[28]A factor of $1/\pi^2$ is missing in Eq. (3.43) *et seq.* in Ref. 3, so that in fact Wilson also obtains the factor $-\frac{9}{2}(\lambda^2/\pi^2)$.

[29]See Schwinger's book referenced in Ref. 13, Vol. II, Sec. 4-7.

[30]C. R. Hagen, Phys. Rev. D 5, 389 (1972).

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# Quantum Field Theories in the Infinite-Momentum Frame. I. Quantization of Scalar and Dirac Fields

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Renormalizable coupled scalar and Dirac field theories are quantized in equal-$x^+$ surfaces (called light fronts). Schwinger's action principle is employed to deduce the correct canonical equal-$x^+$ (anti-) commutation relations. These theories are shown to be Lorentz-invariant. Generalized Schwinger conditions for a quantum field theory to be Lorentz-invariant are given and discussed in an appendix. Spectral sum rules are derived. Leading singularities of Green's functions and products of field operators near the light cone are studied and the implications to current algebra sum rules are discussed. We also discuss some of the delicate features of the light-front formulation.

## I. INTRODUCTION

Using the variables suggested by Susskind,[1] and Bardakci and Halpern,[2] and Chang and Ma,[3]

$$x^{\pm}=x^0 \pm x^3\,,\qquad p^{\pm}=p^0 \pm p^3\,, \tag{1.1}$$

in coordinate and momentum space, respectively; Kogut and Soper,[4] Bjorken, Kogut, and Soper,[5] and Neville and Rohrlich[6] showed that quantum electrodynamics can be formulated consistently by quantizing on an $x^+$ = constant surface, called a light front.[7] Dynamics is assumed to evolve in $x^+$ rather than in time. The first few terms in the $x^+$-ordered perturbation series[4,5] are shown to reproduce automatically Weinberg's rule[8] for ordinary old-fashioned time-ordered perturbation series in the infinite-momentum limit. This work raises the interesting possibility of quantizing field theories on a nonspacelike surface, and provides a possible theoretical basis for the wide applications of infinite-momentum technique to current algebra,[9] deep-inelastic lepton-induced processes[10] and others.[11] It is also relevant to the recent interest in the singular behavior of current commutators near the light cone.[12] In view of the great importance of the problem, we have reexamined all popular renormalizable field theories in this new quantization scheme and searched for a selfcontained framework for this formalism. To indicate the kind of questions we have in mind we make the following remarks. Firstly, the mere change of variables (1.1) alters the fundamental character of the field equations. Consider a classical free scalar field $\varphi(x)$ as an example. It satisfies the familiar Klein-Gordon equation

$$(\partial_0{}^2 - \vec{\nabla}^2 + \mu^2)\varphi(x) = 0\,. \tag{1.2}$$

To solve for $\varphi(x)$, both $\varphi(x)$ and its first time derivative $\dot{\varphi}(x)$ have to be specified at a given time. In terms of the new variables (1.1) the differential equation (1.2) becomes

$$\left(4\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-} - \vec{\nabla}_{\perp}{}^2 + \mu^2\right)\varphi(x) = 0\,. \tag{1.3}$$

Now only $\varphi(x)$ need be specified at a given $x^+$ in order to solve $\varphi(x)$. This difference is reflected in the canonical commutation relations to be imposed in the two quantization schemes. In the conventional quantization, $\varphi(x)$ and $\dot{\varphi}(x)$ are the quantum analogs of $p$ and $q$ in classical mechanics. The equal-time commutation relations are provided by the canonical quantization rule derived from the correspondence principle. In the light-front quantization, however, no such analogy exists. Naive application of the canonical quantization rule gives incorrect commutation relations. Secondly, the canonical equal-time commutation relations remain the same as in free-field theories if no derivative couplings are introduced. This need not be true in the new quantization scheme since a light front contains a line tangent to the light cone which is not spacelike, and consequently interaction can in principle affect the commutation relations. An interesting example is the interacting system of a spin-$\frac{1}{2}$ field and a spin-1 field studied in a separate paper.[13] Thirdly, if a consistent quantum field theory can be formulated using light-front quantization, its equivalence to the conventional formalism is yet to be established. We have found that Schwinger's quantum action principle[14] provides a natural framework for studying these questions. It supplies correct commutation relations even in the presence of interactions. Furthermore, with the aid of Schwinger's functional method,[15] the covariant perturbation theory in the new quantization can be shown to give the same $S$ matrix to all orders as in the ordinary formulation.

In this and the following papers we treat the self-interacting scalar field theories and the system of spinless field and spin-$\frac{1}{2}$ field interacting via a scalar or pseudoscalar coupling. The interacting system of a Dirac and a vector field is studied separately[13] since it possesses many features not present in systems involving spin-0 and spin-$\frac{1}{2}$ fields. In the present paper we study the quantization and related formal aspects, leaving the study of the $S$ matrix and perturbation theory to the following paper.[16]

In Sec. II, Schwinger's action principle is introduced to study the conventional quantization of a free scalar field. In Schwinger's formalism, Bose and Fermi fields are treated on the same footing; canonical quantization is carried out in a form independent of the nature of the quantization surface.

The action principle is then applied to the light-front quantization of scalar fields with nonderivative couplings. The light-front commutator relations among the scalar fields are found to be the same as those of the noninteracting systems. In particular, we demonstrate that the interacting systems based on the light-front quantization are *formally* Lorentz-invariant. These results are not affected by the introduction of isospin or any other internal quantum numbers.

Analysis of these questions is much more complicated for systems involving Dirac fields. In Sec. III, we consider specifically the interacting system of a scalar and a Dirac field given by $\mathcal{L}_{\text{int}} = -g\bar{\psi}\Gamma\psi\varphi$ with $\Gamma = 1$ or $i\gamma_5$. Application of the action principle shows that the light-front (anti-) commutation relations among the independent components of the Dirac field, $\psi^{(+)} \equiv \frac{1}{2}(1+\alpha^3)\psi$ and $\psi^{(+)\dagger}$, and the scalar field $\varphi$ are still not modified by the interactions. We then proceed to show that the light-front quantization indeed leads to Lorentz-invariant theories (at least formally).

Sec. IV is devoted to the derivation of some formal results in quantum field theory from the light-front formulation. These include spectral sum rules, and the asymptotic behavior of the Green's functions at small distance. It should be pointed out that in these derivations we have relied heavily on physical arguments rather than on mathematical rigor.

In Appendixes A and B, some simple commutation relations which constitute the sufficient conditions for a theory to be Lorentz-invariant are given. These sufficient conditions are indeed satisfied by all the field theories studied in this paper. The commutation relations given in Appendix B are the generalizations of Schwinger's conditions for Lorentz covariance[17] to the light-front formulation. Finally, in Appendix C 2, we study briefly the classical analog of the light-front formulation.

## II. INTERACTING SCALAR FIELDS

In this section, we shall study the interaction among scalar fields. We restrict ourselves to nonderivative couplings. We begin with a very brief introduction to Schwinger's action principle.

### A. Schwinger's Action Principle

Schwinger's action principle will be illustrated with the example of conventional quantization of a free spin-0 (scalar) field. In Schwinger's original formalism Bose and Fermi fields are treated on the same footing so that all field equations are first-order differential equations. A spin-0 system is described by a five-component field comprised of a Lorentz scalar $\varphi(x)$ and a four-vector $\varphi_\mu(x)$. The Lagrange function in its standard form for this system is

$$\mathcal{L} = \tfrac{1}{2}(\varphi^\mu\partial_\mu\varphi - \varphi\partial_\mu\varphi^\mu) - \tfrac{1}{2}\varphi_\mu\varphi^\mu - \tfrac{1}{2}\mu^2\varphi^2 , \qquad (2.1)$$

where $\mu$ is the mass of the field quanta. Field equations follow from the principle of stationary action that the action integral

$$W_{12} = \int_{\sigma_2}^{\sigma_1} (dx)\mathcal{L}(\varphi(x), \varphi_\mu(x)) \qquad (2.2)$$

be stationary with respect to variations of $\varphi$ and $\varphi_\mu$ *inside* the volume bounded by the two spacelike surfaces $\sigma_1$ and $\sigma_2$. They are

$$\partial_\mu \varphi = \varphi_\mu , \quad \partial_\mu \varphi^\mu + \mu^2 \varphi = 0 . \tag{2.3}$$

