$$\ln|p\cdot x|=\ln\left[(\vec{p}^2+m^2)^{1/2}t-\frac{\vec{p}^2}{(\vec{p}^2+m^2)^{1/2}}t\right]=\ln\left[\frac{m^2}{(\vec{p}^2+m^2)^{1/2}}t\right],$$

and similarly for $\ln|p'\cdot x|$.

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Approximate Solutions of Predictive Relativistic Mechanics for the Electromagnetic Interaction

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We solve the equations of predictive relativistic mechanics for the electromagnetic interaction of two structureless point charges, up to second order in the coupling constant $g=e_1e_2$, using as a subsidiary condition the Liénard-Wiechert formulas, for both the advanced and the retarded potentials, separately or in the time-reversal-invariant combination. Our general results reduce in the case of one-dimensional rectilinear motion to those obtained previously by Hill, which, as shown recently by Andersen and von Baeyer, are reliable in the low energy regime. In the time-reversal-invariant combination, if $g<0$, concentric circular motion is possible; and assuming that both charges have equal masses we compare the speed-vs-radius relation obtained in this theory to that obtained in the Breit-Darwin approximation and in Wheeler-Feynman electrodynamics.

I. INTRODUCTION

The equations of predictive relativistic mechanics (P.R.M.) can be written in a time-symmetric formalism,[1-4] or in a manifestly covariant one.[5,4] Up to now no physically meaningful exact solution of the equations of P.R.M. has been obtained in either formalism. In this paper we develop a perturbation technique which permits the recurrent calculation of the four-accelerations of the manifestly covariant formalism in the case of two pointlike structureless particles, by assuming that these functions can be expanded into power series of a coupling constant. From them we obtain very easily the corresponding three-accelerations of the time-symmetric formalism.

We had to solve at each order a very simple linear partial differential equation for each unknown function, whose solutions, after a suitable change of variables, can be obtained by quadratures. This leaves the problem still undetermined because a choice of a subsidiary condition is still possible.

We have considered the interaction of two point charges and used as subsidiary conditions the Liénard-Wiechert formulas for both the advanced and retarded potentials, separately or in the time-reversal-invariant combination.

The first two terms of the series expansions have thus been obtained. The results, if the integrations are actually performed, appear as very cumbersome functions of the independent variables. They take, however, very simple forms, in some cases corresponding to simple relations between the relative position and the velocities of the particles. When these relations are in fact constraints which hold along the motion, the three-accelerations also take simple forms. These cases are the one-dimensional rectilinear motion, using any of the subsidiary conditions mentioned before, and the circular plane concentric motion when $g<0$ and the time-reversal-invariant combination is used. In the first case our general results reduce to those obtained previously by Hill.[2] In the second case we obtain, assuming that both charges have equal masses, the speed-vs-radius relation, and we compare this relation to the corresponding ones obtained from the Breit-Darwin approximation and from Wheeler-Feynman electrodynamics.[6,7]

II. PREDICTIVE RELATIVISTIC MECHANICS

This section is a general review of those definitions and results of P.R.M. which will be needed later.

(a) Manifestly covariant formalism.[5,4] Let us consider in the Minkowskian space-time M_4 the autonomous ordinary first-order system of differential equations:

$$\frac{dx_a^\alpha}{d\tau}=u_a^\alpha\,,\qquad \frac{du_a^\alpha}{d\tau}=\xi_a^\alpha(x_b^\beta,u_c^\gamma)\,, \tag{2.1}$$

$(\alpha,\beta,\gamma,\ldots=0,1,2,3;\ a,b,c,\ldots=1,2,\ldots,N)$, whose general solution is

$$x_a^\alpha=\psi_a^\alpha(x_{0b}^\beta,u_{0c}^\gamma;\tau)\,,\qquad u_a^\alpha=\frac{d\psi_a^\alpha}{d\tau}(x_{0b}^\beta,u_{0c}^\gamma;\tau)\,, \tag{2.2}$$

with

$$\psi_a^\alpha(x_{0b}^\beta,u_{0c}^\gamma;0)=x_{0a}^\alpha\,,\qquad \frac{d\psi_a^\alpha}{d\tau}(x_{0b}^\beta,u_{0c}^\gamma;0)=u_{0a}^\alpha\,. \tag{2.3}$$

System (2.1) is a Poincaré-invariant system (P.I.S.) if and only if

$$\psi_a^\alpha(L_\rho{}^{\beta'}(x_{0b}^\rho-A^\rho),L_\sigma{}^{\gamma'}u_{0c}^\sigma;\tau)=L_\mu{}^{\alpha'}[\psi_a^\mu(x_{0b}^\beta,u_{0c}^\gamma;\tau)-A^\mu]\qquad(\alpha=\alpha') \tag{2.4}$$

for each Poincaré transformation $(L_\beta{}^{\alpha'},A^\gamma)$. System (2.1) is a predictive system if and only if

$$u_a^\gamma\frac{\partial\xi_b^\sigma}{\partial x_a^\gamma}+\xi_a^\gamma\frac{\partial\xi_b^\sigma}{\partial u_a^\gamma}=0\qquad(a\neq b;\ \text{no summation over } a) \tag{2.5}$$

and

$$\xi_a^\alpha u_{a\alpha}=0\,. \tag{2.6}$$

Obviously (2.6) implies that $u_a^\alpha u_{a\alpha}$ is constant along each trajectory of (2.2).

