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Quantum Field Theories in the Infinite-Momentum Frame. II. Scattering Matrices of Scalar and Dirac Fields

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The scattering matrices of field theories formulated in the light-front quantization of the preceding paper are studied. Reduction formulas for scalar and Dirac particles are derived. The scattering matrices in this new formulation are shown to give the same predictions as in the equal-time formulation to all orders in perturbations. Second-order renormalization is carried out and it gives well-known results. New Feynman rules of the perturbation theory are given and their peculiarities are discussed.

I. INTRODUCTION

In the preceding paper[1] we have studied the quantization of interacting scalar and Dirac field theories in an infinite-momentum frame. It was shown that the formal structure is consistent with Lorentz invariance, leads to sum rules for the spectral functions of various Green's functions, and provides current commutation relations on the light front from which current-algebra sum rules like those of Adler[2] and Dashen, Gell-Mann, and Fubini[3] can be derived without reference to the infinite-momentum limit.

On the other hand, quantum field theories have been traditionally formulated on equal-time surfaces, and the equal-time current algebra of Gell-Mann[4] has played an essential role in the development of particle physics. We may ask: Are the light-front formulation and the equal-time formulation of quantum field theories equivalent to each other? In other words, do these two formulations give the same scattering matrix for the same physical processes? This is obviously a very important and interesting question. It has been partially answered by Kogut and Soper,[5] and Bjorken, Kogut, and Soper,[6] who demonstrated that certain classes of diagrams in quantum electrodynamics give results identical to those in ordinary formulation. The main purpose of this paper is to supply a formal proof of the equivalence of the S matrices given by the two formulations to *all* orders in perturbation. In the present paper we shall consider only self-interacting scalar field theories and coupled scalar-Dirac field theories. Field theories involving spin-1 particles will be studied separately[7] as they are more complicated and require special treatment. Our proof is based on the functional-derivative method due to Schwinger[8] which has been recently utilized by Gerstein, Jackiw, Lee and Weinberg[9] in their study of nonlinear chiral Lagrangians.

In Sec. II we derive the reduction formulas for both scalar and Dirac particles. In the case of Dirac particles it is shown that either the full in-field $\psi_{\rm in}$ (or $\psi_{\rm out}$) or its independent part $\psi_{\rm in}^{(+)}$ (or $\psi_{\rm out}^{(+)}$) can be used in these formulas. In Sec. III the S matrix for a self-interacting scalar field theory is studied in the interaction representation. In this simple theory the interaction Hamiltonian and the negative of the interaction Lagrangian are identical and covariant; time-ordered products and x^+-ordered products are equivalent. The light-front formulation of this theory and the equal-time formulation are trivially equivalent. The S matrix for a coupled scalar-Dirac field theory is constructed and studied in Secs. IV and V. The interaction Hamiltonian in this case is no longer covariant and is not simply related to the interaction Lagrangian. Furthermore, the fermion propagator acquires a noncovariant term. In Sec. IV the new Feynman rules are presented and applied to the second-order expansion. It is demonstrated to this order that the noncovariant terms in the Hamiltonian and the fermion propagator explicitly cancel each other. The resulting S matrix is covariant and the same as in the ordinary formulation. Section V is devoted to the proof of the equivalence of the light-front formulation of the coupled scalar-Dirac system to the corresponding equal-time formulation to all orders of perturbation. Second-order renormalization for pseudoscalar coupling theory is carried out in Sec. VI, using the new Feynman rules. The well-known renormalized Green's functions and vertex function of this theory are reproduced.

II. REDUCTION FORMULAS

In this section reduction formulas for scalar and Dirac particles will be derived.[10] In the derivation it is assumed that all surface terms can be ignored without any detailed justification. A mathematically more rigorous derivation is certainly desirable.

As long as there are no massless particles, in-states and out-states can be introduced just as in the usual formulation. This follows from the fact that free-particle states at $t\to\infty$ and $t\to-\infty$ coincide with free-particle states at $x^+\to+\infty$ and $x^+\to-\infty$. Even in the presence of massless particles and as long as there is no in- and out-particles moving in the x^- direction, the asymptotic particle states can be defined as well. For simplicity, however, we shall not discuss systems involving massless particles.

For the derivation of the reduction formulas involving scalar particles, the method given in a standard textbook such as Bjorken and Drell goes through. We shall not repeat the derivation here. The result for reducing in one in-particle is given by

$$\langle\beta\text{ out}|\alpha,k\text{ in}\rangle = \langle\beta-k\text{ out}|\alpha\text{ in}\rangle + \frac{i}{\sqrt{Z}}\int d^4x\, f_k(x)(\partial^2+m^2)\langle\beta\text{ out}|\varphi(x)|\alpha\text{ in}\rangle, \tag{2.1}$$

where

$$f_k(x)=\frac{1}{[(2\pi)^3 2k^+]^{1/2}}e^{-ik\cdot x} \tag{2.2}$$

is the one-particle wave function in the light-front formulation. The one-particle states are normalized according to

$$\langle k'|k\rangle=\delta(k^+-k'^+)\delta^2(k-k'). \tag{2.3}$$

This is precisely the well-known reduction formula using a slightly different normalization condition. Other reduction formulas involving scalar fields can be obtained by a similar method.

The derivation for the reduction formula for Dirac fields is more subtle. This is related to the fact that not all components of a Dirac field ψ are dynamical variables on the light front. Only $\psi^{(+)}\equiv\frac{1}{2}(1+\alpha_3)\psi$ and $\psi^{(+)\dagger}$ are independent fields. Thus, $\psi^{(+)}$ and $\psi^{(+)\dagger}$ alone are associated with the in- and out-particle states. They are related to the corresponding creation and annihilation operators through

$$\psi_{\text{in}}^{(+)}(x)=\sum_s\int\frac{d^2p\,dp^+}{(2\pi)^{3/2}}\left(\frac{m}{p^+}\right)^{1/2}\theta(p^+)[b_{\text{in}}(p,s)u^{(+)}(p,s)e^{-ip\cdot x}+d_{\text{in}}^\dagger(p,s)v^{(+)}(p,s)e^{ip\cdot x}] \tag{2.4}$$

and its Hermitian conjugate equation with similar equations for out-fields where $u^{(+)}(p,s)\equiv\frac{1}{2}(1+\alpha_3)u(p,s)$ and $v^{(+)}(p,s)\equiv\frac{1}{2}(1+\alpha_3)v(p,s)$ are projections of the usual spinor wave functions u and v into the upper rapidity space defined by the eigenvalue $\alpha_3'=1$. The spinors $u^{(+)}$ and $v^{(+)}$ are normalized according to

$$u^{(+)\dagger}(p,s')u^{(+)}(p,s)=\tfrac{1}{2}\bar{u}(p,s')\gamma^+u(p,s) = \frac{p^+}{2m}\delta_{s's}, \tag{2.5a}$$

$$v^{(+)\dagger}(p,s')v^{(+)}(p,s)=\frac{p^+}{2m}\delta_{s's}, \tag{2.5b}$$

while the creation and annihilation operators obey

$$\{b_{\text{in}}(p,s),b_{\text{in}}^\dagger(p',s')\}=\delta_{ss'}\delta(p^+-p'^+)\delta^2(p-p'), \tag{2.6a}$$

$$\{d_{\text{in}}(p,s),d_{\text{in}}^\dagger(p',s')\}=\delta_{ss'}\delta(p^+-p'^+)\delta^2(p-p'), \tag{2.6b}$$

and

$$\{b_{\text{in}},b_{\text{in}}\}=\{d_{\text{in}},d_{\text{in}}\}=\{b_{\text{in}},d_{\text{in}}\}=\{b_{\text{in}},d_{\text{in}}^\dagger\}=\cdots=0. \tag{2.6c}$$