Elimination of $\varphi_\mu$ recovers the Klein-Gordon equation

$$(\partial^2 + \mu^2)\varphi = 0 . \tag{2.4}$$

An important aspect of Schwinger's action principle is that it also supplies canonical commutation relations. This is provided by the surface terms of the variation of $W_{12}$ which are ignored in the conventional classical action principle. These surface terms are interpreted by Schwinger as the generators for the unitary transformations which induce the variations $\delta\varphi$ and $\delta\varphi_\mu$. Thus

$$\delta W_{12} = G(\sigma_1) - G(\sigma_2) , \tag{2.5}$$

where

$$G(\sigma) = \tfrac{1}{2} \int d\sigma_\mu (\varphi^\mu \delta\varphi - \varphi \delta\varphi^\mu) \tag{2.6}$$

or

$$G(t) = \tfrac{1}{2} \int d^3x \, (\varphi^0 \delta\varphi - \varphi\delta\varphi^0) . \tag{2.7}$$

The latter form is appropriate for a flat equal-time surface. The equal-time commutation relations

$$x^0 = x^{0\prime}: \quad [\varphi(x), \varphi^0(x')] = i\,\delta^3(x - x') , \quad [\varphi(x), \varphi(x')] = [\varphi^0(x), \varphi^0(x')] = 0 , \tag{2.8}$$

follow from the relations between the field variations and the generator

$$\tfrac{1}{2} i\,\delta\varphi(x) = [\varphi(x), G(t)] , \quad \tfrac{1}{2} i\,\delta\varphi^0(x) = [\varphi^0(x), G(t)] , \tag{2.9}$$

provided the variations $\delta\varphi$ and $\delta\varphi^0$ are postulated to be commuting $c$ numbers in accordance with the Bose statistics of integer-spin fields. The factor $\frac{1}{2}$ in (2.9) stems from the treatment[14,18] of all field components appearing in the generator on the same footing. In the present case it is possible to divide these fields into two sets of which one is fixed and the other varied. This, for example, can be achieved by adding a total divergence $\frac{1}{2}\partial_\mu(\varphi\varphi^\mu)$ to the Lagrange function (2.1) giving

$$\mathfrak{L} = \varphi^\mu \partial_\mu \varphi - \tfrac{1}{2}\varphi_\mu\varphi^\mu - \tfrac{1}{2}\mu^2\varphi^2 . \tag{2.10}$$

This addition does not change the field equations but alters the surface term to $G'(t)$.

$$G'(t) = \int d^3x \, \varphi^0 \delta\varphi , \tag{2.11}$$

which corresponds to fixing $\varphi^0$ while varying $\varphi$. The canonical commutation relation now follows from

$$i\,\delta\varphi(x) = [\varphi(x), G'(t)] \tag{2.12}$$

without a factor $\frac{1}{2}$ on the left-hand side. For a more detailed discussion concerning the proper interpretation of $G(\sigma)$ the reader is referred to the original paper by Schwinger.[18] It should be emphasized, however, that the possibility of dividing the dynamical variables into two independent sets cannot be determined *a priori*. It depends on the nature of the field equations. In the case of a Hermitian spin-$\frac{1}{2}$ field, the conjugate momentum to the field operator $\psi$ is $\psi$ itself, and such a division is not possible. Thus the interpretation (2.7) and (2.9) is more generally valid than that of (2.11) and (2.12). As will be seen later, in light-front quantization the field operators and their conjugate momenta are related by constrained equations and therefore the symmetrized Lagrange function (2.1) and the interpretation (2.9) must be adopted.

In the following sections for the sake of notational simplicity we shall use the quadratic Lagrange function for scalar fields. However, when we calculate the canonical commutator relations by the action principle, proper symmetrizations among the independent fields in $G$ are understood.

### B. Light-Front Quantization

We now introduce the new variables[19]

$$x^+ = x^0 + x^3 , \quad x^- = x^0 - x^3 , \quad \tilde{\mathbf{x}} = (x^1, x^2) . \tag{2.13}$$

The scalar product of two 4-vectors becomes

$$a \cdot b = a^\mu b_\mu = \tfrac{1}{2}a^+ b^- + \tfrac{1}{2}a^- b^+ - a_i b_i , \tag{2.14}$$

where the Latin indices run from 1 to 2. In particular

$$\partial^\mu \varphi_\mu = \tfrac{1}{2}\partial^+ \varphi^- + \tfrac{1}{2}\partial^- \varphi^+ - \partial_i \varphi_i , \tag{2.15}$$

where $\partial^\pm$ are defined as

$$\partial^\pm = 2\frac{\partial}{\partial x^\mp} . \tag{2.16}$$

The 4-dimensional volume element appears as

$$(dx) = \tfrac{1}{2} dx^+ dx^- d^2x. \tag{2.17}$$

The Lagrange function for a self-interacting scalar field can be written as

$$\mathfrak{L} = \tfrac{1}{2}[(\partial_\mu\varphi)^2 - \mu^2\varphi^2] + \mathfrak{L}_{\rm int}(\varphi) , \tag{2.18}$$

where $\mathfrak{L}_{\rm int}(\varphi)$ is a polynomial function of $\varphi$. By action principle, we obtain both the field equation

$$(\partial^2+\mu^2)\varphi=(\partial^+\partial^- - \vec{\nabla}^2+\mu^2)\varphi = \frac{\partial \mathcal{L}_{\rm int}(\varphi)}{\partial\varphi}, \quad (2.19)$$

and the generator on a light front $x^+=$const,

$$G=\tfrac{1}{2}\int d^2x dx^-\, \partial^+\varphi(x)\delta\varphi(x). \quad (2.20)$$

Note that the nonderivative nature of $\mathcal{L}_{\rm int}$ is crucial for deriving (2.20).

The commutation relation is implicit in the relation

$$\tfrac{1}{2}i\,\delta\varphi(x)=[\varphi(x), G(x^+)], \quad (2.21)$$

i.e.,

$$x^+=x'^+: \quad [\varphi(x),\varphi^+(x')]=i\delta(x^- - x'^-)\delta^2(x-x'). \quad (2.22)$$

### C. Lorentz Invariance

To demonstrate that the quantization on the light front leads to a Lorentz-invariant theory, we should first construct the generators of the Lorentz group, and hence the stress tensor. The stress tensor $T^{\mu\nu}$ can be obtained from (2.18) through the standard method, giving

$$T^{\mu\nu}=\partial^\mu\varphi\partial^\nu\varphi - g^{\mu\nu}\mathcal{L}. \quad (2.23)$$

The requirement of Lorentz invariance implies that the energy-momentum and angular momentum operators, defined by

$$P^\mu \equiv \tfrac{1}{2}\int d^2x dx^-\, T^{+\mu}(x), \quad (2.24)$$

$$J^{\mu\nu}\equiv \tfrac{1}{2}\int d^2x dx^-\,(x^\mu T^{+\nu} - x^\nu T^{+\mu}) \quad (2.25)$$

should generate the correct commutation relations with respect to $\varphi$ as well as among themselves. Since only $T^{+\mu}$ enters the definitions of $P^\mu$ and $J^{\mu\nu}$, we shall concentrate on it. According to (2.23), we have

$$T^{++}=(\partial^+\varphi)^2>0, \quad (2.26)$$

$$T^{+i}=\partial^+\varphi\partial^i\varphi \quad (i=1,2) \quad (2.27)$$

and

$$T^{+-}=\partial^+\varphi\partial^-\varphi - 2\mathcal{L} = (\vec{\nabla}\varphi)^2+\mu^2\varphi^2-2\mathcal{L}_{\rm int}(\varphi). \quad (2.28)$$

$T^{+-}$ is positive-definite if $\mathcal{H}_{\rm int}=-\mathcal{L}_{\rm int}$ is. Note that only $T^{+-}$ depends on $\mathcal{L}_{\rm int}$; $T^{++}$ and $T^{+i}$ are interaction-independent and are the same as those of a free field. Hence, the operators $P^+, P^i, E^i \equiv J^{+i}$, and $L^3\equiv J^{12}$, which depend only on $T^{++}$ and $T^{+i}$, behave exactly like those of a free field under commutations. Thus, they generate the correct commutation relations with respect to $\varphi$ and among themselves.