Let us consider a predictive P.I.S. for two particles ($N=2$). In the generic case where $x^\alpha\equiv x_1^\alpha-x_2^\alpha$, the relative position, and u_1^α and u_2^α, the tangent vectors to the world lines of the particles, are three linearly independent vectors, the following decomposition is always possible and unique:

$$\xi_a^\alpha=a_a x^\alpha+b_{a1}u_1^\alpha+b_{a2}u_2^\alpha+c_a n^\alpha\,,\qquad n^\alpha=\eta^{\alpha\beta\gamma\delta}x_\beta u_{1\gamma}u_{2\delta}\,, \tag{2.7}$$

where η is the Levi-Civita tensor. If x^α, u_1^α, and u_2^α are not linearly independent, decomposition (2.7) is not unique, but this case can always be considered as a continuous limit of the generic one.

As a consequence of Poincaré invariance, the coefficients a, b, and c, which we shall call generically a, must be scalar invariants; therefore,

$$a(L_\lambda{}^{\alpha'}x^\lambda,L_\mu{}^{\beta'}u_1^\mu,L_\nu{}^{\gamma'}u_2^\nu)=a(x^\alpha,u_1^\beta,u_2^\gamma)\,. \tag{2.8}$$

A necessary and sufficient condition for (2.8) to hold is that the a's be functions only of the six scalars

$$x^2\equiv x^\alpha x_\alpha\,,\quad u_1\equiv+(-u_1^\alpha u_{1\alpha})^{1/2}\,,\quad u_2\equiv+(-u_2^\alpha u_{2\alpha})^{1/2}$$
$$(xu_1)\equiv x^\alpha u_{1\alpha}\,,\quad (xu_2)\equiv x^\alpha u_{2\alpha}\,,\quad k\equiv-u_1^\alpha u_{2\alpha}\,. \tag{2.9}$$

(We use the signature $+2$ for M_4.) Since u_1^α and u_2^α will be taken as future-oriented ($u_1^0>0$, $u_2^0>0$), timelike vectors, u_1 and u_2 are real and k is positive.

Unless the ξ_a^α's are homogeneous functions of degree two of the u_a^β's and homogeneous functions of degree zero of the u_b^γ's ($b\neq a$), the world-line solutions of (2.1) depend on the initial values of the u_a's and not only on the slopes. It is a matter of conjecture whether it is possible in general to identify u_a and m_a, the mass of the particle a.

We shall take in the next sections a conservative point of view by taking $u_a=1$ and introducing the masses m_a explicitly in the formalism. These constraints can be used from the outset and cause no trouble when solving Eqs. (2.5) because of Eqs. (2.6) and the scalar character of the coefficients of (2.7).

(b) The time-symmetric formalism.[1-4] Let us consider in Euclidean space E_3 the ordinary system of differential equations:

$$\frac{dx_a^i}{dt}=v_a^i\,,\qquad \frac{dv_a^i}{dt}=\mu_a^i(x_b^j,v_c^k;t) \tag{2.10}$$

($i,j,k=1,2,3;\ a,b,c=1,2,\ldots,N$) whose general solution is

$$x_a^i=\theta_a^i(x_{0b}^j,v_{0c}^k;t)\,,\qquad v_a^i=\frac{d\theta_a^i}{dt}(x_{0b}^j,v_{0c}^k;t)\,, \tag{2.11}$$

with

$$\theta_a^i(x_{0b}^j,v_{0c}^k;0)=x_{0a}^i\,,\qquad \frac{d\theta_a^i}{dt}(x_{0b}^j,v_{0c}^k;0)=v_{0a}^i\,.$$

System (2.10) is a Poincaré-invariant system (P.I.S.) if and only if

$$\theta_a^i(R_l{}^{j'}(x_{0b}^l-A^l),R_r{}^{k'}v_{0c}^r;t)=R_s^{i'}[\theta_a^s(x_{0b}^j,v_{0c}^k;t)-A^s] \tag{2.12}$$

for each Euclidean transformation $(R_s{}^{i'},A^j)$, and

$$\frac{\partial\mu_a^i}{\partial t}=0\,, \tag{2.13}$$

$$v_a^s(x_{ar}-x_{br})\frac{\partial\mu_b^i}{\partial x_a^s}+[v_a^s v_{ar}+\mu_a^s(x_{ar}-x_{br})-\epsilon_a\delta_r^s]\frac{\partial\mu_b^i}{\partial v_a^s}=2\mu_b^i v_{br}+v_b^i\mu_{br} \tag{2.14}$$

(summation over a is assumed).

