t)=\int\frac{d^2k}{2\pi\sqrt{2\omega_k}}[a(k)e^{-ik\cdot x}+a^\dagger(k)e^{+ik\cdot x}], \tag{A2}$$

where

$$\omega_k=|\vec{k}|^2+\mu^2$$

and

$$\sqrt{2\omega_k}\,a(k)|k'\rangle=(2\pi)^2 2\omega_k\delta^2(\vec{k}-\vec{k}')|0\rangle\,.$$

Quantization is expressed by

$$[a(k),a^\dagger(k')]=\delta^2(\vec{k}-\vec{k}')$$

or

$$[\dot{\phi}(x),\phi(y)]_{x^0=y^0}=-i\delta^2(\vec{x}-\vec{y})$$

leading to

$$[\phi(x),\phi(y)]=\int\frac{d^2k}{(2\pi)^2 2\omega_k}(e^{-ik\cdot(x-y)}-e^{ik\cdot(x-y)})$$
$$=-\frac{i}{(2\pi)^2}\int\frac{d^2k}{\omega_k}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\sin\omega_k(x_0-y_0)$$
$$=i\Delta(x-y)\,. \tag{A3}$$

$\Delta(x-y)$ may be written as

$$\Delta(x-y)=-i\int\frac{d^3k}{(2\pi)^2}\delta(k^2-\mu^2)\epsilon(k_0)e^{-ik\cdot(x-y)},$$

where

$$\epsilon(k_0)=\begin{cases}+1, & k_0>0\\ -1, & k_0<0\end{cases}. \tag{A4}$$

It follows then that

$$(\partial^\mu\partial_\mu+\mu^2)\Delta(x-y)=0\,,$$
$$\Delta(x-y)=-\Delta(y-x)\,. \tag{A5}$$

One also has

$$\Delta(\vec{x}-\vec{y},0)=0, \tag{A6}$$

leading, by Lorentz covariance, to

$$\Delta(x-y)=0, \quad (x-y)^2<0.$$

One also has

$$\partial_0\Delta(x-y)|_{x^0=y^0}=-\delta^2(\vec{x}-\vec{y}). \tag{A7}$$

The Feynman propagator is finally found to be

$$i\Delta_F(q)=\frac{i}{q^2-\mu^2+i\epsilon}. \tag{A8}$$

2. Dirac Field $\psi(x)$

For a multicomponent field we assume an equation of the form

$$H\psi=i\frac{\partial\psi}{\partial t}=\frac{1}{i}\left(\alpha_1\frac{\partial\psi}{\partial x^1}+\alpha_2\frac{\partial\psi}{\partial x^2}\right)+\beta M\psi, \tag{A9}$$

where H is the Hamiltonian; α_1, α_2, β are Hermitian matrices; and ψ is a multicomponent spinor. In order that the correct energy-momentum relation emerge from Eq. (A9) we must have

$$-\frac{\partial^2\psi_\sigma}{\partial t^2}=(-\nabla^2+M^2)\psi_\sigma, \tag{A10}$$

where

$$\nabla^2=\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial x_2{}^2}.$$

Squaring Eq. (A9) and comparing with Eq. (A10) one immediately obtains the following familiar conditions on the matrices α_1, α_2, and β:

$$\alpha_i{}^2=\beta^2=1,$$
$$\{\alpha_i,\alpha_j\}=2\delta_{ij}, \quad i,j=1,2 \tag{A11}$$
$$\{\alpha_i,\beta\}=0.$$

From Eq. (A11) we can immediately conclude that the α_i are traceless and have eigenvalues of ±1. This implies that they are matrices of even dimension.

The simplest solution to Eq. (A11) is provided by the two-dimensional Pauli matrices and is given by

$$\alpha_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad \alpha_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \quad \beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{A12}$$

ψ_σ in this case is a two-component spinor.