The remaining operators, $P^-$, $F^i\equiv J^{-i}$, and $K^3 \equiv -\frac{1}{2}J^{+-}(=J^{03})$, depend on $T^{+-}$ explicitly. To study their commutation relations, we need to know the following equal-$x^+$ commutator,

$$i[T^{+-}(x),\varphi(x')] = \tfrac{1}{2}\overleftrightarrow{\nabla}\varphi(x)\,\overleftrightarrow{\nabla}(\delta^2(x-x')\epsilon(x^- - x'^-)) + \tfrac{1}{2}\left(\mu^2\varphi(x) - \frac{\partial\mathcal{L}_{\rm int}}{\partial\varphi}\right)\delta^2(x^- - x'^-)\epsilon(x^- - x'^-). \quad (2.29)$$

By the use of (2.19), we can rewrite (2.29) as

$$i[T^{+-}(x),\varphi(x')] = -\tfrac{1}{2}\partial^+\partial^-\varphi(x)\delta^2(x-x')\epsilon(x^- - x'^-) + \tfrac{1}{2}\overleftrightarrow{\nabla}\cdot[\overleftrightarrow{\nabla}\varphi(x)\delta^2(x-x')\epsilon(x^- - x'^-)]. \quad (2.30)$$

Equations (2.29) and (2.30) can be readily integrated, giving

$$i[P^-,\varphi(x')]=\partial'^-\varphi(x'), \quad (2.31)$$

$$i[F^i,\varphi(x')]=(x^i\partial^- - x^-\partial^i)'\varphi(x'), \quad (2.32)$$

$$i[K^3,\varphi(x')]=-\tfrac{1}{2}(x^+\partial^- - x^-\partial^+)'\varphi(x'). \quad (2.33)$$

These relations, together with the analogous relations for $P^+$, $P^i$, $E^i$, and $L^3$, constitute the transformation laws of $\varphi$ under $P^\mu$ and $J^{\mu\nu}$,

$$[\varphi(x),P^\mu]=i\partial^\mu\varphi(x), \quad (2.34)$$

$$[\varphi(x),J^{\mu\nu}]=i(x^\mu\partial^\nu - x^\nu\partial^\mu)\varphi(x). \quad (2.35)$$

Given (2.34) and (2.35), the transformation law of $T^{\lambda\sigma}$ under $P^\mu$ and $J^{\mu\nu}$ can be computed easily as

$$[T^{\lambda\sigma}(x),P^\mu]=i\partial^\mu T^{\lambda\sigma}(x), \quad (2.36)$$

$$[T^{\lambda\sigma}(x),J^{\mu\nu}]=i(x^\mu\partial^\nu - x^\nu\partial^\mu)T^{\lambda\sigma} + i(g^{\lambda\mu}T^{\nu\sigma} - g^{\lambda\nu}T^{\mu\sigma} + g^{\sigma\mu}T^{\lambda\nu} - g^{\sigma\nu}T^{\lambda\mu}). \quad (2.37)$$

As demonstrated in Appendix A, Eqs. (2.36) and (2.37) imply the correct commutation relations among the $P_\mu$ and $J^{\lambda\sigma}$, and hence the Lorentz invariance. A direct verification of the Lorentz covariance in terms of the equal -$x^+$ commutators among the densities $T^{+\mu}$ are also given in Appendix B. These commutation relations are the generalizations of Schwinger relations to the light-front formulation.

## III. $\bar{\psi}\Gamma\psi\varphi$ THEORY, QUANTIZATION

In this section we shall investigate the formal structure of the canonical quantization on the light front and the Lorentz invariance of the scalar and the pseudoscalar coupling theory described by

$$\mathcal{L}_{\rm int} = -g\bar{\psi}\psi\varphi \tag{3.1}$$

and

$$\mathcal{L}_{\rm int} = -ig\bar{\psi}\gamma_5\psi\varphi\,, \tag{3.2}$$

respectively. Since the formal structure of these two theories are similar, we shall treat them collectively as

$$\mathcal{L}_{\rm int} = -g\bar{\psi}\Gamma\psi\varphi\,, \tag{3.3}$$

with

$$\Gamma = 1 \text{ or } i\gamma_5\,. \tag{3.4}$$

### A. Light-Front Quantization

The Lagrange function for the coupled scalar and Dirac system is

$$\mathcal{L} = \tfrac{1}{2}[(\partial_\mu\varphi)^2 - \mu^2\varphi^2] + \bar{\psi}(i\partial\!\!\!/ - m)\psi - g\bar{\psi}\Gamma\psi\varphi\,. \tag{3.5}$$

By the action principle, we obtain both the field equations

$$(\partial^2 + \mu^2)\varphi + g\bar{\psi}\Gamma\psi = 0\,, \tag{3.6}$$

$$(i\partial\!\!\!/ - m)\psi - g\Gamma\psi\varphi = 0\,, \tag{3.7}$$

$$\bar{\psi}(-i\overleftarrow{\partial\!\!\!/} - m) - g\bar{\psi}\Gamma\varphi = 0\,, \tag{3.8}$$

and the generator

$$G = \tfrac{1}{2}\int d^2x dx^-\,(\partial^+\varphi\delta\varphi + i\bar{\psi}\gamma^+\delta\psi)$$
$$= \int d^2x dx^-\,[\tfrac{1}{2}\partial^+\varphi\delta\varphi + i\psi^{(+)\dagger}\delta\psi^{(+)}]\,. \tag{3.9}$$

The field variations $\delta\psi^{(+)}$, $\delta\psi^{(+)\dagger}$ are now assumed to be anticommuting $c$ numbers in accordance with the Fermi statistics of half-integer-spin fields. Also

$$\psi^{(\pm)} = \Lambda^{(\pm)}\psi\,,$$
$$\Lambda^{(\pm)} = \tfrac{1}{2}(1 \pm \gamma^0\gamma^3) = \tfrac{1}{2}(1 + \alpha^3)\,. \tag{3.10}$$

Note that only $\varphi$, $\psi^{(+)}$, and $\psi^{(+)\dagger}$ enter the generator suggesting that they are the only independent components in the present formulation. As we mentioned earlier, the number of independent variables in the light-front formulation is only one-half of that given in the conventional formulation. For the scalar field $\varphi$ is concerned, this is due to the fact that (3.6) is only linear, rather than quadratic, in the "time" derivative $\partial^-$

$$\partial^+\partial^-\varphi(x) = -[(\mu^2 - \vec{\nabla}^2)\varphi + g\bar{\psi}\Gamma\psi]\,. \tag{3.11}$$

Hence, we have

$$\partial^-\varphi(x) = -\int d^4x'\,\tfrac{1}{4}\delta(x^+ - x'^+)\delta^2(x - x')\epsilon(x^- - x'^-)[(\mu^2 - \vec{\nabla}'^2)\varphi(x') + g\bar{\psi}(x')\Gamma\psi(x')]\,. \tag{3.12}$$

It is only slightly more complicated to see that $\psi^{(-)}$ and $\psi^{(-)\dagger}$ are no longer independent variables. To see this explicitly in the present case, we separate Eq. (3.7) or (3.8) into two two-component equations,

$$i\partial^-\psi^{(+)} - (i\vec{\alpha}\cdot\vec{\nabla} + \gamma^0\mathfrak{M})\psi^{(-)} = 0\,, \tag{3.13}$$

$$i\partial^+\psi^{(-)} - (i\vec{\alpha}\cdot\vec{\nabla} + \gamma^0\mathfrak{M})\psi^{(+)} = 0\,, \tag{3.14}$$

where

$$\mathfrak{M} = m + g\Gamma\varphi\,. \tag{3.15}$$

It is easy to see that $\gamma^0\mathfrak{M}$ is Hermitian. Equation (3.14) indicates that $\psi^{(-)}$ is a function of $\psi^{(+)}$ and $\varphi$ on the light front $x^+ = \text{const}$

$$\psi^{(-)}(x) = -\tfrac{1}{4}i\int d^2x'dx'^-\,\delta^2(x - x')\epsilon(x^- - x'^-)(i\vec{\alpha}\cdot\vec{\nabla}' + \gamma^0\mathfrak{M})\psi^{(+)}(x')\,. \tag{3.16}$$

After verifying that only $\varphi$, $\psi^{(+)}$, and $\psi^{(+)\dagger}$ are indeed the independent components, we can determine the canonical commutation (or anticommutation) relations among them easily. From the generator function (3.9), we obtain at $x^+ = x'^+$

$$i[\varphi(x), \varphi(x')] = \tfrac{1}{4}\delta^2(x - x')\epsilon(x^- - x'^-)\,, \tag{3.17}$$

$$\{\psi^{(+)}(x), \psi^{(+)\dagger}(x')\} = \Lambda^{(+)}\delta^2(x - x')\delta(x^- - x'^-)\,, \tag{3.18}$$

and

$$\{\psi^{(+)}(x), \psi^{(+)}(x')\} = \{\psi^{(+)\dagger}(x), \psi^{(+)\dagger}(x')\} = 0\,, \tag{3.19}$$

$$[\psi^{(+)}(x), \varphi(x')] = [\psi^{(+)\dagger}(x), \varphi(x')] = 0\,. \tag{3.20}$$

These commutation relations (3.17)–(3.20) are identical to those of free fields. The commutation relations (3.17)–(3.20) and the field equations (3.6)–(3.8) specify our system completely.