It can be proved that there is a local equivalence between the definition of predictive P.I.S. given in (a) and the preceding one. The functions μ_a^i are obtained from the ξ_a^α's by the formulas

$$\mu_a^i=(1-v_a{}^2)(\xi_a^i-\xi_a^0v_a^i)\,,\qquad v_a{}^2=v_a^iv_{ai}\,, \tag{2.15}$$

where in the ξ_a^α's we have to take $x_a^0-x_b^0=0$, $u_a^0=(1-v_a{}^2)^{-1/2}$, and $u_a^i=(1-v_a{}^2)^{-1/2}v_a^i$. ($c$, the speed of light in vacuum, is taken equal to unity.) The reciprocal formulas to obtain the ξ_a^α's from the μ_a^i's are more involved. We shall not need them.

III. THE TWO-BODY PROBLEM: RECURRENT METHOD TO OBTAIN THE ACCELERATIONS

In order to determine the ξ_a^α's for $N=2$, an algorithm of perturbations will be used. We shall assume that these functions can be expanded into power series of a coupling constant g:

$$\xi_a^\alpha=\sum_{n=1}^\infty g^n\xi_a^{(n)\alpha}\,,\qquad \mu_a^i=\sum_{n=1}^\infty g^n\mu_a^{(n)i}\,. \tag{3.1}$$

Substituting (3.1) into (2.5) and equating each term of the series we obtain

$$D_b\xi_a^{(n)\alpha}=A_a^{(n)\alpha}\qquad\left(D_b\equiv u_b^\rho\frac{\partial}{\partial x^\rho};\ a\neq b\right), \tag{3.2}$$

where

$$A_a^{(1)\alpha}=0\,, \tag{3.3}$$

$$A_1^{(n)\alpha}=\sum_{p+q=n}\xi_2^{(p)\rho}\frac{\partial\xi_1^{(q)\alpha}}{\partial u_2^\rho}\,,\qquad A_2^{(n)\alpha}=-\sum_{p+q=n}\xi_1^{(p)\rho}\frac{\partial\xi_2^{(q)\alpha}}{\partial u_1^\rho}\quad(n>1)\,. \tag{3.4}$$

Similarly to (2.7) we can write

$$A_a^{(n)\alpha}=A_a^{(n)}x^\alpha+B_{a1}^{(n)}u_1^\alpha+B_{a2}^{(n)}u_2^\alpha+C_a^{(n)}n^\alpha\,, \tag{3.5}$$

and then from (3.2) and (2.6), supposed to hold order by order, we get

$$D_b a_a^{(n)}=A_a^{(n)}\,,\quad D_b b_{aa}^{(n)}=B_{aa}^{(n)}\,,\quad D_b c_a^{(n)}=C_a^{(n)}\,,\quad b_{ab}^{(n)}=k^{-1}[(xu_a)a_a^{(n)}-b_{aa}^{(n)}]\,. \tag{3.6}$$

[The equation $D_b b_{ab}^{(n)}+a_a^{(n)}=B_{ab}^{(n)}$ is a consequence of (3.6) and therefore can be omitted.] Since the right-hand terms depend on lower-order coefficients, these equations are the basis of a recurrent method to obtain each unknown to any order.

Equations (3.6) may be simplified by a change of variables. Instead of using the four scalars (2.9) (we take $u_a=1$), it is more convenient to choose a set of independent solutions of equations $D_b=0$, e.g.,

$$k\,,\quad r_b\equiv[x^2+(xu_b)^2]^{1/2}\,,\quad s_b\equiv(xu_a)-k(xu_b)\,, \tag{3.7}$$

together with (xu_b), which is not a solution. With these new variables we have

$$D_b = -\frac{\partial}{\partial(xu_b)},$$

and equations (3.6) are readily solved. We shall write the general solution in two different forms ($\epsilon = \pm 1$):

$$\begin{aligned}
a_a^{(n)}(\epsilon) &= -\int_{(-1)^{a+1}\epsilon r_b}^{(xu_b)} A_a^{(n)}\,dy + a_a^{(n)*}(\epsilon),\\
b_{aa}^{(n)}(\epsilon) &= -\int_{(-1)^{a+1}\epsilon r_b}^{(xu_b)} B_{aa}^{(n)}\,dy + b_{aa}^{(n)*}(\epsilon),\\
c_a^{(n)}(\epsilon) &= -\int_{(-1)^{a+1}\epsilon r_b}^{(xu_b)} C_a^{(n)}\,dy + c_a^{(n)*}(\epsilon),\\
b_{ab}^{(n)}(\epsilon) &= k^{-1}[(xu_a)a_a^{(n)}(\epsilon) - b_{aa}^{(n)}(\epsilon)].
\end{aligned} \tag{3.8}$$

In these formulas the integrands are considered as functions of (3.7) and (xu_b), the former being kept constant during the integration. $a_a^{(n)*}(\epsilon)$, $b_{aa}^{(n)*}(\epsilon)$, and $c_a^{(n)*}(\epsilon)$, are arbitrary functions of (3.7).