If we now define the γ matrices by

$$\gamma^1=\beta\alpha_1=\begin{pmatrix}0&1\\-1&0\end{pmatrix},$$
$$\gamma^2=\beta\alpha_2=\begin{pmatrix}0&-i\\-i&0\end{pmatrix}, \tag{A13}$$
$$\gamma^0=\beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

then the algebra of Eq. (A11) becomes

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}1, \quad \mu=0,1,2 \tag{A14}$$

where the metric tensor is given by

$$g^{\mu\nu}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}, \tag{A15}$$

and γ_ν is then defined by

$$\gamma^\mu=g^{\mu\nu}\gamma_\nu. \tag{A16}$$

The equation of motion now reads

$$(\gamma^\mu\partial_\mu+M)\psi(x)=0, \tag{A17}$$

in analogy with the Dirac equation except that $\mu=0,1,2$, γ^μ are two-by-two matrices, and $\psi(x)$ is a two-spinor.

In analogy with the four-dimensional case one can show that in the space of these spinors the generators of Lorentz transformations $J_{\mu\nu}$ may be represented using

$$\sigma_{\mu\nu}=\tfrac{1}{2}i[\gamma_\mu,\gamma_\nu]. \tag{A18}$$

Consequently Eq. (12) is found to be covariant. If we define

$$\bar{\psi}(x)=\psi^\dagger(x)\gamma^0, \tag{A19}$$

we can then show that $\bar{\psi}\psi$ is a scalar, $\bar{\psi}\gamma^\mu\psi$ is a vector, and $\bar{\psi}\sigma^{\mu\nu}\psi$ is an antisymmetric second-rank tensor under Lorentz transformations in three dimensions. It may be easily shown that $\gamma^0\gamma^1\gamma^2=-i$ and therefore that there is no independent matrix analogous to γ_5 in three dimensions.

The free single-particle wave functions may be obtained by finding them first for particles at rest and then boosting to arbitrary momentum. We thus first solve

$$i\frac{\partial\psi}{\partial t}=\beta M\psi, \tag{A20}$$

and find two solutions

$$\psi^{(r)}(x)=\omega^{(r)}(0)e^{-i(\epsilon_r Mt)}, \quad r=1,2$$

where

$$\omega^{(1)}(0)=\begin{pmatrix}1\\0\end{pmatrix}, \quad \omega^{(2)}(0)=\begin{pmatrix}0\\1\end{pmatrix},$$
$$\epsilon_r=\begin{cases}+1, & r=1\\-1, & r=2\end{cases}. \tag{A21}$$

The solution with $r=1$ is the positive-energy solution, and that with $r=2$ the negative-energy solution. These also correspond to eigenvalues of $\gamma^0=\beta=\sigma_{12}=S_3$ of $+1$ and -1, respectively. Thus only one "spin" state occurs for each solution in contrast to the ordinary Dirac particle.

The general free wave function may then be found by boosting $\psi^r(x)$ to some arbitrary momentum $\vec{p}$. Taking $\vec{p}$ to lie along the x_1 direction we have

$$\omega^{(r)}(p) = e^{-i(\kappa/2)\sigma_{01}}\omega^{(r)}(0), \tag{A22}$$

where

$$\kappa = -\tanh^{-1}(v/c);$$

v is the speed of the particle. We thus have

$$\begin{aligned} \psi^{(1)}(x) &= e^{-ip\cdot x}\begin{pmatrix} 1 \\ -\tanh\frac{1}{2}\kappa \end{pmatrix}\cosh\tfrac{1}{2}\kappa \\ &= e^{-ip\cdot x}\omega^{(1)}(p), \\ \psi^{(2)}(x) &= e^{+ip\cdot x}\begin{pmatrix} -\tanh\frac{1}{2}\kappa \\ 1 \end{pmatrix}\cosh\tfrac{1}{2}\kappa \\ &= e^{+ip\cdot x}\omega^{(2)}(p), \end{aligned} \tag{A23}$$

where

$$-\tanh\tfrac{1}{2}\kappa = \frac{pc}{E+Mc^2},$$

$$\cosh\tfrac{1}{2}\kappa = \left(\frac{E+Mc^2}{2Mc^2}\right)^{1/2},$$

$p = |\vec{p}|$, and E is the energy of the particle.