Equations similar to (2.6a)–(2.6c) hold for b_{out}, $b_{\text{out}}^\dagger$, d_{out}, and $d_{\text{out}}^\dagger$. Equation (2.4) can be inverted to give

$$b_{\text{in}}(p,s)=\int d^2x\,dx^-\frac{e^{ip\cdot x}}{(2\pi)^{3/2}}\left(\frac{m}{p^+}\right)^{1/2}u^{(+)\dagger}(p,s)\psi_{\text{in}}^{(+)}(x) = \int d^2x\,dx^-\,U_{ps}^{(+)\dagger}(x)\psi_{\text{in}}^{(+)}(x), \tag{2.7}$$

$$d_{\rm in}^\dagger(p,s)=\int d^2x\,dx^-\,\frac{e^{-ip\cdot x}}{(2\pi)^{3/2}}\left(\frac{m}{p^+}\right)^{1/2}v^{(+)\dagger}(p,s)\psi_{\rm in}^{(+)}(x)$$

$$=\int d^2x\,dx^-\,V_{ps}^{(+)\dagger}(x)\psi_{\rm in}^{(+)}(x),\tag{2.8}$$

and similarly for $b_{\rm in}^\dagger$, $d_{\rm in}$, $b_{\rm out}$, $b_{\rm out}^\dagger$, $d_{\rm out}$, and $d_{\rm out}^\dagger$, where

$$U_{ps}^{(+)}(x)=\frac{e^{-ip\cdot x}}{(2\pi)^{3/2}}\left(\frac{m}{p^+}\right)^{1/2}u^{(+)}(p,s)$$

$$=\tfrac{1}{2}(1+\alpha_3)U_{ps}(x),\tag{2.9}$$

$$V_{ps}^{(+)}(x)=\frac{e^{ip\cdot x}}{(2\pi)^{3/2}}\left(\frac{m}{p^+}\right)^{1/2}v^{(+)}(p,s)$$

$$=\tfrac{1}{2}(1+\alpha_3)V_{ps}(x),\tag{2.10}$$

and $U_{ps}(x)$ and $V_{ps}(x)$ are solutions of the free Dirac equation.

The simple reduction formula for removing a Dirac particle (p,s) from the in-state can be computed as follows:

$$\langle\beta\text{ out}|(ps),\alpha\text{ in}\rangle=\langle\beta-(ps)\text{ out}|\alpha\text{ in}\rangle+\langle\beta\text{ out}|b_{\rm in}^\dagger(p,s)-b_{\rm out}^\dagger(p,s)|\alpha\text{ in}\rangle$$

$$=\langle\beta-(ps)\text{ out}|\alpha\text{ in}\rangle+\int d^2x\,dx^-\langle\beta\text{ out}|\psi_{\rm in}^{(+)\dagger}(x)-\psi_{\rm out}^{(+)\dagger}(x)|\alpha\text{ in}\rangle U_{ps}^{(+)}(x)$$

$$=\langle\beta-(ps)\text{ out}|\alpha\text{ in}\rangle-\frac{1}{\sqrt{Z_2}}\int d^4x\,\langle\beta\text{ out}|\partial^-(\psi^{(+)\dagger}(x)U_{ps}^{(+)}(x))|\alpha\text{ in}\rangle\,.\tag{2.11}$$

By the use of the Dirac equation

$$(i\partial\!\!\!/-m)U_{ps}(x)=0\,,\tag{2.12}$$

we have

$$\int d^4x\,\partial^-[\psi^{(+)\dagger}(x)U_{ps}^{(+)}(x)]=\int d^4x\,i\bar\psi^{(+)}(-i\overleftarrow{\partial\!\!\!/}-m)U_{ps}(x),\tag{2.13}$$

where

$$\bar\psi^{(+)}=\psi^{(+)\dagger}\gamma^0\,.\tag{2.14}$$

Substituting (2.13) into (2.11), we have the reduction formula

$$\langle\beta\text{ out}|(ps),\alpha\text{ in}\rangle=\langle\beta-(ps)\text{ out}|\alpha\text{ in}\rangle-\frac{i}{\sqrt{Z_2}}\int d^4x\langle\beta\text{ out}|\bar\psi^{(+)}(x)|\alpha\text{ in}\rangle(-i\overleftarrow{\partial\!\!\!/}-m)U_{ps}(x)\,.\tag{2.15}$$

At first sight, Eq. (2.15) does not appear to be the same as the usual one-particle reduction formula

$$\langle\beta\text{ out}|(ps),\alpha\text{ in}\rangle=\langle\beta-(ps)\text{ out}|\alpha\text{ in}\rangle-\frac{i}{\sqrt{Z_2}}\int d^4x\,\langle\beta\text{ out}|\bar\psi(x)|\alpha\text{ in}\rangle(-i\overleftarrow{\partial\!\!\!/}-m)U_{ps}(x)\,.\tag{2.16}$$

However, a straightforward calculation extablishes that Eqs. (2.15) and (2.16) are actually the same.

The reduction formula for removing two or more particles from the in- and out-states can be worked out in an analogous way. One can continue the reduction process until all particles are removed from the state vectors. For a given process, the connected contribution is simply given by the product of individual one-particle factors operating on the vacuum expectation value of fields

$$\langle 0|T^+(\varphi(x_1)\cdots\psi^{(+)}(y_1)\cdots\bar\psi^{(+)}(z_1)\cdots)|0\rangle\,.$$

We note in our formulation that (i) the field operators in the above vacuum expectation value are x^+-ordered; (ii) one can use either $\psi^{(+)}$ or ψ in the reduction formulas; (iii) the noncovariant term in the Wick's contraction $\psi^\bullet\bar\psi^\bullet$ does not contribute at all in the reduction formula.[11]

III. SCATTERING MATRIX OF SELF-INTERACTING SCALAR FIELD THEORY

The S matrix of the simpler case of a self-coupled scalar field theory will be studied in this section. It will be established that the Feynman rules for computing the S matrix are the same in the light-front formulation as in the conventional quantization.

The S-matrix can be computed with the help of the interaction representation and the Wick expansion. The free and the perturbed Hamiltonian density in the *interaction representation* are

$$\mathcal{H}_0 = \tfrac{1}{2} T_0^{+-} = \tfrac{1}{2}[(\vec{\nabla}\varphi)^2 + \mu^2\varphi^2] \tag{3.1}$$

and

$$\mathcal{H}_I = -\mathcal{L}_I(\varphi), \tag{3.2}$$

respectively, where in (3.1) and (3.2) φ and $\mathcal{L}_I(\varphi)$ are in terms of interaction representation field operators. Of course, φ obeys the equal-x^+ commutation relation of a free scalar field. The equal-x^+ commutation relation for φ, (3.1), and the Heisenberg equation of motion,

$$i[P_0^-, \varphi(x)] = \partial^-\varphi(x), \tag{3.3}$$

with

$$P_0^- = \int d^2x\, dx^- \mathcal{H}_0(x), \tag{3.4}$$

are sufficient to establish that $\varphi(x)$ is in fact a free field operator.

The S matrix can be expressed simply by

$$S = T^+ \exp\left[-i\int d^4x\, \mathcal{H}_I(x)\right], \tag{3.5}$$

where T^+ stands for x^+-ordered. We can evaluate the S matrix (3.5) by means of perturbation theory through Wick's expansion. We then obtain a set of Feynman rules for this perturbation theory: (1) The vertex is given by $\mathcal{L}_I$ just as in the conventional theory; (2) the propagator is given by

$$\varphi(x)^\bullet\varphi(0)^\bullet = T^+\langle 0|(\varphi(x)\varphi(0))|0\rangle$$

$$= \theta(x^+)\int \frac{d^3k}{(2\pi)^3}\frac{1}{2k^0}e^{-ik\cdot x} + \theta(-x^+)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2k^0}e^{ik\cdot x} \tag{3.6}$$

$$= \theta(x^+)\int\frac{d^2k}{(2\pi)^3}\frac{dk^+}{2k^+}e^{-ik\cdot x} + \theta(-x^+)\int\frac{d^2k}{(2\pi)^3}\frac{dk^+}{2k^+}e^{ik\cdot x}$$

$$= \int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\frac{i}{k^2-\mu^2+i\epsilon}\,; \tag{3.7}$$

(3) one has to sum and integrate over all intermediate states. These Feynman rules are precisely the ones which are obtained in the conventional theory. Thus, the two theories give rise to the same S matrix.

IV. $\bar{\psi}\Gamma\psi\varphi$ THEORY, S MATRIX

For a theory involving fermions, the perturbation calculational procedure of the S matrix in the light-front formulation is different from that of the usual formulation. We would like to point out that there is no *a priori* reason why the S matrix computed from these two formulations should be the same. For instance, one can never bring a spacelike surface into a light-front surface and vice versa by any finite Lorentz transformation. This would seem to suggest that these two formulations might be intrinsically different. In fact, as we shall see, the interaction Hamiltonian and the fermion propagator obtained in the light-front formulation are *different* from those obtained in the conventional formulation. It is therefore rather remarkable to see that the S matrices computed in these two formulations can be brought into each other after a regrouping of terms in the perturbation series.