### B. Lorentz Invariance

The stress tensor $T^{\mu\nu}$ of the system can be computed by the standard method. It gives

$$T^{\mu\nu}=\partial^{\mu}\varphi\partial^{\nu}\varphi+\tfrac{1}{2}i[\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi-\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi]-\tfrac{1}{2}g^{\mu\nu}[(\partial_{\lambda}\varphi)^2-\mu^2\varphi^2]. \tag{3.21}$$

A symmetrization for the indices $\mu$, $\nu$ in the parenthesis is understood.

The components which are important for the verification of Lorentz invariance are

$$T^{++}=(\partial^{+}\varphi)^2+i(\psi^{(+)\dagger}\partial^{+}\psi^{(+)}-\partial^{+}\psi^{(+)\dagger}\psi^{(+)}), \tag{3.22}$$

$$T^{+i}=\partial^{+}\varphi\partial^{i}\varphi+i(\psi^{(+)\dagger}\partial^{i}\psi^{(+)}-\partial^{i}\psi^{(+)\dagger}\psi^{(+)})+\tfrac{1}{4}i\vec{\nabla}\cdot[\psi^{(+)\dagger}(\alpha^{i}\vec{\alpha}-\vec{\alpha}\alpha^{i})\psi^{(+)}]-\tfrac{1}{4}i\partial^{+}(\psi^{(+)\dagger}\alpha^{i}\psi^{(-)}-\psi^{(-)\dagger}\alpha^{i}\psi^{(+)}), \tag{3.23}$$

and

$$T^{+-}=(\vec{\nabla}\varphi)^2+\mu^2\varphi^2+\tfrac{1}{2}i[\psi^{(+)\dagger}\partial^{-}\psi^{(+)}-\partial^{-}\psi^{(+)\dagger}\psi^{(+)}+\psi^{(-)\dagger}\partial^{+}\psi^{(-)}-\partial^{+}\psi^{(-)\dagger}\psi^{(-)}]. \tag{3.24}$$

In deriving (3.23), we have used the constraint equation (3.14). Note that operators $T^{++}$ and $T^{+i}$ given in (3.22) and (3.23) are of the same form as in the noninteracting system. In particular, the integrated operators $P^{+}$, $P^{i}$, $E^{i}$, and $L^3$ contain the independent components $\varphi$, $\psi^{(+)}$, and $\psi^{(+)\dagger}$ only. Hence, they give rise to the same commutator relations among themselves and with respect to $\psi^{(+)}$, $\psi^{(+)\dagger}$, and $\varphi$ as in the case of free fields. Thus, Lorentz covariance is manifest for these transformations.

The remaining generators of the Lorentz group can be computed straightforwardly. With the help of the field equations (3.13) and (3.14), we have

$$P^{-}=\tfrac{1}{2}\int d^2x\,dx^{-}[(\nabla\varphi)^2+\mu^2\varphi^2+i(\psi^{(+)\dagger}\partial^{-}\psi^{(+)}-\partial^{-}\psi^{(+)\dagger}\psi^{(+)})], \tag{3.25}$$

$$K^3=-\tfrac{1}{4}\int d^2x\,dx^{-}\{x^{+}[(\vec{\nabla}\varphi)^2+\mu^2\varphi^2]-x^{-}(\partial^{+}\varphi)^2+2i(x^{+}\psi^{(+)\dagger}\partial^{-}\psi^{(+)}-x^{-}\psi^{(+)\dagger}\partial^{+}\psi^{(+)}-\psi^{(+)\dagger}\psi^{(+)})\}, \tag{3.26}$$

and

$$F^{i}=\tfrac{1}{2}\int d^2x\,dx^{-}\{x^{-}\partial^{+}\varphi\partial^{i}\varphi-x^{i}[(\vec{\nabla}\varphi)^2+\mu^2\varphi^2]+2i[\psi^{(+)\dagger}(x^{-}\partial^{i}-x^{i}\partial^{-})\psi^{(+)}+\psi^{(+)\dagger}\alpha^{i}\psi^{(-)}]\}. \tag{3.27}$$

In deriving these equations, we have made use of the integration by parts freely and dropped all surface terms. The validity of these operations are based on the assertion that the field operator products in the Lorentz generators are interpreted as distributions, and their operations have meaning only after they are smeared by proper test functions.

From (3.25)–(3.27), we obtain, after some rather complicated algebra, that

$$[\varphi(x),P^{-}]=i\partial^{-}\varphi(x), \tag{3.28a}$$

$$[\varphi(x),K^3]=-\tfrac{1}{2}i(x^{+}\partial^{-}-x^{-}\partial^{+})\varphi(x), \tag{3.28b}$$

$$[\varphi(x),F^{i}]=i(x^{-}\partial^{i}-x^{i}\partial^{-})\varphi(x), \tag{3.28c}$$

$$[\psi^{(+)}(x),P^{-}]=i\partial^{-}\psi^{(+)}(x), \tag{3.29a}$$

$$[\psi^{(+)}(x),K^3]=-\tfrac{1}{2}i(x^{+}\partial^{-}-x^{-}\psi^{+})\psi^{(+)}(x)+\tfrac{1}{2}i\psi^{(+)}(x), \tag{3.29b}$$

$$[\psi^{(+)}(x),F^{i}]=i(x^{-}\partial^{i}-x^{i}\partial^{-})\psi^{(+)}(x)+i\alpha^{i}\psi^{(-)}(x), \tag{3.29c}$$

where $\psi^{(-)}$ is given by (3.16). It is easy to see that (3.28) and (3.29) are indeed the correct transformation law of $\varphi$ and $\psi^{(+)}$ under $P^{-}$, $K^3$, and $F^{i}$. Equations (3.28) and (3.29), together the transformation law of $\varphi$, $\psi^{(+)}$ and $\psi^{(+)\dagger}$ under $P^{+}$, $P^{i}$, $E^{i}$, and $L^3$, describe the entire Lorentz transformation properties of the independent field operators. The transformation law of the dependent field $\psi^{(-)}$ can be obtained from the constraint equation (3.16). These transformation properties can be combined into simple covariant forms,

$$[\varphi(x),P^{\mu}]=i\partial^{\mu}\varphi(x), \tag{3.30a}$$

$$[\varphi(x),J^{\mu\nu}]=i(x^{\mu}\partial^{\nu}-x^{\nu}\partial^{\mu})\varphi(x), \tag{3.30b}$$

and

$$[\psi(x),P^{\mu}]=i\partial^{\mu}\psi(x), \tag{3.31a}$$

$$[\psi(x), J^{\mu\nu}] = [i(x^\mu\partial^\nu - x^\nu\partial^\mu) + \tfrac{1}{2}\sigma^{\mu\nu}]\psi(x). \quad (3.31b)$$

Knowing (3.30), (3.31), and the definition of $T^{\mu\nu}$, we can derive the transformation laws of the stress tensor under any Lorentz transformation. It is now easy to see that the transformation laws are indeed given by Eqs. (2.36) and (2.37). According to the derivation given in Appendix A, we can proceed to verify the commutator relations among $P^\mu$ and $J^{\mu\nu}$, and thereby complete the verification of the Lorentz invariance of the theory.

As demonstrated in Appendix B, we can also establish the Lorentz invariance by verifying a set of generalized Schwinger relations. These relations are verified explicitly in the present theory.

## IV. VACUUM EXPECTATION VALUES AND ALGEBRAIC PROPERTIES ON THE LIGHT FRONT

In this section we wish to mention that many important results in quantum field theory can be established in the light-front formulation as well. Among these we will discuss the spectral sum rules, and the asymptotic behavior of the Green's functions at small distances.