If $\not{p} = p^\mu\gamma_\mu$ we also have

$$(\not{p} - M\epsilon_r)\omega^{(r)}(p) = 0. \tag{A24}$$

One may also demonstrate easily that

$$\overline{\omega}^{(r)}(p)\omega^{(r')}(p) = \delta_{rr'}\epsilon_r \tag{A25}$$

and that

$$\sum_{r=1,2}\epsilon_r\omega_\alpha^{(r)}(p)\overline{\omega}_\beta^{(r)}(p) = \delta_{\alpha\beta}.$$

If we define

$$U(p) = \omega^{(1)}(p), \qquad V(p) = \omega^{(2)}(p)$$

we then find the energy projection operators

$$\begin{aligned} U_\alpha(p)\overline{U}_\beta(p) &= \left(\frac{\not{p}+M}{2M}\right)_{\alpha\beta}, \\ V_\alpha(p)\overline{V}_\beta(p) &= \left(\frac{\not{p}-M}{2M}\right)_{\alpha\beta}. \end{aligned} \tag{A26}$$

FIG. 3. Contour for the integral in Eq. (B5).

The momentum-space expansion of the free field is

$$\psi(\bar{x}, t) = \int \frac{d^2p}{(2\pi)^2}\left(\frac{M}{E}\right)^{1/2}[b(p)U(p)e^{-ip\cdot x} + d^\dagger(p)\overline{V}(p)e^{ip\cdot x}], \tag{A27}$$

with $E = (|\vec{p}|^2 + M^2)^{1/2}$. $b(p)$ is a destruction operator of a particle and $d(p)$ that of the antiparticle. The normalization of states is such that

$$\sqrt{2E}\,b(p)|p'\rangle = (2\pi)^2(2E)\delta^2(\vec{p}-\vec{p}')|0\rangle.$$

Quantization is implemented by

$$\begin{aligned} \{b(p), b^\dagger(p')\} &= \delta^2(\vec{p}-\vec{p}'), \\ \{d(p), d^\dagger(p')\} &= \delta^2(\vec{p}-\vec{p}'), \end{aligned} \tag{A28}$$

or equivalently

$$\{\psi_\alpha(\vec{x}, t), \psi_\beta^\dagger(\vec{x}', t)\} = \delta^2(\vec{x}-\vec{x}')\delta_{\alpha\beta}. \tag{A29}$$

From Eqs. (A27) and (A28) and using Eq. (A3) we then find

$$\begin{aligned} \{\psi_\alpha(x), \overline{\psi}_\beta(x')\} &= i(i\not{\partial}+M)_{\alpha\beta}\Delta(x-x') \\ &= -iS_{\alpha\beta}(x-x'). \end{aligned} \tag{A30}$$

The Feynman propagator then takes the form

$$iS_F(p) = \frac{i(\not{p}+M)}{p^2 - M^2 + i\epsilon}. \tag{A31}$$

Some trace theorems and γ identities are

$$\not{a}\not{a} = a^2,$$

$$\mathrm{tr}\not{a}\not{b} = 2a\cdot b,$$

$$\mathrm{tr}\not{a}_1\not{a}_2\not{a}_3\not{a}_4 = 2(a_1\cdot a_2 a_3\cdot a_4 - a_1\cdot a_3 a_2\cdot a_4 + a_1\cdot a_4 a_2\cdot a_3),$$

and the trace of an odd number of γ's vanishes.

APPENDIX B

We discuss in this appendix the calculation of equal-time commutators from short-distance expansions in three dimensions.