Using the notation $f_{0a}(\epsilon;\,\epsilon')$, or $f_{0aa}(\epsilon;\,\epsilon')$, or $f_{0ab}(\epsilon;\,\epsilon')$ to indicate the values of given functions $f_a(\epsilon)$ or $f_{aa}(\epsilon)$, or $f_{ab}(\epsilon)$ when the vector $(-1)^{a+1}x^\alpha$ is null and future-oriented ($\epsilon' = -1$) or null and past-oriented ($\epsilon' = +1$), from (3.8) there follows

$$\begin{aligned}
a_{0a}^{(n)}(\epsilon;\,\epsilon) &= a_{0a}^{(n)*}(\epsilon;\,\epsilon),\\
b_{0aa}^{(n)}(\epsilon;\,\epsilon) &= b_{0aa}^{(n)*}(\epsilon;\,\epsilon),\\
c_{0a}^{(n)}(\epsilon;\,\epsilon) &= c_{0a}^{(n)*}(\epsilon;\,\epsilon),
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
a_{0a}^{(n)}(\epsilon;\,-\epsilon) &= -\int_{(-1)^{a+1}\epsilon r_b}^{(-1)^{a}\epsilon r_b} A_a^{(n)}\,dy + a_{0a}^{(n)*}(\epsilon;\,-\epsilon),\\
b_{0aa}^{(n)}(\epsilon;\,-\epsilon) &= -\int_{(-1)^{a+1}\epsilon r_b}^{(-1)^{a}\epsilon r_b} B_{aa}^{(n)}\,dy + b_{0aa}^{(n)*}(\epsilon;\,-\epsilon),\\
c_{0a}^{(n)}(\epsilon;\,-\epsilon) &= -\int_{(-1)^{a+1}\epsilon r_b}^{(-1)^{a}\epsilon r_b} C_a^{(n)}\,dy + C_{0a}^{(n)*}(\epsilon;\,-\epsilon).
\end{aligned} \tag{3.10}$$

Having selected two particular sets of functions $a_a^{(n)*}(\epsilon)$, and so on, the general solution can also be written

$$a^{(n)} = \lambda a^{(n)}(-1) + (1-\lambda)a^{(n)}(+1), \tag{3.11}$$

where λ is any function of (3.7).

We have thus formally solved the interesting equations of P.R.M. Whether or not the series converge is of course an open question.

In Sec. IV we shall consider the problem of obtaining the $a_a^{(n)*}(\epsilon)$ and the remaining arbitrary functions, by applying, up to the second order, this general method to a particular problem.

IV. THE ELECTROMAGNETIC INTERACTION

The problem of constructing numerical solutions for the electromagnetic interaction of two charges using a conjecture by Synge has been considered recently by Andersen and von Baeyer in the very particular case of one-dimensional rectilinear motion. Further progress on this field may come from extending these results and from the elaboration of constructive theoretical methods. In this paper we follow this second course using techniques of P.R.M. as previously developed.

Until now we have not specified any kind of interaction between the particles; henceforth we shall consider the electromagnetic interaction between two structureless pointlike particles with charges e_a. We shall take $g = e_1 e_2$.

(a) In order to determine $a_b^{(n)*}(\epsilon)$ and the remaining arbitrary functions, we require that when $(-1)^{b+1}x^\alpha$ is a future- (past-) oriented null vector, the values of ξ_a^α, to each order, agree with those obtained from the retarded (advanced) Liénard-Wiechert potentials.

The retarded, $\epsilon = -1$, and the advanced, $\epsilon = +1$, electromagnetic field at x_1^α is

$$F_{1\alpha\beta} = \epsilon e_2 r_2^{-2}[x_\alpha \xi_{2\beta} - x_\beta \xi_{2\alpha} - \epsilon r_2^{-1}(1 + x^\rho \xi_{2\rho})(x_\alpha u_{2\beta} - x_\beta u_{2\alpha})], \tag{4.1}$$

where $x^\alpha x_\alpha = 0$, $x^0 = -\epsilon|x^0|$. Taking into account the equations of motion

$$\frac{du_1^\alpha}{d\tau} = m_1^{-1} e_1 F_1{}^\alpha{}_\beta u_1^\beta,$$

the four-accelerations in each case are

$$\xi_1^\alpha = \epsilon g m_1^{-1} r_2^{-2}[(\xi_2{}^\rho u_{1\rho})x^\alpha - \epsilon r_1 \xi_2^\alpha + \epsilon r_2^{-1}(1 + x^\rho \xi_{2\rho})(kx^\alpha + \epsilon r_1 u_2^\alpha)]. \tag{4.2}$$

Substituting the expressions (2.7) into (4.2), expanding into power series of g, and equating the corresponding terms, there follows

$$a_{01}^{(1)*}(\epsilon;\epsilon)=m_1^{-1}kr_2^{-3}, \quad a_{01}^{(n)*}(\epsilon;\epsilon)=\epsilon m_1^{-1}r_2^{-2}(kr_1r_2^{-1}-1)b_{021}^{(n-1)}(\epsilon;-\epsilon),$$
$$b_{011}^{(1)*}(\epsilon;\epsilon)=0, \quad b_{011}^{(n)*}(\epsilon;\epsilon)=-m_1^{-1}r_1r_2^{-2}b_{021}^{(n-1)}(\epsilon;-\epsilon) \quad (n>1), \tag{4.3}$$
$$b_{012}^{(1)}(\epsilon;\epsilon)=\epsilon m_1^{-1}r_1r_2^{-3}, \quad b_{012}^{(n)}(\epsilon;\epsilon)=m_1^{-1}r_1^{2}r_2^{-3}b_{021}^{(n-1)}(\epsilon;-\epsilon),$$
$$c_{01}^{(1)*}(\epsilon;\epsilon)=0, \quad c_{01}^{(n)*}(\epsilon;\epsilon)=0,$$

where use has been made of (3.9). The corresponding quantities for particle 2 would be obtained by systematically interchanging subindices 1 and 2, (xu_a) with $-(xu_a)$, and changing the sign in the expression of $a_{02}^{(n)}$. This symmetry transformation will be used next several times, without mentioning it explicitly.