### A. Spectral Sum Rules and the Leading Light-Cone Singularities

To derive the spectral sum rule for a scalar field, we consider the spectral representation of the vacuum expectation value of the commutator

$$\Delta'(x) \equiv -i\langle 0|[\varphi(x), \varphi(0)]|0\rangle = \int_0^\infty d\mu^2\, \rho(\mu^2)\Delta(x, \mu^2), \quad (4.1)$$

where $\varphi(x)$ is a neutral scalar field, $\rho(\mu^2)$ is a nonnegative spectral function, and $\Delta(x,\mu^2)$ is the invariant function for a free particle of mass $\mu$. The important fact is that the invariant function $\Delta(x,\mu^2)$ has a $\mu^2$-*independent* value on the light front

$$\Delta(x,\mu^2)|_{x^+=0} = -\tfrac{1}{4}\delta^2(x)\epsilon(x^-). \quad (4.2)$$

Then, the light-front canonical commutation relation

$$\frac{1}{i}[\varphi(x), \varphi(0)]|_{x^+=0} = -\tfrac{1}{4}\delta^2(x)\epsilon(x^-) \quad (4.3)$$

leads easily to the spectral sum rule

$$\int_0^\infty d\mu^2 \rho(\mu^2) = 1. \quad (4.4)$$

The spectral sum rule for a Dirac field $\psi(x)$ can be obtained analogously.

Now, we consider the small $x^2$ behavior of the operator commutator function $\Delta_{\rm op}(x)$ $= -i[\varphi(x), \varphi(0)]$ and the Green's function

$$\Delta_F'(x) = \int_0^\infty d\mu^2 \rho(\mu^2)\Delta_F(x,\mu^2), \quad (4.5)$$

where $\Delta_F(x,\mu^2)$ is the propagator function for a free particle with mass $\mu$. At small $x^2$, the leading behavior of $\Delta_F(x,\mu^2)$ is

$$\Delta_F(x,\mu^2) \approx \frac{i}{4\pi^2}\frac{1}{x^2 - i\epsilon}, \quad (4.6)$$

independent of $\mu^2$. Using (5.6) and the spectral sum rule, we obtain the leading asymptotic behavior of $\Delta_F'(x)$ as

$$\Delta_F'(x)|_{x^2\to 0} \approx \frac{i}{4\pi^2}\frac{1}{x^2 - i\epsilon}\int_0^\infty d\mu^2\rho(\mu^2) = \frac{i}{4\pi^2}\frac{1}{x^2-i\epsilon}, \quad (4.7)$$

i.e., $\Delta_F'(x)$ behaves as a free-particle Green's function near the light cone, (i.e., near $x^2=0$). Similarly, we can compute the leading behavior of $\Delta_{\rm op}(x)$ near $x^2=0$. Note that as $x^2\to 0$, we can always choose a frame such that $x^+$, $\vec{x}\to 0$, and $x^-$ remains finite. In this frame, we can use the light-front quantization relation (4.3) to obtain[20]

$$\Delta_{\rm op}(x)|_{x^2\to 0} \to -\tfrac{1}{4}\delta^2(x)\epsilon(x^-)1_{\rm op} = \Delta(x)|_{x^2\to 0}1_{\rm op}, \quad (4.8)$$

where $1_{\rm op}$ is the unit operator in the physical Hilbert space.

The above method can be applied equally well to study the leading behavior of the anticommutator $S_{\rm op}(x) = \{\psi(x), \bar\psi(0)\}$ and the fermion Green's function $S_F'(x)$.

In the theory of a charged scalar particle or a Dirac particle interacting with a neutral vector gluon field $A_\mu$, the small-distance behaviors of the (anti-) commutators are known to be modified. These modifications should be reflected in the quantization rules. In a subsequent paper it will be shown in detail how the canonical commutation relations on the light front between the charged fields,[13] $\varphi$ and/or $\psi$ can acquire an $A^\mu$ dependence. Here, we simply summarize our findings: In a charged-scalar-meson theory with the interaction

$$\mathcal{L}_{\rm int} = ig\varphi^*\overleftrightarrow{\partial}^\mu\varphi A_\mu + g^2\varphi^*\varphi A^\mu A_\mu, \quad (4.9)$$

we find that the canonical commutator relations at $x^+ = $const are simple only between the modified charged fields

$$\varphi'(x) = \varphi(x)\exp\left[-ig\int dx'^-\,\tfrac{1}{4}\epsilon(x^- - x'^-)A^+(x^+, \vec{x}, x'^-)\right] \quad (4.10)$$

and its adjoint $\varphi'^*(x)$, giving

$$[\varphi'^*(x), \varphi'(y)] = -\tfrac{1}{4} i\,\delta^2(x-y)\epsilon(x^- - y^-)\,, \quad (4.11a)$$

$$[\varphi'(x), \varphi'(y)] = [\varphi'^*(x), \varphi'^*(y)] = 0\,. \quad (4.11b)$$

Similar modification occurs in the theory of a Dirac field interacting with a vector-gluon field. The (anti-) commutator relations and the Green's functions associated with the new fields $\varphi'$ and $\psi'$ behave as those of free fields near the light front $x^+ = y^+$ and as $(x-y)^2 \to 0$. Thus, we conclude that as $x^+ - y^+$, $\vec{x} - \vec{y} \to 0$ but with a finite $x^- - y^-$,

$$\begin{aligned}\Delta_{\rm op}(x-y) &\equiv -i[\varphi(x), \varphi^*(y)] \\ &= \exp\left[ ig \int dx'^- \tfrac{1}{4}\epsilon(x^- - x'^-) A^+(x^+, \vec{x}, x'^-) - ig\int dy'^- \tfrac{1}{4}\epsilon(y^- - y'^-)A^+(y^+, \vec{y}, y'^-)\right](-i)[\varphi'(x), \varphi'^*(y)] \\ &\to \exp\left( ig\int_y^x dz^\mu A_\mu(z)\right)\Delta(x-y)\,, \end{aligned} \quad (4.12)$$

and similarly

$$S_{\rm op}(x-y) \equiv -i\{\psi(x), \bar{\psi}(y)\} \to \exp\left( ig\int_y^x dz^\mu A_\mu(z)\right) S(x-y)\,, \quad (4.13)$$

where the line integral $dz^\mu$ is along a straight lightlike path from $y$ to $x$. In deriving (4.12) and (4.13), we have used the fact that at $x^+ = \text{const}$, $A^+$ commutes among itself as well as with $\varphi'$ and $\psi'^{(+)}$.[21] Equation (4.13) is precisely the expression obtained by Gross and Treiman.[12] Similarly, in the presence of an *external* $A$ field, the Green's function will be modified by a phase just as in (4.12). It is interesting to point out that this phase factor is precisely the eikonal phase which appears in the relativistic potential scattering. As $x^+ - y^+$, $\vec{x} - \vec{y} \to 0$ and with a finite $x^- - y^-$, the space-time points $x$ and $y$ can still be linked by a high-energy particle moving in the path of $z^\mu$. It is known that the eikonal phase of a charged particle moving in an external vector gluon field $A_\mu$ approaches an energy-independent value given by (4.12) at very high energy; while the eikonal phase picked up by a particle moving in an external scalar field will vanish as the energy of the charged particle tends to infinity. This supplies a simple physical reason why the canonical (anti-) commutator relations and the Green's functions on the light cone in the $\bar{\psi}\Gamma\psi\varphi$ theory are not changed by the interaction, while the corresponding expressions in the vector-gluon model are modified.

We shall conclude this section by mentioning that many of the important current-algebra results – e.g., the Adler-Weisberger relation,[22,23] the Adler's sum rules on neutrino scatterings,[24] and more generally, the Fubini–Dashen–Gell-Mann sum rules[9] – originally obtained from the infinite-momentum technique, can be obtained from the light-front commutator rules without referring to the infinite-momentum limit. For many interesting applications of the light-front current algebra, we refer the readers to the work of Brandt, Cornwall, and Jackiw,[25] and Dicus, Jackiw, and Teplitz.[26]

## V. DISCUSSION

In this paper, formally consistent quantum field theories of interacting scalar and Dirac systems are formulated by light-front formulation. The scattering matrix in this formulation can be shown to reproduce the well-known results. Thus, light-front formulation of quantum field theories can be taken as an alternative to the equal-time formulation as the basis for general studies of relativistic quantum systems. In this formulation dynamics is organized in a very novel fashion. Consequently, many nontrivial predictions which follow immediately here are not readily accessible in the conventional formalism, such as current algebra sum rules mentioned before; it is also particularly suited for study of deep-inelastic processes.[25,26]

However, this new formulation is not without its delicate features. Operator products near the light cone are highly singular. Previous experience with equal-time formulation shows that results obtained by formal manipulation often cannot be verified by explicit perturbation calculations.[27] As the exact solution to a complete theory is not available, it is still controversial whether the formal results or the perturbation calculations should be trusted. This is acutely so in view of the Bjorken's scaling behavior[28] found in deep-inelastic electron scattering experiments which follows from naive light-cone current algebra but is violated in perturbation calculation of any renormalizable field theories.[29] We can only emphasize that the formal results obtained here, as any formal result obtained anywhere else, must be treated with due caution.