If $C_n(x)$ is the singular c-number function, then using the definition of the δ function we have

$$C_n(x) = F_n(x^0)\delta^2(\vec{x}) + \vec{F}_n(x^0)\cdot\vec{\nabla}\delta(\vec{x}) + \cdots, \tag{B1}$$

with

$$\begin{aligned} F_n(x^0) &= \int d^2x\, C_n(x^0, \vec{x}), \\ \vec{F}_n(x^0) &= -\int d^2x\, \vec{x}C_n(x^0, \vec{x}), \end{aligned} \tag{B2}$$

etc. In the limit $x^0 \to 0$, $F_n(x^0)$ and $\vec{F}_n(x^0)$ give, respectively, the regular part and the Schwinger term of the equal-time commutator.

In evaluating integrals of the form given in Eq. (B2) we are usually faced with expressions of the form

$$F_{\mu\cdots\nu}(x^0)=\int d^2x\left(\frac{1}{(x^2-i\epsilon x^0)^\alpha}-\frac{1}{(x^2+i\epsilon x^0)^\alpha}\right)x_\mu\cdots x_\nu\ . \tag{B3}$$

If the x_n factors are such that any component of $\vec{x}$ is left unsquared, the integral is zero from rotational invariance. Using polar coordinates one then first performs the angular integration and is left with

$$f_{\mu\cdots\nu}(x^0)\propto\int_0^\infty r\,dr F_{\mu\cdots\nu}(r^2,x^0)\left(\frac{1}{(r-x_0-\frac{1}{2}i\epsilon)^p(r+x_0+\frac{1}{2}i\epsilon)^p}-\text{c.c.}\right). \tag{B4}$$

In the complex r plane and for arbitrary p the integrand displays cuts from x^0 to $+\infty$ and $-x^0$ to $-\infty$. The integral in Eq. (B4) is then given by

$$f_{\mu\cdots\nu}(x^0)\propto\int_c r\,dr F_{\mu\cdots\nu}(r^2,x^0)\frac{1}{(r-x^0)^p(r+x^0)^p}\ , \tag{B5}$$

where c is the contour shown in Fig. 3.

For integer p one uses the residue theorem, and for noninteger α one integrates the discontinuity across the cut from $r=x^0$ to $r=\infty$.

APPENDIX C

The contribution of the single-particle intermediate-state Born term to $W_{\mu\nu}$ and $\overline{W}_{\mu\nu}$ is well known and will not be reproduced here. Its relevance to the results of Ref. 1 is discussed in Appendix B of that article and the reader is referred to it for details.

The calculation of the two-particle intermediate-state contributions is tedious but straightforward and proceeds as follows.

1. Calculation of $W_{\mu\nu}$

Upon referring to Fig. 1 we may then write down the explicitly gauge-invariant expression

$$W_{\mu\nu}=g^2e^2\frac{M}{p^0}\int d^3p_1\int d^3k\,\delta(k^2-\mu^2)\delta(p_1{}^2-M^2)\delta(q+p-k-p_1)\theta(k^0-\mu)\theta(p_1^0-M)\Lambda_{\mu\nu}\ , \tag{C1}$$

where

$$\Lambda_{\mu\nu}=\frac{1}{4M^2}\operatorname{tr}\left[(\not p+M)\left(\frac{2p_{1\mu}-\not q\gamma_\mu}{q^2-2q\cdot p_1+i\epsilon}+\frac{2p_\mu+\gamma_\mu\not q}{q^2+2q\cdot p+i\epsilon}\right)(\not p_1+M)\left(\frac{2p_{1\nu}-\gamma_\nu\not q}{q^2-2q\cdot p_1-i\epsilon}+\frac{2p_\nu+\not q\gamma_\nu}{q^2+2q\cdot p-i\epsilon}\right)\right]\ , \tag{C2}$$

$q^\mu\Lambda_{\mu\nu}=0$, and $q^\nu\Lambda_{\mu\nu}=0$ explicitly. Using the vectors η_L^μ and η_T^μ we define

$$\begin{aligned}\Lambda^L&=\eta_L^\mu\eta_L^\nu\Lambda_{\mu\nu}\ ,\\ \Lambda^T&=\eta_T^\mu\eta_T^\nu\Lambda_{\mu\nu}\ .\end{aligned} \tag{C3}$$

The trace is then taken with simplifications occurring due to the properties of the vectors η_L^μ and η_T^μ.