Dropping the constraint $x^\alpha x_\alpha=0$ and substituting in the starred quantities of (4.3) r_1 for ϵs_2+kr_2, and s_1 for $-[ks_2+\epsilon r(k^2-1)]$, and substituting in the analogs to (4.3) for particle 2, r_2 for ϵs_1+kr_1 and s_2 for $-[ks_1+\epsilon r_2(k^2-1)]$, we obtain a recurrent system of equations to calculate the arbitrary functions $a_a^{(n)*}(\epsilon)$ and $b_{ab}^{(n)*}(\epsilon)$ of Sec. III.

(b) Let us now apply our general method to calculate $\xi_1^{(1)\alpha}$ and $\xi_1^{(2)\alpha}$. From (3.3), (3.8), and (4.3) we get

$$a_1^{(1)}(\epsilon)=m_1^{-1}r_2^{-3}k, \quad b_{11}^{(1)}(\epsilon)=0,$$
$$b_{12}^{(1)}(\epsilon)=m_1^{-1}r_2^{-3}(xu_1), \quad c_1^{(1)}(\epsilon)=0. \tag{4.4}$$

$g\xi_1^{(1)\alpha}$, which proves to be independent of ϵ, is precisely the four-acceleration of particle 1 moving in the retarded or advanced field of particle 2 if the motion of the latter were rectilinear and uniform.

To calculate $\xi_1^{(2)\alpha}$ we need first to calculate $A_1^{(2)}$, $B_{11}^{(2)}$, and $C_1^{(2)}$. From (3.4), (3.5), and the expressions for $\xi_1^{(1)\alpha}$ and $\xi_2^{(1)\alpha}$ already known, we get

$$A_1^{(2)}=+m_1^{-1}m_2^{-1}r_2^{-5}r_1^{-3}(xu_2)\{3k[(xu_1)(xu_2)+kx^2]-r_2^{2}\},$$
$$B_{11}^{(2)}=-m_1^{-1}m_2^{-1}r_2^{-3}r_1^{-3}(xu_1)(xu_2), \tag{4.5}$$
$$C_1^{(2)}=0.$$

These quantities are used in (3.8) as integrands where k, r_2, and s_2 are kept fixed and (xu_2) is the variable of integration. Therefore the relevant expressions to be used in (3.8) are

$$A_1^{(2)}=m_1^{-1}m_2^{-1}r_2^{-5}[r_2^{2}+s_2^{2}+2ks_2y+(k^2-1)y^2]^{-3/2}[3ks_2y^2+(3k^2-1)r_2^{2}y],$$
$$B_{11}^{(2)}=-m_1^{-1}m_2^{-1}r_2^{-3}[r_2^{2}+s_2^{2}+2ks_2y+(k^2-1)y^2]^{-3/2}(s_2+ky)y. \tag{4.6}$$

Finally from (3.8), (4.6), and (4.3), after the substitution of r_1 for ϵs_2+kr_2, we obtain

$$a_1^{(2)}(\epsilon)=-m_1^{-1}m_2^{-1}r_2^{-5}\int_{\epsilon r_2}^{(xu_2)}[r_2^{2}+s_2^{2}+2ks_2y+(k^2-1)y^2]^{-3/2}[3ks_2y^2+(3k^2-1)r_2^{2}y]dy$$
$$-m_1^{-1}m_2^{-1}r_2^{-2}[(k^2-1)r_2+\epsilon ks_2](kr_2+\epsilon s_2)^{-3}, \tag{4.7}$$

$$b_{11}^{(2)}(\epsilon)=m_1^{-1}m_2^{-1}r_2^{-3}\int_{\epsilon r_2}^{(xu_2)}[r_2^{2}+s_2^{2}+2ks_2y+(k^2-1)y^2]^{-3/2}(ky^2+s_2y)dy+\epsilon m_1^{-1}m_2^{-1}r_2^{-1}(kr_2+\epsilon s_2)^{-2}, \tag{4.8}$$

and

$$b_{12}^{(2)}(\epsilon)=k^{-1}[a_1^{(2)}(\epsilon)(xu_1)-b_{11}^{(1)}(\epsilon)], \quad c_1^{(2)}(\epsilon)=0, \tag{4.9}$$

from which we know $\xi_1^{(2)\alpha}$ and $\xi_2^{(2)\alpha}$. The integrals of (4.7) and (4.8) can be easily calculated but the resulting expressions are rather cumbersome.