To conclude we make two remarks. Firstly, light-front formulation shares the failure of equal-time formulation to produce the necessary Schwinger terms in the commutation relations of fermion currents. Secondly, it is generally believed that use of the infinite-momentum limit to provide current-algebra sum rules is equivalent to the assumption of unsubtracted dispersion relations. In the light-front formulation, these sum rules follow without reference to any infinite-momentum limit. Although the light-front formulation is not identical to the infinite-momentum limit, where does the need of unsubtracted dispersion relations enter in the derivations?[30]

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## APPENDIX A

Knowing the transformation properties of the stress tensor $T^{\mu\nu}$, we can establish the correct commutation relations among the generators of the Lorentz group.

Assume that $T^{\mu\nu}$ has the following properties

$$T^{\mu\nu}=T^{\nu\mu}, \tag{A1}$$

$$\partial_\mu T^{\mu\nu}=0, \tag{A2}$$

$$[T^{\lambda\sigma},P^\mu]=i\partial^\mu T^{\lambda\sigma}, \tag{A3}$$

and

$$\begin{aligned}[T^{\lambda\sigma},J^{\mu\nu}]=&\,i(x^\mu\partial^\nu-x^\nu\partial^\mu)T^{\lambda\sigma}\\&+ig^{\mu\sigma}T^{\lambda\nu}-ig^{\sigma\nu}T^{\lambda\mu}\\&+ig^{\lambda\mu}T^{\nu\sigma}-ig^{\lambda\nu}T^{\mu\sigma}.\end{aligned} \tag{A4}$$

The Lorentz generators $P^\mu$ and $J^{\mu\nu}$ can be obtained from the stress tensor by integrating over either a spacelike surface

$$P^\mu=\int d^3x\,T^{0\mu}, \tag{A5}$$

$$J^{\mu\nu}=\int d^3x\,(x^\mu T^{0\nu}-x^\nu T^{0\mu}), \tag{A6}$$

or a light-front surface

$$P^\mu=\tfrac{1}{2}\int d^2x dx^-\,T^{+\mu}, \tag{A7}$$

$$J^{\mu\nu}=\tfrac{1}{2}\int d^2x dx^-\,(x^\mu T^{+\nu}-x^\nu T^{+\mu}). \tag{A8}$$

Since the stress tensor is conserved, Eqs. (A5) and (A6) and Eqs. (A7) and (A8) give the same generators. For simplicity, let us use the definitions (A5) and (A6).

To demonstrate that (A1)–(A8) imply the correct commutator relations among the Lorentz generators, we work out the commutator between a $J$ and a $P$:

$$[J^{\lambda\sigma},P^\mu]=i\int d^3x\,(x^\lambda\partial^\mu T^{0\sigma}-x^\sigma\partial^\mu T^{0\lambda}). \tag{A9}$$

For $\mu=k\neq 0$, we have immediately

$$\begin{aligned}[J^{\lambda\sigma},P^k]&=i\int d^3x\,(-g^{\lambda k}T^{0\sigma}+g^{\sigma k}T^{0\lambda})\\&=ig^{\sigma k}P^\lambda-ig^{\lambda k}P^\sigma.\end{aligned} \tag{A10}$$

For $\mu=0$, we can use (A2) to simplify our integration, giving

$$\begin{aligned}[J^{\lambda\sigma},P^0]&=i\int d^3x\,(x^\lambda\partial^m T^{m\sigma}-x^\sigma\partial^m T^{m\lambda})\\&=i\int d^3x\,(-g^{m\lambda}T^{m\sigma}+g^{\sigma m}T^{m\lambda})\\&=i\int d^3x\,(T^{\lambda\sigma}-g^{0\lambda}T^{0\sigma}-T^{\sigma\lambda}+g^{0\sigma}T^{0\lambda})\\&=-ig^{0\lambda}P^\sigma+ig^{0\sigma}P^\lambda.\end{aligned} \tag{A11}$$

Equations (A10) and (A11) can be written covariantly as

$$[J^{\lambda\sigma},P^\mu]=i(g^{\sigma\mu}P^\lambda-g^{\lambda\mu}P^\sigma). \tag{A12}$$

The commutators between two $P$'s and two $J$'s can be worked out in a similar way, giving

$$[P^\sigma,P^\mu]=0, \tag{A13}$$

$$\begin{aligned}[J^{\lambda\sigma},J^{\mu\nu}]=&\,ig^{\lambda\mu}J^{\nu\sigma}-ig^{\lambda\nu}J^{\mu\sigma}\\&+ig^{\mu\sigma}J^{\lambda\nu}-ig^{\nu\sigma}J^{\lambda\mu}.\end{aligned} \tag{A14}$$

Equations (A12)–(A14) are the required commutators among the Lorentz generators.

## APPENDIX B

Since the stress tensor $T^{\mu\nu}$ is symmetric and conserved, the generators $P^\mu$ and $J^{\mu\nu}$ defined in Appendix A, Eqs. (A7) and (A8), are independent of $x^+$. The commutators of $P^\mu$ and $J^{\mu\nu}$ may be evaluated once we know the commutators of the stress tensor on the light front. We wish to point out that $J^{\mu\nu}$ and $P^\mu$ will satisfy the commutation relations for the generators of the Lorentz group if the equal-$x^+$ commutators for the densities obey the generalized Schwinger conditions[17]

$$[T^{+\mu}(x), T^{+\nu}(y)] = 2i(T^{+\nu}(x)\partial_x^{\mu} - T^{+\mu}(y)\partial_y^{\nu})\delta^2(x-y)\delta(x^- - y^-), \tag{B1}$$

$$[T^{++}(x), T^{+-}(y)] = 2i(T^{+-}(x)\partial_x^{+} + T^{+-}(y)\partial_y^{+} - 2T^{+i}(y)\partial_y^{i})\delta^2(x-y)\delta(x^- - y^-), \tag{B2}$$

$$[T^{+i}(x), T^{+-}(y)] = 2iT^{+-}(x)\partial_x^{i}\delta^2(x-y)\delta(x^- - y^-), \tag{B3}$$

$$[T^{+-}(x), T^{+-}(y)] = 0. \tag{B4}$$

In the above expressions, as well as throughout the rest of this appendix, unless explicitly stated otherwise, $\mu$, $\nu$, $\sigma$, and $\rho$ may take on any value but "$-$," while $i$, $k$, $j$, and $l$ are transverse components. We must hasten to emphasize that (B1)–(B4) are sufficient conditions for Lorentz covariance, and in general there are extra terms which do not contribute to the commutators of the Lorentz generators. Existence of these extra terms will be demonstrated in free-field theories and their properties will be discussed.

The derivation of the Lorentz group commutator relations from (B1)–(B4) is tedious but straightforward. Hence we omit it. The detailed derivation will be given in the thesis of one of the authors (R.G.R.).