Recently, Kogut and Soper[5] have studied the S matrix of quantum electrodynamics in an infinite-momentum frame choosing a special gauge $A^+=0$. They have constructed the interaction Hamiltonian in this infinite-momentum frame, and have analyzed the S matrix by means of the "old-fashioned" perturbation theory. They have reached many conclusions similar to ours obtained in the $\bar{\psi}\Gamma\psi\varphi$ theory.

(a) Interaction representation. The Hamiltonian density of the interaction system in the Heisenberg picture is given by (3.24) of paper I as

$$\tfrac{1}{2}T^{+-}=\tfrac{1}{2}[(\vec{\nabla}\varphi_H)^2+\mu^2\varphi_H{}^2]+\tfrac{1}{4}i[\psi_H^{(+)\dagger}\partial^-\psi_H^{(+)}-\partial^-\psi_H^{(+)\dagger}\psi_H^{(+)}+\psi_H^{(-)\dagger}\partial^+\psi_H^{(-)}-\partial^+\psi_H^{(-)\dagger}\psi_H^{(-)}], \tag{4.1}$$

where for clarity we denote all Heisenberg operators by a subscript H. For definiteness, we assume that all field operators are normal-ordered. The Hamiltonian is obtained by integrating $\frac{1}{2}T^{+-}$ over $d^2x\,dx^-$, giving

$$\begin{aligned}H&=P^-\\&=\tfrac{1}{2}\int d^2x\,dx^-\,T^{+-}\\&=\tfrac{1}{2}\int d^2x\,dx^-\,[(\vec{\nabla}\varphi_H)^2+\mu^2\varphi_H{}^2]\\&\quad+\tfrac{1}{4}\int d^2x\,dx^-\int d^2x'\,dx'^-\,\psi_H^{(+)\dagger}(x)[-i\vec{\alpha}\cdot\vec{\nabla}+\gamma^0\mathfrak{M}_H(x)][i\vec{\alpha}\cdot\vec{\nabla}'+\gamma^0\mathfrak{M}_H(x')]\psi_H^{(+)}(x')(-i)\delta^2(x-x')\epsilon(x^--x'^-) &(4.2)\\&=H_0+H_{\text{int}}\ , &(4.3)\end{aligned}$$

where

$$\mathfrak{M}_H(x)\equiv m+g\Gamma\varphi_H(x), \tag{4.4}$$

$$\begin{aligned}H_0(\varphi_H,\psi_H^{(+)},\psi_H^{(+)\dagger})&=\tfrac{1}{2}\int d^2x\,dx^-\,[(\vec{\nabla}\varphi_H)^2+\mu^2\varphi_H{}^2]\\&\quad+\tfrac{1}{4}\int d^2x\,dx^-\int d^2x'\,dx'^-\,\psi_H^{(+)\dagger}(x)(-i\vec{\alpha}\cdot\vec{\nabla}+\gamma^0m)(i\vec{\alpha}\cdot\vec{\nabla}'+\gamma^0m)\psi_H^{(+)}(x')(-i)\delta^2(x-x')\epsilon(x^--x'^-),\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}H_{\text{int}}(\varphi_H,\psi_H^{(+)},\psi_H^{(+)\dagger})=\tfrac{1}{4}\int d^2x\,dx^-\int d^2x'\,dx'^-\,&[\psi_H^{(+)}(x)^\dagger\gamma^0g\Gamma\varphi_H(x)(i\vec{\alpha}\cdot\vec{\nabla}'+\gamma^0m)\psi_H^{(+)}(x')\\&+\psi_H^{(+)}(x)^\dagger(-i\vec{\alpha}\cdot\vec{\nabla}+\gamma^0m)\gamma^0g\Gamma\varphi_H(x')\psi_H^{(+)}(x')\\&+g^2\psi_H^{(+)\dagger}(x)\varphi_H(x)\varphi_H(x')\psi_H^{(+)}(x')](-i)\delta^2(x-x')\epsilon(x^--x'^-).\end{aligned} \tag{4.6}$$

We introduce the interaction representation by the standard method and denote every operator in the interaction picture by a subscript I. The field operators in the interaction representation are related to the corresponding operators in the Heisenberg representation by a unitary transformation,

$$\varphi_I(x)=U(x^+,0)\varphi_H(x)U(x^+,0)^{-1}, \tag{4.7a}$$

$$\psi_I^{(+)}(x)=U(x^+,0)\psi_H^{(+)}(x)U(x^+,0)^{-1}. \tag{4.7b}$$

Among the many properties of φ_I, ψ_I, and U in the interaction picture, we list the following: (i) φ_I, $\psi_I^{(+)}$, and $\psi_I^{(+)\dagger}$ obey the Heisenberg equations of motion for H_0,

$$[\varphi_I(x),H_0(\varphi_I,\psi_I^{(+)},\psi_I^{(+)\dagger})]=i\partial^-\varphi_I(x), \tag{4.8a}$$

$$[\psi_I^{(+)}(x),H_0(\varphi_I,\psi_I^{(+)},\psi_I^{(+)\dagger})]=i\partial^-\psi_I^{(+)}(x), \tag{4.8b}$$

and

$$[\psi_I^{(+)\dagger}(x),H_0(\varphi_I,\psi_I^{(+)},\psi_I^{(+)\dagger})]=i\partial^-\psi_I^{(+)\dagger}(x), \tag{4.8c}$$

where $H_0(\varphi_I,\psi_I^{(+)},\psi_I^{(+)\dagger})$ is the free Hamiltonian (4.5) in terms of the interaction representation operators. (ii) φ_I, $\psi_I^{(+)}$, and $\psi_I^{(+)\dagger}$ satisfy the equal-x^+ (anti-) commutator relations,

$$i[\varphi_I(x),\varphi_I(x')]=\tfrac{1}{4}\delta^2(x-x')\epsilon(x^--x'^-), \tag{4.9}$$

$$\{\psi_I^{(+)}(x),\psi_I^{(+)}(x')^\dagger\}=\tfrac{1}{2}(1+\alpha_3)\delta^2(x-x')\delta(x^--x'^-), \tag{4.10}$$

and

$$[\varphi_I, \psi_I^{(+)}] = [\varphi_I, \psi_I^{(+)\dagger}]$$
$$= \{\psi_I^{(+)}, \psi_I^{(+)}\}$$
$$= \{\psi_I^{(+)\dagger}, \psi_I^{(+)\dagger}\}$$
$$= 0 . \tag{4.11}$$

(iii) The U matrix is given by

$$U(x^+, x'^+) = T^+ \exp\left(-\tfrac{1}{2}i\int_{x'^+}^{x^+} dx^+ H_I(x^+)\right) \tag{4.12a}$$

and the S matrix is given by

$$S = U(\infty, -\infty) = T^+ \exp\left(-\tfrac{1}{2}i\int_{-\infty}^{\infty} dx^+ H_I(x^+)\right), \tag{4.12b}$$

where T^+ stands for x^+-ordered. H_I has the same functional form as $H_{\rm int}$ but is expressed as a function of the interaction representation operators,

$$H_I = H_{\rm int}(\varphi_I, \psi_I^{(+)}, \psi_I^{(+)\dagger}) . \tag{4.13}$$

Properties (i) and (ii) imply that φ_I and $\psi_I^{(+)}$ obey the free-field equations. It is easy to see that this is true for φ_I. To show that $\psi_I^{(+)}$ leads to a free Dirac equation, we carry out (4.8b) and obtain

$$i\partial^- \psi_I^{(+)}(x) = \tfrac{1}{2}(1+\alpha_3)\int d^2x' dx'^- (i\vec{\alpha}\cdot\vec{\nabla} + \gamma^0 m)(-\tfrac{1}{4}i)\delta^2(x-x')\epsilon(x^- - x'^-)(i\vec{\alpha}\cdot\vec{\nabla}' + \gamma^0 m)\psi_I^{(+)}(x')$$
$$= (i\vec{\alpha}\cdot\vec{\nabla} + \gamma^0 m)\psi_I^{(-)}(x), \tag{4.14}$$

where $\psi_I^{(-)}(x)$ is defined by

$$\psi_I^{(-)}(x) \equiv \int d^2x' dx'^- (-\tfrac{1}{4}i)\delta^2(x-x')\epsilon(x^- - x'^-)(i\vec{\alpha}\cdot\vec{\nabla}' + \gamma^0 m)\psi_I^{(+)}(x') . \tag{4.15}$$

Equation (4.15) implies

$$i\partial^+ \psi_I^{(-)}(x) = (i\vec{\alpha}\cdot\vec{\nabla} + \gamma^0 m)\psi_I^{(+)}(x) . \tag{4.16}$$

Introducing

$$\psi_I(x) \equiv \psi_I^{(+)}(x) + \psi_I^{(-)}(x), \tag{4.17}$$

and combining (4.14) and (4.16), we obtain the free Dirac equation

$$(i\partial\!\!\!/ - m)\psi_I(x) = 0 \tag{4.18}$$

as desired.