(c) Once we know ξ_a^α to a given order, formulas (2.15) give us immediately the three-accelerations to the same order. In our particular case, it is obvious that $C_a^{(n)}=0$, and the $\mu_a^{(n)i}$'s take the simpler form

$$\mu_a^{(n)k}(\epsilon)=(1-v_a^{2})[a_a^{(n)}(\epsilon)x^k+b_{ab}^{(n)}(\epsilon)(1-v_b^{2})^{-1/2}(v_b^k-v_a^k)] \quad (b\neq a)$$

or (4.10)

$$\mu_a^{(n)k}(\epsilon)=(1-v_a^{2})\{a_a^{(n)}(\epsilon)x^k+k^{-1}[(xu_a)a^{(n)}(\epsilon)-b_{aa}^{(n)}(\epsilon)](1-v_b^{2})^{-1/2}(v_b^{k}-v_a^{k})\},$$

where in the expressions of $a_a^{(n)}$ and $b_{ab}^{(n)}$ as functions of the variables (2.9) we have to take

$$x^2=x^ix_i, \quad (xu_a)=(1-v_a^{2})^{-1/2}(\vec{x}\cdot\vec{v}_a),$$
$$k=(1-v_1^{2})^{-1/2}(1-v_2^{2})^{-1/2}[1-(\vec{v}_1\cdot\vec{v}_2)]. \tag{4.11}$$

V. ONE-DIMENSIONAL RECTILINEAR MOTION

Let us consider the simple problem where because of appropiate initial conditions the two charges move along a straight line. Then

$$\vec{x} \propto \vec{v}_1 \propto \vec{v}_2 . \tag{5.1}$$

In this case (4.11) becomes

$$x = \pm(x^i x_i)^{1/2}, \quad (xu_a) = xv_a(1 - v_a^2)^{-1/2},$$
$$k = (1 - v_1^2)^{-1/2}(1 - v_2^2)^{-1/2}(1 - v_1 v_2), \tag{5.2}$$

where x and v_a are here considered as algebraic quantities on the oriented straight line. Using these expressions in (4.4) and in (4.10), reduced to one dimension, we get

$$\mu_1(\epsilon) = m^{-1}(1 - v_1^2)^{3/2}(1 - v_2^2)x|x|^{-3} . \tag{5.3}$$

The particular relations (5.2) brought into (3.7) give

$$s_a = \alpha r_a(k^2 - 1)^{1/2}, \quad \alpha = \operatorname{sgn}[x(v_1 - v_2)] . \tag{5.4}$$

These constraints greatly simplify the integrals of (4.7), (4.8), and their analogs for particle 2, because the radicand becomes

$$[kr_2 + \alpha(k^2 - 1)^{1/2} y]^2 .$$

The explicit expressions which are thus obtained are

$$a_1^{(1)}(\epsilon) = -m_1^{-1} m_2^{-1} r_2^{-4}[3(1 - k^{-2})^{-1}M + 3(k^2 - 1)k^{-2}(1 - k^{-2})N - P^{-1} + P^{-2}],$$
$$b_{11}^{(2)}(\epsilon) = m_1^{-1} m_2^{-1} r_2^{-3} k^{-2}[\alpha(1 - k^{-2})^{-3/2}M + \alpha(1 - k^{-2})^{-1/2}N + \epsilon P^{-2}], \tag{5.5}$$

where

$$M \equiv \ln(SP^{-1}) + 2(S^{-1} - P^{-1}) - \tfrac{1}{2}(S^{-2} - P^{-2}), \quad N \equiv -S^{-1} + P^{-1} + \tfrac{1}{2}(S^{-2} - P^{-2}),$$
$$P \equiv 1 + \epsilon\alpha(1 - k^{-2})^{1/2}, \quad S \equiv 1 + \alpha r_2^{-1}(1 - k^{-2})^{1/2}(xu_2) . \tag{5.6}$$

Substituting (5.5) into (4.10) with partial use of (5.2) gives

$$\mu_1^{(2)} = -2m_1^{-1} m_2^{-1}(1 - v_1^2)^{5/2}(1 - v_2^2)^{5/2}(v_1 - v_2)^{-2}[\ln(SP^{-1}) + (S^{-1} - P^{-1})]x^{-3},$$

and further use of (5.2) to evaluate the bracket yields

$$\mu_1^{(2)}(\epsilon) = -2m_1^{-1} m_2^{-1}(1 - v_1^2)^{5/2}(1 - v_2^2)^{5/2}(v_1 - v_2)^{-2}$$
$$\times\{\epsilon(1 - v_2^2)^{-1}(1 + \epsilon v_1)^{-1}(1 - v_1 v_2)(v_1 - v_2) + \ln[(1 + \epsilon v_2)(1 + \epsilon v_1)^{-1}]\}\, x^{-3} . \tag{5.7}$$

Formulas (5.3) and (5.7) coincide with those obtained by Hill[2] using a quite independent method and give confidence in the general expressions (4.7) and (4.8).