We first then perform the k integration, getting an integral of the form

$$I=\int_0^\infty|\vec{p}_1|\,d|\vec{p}_1|\int_0^{2\pi}d\theta\int_M^\infty dp_1^0\delta((q+p)^2-2p_1\cdot(q+p)+M^2-\mu^2)\times\delta((p_1^0)^2-|\vec{p}_1|^2-M^2)\frac{1}{(q^2-2q\cdot p_1)^\eta}\ ,\qquad \eta=1,2,0\ . \tag{C4}$$

We choose a frame such that $\vec{q}+\vec{p}=0$. Then the p_1^0 and $|\vec{p}_1|$ integrations become trivial. One is finally left with integrals of the form

$$\int_0^{2\pi}d\theta\,\frac{F(\cos\theta,\sin\theta)}{(\alpha+\cos\theta)^\eta}\ ,\qquad \eta=0,1,2\ . \tag{C5}$$

One finds that $\alpha<-1$ for $0<\omega<1$, and the integrals are well known. One obtains factors of the form

$$\frac{1}{(\alpha^2-1)^{3/2}}\ ,\qquad \frac{1}{(\alpha^2-1)^{1/2}} \tag{C6}$$

multiplying functions of ν and q^2. The evaluation is then done exactly. The scaling limit is now taken. The only point to watch for at this stage is that

$$\alpha^2 \underset{B}{\sim} 1+\frac{a}{\nu}+O\left(\frac{1}{\nu^2}\right), \tag{C7}$$

and hence that the factors in (C5) contribute factors of ν to the scaling limit. This is also why one should not take this limit inside the integrals of (C5) as $\alpha \to -1$ and one may not be able to keep track of the proper factors of ν.

The final expressions for W_L and W_T are not simple and are not illuminating enough to justify writing them in detail here. The resulting scaling functions are reproduced in the text.

2. Calculation of $\overline{W}_{\mu\nu}$

Referring to Fig. 2 we may write

$$\overline{W}_{\mu\nu}=g^2e^2\frac{M}{p^0}\int d^3\overline{p}\int d^3k\,\delta(k^2-\mu^2)\delta(\overline{p}^2-M^2)\delta(q-p-k-\overline{p})\theta(k^0-\mu)\theta(\overline{p}^0-M)\overline{\Lambda}_{\mu\nu}, \tag{C8}$$

where

$$\overline{\Lambda}_{\mu\nu}=\frac{1}{4M^2}\operatorname{tr}\left[(\not p+M)\left(\frac{2\overline{p}_\mu-\not q\gamma_\mu}{q^2-2q\cdot\overline{p}+i\epsilon}+\frac{-2p_\mu+\gamma_\mu\not q}{q^2-2q\cdot p+i\epsilon}\right)(\not{\overline{p}}-M)\left(\frac{2\overline{p}_\nu-\gamma_\nu\not q}{q^2-2q\cdot\overline{p}-i\epsilon}+\frac{-2p_\nu+\not q\gamma_\nu}{q^2-2q\cdot p-i\epsilon}\right)\right]. \tag{C9}$$

Comparing Eq. (C9) with Eq. (C2) we see that

$$\overline{\Lambda}_{\mu\nu}(q,p,\overline{p})=-\Lambda_{\mu\nu}(-q,p,-\overline{p}). \tag{C10}$$

We then find that

$$\Lambda_{\mu\nu}(-q,p,-\overline{p})=\Lambda_{\mu\nu}(q,-p,\overline{p}), \tag{C11}$$

thus leading to

$$\overline{W}_{\mu\nu}(q,p)=-W_{\mu\nu}(q,-p). \tag{C12}$$

Thus

$$\begin{aligned}\overline{F}_L(\omega)&=-F_L(\omega),\\ \overline{F}_T(\omega)&=-F_T(-\omega).\end{aligned} \tag{C13}$$

We then have the results given in the text in Eqs. (57) and (58).

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