In terms of $\psi_I^{(\pm)}$ and finally $\psi_I(x)$, we can rewrite H_I as

$$H_I = g\int d^2x dx^- \bar{\psi}_I(x)\Gamma\psi_I(x)\varphi_I(x) - \tfrac{1}{4}ig^2\int d^2x dx^- d^2x' dx'^- \psi_I^{(+)\dagger}(x)\varphi_I(x)\varphi_I(x')\psi_I^{(+)}(x')\delta^2(x-x')\epsilon(x^- - x'^-) . \tag{4.19}$$

Note that due to the presence of the second term,

$$H_I \neq -\int d^2x dx^- \mathcal{L}_{\rm int}(\varphi_I, \psi_I, \psi_I^\dagger) . \tag{4.20}$$

No such additional term appears in the conventional formulation of the $\bar{\psi}\Gamma\psi\varphi$ theory. This additional term makes the comparison between the two formulations rather complicated. Previous experiences based on the derivative coupling $g\bar{\psi}\gamma^\mu\psi\partial_\mu\varphi$ and the charged-vector-meson theory suggest that the contribution to the S matrix due to this additional term in H_I may be canceled by a corresponding noncovariant term in the fermion propagator. In the rest of this section and the next section, we shall concentrate on these noncovariant terms and the cancellation of these additional terms in the S matrix. For notational simplicity, we shall now omit the subscript I in the interaction representation operators. Unless stated otherwise, all field operators in the remaining part of this paper are in the interaction representation.

(b) Wick's expansion and Feynman rules. We know from (3.7) that the Wick contraction between two x^+-ordered scalar fields indeed leads to the covariant propagator,

$$\varphi(x)^{\bullet}\varphi(0)^{\bullet} = T^+\langle 0|(\varphi(x)\varphi(0))|0\rangle = i\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\frac{i}{k^2-\mu^2+i\epsilon} . \tag{4.21}$$

Now, we investigate the contraction between a ψ and a $\overline{\psi}$,

$$i\overline{S}_F(x) \equiv \psi(x)^{\bullet}\overline{\psi}(0)^{\bullet} = T^+\langle 0|(\psi(x)\overline{\psi}(0))|0\rangle \equiv \theta(x^+)\langle 0|\psi(x)\overline{\psi}(0)|0\rangle - \theta(-x^+)\langle 0|\overline{\psi}(0)\psi(x)|0\rangle . \tag{4.22}$$

Note that only one-particle intermediate states contribute to (4.22). Making use of

$$\langle 0|\psi(x)|p,s\rangle = \left(\frac{m}{p^+(2\pi)^3}\right)^{1/2} e^{-ik\cdot x}u(p,s), \tag{4.23a}$$

$$\langle p,s|\overline{\psi}(0)|0\rangle = \left(\frac{m}{p^+(2\pi)^3}\right)^{1/2} \overline{u}(p,s), \tag{4.23b}$$

etc., we have

$$i\overline{S}_F(x) = \theta(x^+)\sum_s \int \frac{d^3p}{(2\pi)^3}\frac{m}{p^+}u(p,s)\overline{u}(p,s)e^{-ip\cdot x} - \theta(-x^+)\sum_s \int \frac{d^3p}{(2\pi)^3}\frac{m}{p^+}v(p,s)\overline{v}(p,s)e^{ip\cdot x}$$
$$= \theta(x^+)\int \frac{d^2p\,dp^+}{2(2\pi)^3p^+}(\not{p}+m)e^{-ip\cdot x} + \theta(-x^+)\int \frac{d^2p\,dp^+}{2(2\pi)^3p^+}(-\not{p}+m)e^{ip\cdot x} . \tag{4.24}$$

In deriving (4.24), we have used the fact that

$$\sum_s u(p,s)\overline{u}(p,s) = \frac{\not{p}+m}{2m} , \tag{4.25a}$$

$$-\sum_s v(p,s)\overline{v}(p,s) = \frac{-\not{p}+m}{2m} . \tag{4.25b}$$

Of course, the momentum p^μ appearing in (4.24)–(4.25) obeys the mass-shell relation

$$p^2 = m^2 ,$$

or (4.26)

$$p^- = (\vec{p}^{\,2}+m^2)/p^+ .$$

We now try to relate (4.24) as a four-dimensional Fourier transform of the fermion propagator $i/(\not{p}-m+i\epsilon)$. It is now important to distinguish the on-shell momentum from a general 4-momentum p^μ. We introduce the on-shell momentum $\overline{p}^\mu$ for an arbitrary momentum p^μ through

$$\overline{p}^+ = p^+ ,\quad \overline{p}^i = p^i \quad (i=1,2), \tag{4.27}$$

and

$$\overline{p}^- = (\vec{p}^{\,2}+m^2)/p^+ \neq p^- . \tag{4.28}$$

Then, Eq. (4.24) can be written as

$$i\overline{S}_F(x) = i\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\left[\frac{\theta(p^+)}{p^+}\,\frac{\overline{\not{p}}+m}{p^- - \overline{p}^- + i\epsilon} + \frac{\theta(-p^+)}{-p^+}\,\frac{\overline{\not{p}}+m}{-p^-+\overline{p}^-+i\epsilon}\right]$$
$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\left[\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} - \tfrac{1}{2}i\frac{\gamma^+}{p^+}\right]$$
$$= iS_F(x) - \tfrac{1}{4}\gamma^+\delta(x^+)\delta^2(x)\epsilon(x^-) . \tag{4.29}$$

Indeed, the x^+ contraction between a ψ and a $\overline{\psi}$ leads to a noncovariant term in addition to the usual covariant propagator function $iS_F(x)$.

Now we can apply the Wick expansion to the Dyson formula (4.12). This leads to the following Feynman rules for the vertices and the propagators:

(i) There are two types of vertex contribution. We associate a factor $-ig\Gamma$ for each regular trilinear $\bar{\psi}\Gamma\psi\varphi$ vertex, and a factor $\frac{1}{4}(-ig)^2\gamma^+\delta^2(x-x')\times\epsilon(x^- - x'^-)$ for each noncovariant, nonlocal $(\bar{\psi}\varphi(x))(\varphi\psi(x'))$ vertex. As we shall see, the appearance of this noncovariance nonlocal interaction is only superficial; it will be canceled by a similar nonlocal term in the propagator.

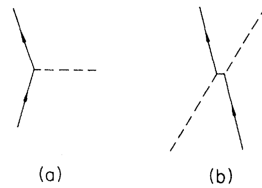

FIG. 1. (a) A regular trilinear $\bar{\psi}\Gamma\psi\varphi$ vertex; (b) a noncovariant 4-point $(\bar{\psi}\varphi)(\varphi\psi)$ vertex.

(ii) A factor $i\Delta_F(x-y)$ for each internal boson line connecting the points x and y.

(iii) A factor

$$i\bar{S}_F(x-y)=iS_F(x-y)-\tfrac{1}{4}\gamma^+\delta(x^+-y^+)\delta^2(x-y)\epsilon(x^- - y^-)$$

for each internal fermion line directed from y to x. In addition, there are the usual kinematical factors, loop integrals, and the sign conventions.

These rules can be translated easily to the momentum space, giving the following:

(i) For a vertex such as in Fig. 1(a), we associate a factor $-ig\Gamma$. For the noncovariant 4-point vertex in Fig. 1(b), we associate a factor $-ig^2\gamma^+/2p^+$.