VI. CIRCULAR MOTION

(a) Let us consider now particular initial conditions for which we have

$$\vec{x} \cdot \vec{v}_1 = \vec{x} \cdot \vec{v}_2 = 0 . \tag{6.1}$$

In this case from (4.11) we get $(xu_a) = 0$. Using this result in (4.4) and in (4.10), we get

$$\mu_1^{(1)k}(\epsilon) = m_1^{-1}(1 - v_1^2)kx^{-3}x^k . \tag{6.2}$$

Since $(xu_a) = 0$, from (3.7) we get $s_a = 0$. This greatly simplifies the integrals of (4.7) and (4.8). The final results for $a_1^{(2)}(\epsilon)$ and $b_{11}^{(2)}(\epsilon)$ are

$$a_1^{(2)}(\epsilon) = m_1^{-1} m_2^{-1} x^{-4} k^{-3}(k + 1)^{-1}(3k^4 - k^3 - 2k^2 + k + 1),$$
$$b_{11}^{(2)}(\epsilon) = \epsilon m_1^{-1} m_2^{-1} x^{-3} k^{-2}(k^2 - 1)^{-1}\{2k^2 - 1 - k^3(k^2 - 1)^{-1/2} \ln[k + (k^2 - 1)^{1/2}]\}. \tag{6.3}$$

Using again $(xu_a) = 0$, (4.10) becomes

$$\mu_1^{(2)k}(\epsilon)=(1-v_1^2)[a_1^{(2)}(\epsilon)x^k+k^{-1}b_{11}^{(2)}(\epsilon)(1-v_2^2)^{-1/2}(v_1^k-v_2^k)]. \tag{6.4}$$

Therefore, instead of what happens with $\mu_1^{(1)k}$, $\mu_1^{(2)k}$ has in general a radial and a tangential component.

(b) Let us examine now the possibility of having particular solutions of (2.10), up to the second order, corresponding to uniform concentric circular motion. More precisely, let us assume that

$$x_2^i=-\rho^2x_1^i, \quad v_1=\text{const}, \quad \rho^2=\text{const}, \quad |\vec{x}_1|=\text{const}. \tag{6.5}$$

These conditions can hold only if (6.1) are satisfied and if the tangential component of (6.4) is zero. Therefore (6.5) can hold only if $b_{11}^{(2)}(\epsilon)$ is zero. This condition is meaningless for small velocities where $\epsilon b_{11}^{(2)}(\epsilon)>0$ and we shall not consider it further. We conclude, then, that whatever the value of ϵ, uniform concentric circular motion is not possible. This result is in agreement with an exact result of Schild.[6]

If instead of (6.3) we consider the time-reversal-invariant solutions given by (3.11) with $\lambda=\frac{1}{2}$, then $a_1^{(2)}=a_1^{(2)}(\epsilon)$, and $b_{11}^{(2)}=0$. Therefore in this case $\mu_1^{(2)k}$ does not have a tangential component. More explicitly

$$\mu_1^{(2)k}=m_1^{-1}m_2^{-1}k^{-3}(k+1)^{-1}(3k^4-k^3-2k^2+k+1)(1-v_1^2)x^{-4}x^k, \tag{6.6}$$

where k, as in the corresponding formula (6.2), takes the value

$$k=(1-v_1^2)^{-1/2}(1-v_2^2)^{-1/2}(1+v_1v_2). \tag{6.7}$$

When the speed of light c is restored in (6.2), (6.6), and (6.7), the expansions into the power series of $1/c$ up to the second order yield

$$\mu_1^k\simeq+gm_1^{-1}x^{-3}\{[1+(1/2c^2)(2v_1v_2-v_1^2+v_2^2)]+(1/c^2)gm_2^{-1}x^{-1}\}x^k. \tag{6.8}$$

This result coincides with that obtained in the Breit-Darwin approximation.

For (6.5) to hold at any time $\vec{\mu}_1$ has to be opposite to $\vec{x}$, and therefore g has to be negative, and its absolute value has to be $v_1^2/|\vec{x}_1|$. This last condition gives the speed-vs-radius relation. We give this relation below for the $m_1=m_2\equiv m$ case, which implies $v_1=v_2\equiv\beta$ and $|\vec{x}_1|=|\vec{x}_2|\equiv R$. We list also the analogous relation obtained from the Breit-Darwin approximation and the exact one obtained by Schild using Wheeler-Feynman electrodynamics:

first order: $$R^{(1)}=-(\tfrac{1}{4})gm^{-1}\beta^{-2}(1+\beta^2), \tag{6.9}$$

second order: $$R^{(2)}=-(\tfrac{1}{8})gm^{-1}\beta^{-2}(1+\beta^2)\{1+[1-8(1+\beta^2)^{-5}(1-\beta^2)\beta^2(1+2\beta^2+14\beta^4+6\beta^6+\beta^8)]^{1/2}\}, \tag{6.10}$$

Breit-Darwin: $$R_{B-D}=-(\tfrac{1}{4})gm^{-1}\beta^{-2}(1-\beta^2), \tag{6.11}$$

Wheeler-Feynman: $$R_{W-F}=-gm^{-1}\beta(1-\beta^2)^{1/2}(\theta+\beta^2\sin\theta)^{-3}[(1+\cos\theta)(1-\beta^2)^2+(\theta+\sin\theta)(\theta+\beta^2\sin\theta)] \quad [\text{where } 2\beta^2=\theta^2/(1+\cos\theta)]. \tag{6.12}$$

The positive sign in the square root of (6.10) has been chosen to ensure the correct limit for small β. The radicand of this equation becomes negative for $\beta^2\simeq 0.19$ and becomes again positive for $\beta^2\simeq 0.8$. But anyway, the expression (6.10) is reliable at most for values of β^2 smaller than 0.19.