In any specific theory, the stress tensor commutators at $x^+ = y^+$ can be calculated from the known commutators among the fields. As an example, consider the free scalar field where the stress tensor is given by

$$T^{++}(x) = [\varphi^+(x)]^2, \tag{B5a}$$

$$T^{+i}(x) = \varphi^+(x)\varphi^i(x), \tag{B5b}$$

$$T^{+-}(x) = (\varphi^k)^2 + \mu^2\varphi^2(x). \tag{B5c}$$

Then the equal-$x^+$ commutators of the stress tensor components are:

$$[T^{++}(x), T^{++}(y)] = 2i(T^{++}(x)\partial_x^{+} - T^{++}(y)\partial_y^{+})\delta^2(x-y)\delta(x^- - y^-), \tag{B6}$$

$$[T^{++}(x), T^{+i}(y)] = 2i(T^{+i}(x)\partial_x^{+} - T^{++}(y)\partial_y^{+})\delta^2(x-y)\delta(x^- - y^-), \tag{B7}$$

$$[T^{++}(x), T^{+-}(y)] = 2i(T^{+-}(x)\partial_x^{+} + T^{+-}(y)\partial_y^{+} - 2T^{+i}(y)\partial_y^{i})\delta^2(x-y)\delta(x^- - y^-), \tag{B8}$$

$$[T^{+i}(x), T^{+j}(y)] = 2i(T^{+j}(x)\partial_x^{i} - T^{+i}(y)\partial_y^{j})\delta^2(x-y)\delta(x^- - y^-) + \partial_x^+\partial_y^+\alpha^{ij}(x,y) + \partial_x^i A^j(x,y) - \partial_y^j A^i(y,x), \tag{B9}$$

$$[T^{+i}(x), T^{+-}(y)] = 2iT^{+-}(x)\partial_x^{i}\delta^2(x-y)\delta(x^- - y^-) - 2\partial_x^+\partial_y^k\alpha^{ik}(x,y) + 2\partial_x^+\beta^i(x,y) + \partial_x^k\gamma^{ik}(x,y) + \partial_x^i B(x,y), \tag{B10}$$

$$[T^{+-}(x), T^{+-}(y)] = 0 + 4\partial_x^k\partial_y^l\alpha^{kl}(x,y) + 4\partial_x^l\beta^l(y,x) - 4\partial_y^k\beta^k(x,y) + \partial_x^+C(x,y) - \partial_y^+C(y,x), \tag{B11}$$

where

$$\alpha^{ij}(x,y) = \varphi^i(x)\varphi^j(y)[-\tfrac{1}{4}i\epsilon(x^- - y^-)\delta^2(x-y)], \tag{B12a}$$

$$\beta^i(x,y) = \varphi^i(x)[\partial_y^+\partial_y^-\varphi(y)][-\tfrac{1}{4}i\epsilon(x^- - y^-)\delta^2(x-y)], \tag{B12b}$$

$$\gamma^{ik}(x,y) = 4i\varphi^i(x)\varphi^k(y)\delta^2(x-y)\delta(x^- - y^-), \tag{B12c}$$

and

$$\begin{aligned} A^j(x,y) = {} & -\partial_x^+\partial_y^+\{\varphi(x)\varphi^j(y)[-\tfrac{1}{4}i\epsilon(x^- - y^-)\delta^2(x-y)]\} \\ & -\partial_x^j[\varphi^+(x)\varphi(y)i\delta^2(x-y)\delta(x^- - y^-)] + \partial_x^+[\varphi(x)\varphi^j(x)i\delta^2(x-y)\delta(x^- - y^-)] \\ & + \tfrac{1}{2}\partial_y^j\partial_y^+[\varphi(y)\varphi(y)i\delta^2(x-y)\delta(x^- - y^-)] + \tfrac{1}{2}\partial_x^+\partial_y^+\partial_y^j\{\varphi(x)\varphi(y)[-\tfrac{1}{4}i\epsilon(x^- - y^-)\delta^2(x-y)]\}, \end{aligned} \tag{B13a}$$

$$\begin{aligned} B(x,y) = {} & 2\partial_x^+\{\varphi(x)[-\partial_y^+\partial_y^-\varphi(y)][-\tfrac{1}{4}i\epsilon(x^- - y^-)\delta^2(x-y)]\} \\ & + 2\partial_x^+\partial_y^k\{\varphi(x)\varphi^k(y)[-\tfrac{1}{4}i\epsilon(x^- - y^-)\delta^2(x-y)]\} - 2\partial_x^k[\varphi(y)\varphi^k(y)i\delta^2(x-y)\delta(x^- - y^-)], \end{aligned} \tag{B13b}$$

$$C(x,y) = 2[\partial_x^-\varphi(x)][\partial_y^+\partial_y^-\varphi(y)][-\tfrac{1}{4}i\epsilon(x^- - y^-)\delta^2(x-y)]. \tag{B13c}$$

The commutation relations (B6)–(B11) contain not only the terms in (B1)–(B4) which are required for Lorentz covariance, but also extra total derivative terms which do not contribute when we integrate to find the commutators among the generators.

For the self-interacting scalar field with no derivative coupling, the stress-tensor components are the same as in (B5a)–(B5c) except for

$$T^{+-}(x) = [\varphi^k(x)]^2 + \mu^2\varphi^2(x) - 2\mathcal{L}_{\text{int}}(x). \tag{B14}$$

The commutators are still given by (B6)–(B11), the interaction being contained entirely in $[\partial_x^- \varphi(x)]$.

We have also checked the free Dirac field, and the interaction of a Dirac field with a scalar field through $g\bar{\psi}\Gamma\psi\varphi$ where $\Gamma$ is either 1 or $i\gamma^5$. The basic structure of Eqs. (B1)–(B4) remains, but with nontrivial extra terms which do not contribute after integrations to obtain the generators. The details of the calculation will appear elsewhere.

It is worth mentioning that among all the interacting field theories we have investigated, only the equal-$x^+$ commutator of $T^{++}$ with itself is simple as stated in (B1), free of extra terms. It is

$$[T^{++}(x), T^{++}(y)] = 2i(T^{++}(x) + T^{++}(y))\partial_x^+ \delta^2(x-y)\delta(x^- - y^-).$$

## APPENDIX C (added in proof)

In this appendix we first discuss and explain in more detail why the initial-value problem differs in the light-front and equal-time formulations, we study briefly the classical analog of the light-front formulation, and finally we consider some special features of the zero-mass particles in the light-front formulation.

### 1. Initial-Value Problem in the Light-Front and Equal-Time Formulations

It was pointed out in the text that the number of independent variables describing a dynamical system is reduced by $\frac{1}{2}$ in the light-front formulation as compared with the conventional equal-time quantization. This is reflected in the initial conditions required to determine the solutions of the corresponding wave equations. In the case of a spinless particle this can be understood, as mentioned in the Introduction, by the fact that the Klein-Gordon equation is second order in the time variable but only first order in $x^+$. It is less obvious why the reduction of variables should also occur in the case of a spin-$\frac{1}{2}$ particle since the Dirac equation is first order in both sets of variables.

We would like to discuss this point further in the context of the initial-value problem. We will see that this phenomenon is intimately connected with the fact that the mass-shell condition

$$p^2 = m^2 \tag{C1}$$

fixes the energy $p^0$ only up to a sign for a given $\vec{p}$:

$$p^0 = \pm(\vec{p}^2 + m^2)^{1/2}; \tag{C2}$$

but it uniquely determines $p^-$ in terms of $\vec{p}_\perp$ and $p^+$:

$$p^- = \frac{\vec{p}_\perp{}^2 + m^2}{p^+}. \tag{C3}$$

Let us begin our discussion by writing the general solution of the Klein-Gordon equation as

$$\phi(x) = \int d^4k\, \delta(k^2 - \mu^2)\, a(k) e^{-ik\cdot x}. \tag{C4}$$

The question is: How much information does one need in order to determine $a(k)$? To see this we invert Eq. (C4) to obtain

$$\delta(k^2 - \mu^2) a(k) = \frac{1}{(2\pi)^4} \int d^4x\, e^{ik\cdot x} \phi(x). \tag{C5}$$

Because of the mass-shell $\delta$ function on the left-hand side, one does not have to know $\phi(x)$ for all $x$ to fit $a(k)$. For example, if the information is given on the surface $x^0 = 0$, we can integrate over $k^0$ on both sides. Making use of

$$\delta(k^{02} - \omega^2) = \frac{1}{2\omega}[\delta(k^0 - \omega) + \delta(k^0 + \omega)], \tag{C6}$$

we obtain

$$\frac{1}{2\omega}[a(\vec{k}, +\omega) + a(\vec{k}, -\omega)] = \frac{1}{(2\pi)^3} \int d^3x\, e^{-ik\cdot x} \phi(\vec{x}, 0), \tag{C7}$$

where

$$\omega = +(\vec{k}^2 + \mu^2)^{1/2}. \tag{C8}$$

It is clear that Eq. (C7) alone is not sufficient to solve for $a(\vec{k}, +\omega)$ and $a(\vec{k}, -\omega)$ separately. To obtain another equation we multiply Eq. (C5) by $k^0$ and then integrate; we get

$$\tfrac{1}{2}[a(\vec{k}, +\omega) - a(\vec{k}, -\omega)] = \frac{i}{(2\pi)^3} \int d^3x\, e^{-ik\cdot x} \dot{\phi}(\vec{x}, 0). \tag{C9}$$

Combining Eqs. (C7) and (C9) we have

$$a(\vec{k}, +\omega) = \frac{1}{(2\pi)^3} \int d^3x\, e^{+ik\cdot x} \overleftrightarrow{\partial_0} \phi(x)|_{x^0=0}, \tag{C10}$$

$$a(-\vec{k}, -\omega) = \frac{1}{(2\pi)^3} \int d^3x\, e^{-ik\cdot x} \overleftrightarrow{\partial_0} \phi(x)|_{x^0=0}. \tag{C11}$$

Thus we need both $\phi(x)$ and $\dot{\phi}(x)$ at $x^2 = 0$ to determine $a(k)$.