(ii) A factor $i/(q^2-\mu^2+i\epsilon)$ for each internal boson line.

(iii) A factor

$$i\bar{S}_F(p)=i\left(\frac{1}{\not{p}-m+i\epsilon}-\frac{\gamma^+}{2p^+}\right)$$

$$=i\,\frac{\bar{\not{p}}+m}{p^2-m^2+i\epsilon}$$

for each fermion line.

As in the coordinate space, there are the usual rules for internal loop integrations, sign conventions, and other kinematical factors.

Even though these rules look quite different from the usual Feynman rules, we shall demonstrate that they lead to results which are identical to those obtained from the conventional rules.

Instead of using Wick contractions as described above, one can also directly expand the x^+-ordered perturbation series analogous to the "old-fashioned" time-ordered perturbation expansion. Here one has p^- denominators rather than energy denominators. Conservation of p^+'s implies that each intermediate particle can only have positive p^+ less than the total p^+ of the system since the expansions of field operators (2.1) and (2.10) contain only $p^+>0$. These rules are Weinberg's prescription for infinite-momentum-frame calculations using "old-fashioned" time-ordered perturbation diagrams. These same remarks apply to the self-interacting scalar field theory discussed in Sec. III.

(c) Second-order expansion. To understand the mechanism of detailed cancellation of the noncovariant terms, we work out the Wick's expansion to order g^2 explicitly. From (4.12) and (4.19), we find that the second-order expansion of the S matrix is

$$T^+\tfrac{1}{2}(-ig)^2\int d^4x\,d^4x'\,[:\bar{\psi}(x)\Gamma\psi(x)\varphi(x)::\bar{\psi}(x')\Gamma\psi(x')\varphi(x'):$$
$$+\tfrac{1}{2}:\varphi(x)\bar{\psi}(x)\gamma^+\psi(x')\varphi(x'):\delta(x^+-x'^+)\delta^2(x-x')\epsilon(x^- - x'^-)]\,. \tag{4.30}$$

We have put in : : explicitly in (4.30) to indicate the normal-ordering of the products. For a properly renormalized theory, we can replace (4.30) by

$$T^+\tfrac{1}{2}(-ig)^2\int d^4x\,d^4x'\,[:\bar{\psi}(x)\Gamma\psi(x)\varphi(x)::\bar{\psi}(x')\Gamma\psi(x')\varphi(x'):$$
$$+\tfrac{1}{2}:\varphi(x)\bar{\psi}(x):\gamma^+:\psi(x')\varphi(x'):\delta(x^+-x'^+)\delta^2(x-x')\epsilon(x^- - x'^-)]\,, \tag{4.31}$$

where the factors $\varphi\bar{\psi}$ and $\psi\varphi$ in the second term are x^+-ordered before we average over the limits $x^+\to x'^+\pm 0$. Note that the difference between the integrands in (4.31) and in (4.30) consists of only the contraction terms,

$$\tfrac{1}{2}\varphi(x)^{\bullet}\varphi(x')^{\bullet}:\bar{\psi}(x)\gamma^{+}\psi(x'):\delta(x^{+}-x'^{+})\delta^{2}(x-x')\epsilon(x^{-}-x'^{-}), \tag{4.32a}$$

$$-\tfrac{1}{2}\,\mathrm{Tr}(\psi(x')^{\bullet}\bar{\psi}(x)^{\bullet}\gamma^{+}):\varphi(x)\varphi(x'):\delta(x^{+}-x'^{+})\delta^{2}(x-x')\epsilon(x^{-}-x'^{-}), \tag{4.32b}$$

and

$$-\tfrac{1}{2}\,\mathrm{Tr}(\psi(x')^{\bullet}\bar{\psi}(x)^{\bullet}\gamma^{+})\varphi(x)^{\bullet\bullet}\varphi(x')^{\bullet\bullet}\delta(x^{+}-x'^{+})\delta^{2}(x-x')\epsilon(x^{-}-x'^{-}). \tag{4.32c}$$

The first two terms (4.32a) and (4.32b) lead only to a fermion and a meson self-energy contribution. A simple power counting indicates that both expressions lead to divergent results. However, these divergent terms can be readily removed after proper renormalizations, and the remaining terms are found to vanish identically (see Sec. VI). Expression (4.32c) contributes only to the vacuum amplitude, and will not lead to any observable effect either.

Knowing that we can replace (4.30) by (4.31), it is straightforward to see the cancellation of noncovariant terms from (4.31) explicitly. Let us concentrate on the various amplitudes in the integrand of (4.31):

(i) Amplitude with no contraction. This is given by $:\bar{\psi}(x)\Gamma\psi(x)\varphi(x)\bar{\psi}(x')\Gamma\psi(x')\varphi(x'):$ which involves no noncovariant term and hence is already the correct answer.

(ii) Amplitude with a single $\psi\bar{\psi}$ contraction. The amplitude is

$$\begin{aligned}
2:\bar{\psi}(x)\Gamma\psi(x)^{\bullet}\bar{\psi}(x')^{\bullet}\Gamma\psi(x')\varphi(x)\varphi(x'):+\tfrac{1}{2}:\bar{\psi}(x)\gamma^{+}\psi(x')\varphi(x)\varphi(x'):\delta(x^{+}-x'^{+})\delta^{2}(x-x')\epsilon(x^{-}-x'^{-})\\
=2:\bar{\psi}(x)\Gamma[iS_{F}(x-x')-\tfrac{1}{4}\gamma^{+}\delta(x^{+}-x'^{+})\delta^{2}(x-x')\epsilon(x^{-}-x'^{-})]\Gamma\psi(x')\varphi(x)\varphi(x')\\
+\tfrac{1}{2}:\bar{\psi}(x)\gamma^{+}\psi(x')\varphi(x)\varphi(x'):\delta(x^{+}-x'^{+})\delta^{2}(x-x')\epsilon(x^{-}-x'^{-})\\
=2:\bar{\psi}(x)\Gamma iS_{F}(x-x')\Gamma\psi(x')\varphi(x)\varphi(x'):
\end{aligned} \tag{4.33}$$

as required. The factors 2 in (4.33) account for the two ways of making the $\psi\bar{\psi}$ contraction.

(iii) Amplitude with two $\psi\bar{\psi}$ contractions. This amplitude leads to

$$\begin{aligned}
&\bar{\psi}(x)^{\bullet}\Gamma\psi(x)^{\bullet\bullet}\bar{\psi}(x')^{\bullet\bullet}\Gamma\psi(x')^{\bullet}:\varphi(x)\varphi(x'):+\tfrac{1}{2}\bar{\psi}(x)^{\bullet}\gamma^{+}\psi(x)^{\bullet}:\varphi(x)\varphi(x'):\\
&\quad=(-1)\,\mathrm{Tr}([iS_{F}(x-x')-\tfrac{1}{4}\gamma^{+}\delta(x^{+}-x'^{+})\delta^{2}(x-x')\epsilon(x^{-}-x'^{-})]\Gamma\\
&\qquad\times[iS_{F}(x'-x)-\tfrac{1}{4}\gamma^{+}\delta(x'^{+}-x^{+})\delta^{2}(x'-x)\epsilon(x'^{-}-x^{-}]\Gamma):\varphi(x)\varphi(x'):\\
&\qquad+\tfrac{1}{2}(-1)\,\mathrm{Tr}([iS_{F}(x'-x)-\tfrac{1}{4}\gamma^{+}\delta(x'^{+}-x^{+})\delta^{2}(x'-x)\epsilon(x'^{-}-x^{-})]\gamma^{+}):\varphi(x)\varphi(x'):\delta(x^{+}-x'^{+})\delta^{2}(x-x')\epsilon(x^{-}-x'^{-}).
\end{aligned} \tag{4.34}$$

One immediately sees from (4.34) that the contribution from the crossed terms in the first trace cancels the contribution from the iS_F term in the second trace,

$$2\times\tfrac{1}{4}\,\mathrm{Tr}(iS_{F}(x'-x)\Gamma\gamma^{+}\Gamma)-\tfrac{1}{2}\,\mathrm{Tr}(iS_{F}(x'-x)\gamma^{+})=0. \tag{4.35}$$

A factor 2 is included in (4.35) because there are two crossed terms. The remaining $(\gamma^{+})^{2}$ terms in (4.34) also drop out owing to the identity $(\gamma^{+})^{2}=0$. Thus (4.34) reduces to

$$(-1)\,\mathrm{Tr}(iS_{F}(x-x')\Gamma iS_{F}(x'-x)\Gamma) \tag{4.36}$$

as given by the conventional second-order perturbation theory.