VII. CONCLUDING REMARKS

Up to now, numerical solutions of the relativistic two-body problem have been discussed only in two particular cases: the uniform circular motion case, which is possible only with the time-reversal-invariant version of the theories that have been considered, and the one-dimensional rectilinear motion case for the electromagnetic interaction of two charges.[8] The results of this paper will permit the numerical study of this problem in the general case for a given range of the velocities. The numerical result mentioned before, as well as the recent calculation of Andersen and von Baeyer, indicate, that at least for the equal-mass case, the approximation considered in this paper will be reliable at most for squared center-of-mass velocities β^2 smaller than 0.19. Nevertheless, velocities approaching this order might already justify the relativistic, but classical, calculation of scattering cross sections. It would be most interesting to obtain these cross sections in the three cases: the retarded, the advanced, and the time-reversal-invariant version of electrodynamics.

It is, of course, not difficult to go to higher orders in the perturbation calculations. This does not introduce new concepts and it is only a matter of writing, even though the numerical calculations will certainly become more involved.

We have taken in this paper, as an example, the electromagnetic interaction. The application of our general method to a massless scalar field is, of course, obvious. Instead, a similar discussion with massive fields, vector or scalar, will certainly need more consideration.

In some cases it might be better to use the ratio m_1/m_2 as the parameter in the perturbation calculation, if m_2 is much greater than m_1. This introduces a few, but not major, difficulties, at least at the theoretical level. The results will be published in a separate paper.

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Role of the Fundamental Tachyon Field in the Elastic Scattering Amplitude

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In previous papers a fundamental field $\phi_\alpha(x)$ was determined giving a nonperturbative solution of a nonlinear field equation of motion. In one work it was shown how the observable scattering processes corresponding to the fundamental field could be determined in general by applying unitarity to the function $\Delta^+_{\alpha\beta}(x-y)=\langle 0|\phi_\alpha(x)\phi^\dagger_\beta(y)|0\rangle$. In particular the contribution of the fundamental field components with positive mass-squared values to the direct channel of the elastic scattering amplitude was determined. In the present paper it is shown how the contribution of the fundamental tachyon field component (with negative mass-squared values) to the momentum-transfer channel of the elastic scattering amplitude may be determined.

In previous papers[1,2] we determined a fundamental field $\phi_\alpha(x)$ giving a nonperturbative solution of a nonlinear field equation of motion.[2] The solution of a nonlinear field equation of motion does not obey the superposition principle, and hence $\phi_\alpha(x)$ should be considered *in toto*, whereas its separate parts have no relevant meaning and are not expected to be directly observable, in general. Since $\phi_\alpha(x)$ was written as an infinite-component sum of free fields, it automatically obeys unitarity. Hence its observable consequences can be indirectly determined from $\phi_\alpha(x)$ by applying[3] unitarity to the function

$$\Delta^+_{\alpha\beta}(x-y)=\langle 0|\phi_\alpha(x)\phi^\dagger_\beta(y)|0\rangle. \quad (1)$$

To do this one forms a complete set of scattering states having all possible numbers of particles with mass and spin m_0 and 0, respectively, corresponding to the stable component of the fundamental field, in both the direct in-channel and the momentum-transfer "in"-channel. If one uses the complete set of scattering states as intermediate states in the unitarity sum for $\Delta^+_{\alpha\beta}(x-y)$ one can kinematically disentangle the individual contributions and hence determine all the vertex functions[3] between the scattering states and the fundamental field. A second application of unitarity, in terms of the fundamental field, now yields the scattering amplitudes. In this way, using two-particle in-states in the direct channel, such as $|p_1,p_2\rangle^{++}_{\text{in}}$, where the two particles have momenta p_1 and p_2 and both have mass and spin m_0 and 0, respectively, we determined[3] the direct-channel contribution to the elastic scattering of two particles of mass and spin m_0 and 0, respectively. States such as $|p_1,p_2\rangle^{++}_{\text{in}}$ only couple to the positive mass-squared continuum of $\phi_\alpha(x)$, called[3] $\phi'_\alpha(x)$. To couple to the fundamental tachyon field, called[3] $\phi''_\alpha(x)$, or the negative mass-squared continuum contribution to $\Delta^+_{\alpha\beta}(x-y)$, one must use scattering states in the momentum-transfer "in"-channel, such as $|p_1,p_2\rangle^{-+}_{\text{in}}$, as intermediate states in the unitarity calculation.[3] $|p_1,p_2\rangle^{-+}_{\text{in}}$ has an incoming particle with momentum p_2 and an outgoing particle of momentum p_1, both with mass and spin m_0 and 0, respectively. This will contribute then (as