Alternatively, if the information is given on the surface $x^+ = 0$, we can integrate (C5) over $k^-$. The result is

$$\frac{1}{k^+}\,a\left(k^+,\vec{k}_\perp,\frac{\vec{k}_\perp{}^2+\mu^2}{k^+}\right) = \frac{1}{(2\pi)^3}\int dx^- d^2x\, e^{ik^+x^-/2-i\vec{k}_\perp\cdot\vec{x}_\perp}\phi(x,x^+=0)\,, \tag{C12}$$

which determines $a(k)$ uniquely if $\phi(x)$ is known at $x^+=0$.

It is now an easy matter to understand the spin-$\frac{1}{2}$ case. The Dirac equation

$$(\gamma^\mu i\partial_\mu - M)\psi=0 \tag{C13}$$

implies that $\psi(x)$ must also satisfy the Klein-Gordon equation

$$(\partial^2+M^2)\psi=0\,. \tag{C14}$$

The general solution for $\psi$ can then be written as

$$\psi(x)=\sum_s\int d^4p\,\delta(p^2-M^2)u(p,s)b(p,s)e^{-ip\cdot x}\,, \tag{C15}$$

where $u(p,s)$ is the properly normalized solution of the Dirac equation in momentum space,

$$(\gamma^\mu p_\mu - M)u(p,s)=0\,,\quad p^2-M^2=0\,. \tag{C16}$$

As in the spin-0 case, Eq. (C15) can be inverted to obtain $b(p,s)$. If $\psi(x)$ is known at $x^0=0$, then we need all the four components of $\psi(x)$ to project out the $b(p,s)$ associated with $p^0=\pm(\vec{p}^2+M^2)^{1/2}$ and $s=\pm\frac{1}{2}$ for a given momentum $\vec{p}$. On the other hand, if $\psi(x)$ is given at $x^+=0$, we need only two components of $\psi(x)$ to separate out the $b(p,s)$ associated with the two spin states since $p^-=(\vec{p}_\perp{}^2+M^2)/p^+$ is single valued for an assigned set of $\vec{p}_\perp$ and $p^+$.

### 2. Classical Analog of the Light-Front Formulation

We now investigate the classical analogs of the light-front formulation of quantum field theories, namely, those systems for which the conjugate pairs of variables are functions of each other. For theories in which $p_i$ and $q_i$ are independent, the Hamilton equations are

$$\dot{q}_i=\frac{\partial H}{\partial p_i}\,,\quad \dot{p}_i=-\frac{\partial H}{\partial q_i}\,. \tag{C17}$$

For systems in which $p_i$ and $q_i$ are related and not independent, the Hamiltonian is actually a function of $q$ only (or of $p$ only). The question is: What are the analogs of Eq. (C17) in such a case?

Consider the following example[31]:

$$\mathcal{L}=iq\beta\dot{q}-\omega qq\,, \tag{C18}$$

where $q$ is a column vector and $\beta$ is an antisymmetric matrix with the properties

$$\beta^T=-\beta\,,\quad \beta^2=1\,. \tag{C19}$$

The conjugate momentum to $q$ is

$$p=\frac{\partial\mathcal{L}}{\partial\dot{q}} = iq\beta = -i\beta q\,. \tag{C20}$$

$p$ is obviously linearly related to $q$. The equation of motion is

$$i\beta\dot{q}=\omega q \tag{C21}$$

from which it follows that

$$\ddot{q}+\omega^2q=0\,. \tag{C22}$$

Thus, $\mathcal{L}$ of Eq. (C18) describes a system of uncoupled harmonic oscillators. The Hamiltonian is constructed by the standard formula

$$H=p\dot{q}-\mathcal{L} = \omega qq\,, \tag{C23}$$

which is a function of $q$ only.

It can be readily verified that the Hamilton equations Eq. (C17) are not satisfied. They are replaced by

$$\dot{p}=-\frac{1}{2}\frac{\partial H}{\partial q}\,,\qquad \dot{q}=\frac{1}{2}\frac{\partial H}{\partial p}\equiv\frac{1}{2}\frac{\partial H}{\partial q}\frac{\partial q}{\partial p}\,, \tag{C24}$$

where

$$\partial q/\partial p=i\beta \tag{C25}$$

and similarly

$$\partial p/\partial q=-i\beta\,. \tag{C26}$$

We now define the modified Poisson bracket as

$$\{A,B\}\equiv\frac{1}{4}\sum_i\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right), \tag{C27}$$

where $A$ and $B$ are expressed as functions of $q$ only, and

$$\frac{\partial A}{\partial p_i}=\sum_j\frac{\partial A}{\partial q_j}\frac{\partial q_j}{\partial p_i}\,,\qquad \frac{\partial B}{\partial p_i}=\sum_j\frac{\partial B}{\partial q_j}\frac{\partial q_j}{\partial p_i}\,. \tag{C28}$$

In particular,

$$\{q_k,p_l\}=\tfrac{1}{2}\delta_{kl} \tag{C29}$$

or

$$\{q_k, q_l\} = -\tfrac{1}{2}i\beta_{kl} . \tag{C30}$$

The factor $\frac{1}{2}$ should be noticed. Now

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i = \frac{\partial F}{\partial t} + \sum_i \frac{\partial F}{\partial p_i}\dot{p}_i . \tag{C31}$$

The two versions follow from regarding $F$ as either a function of $q$ or a function of $p$. The symmetrized form is

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{1}{2}\sum_i \left(\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i\right) = \frac{\partial F}{\partial t} + \frac{1}{4}\sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \tag{C32}$$

or

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F,H\} . \tag{C33}$$

Thus, the particular definition, Eq. (C27), of the modified Poisson bracket is designed to preserve the familiar formal structure (C33) as in the case where $p$ and $q$ are independent. For a quantum-mechanical system, one can simply replace the Poisson brackets by the corresponding commutators.

### 3. Special Features of the Massless Case

We discuss here the special features of the massless case. Let us start with the 1+1 dimensional space-time. The Klein-Gordon equation degenerates into the one-dimensional wave equation

$$(\partial_0{}^2 - \partial_1{}^2)\phi(x) = 0 ; \tag{C34}$$

the general solution is

$$\phi(x) = f(x^+) + g(x^-) , \tag{C35}$$

where $x^{\pm} = x^0 \pm x^1$ and $f$ and $g$ are arbitrary functions. The information on $\phi$ and $\dot{\phi}$ at a particular time, say $x^0 = 0$, determines $f$ and $g$ completely. But suppose only $\phi(x)$ is specified at $x^+ = 0$,

$$x^+ = 0 , \quad \phi(x) = h(x^-) , \tag{C36}$$

then

$$\phi(x) = h(x^-) + f(x^+) - f(0) \tag{C37}$$

satisfies the boundary condition (C36) but it leaves $f(x^+)$ arbitrary. To specify $\phi(x)$ uniquely, one can require that the acceptable solution must vanish as $x^- \to \infty$. Then

$$\phi(x) = h(x^-) . \tag{C38}$$

Half of the solutions to the wave equation (C34) are lost. There are only waves advancing in the positive $x^1$ axis but not those advancing in the negative $x^1$ axis.

Since the one-dimensional waves also satisfy the two-dimensional or three-dimensional wave equation, the arbitrariness mentioned above also exists in higher dimensions for zero-mass particles. The additional boundary condition that $\phi(x)$ must vanish as $x^-, \vec{x}_\perp \to \infty$ must be imposed in order to remove the arbitrariness.

This arbitrariness associated with massless particles can easily be understood in momentum space. The mass-shell condition

$$k^2 = 0 \tag{C39}$$

becomes

$$k^+ k^- = 0 \tag{C40}$$

in 1+1 dimension. As long as $k^+ = 0$ (or $k^- = 0$), $k^-$ (or $k^+$) can be arbitrary, i.e., there is no unique relation between $k^+$ and $k^-$. This arbitrariness occurs in higher dimension if $\vec{k}_\perp \equiv 0$, since

$$k^- = \frac{\vec{k}^2}{k^+} .$$

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