We can show that further contractions on φ's cannot introduce any discrepancy between the two formulations. This follows from the simple fact that the $\varphi\varphi$ contractions are the same in both formulations. Thus, to establish the equivalence of these two formulations, we only need to show that the various $\psi\bar{\psi}$ contractions in both formulations give rise to the same result to all orders in g.

V. EQUIVALENCE THEOREM

In Sec. IV we have demonstrated to second order that the noncovariant term in the interaction Hamiltonian of the light-front formulation cancels that of the fermion propagator in all possible matrix elements. The final results are covariant and identical to those found in the ordinary formulation. We would like now to establish that this cancellation occurs to all orders in the perturbation, leading to a covariant scattering matrix equivalent to the one in conventional theory. We conclude, at least formally, that the two formulations give the same physical predictions.

Our proof is based on the functional-derivative method developed by Schwinger.[8] The technique suitable for our application was illustrated in a recent paper by Gerstein, Jackiw, Lee, and Weinberg.[9]

The S matrix computed in the light-front formulation is given by Dyson's formula

$$S = T^+ \exp\left[-\tfrac{1}{2}i\int dx^+ H_I(x^+)\right]$$
$$= T^+ \exp\left[-i\int d^4x\, d^4x'\, \overline{\psi}(x)\overline{H}(x,x')\psi(x')\right], \tag{5.1}$$

with

$$\overline{H}(x,x') = g\Gamma\varphi\delta^4(x-x') - \tfrac{1}{4}ig^2\gamma^+\varphi(x)\varphi(x')\delta(x^+-x'^+)\delta^2(x-x')\epsilon(x^- - x'^-)\,. \tag{5.2}$$

Strictly speaking, all operators in H_I should be normal-ordered. As we have demonstrated in the previous subsection, we can assume H_I to be x^+-ordered without affecting the *renormalized* scattering amplitudes.

The presence of an explicit nilpotent factor γ^+ $[(\gamma^+)^2=0]$ in the second term of (5.2) greatly simplifies our calculation. In particular, the inverse $\overline{H}(x,x')$, which is defined by

$$\int d^4y\, \overline{H}(x,y)\overline{H}^{-1}(y,x') = \int d^4y\, \overline{H}^{-1}(x,y)\overline{H}(y,x')$$
$$= \delta^4(x-x')\,, \tag{5.3}$$

is given simply by

$$\overline{H}(x,x')^{-1} = (g\Gamma\varphi)^{-1}\delta^4(x-x') + \tfrac{1}{4}i\gamma^+\delta(x^+-x'^+)\delta^2(x-x')\epsilon(x^- - x'^-)\,. \tag{5.4}$$

As mentioned earlier, we denote the noncovariant fermion propagator by

$$i\overline{S}_F(x-x') = iS_F(x-x') - \tfrac{1}{4}\gamma^+\delta(x^+-x'^+)\delta^2(x-x')\epsilon(x^- - x'^-)\,. \tag{5.5}$$

Note that the noncovariant term in $\overline{H}^{-1}$ matches with the noncovariant term in $\overline{S}_F$.

Now, we consider all possible contractions $\psi\overline{\psi}$ of internal fermion lines in the S matrix. According to Schwinger's formalism, this can be taken care of systematically by replacing the T^+-ordered equation by

$$S = :\exp\left[-i\int d^4x\, d^4x'\left(\overline{\psi}(x) - i\overline{S}_F{}^T\frac{\delta}{\delta\psi(x)}\right)\overline{H}(x,x')\left(\psi(x') + i\overline{S}_F\frac{\delta}{\delta\overline{\psi}(x')}\right)\right]: \tag{5.6a}$$

$$= :\exp\left[-i\left(\overline{\psi} - i\overline{S}_F{}^T\frac{\delta}{\delta\psi}\right)\overline{H}\left(\psi + i\overline{S}_F\frac{\delta}{\delta\overline{\psi}}\right)\right]:, \tag{5.6b}$$

with $i\overline{S}_F{}^T\delta/\delta\psi(x)$ being the shorthand for

$$\int d^4y\, i\overline{S}_F{}^T(x-y)\frac{\delta}{\delta\psi(y)}\,,$$

etc., and $\overline{S}_{F\alpha\beta}{}^T(x-y) = \overline{S}_{F\beta\alpha}(y-x)$. In (5.6) the notation : : stands for the normal-ordered among the fermion fields $\psi, \overline{\psi}$ only.[12] The meson field operators are still in their properly defined T^+-ordered form. Unless stated otherwise, all the subsequent equations in this section are assumed to be ordered as stated above. Equation (5.6b) is a matrix representation of (5.6a). The integrations over x and x' are now understood as matrix multiplications. It is not difficult to convince oneself by working out a few lower-order expansions explicitly that Eq. (5.6) gives rise to the correct Dyson-Wick expansion.

Since ψ is an anticommuting field, it is necessary to distinguish between a left and a right functional derivative. The left and the right derivatives $\delta S/\delta\psi$ are defined through the change of S due to a variation of ψ,

$$\delta S = \delta\psi\frac{\delta_l S}{\delta\psi} = \frac{\delta_r S}{\delta\psi}\delta\psi\,. \tag{5.7}$$

Then, we obtain from (5.6)

$$\frac{\delta_l S}{\delta\psi} = i\left(\overline{\psi} - i\overline{S}_F{}^T\frac{\delta_l}{\delta\psi}\right)\overline{H}S\,, \tag{5.8}$$

$$\frac{\delta_l S}{\delta\overline{\psi}} = -i\overline{H}\left(\psi + i\overline{S}_F\frac{\delta_l}{\delta\overline{\psi}}\right)S\,, \tag{5.9}$$

where in (5.8) and (5.9) we interpret $\overline{\psi}$, H, and ψ as matrices in the space of x and x'. Equations (5.8) and (5.9) can be rewritten as

$$\frac{\delta_l S}{\delta\psi} = i\overline{\psi}(\overline{H}^{-1} - \overline{S}_F)^{-1}S \tag{5.10}$$

and

$$\frac{\delta_l S}{\delta\overline{\psi}} = -i(\overline{H}^{-1} - \overline{S}_F)^{-1}\psi S\,, \tag{5.11}$$

respectively. Equations (5.10) and (5.11) can be integrated readily, giving

$$S = C : \exp[-i\bar{\psi}(\bar{H}^{-1} - \bar{S}_F)^{-1}\psi] : , \tag{5.12}$$

where we have suppressed the x integrations and interpreted (5.12) as a matrix equation.

The constant C can be determined through

$$\frac{\delta C}{\delta \bar{H}} = \frac{\delta S}{\delta \bar{H}}\bigg|_{\psi,\bar{\psi}=0} \tag{5.13}$$

and by (5.6)

$$\begin{aligned}\frac{\delta C}{\delta \bar{H}} &= \left(-\bar{S}_F{}^T - i\bar{S}_F{}^T \frac{\delta}{\delta\psi}\bar{S}_F\frac{\delta}{\delta\bar{\psi}}\right)S\bigg|_{\psi=\bar{\psi}=0}\\ &= -\bar{S}_F{}^T C - \bar{S}_F{}^T(\bar{H}^{-1} - \bar{S}_F)^{-1}\bar{S}_F C\\ &= -(\bar{S}_F{}^{-1} - \bar{H})^{-1}C.\end{aligned} \tag{5.14}$$

One further integration leads to

$$C = \det(1 - \bar{S}_F\bar{H}), \tag{5.15}$$

and consequently

$$S = \det(1 - \bar{S}_F\bar{H}) : \exp[-i\bar{\psi}(\bar{H}^{-1} - \bar{S}_F)^{-1}\psi] : . \tag{5.16}$$

In terms of the covariant interaction Hamiltonian and the covariant propagator

$$H(x - x') = g\Gamma\varphi\delta^4(x - x') \tag{5.17}$$

and $S_F(x - x')$, we find

$$\bar{H}^{-1} - \bar{S}_F = H^{-1} - S_F . \tag{5.18}$$

Similarly, we have

$$\begin{aligned}\det(1 - \bar{S}_F\bar{H}) &= \det(\bar{H}^{-1} - \bar{S}_F)\det\bar{H}\\ &= \det(H^{-1} - S_F)\det\bar{H}\\ &= \det(1 - S_F H)\det(\bar{H}H^{-1})\\ &= \det(1 - S_F H)\exp[-\ln\det(H\bar{H}^{-1})]\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}\ln\det(H\bar{H}^{-1}) &= \mathrm{Tr}\ln(1 + i\tfrac{1}{4}\gamma^+\epsilon H)\\ &= \mathrm{Tr}(i\tfrac{1}{4}\gamma^+\epsilon H)\\ &= 0 .\end{aligned} \tag{5.20}$$

In (5.20), ϵ stands for $\delta(x^+ - x'^+)\delta^2(\vec{x} - \vec{x}')\epsilon(x^- - x'^-)$, and we have used the fact that $(\gamma^+)^2 = \mathrm{Tr}\gamma^+ = 0$. Thus, we have the interesting result

$$\det(1 - \bar{S}_F\bar{H}) = \det(1 - S_F H) \tag{5.21}$$

and

$$S = \det(1 - S_F H) : \exp[-i\bar{\psi}(H^{-1} - S_F)^{-1}\psi] : . \tag{5.22}$$

Equation (5.22) is precisely (5.16) with the substitution $\bar{S}_F \to S_F$ and $\bar{H} \to H$. In other words, the S matrix will be the same if we replace simultaneously $\bar{H}_I$ by the covariant interaction Hamiltonian

$$H_I = -\int d^2x\,dx^-\,\mathcal{L}_{\rm int}$$

and $i\bar{S}_F$ by the covariant contraction $\langle 0|T^*(\psi\bar{\psi})|0\rangle = iS_F$. Therefore the S matrix in the light-front formulation can be reduced to

$$S = T^*\exp\left(-i\int d^4x\, g\bar{\psi}\Gamma\psi\varphi\right), \tag{5.23}$$

where T^* stands for the covariant x^+-ordering. Equation (5.23) is precisely the S matrix obtained in the conventional theory.

We conclude this section by mentioning that with very little effort the above equivalence theorem can be extended to include the isospin-symmetric interaction

$$\mathcal{L}_{\rm int} = -g\bar{\psi}\Gamma\tfrac{1}{2}\vec{\tau}\cdot\psi\vec{\varphi} \tag{5.24}$$

as well.

VI. SECOND-ORDER CALCULATIONS

In this section, we shall test the light-front Feynman rules by calculating the lowest-order contributions of the fermion and the meson self-energy diagrams, and of the $\bar{\psi}\Gamma\psi\varphi$ vertex corrections. For a specific example, we consider the neutral psuedoscalar-pseudoscalar theory with $\Gamma = i\gamma_5$.

(a) Fermion self-energy diagram. According to our new rules, the self-energy contribution corresponding to Fig. 2 is

$$-i\Sigma(p) = \int \frac{d^4q}{(2\pi)^4}\, g\gamma_5 i\bar{S}_F(p-q) g\gamma_5 \frac{i}{q^2 - \mu^2 + i\epsilon}\,, \tag{6.1}$$

with

$$\bar{S}_F(p-q) = ((\bar{\not p} - \bar{\not q}) + m)/[(p-q)^2 - m^2 + i\epsilon].$$

The 4-vector $(\bar{p} - \bar{q})^\mu$ is given by

$$(\bar{p} - \bar{q})^+ = (p - q)^+ ,$$

$$(\bar{p} - \bar{q})^i = (p - q)^i \quad (i = 1, 2),$$

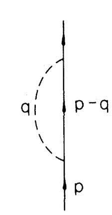

FIG. 2. Second-order fermion self-energy diagram.

and

$$(\bar{p}-\bar{q})^- = [(\vec{p}-\vec{q})^2+m^2]/(p-q)^+ \neq (p-q)^- .$$

[This on-shell 4-vector should be represented by a bar over the entire vector $p-q$. For typographical reasons, we denote it throughout this paper by $(\bar{p}-\bar{q})$.]

Note that the noncovariant g^2 term in H_1 does not contribute because the field operators are normal-ordered. One can see by direct comparison that the unrenormalized amplitude (6.1) is in fact different from the corresponding second-order self-energy contribution in the usual formulation because of $\bar{S}_F \neq S_F$. As we have pointed out in Sec. IV, however, the difference between these two unrenormalized amplitudes is a p-independent (but infinite) constant, and can be removed by the mass and wave-function renormalizations. Hence, we expect that the renormalized amplitudes calculated from both formulations should give the same answer.

Simple power counting reveals that $\Sigma(p)$ diverges linearly. To separate out the divergent part, we follow the standard regulator method by introducing a cutoff covariantly. The regulated self-energy part $\Sigma^R(p)$ is given by

$$\Sigma^R(p)=\Sigma(p)-\sum_i [c_i\,\Sigma(\mu\to\mu_i)], \tag{6.2}$$

where $\Sigma(\mu\to\mu_i)$ is the self-energy part associated with a "heavy" pion with mass μ_i. The coefficients c_i's and the regulator masses μ_i are chosen to make the q integration in $\Sigma(p)$ absolutely convergent.

Since there is no q^- factor in the numerator function, it is convenient to evaluate the q^- integral first. As it was pointed out in Ref. 13, the remaining q^+ integral will have a finite integration range which can be associated with the usual Feynman-parameter integral. Explicit q^- integration leads to

$$\Sigma^R(p)=g^2\int_0^{p^+} dq^+ \frac{d^2q}{2(2\pi)^3} \times\left[((\bar{\not p}-\bar{\not q})-m)\frac{1}{(p-q)^+q^+}\,\frac{1}{p^- -(\vec{q}^{\,2}+\mu^2)/q^+ -[(\vec{p}-\vec{q})^2+m^2]/(p^+-q^+)+i\epsilon} - \text{regulator terms}\right]. \tag{6.3}$$

Introducing

$$q^+=xp^+, \qquad \vec{q}=\vec{q}\,'+x\vec{p}, \tag{6.4}$$

we can rewrite (6.3) as

$$\Sigma^R(p)=-g^2\int_0^1 dx\int\frac{d^2q'}{2(2\pi)^3}\,\frac{(1-x)\not p-m}{\vec{q}\,'^2+xm^2+(1-x)\mu^2-x(1-x)p^2-i\epsilon} -g^2\int_0^1 dx\int\frac{d^2q'}{2(2\pi)^3}\,\frac{\gamma^+}{2p^+}\left(\frac{\vec{q}\,'^2+m^2}{1-x}-xp^2\right)\frac{1}{\vec{q}\,'^2+xm^2+(1-x)\mu^2-x(1-x)p^2-i\epsilon} - \text{regulator terms}. \tag{6.5}$$

It is straightforward to see that the second term (i.e., the noncovariant term) in (6.5) drops out after x and q' integrations if the regulator masses are chosen to satisfy

$$\sum c_i=1, \qquad \sum c_i\mu_i^2=\mu^2, \qquad \sum c_i\mu_i^2\ln\mu_i^2=\mu^2\ln\mu^2. \tag{6.6}$$

The remaining terms in (6.5) can be integrated readily, giving

$$\Sigma^R(p)=-\frac{g^2}{16\pi^2}\int_0^1 dx[(1-x)\not p-m]\left[\ln\frac{\Lambda^2}{xm^2+(1-x)\mu^2-x(1-x)p^2+i\epsilon}-\text{regulator terms}\right], \tag{6.7}$$

where Λ is a cutoff on the q' integration. These $\ln\Lambda$ terms will be canceled out after the contributions from the regulator terms are included.

Given the regulated (finite) part, we can recover the original amplitude easily as

$$\Sigma(p)=a\not{p}+b-\frac{g^2}{16\pi^2}\int_0^1 dx[(1-x)\not{p}-m]\ln\frac{x^2m^2+(1-x)\mu^2}{xm^2+(1-x)\mu^2-x(1-x)p^2-i\epsilon}\ , \tag{6.8}$$

where a and b are infinite but p-independent constants. As is well known, these infinite constants can be absorbed into the mass and wave-function renormalization constants. Equations (6.7) and (6.8) are indeed what one finds by using the conventional Feynman technique. We would like to mention that (6.8) can be obtained by using other cutoff methods as well. For instance, we can obtain (6.8) from (6.1) by first making a Taylor expansion around $\not{p}=m$ and then carrying out the q^- and $\vec{q}$ integrals for terms of order $(\not{p}-m)^2$ or higher.

(b) Meson self-energy diagram. The lowest-order meson self-energy diagram is given by Fig. 3. According to our new rules, the amplitude of this diagram is

$$-i\Pi(q^2)=g^2\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}[\gamma_5(\bar{\not{p}}+m)\gamma_5((\bar{\not{p}}-\bar{\not{q}})+m)]}{(p^2-m^2+i\epsilon)[(p-q)^2-m^2+i\epsilon]}\ , \tag{6.9}$$

where $\bar{p}$ and $(\bar{p}-\bar{q})$ are on-shell 4-vectors. Just as the fermion self-energy part, the unrenormalized $\Pi(q^2)$ is different from that of the conventional theory. However, the difference can be removed by renormalizations, and consequently both theories should lead to the same finite answer.

Taking the trace and carrying out the p^- integration, we can reduce (6.9) to

$$\Pi(q^2)=g^2\int_0^{q^+}dp^+\int\frac{d^2p}{2(2\pi)^3}\,4[\bar{p}\cdot(\bar{q}-\bar{p})+m^2]\frac{1}{p^+(q^+-p^+)\{q^- -(\vec{p}^{\,2}+m^2)/p^+ -[(\vec{q}-\vec{p})^2+m^2]/(q^+-p^+)+i\epsilon\}}\,. \tag{6.10}$$

The d^2p integral diverges quadratically. We can remove the divergence by introducing a set of regulators, leading to the regulated amplitude,

$$\Pi^R(q^2)=\Pi(q^2)-\sum c_i\Pi(m\to M_i)\,. \tag{6.11}$$

As usual, c_i and M_i are chosen to make Π^R finite. To evaluate $\Pi^R(q^2)$, we define

$$p^+\equiv xq^+,\qquad \vec{p}=\vec{p}\,'+x\vec{q}\,, \tag{6.12}$$

and obtain

$$\Pi^R(q^2)=-g^2\int_0^1 dx\int\frac{d^2p'}{2(2\pi)^3}\frac{2(\vec{p}\,'^2+m^2)}{x(1-x)}\frac{1}{\vec{p}\,'^2+m^2-x(1-x)q^2-i\epsilon}-\text{regulator terms}$$

$$=-\frac{g^2}{8\pi^2}\int_0^1 dx\,q^2\left[\ln\frac{1}{m^2-x(1-x)q^2-i\epsilon}-\text{regulator terms}\right]. \tag{6.13}$$

The original divergent meson self-energy amplitude can be recovered as

$$\Pi(q^2)=aq^2+b-\frac{g^2}{8\pi^2}\int_0^1 dx\ln\frac{m^2-x(1-x)\mu^2}{m^2-x(1-x)q^2-i\epsilon}\ , \tag{6.14}$$

where a and b are q^2-independent infinite constants. That Eqs. (6.13)–(6.14) are indeed the correct answer can be verified by direct calculation.

(c) Vertex correction. As a final example, we write down the second-order correction to a γ_5 vertex, as given in Figs. 4(a)–4(c). The contributions from Figs. 4(a)–4(c) are given, respectively, by

$$g\Gamma^{(a)}(p',p)=\int\frac{d^4q}{(2\pi)^4}g\gamma_5\left(\frac{i}{\not{p}'-\not{q}-m+i\epsilon}-\frac{i\gamma^+}{2(p'-q)^+}\right)g\gamma_5\left(\frac{i}{\not{p}-\not{q}-m+i\epsilon}-\frac{i\gamma^+}{2(p-q)^+}\right)g\gamma_5\frac{i}{q^2-\mu^2+i\epsilon}, \tag{6.15a}$$

$$g\Gamma^{(b)}(p',p)=\int\frac{d^4q}{(2\pi)^4}(-ig^2)\frac{\gamma^+}{2(p'-q)^+}\left(\frac{i}{\not{p}-\not{q}-m+i\epsilon}-\frac{i\gamma^+}{2(p-q)^+}\right)g\gamma_5\frac{i}{q^2-\mu^2+i\epsilon}\ , \tag{6.15b}$$

and

$$g\Gamma^{(c)}(p',p)=\int \frac{d^4q}{(2\pi)^4}g\gamma_5\left(\frac{i}{\not p'-\not q-m+i\epsilon}-\frac{i\gamma^+}{2(p'-q)^+}\right)(-ig^2)\frac{\gamma^+}{2(p-q)^+}\frac{i}{q^2-\mu^2+i\epsilon}\ . \tag{6.15c}$$

Using the fact that $(\gamma^+)^2=\gamma^+\gamma_5\gamma^+=0$, we indeed have

$$\Gamma^{(a)}+\Gamma^{(b)}+\Gamma^{(c)}=g^2\int \frac{d^4q}{(2\pi)^4}\gamma_5\frac{i}{\not p'-\not q-m+i\epsilon}\gamma_5\frac{i}{\not p-\not q-m+i\epsilon}\gamma_5\frac{i}{q^2-\mu^2+i\epsilon} \tag{6.16}$$

as given by the usual Feynman rules. Unlike the lowest-order self-energy diagrams, the lowest-order vertex contribution obtained from the light-front perturbation theory agrees exactly with that of the usual theory even *before* the renormalizations. As we have shown in Sec. V, this is actually the rule rather than an exception. The lowest-order self-energy diagrams given in Figs. 2 and 3 are the *only* amplitudes where the renormalizations are needed to prove the equivalence of the present theory and the usual perturbation calculation.

VII. DISCUSSION

The S matrix in the light-front formulation of quantum field theories involving scalar and Dirac systems is studied in this paper. The new S matrix is shown formally to give the same predictions as in the ordinary formulation. Taking together the conclusions reached in paper I and this one, we should conclude that the light-front formulation can replace the conventional equal-time formulation as the formal basis of quantum field theories, at least for scalar and Dirac fields. Furthermore, there are certain advantages of using this new formulation to derive current-algebra sum rules and to study deep-inelastic processes induced by leptons as discussed in paper I.

Nevertheless, this formulation has its own subtleties some of which are already mentioned in paper I. Here we would like to make a remark in connection with practical perturbation calculations. A light front (x^+=constant) contains a line on the light cone. Two space-time points on such a line ($\Delta x^+=0, \Delta\vec{x}=0$, and Δx^- arbitrary) are not kinematically independent, since information can reach one point from the other by a particle with infinite p^- or very small p^+. As a result, there are singularities in matrix elements as $p^+\to 0$. These singularities arise from the noncovariant terms in the Hamiltonian and the fermion propagator. If covariant perturbation theory is employed these singularities explicitly cancel in pairs and no difficulty arises. On the other hand, if noncovariant perturbation theory is used (i.e., p^- integrations are carried out first) the cancellation of these singularities is no longer apparent and the resulting momentum integrals are not well-defined. This difficulty is reflected in the calculation of self-energy diagrams of fermions and mesons of Sec. VI. There a regularization method very different from the ordinary one has to be applied in order to reproduce the known results. It must be remembered, however, a similar situation already exists in the equal-time formulation where a consistent renormalization program based on old-fashioned time-ordered perturbation theory is also lacking. Whenever ambiguities in certain diagrams arise, they must be defined in terms of covariant Feynman diagrams.

After this paper was written, we received a report from Bouchiat, Fayet, and Sourlas,[14] which also confirmed that the light-front formulation possesses a consistent regularization prescription for low-order diagrams.

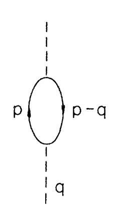

FIG. 3. Second-order meson self-energy diagram.

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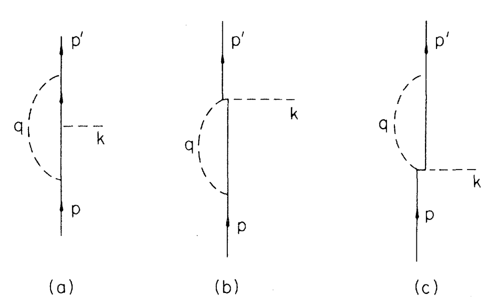

FIG. 4. (a), (b), and (c). Second-order corrections to a γ_5 vertex.

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