*This work is supported in part through funds provided by the Atomic Energy Commission under Contract AT (11-1)-3069.

[1]S. Weinberg, Phys. Rev. Letters 19, 1264 (1967). For subsequent references see the rapporteur's talk by B. W. Lee, in Proceedings of the XVI International Conference on High Energy Nuclear Physics, Batavia, Ill., 1972 (to be published).

[2]Among early papers on the unification of weak and electromagnetic interactions, see J. Schwinger, Ann. Phys. (N.Y.) 2, 407 (1957); A. Salam and J. Ward, Nuovo Cimento 11, 568 (1959); S. L. Glashow, Nucl. Phys. 22, 579 (1961); A. Salam and J. Ward, Phys. Letters 13, 168 (1964); also see A. Salam, in *Elementary Particle Theory*, edited by N. Svartholm (Wiley, New York, 1969).

[3]S. Weinberg, Phys. Rev. Letters 29, 388 (1972); H. Georgi and S. L. Glashow, Phys. Rev. D 6, 2977 (1972).

[4]P. W. Higgs, Phys. Rev. Letters 12, 132 (1964); 13, 508 (1964); Phys. Rev. 145, 1156 (1966); F. Englert and R. Brout, Phys. Rev. Letters 13, 321 (1964); G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, *ibid.* 13, 585 (1965); T. W. Kibble, Phys. Rev. 155, 1554 (1967).

[5]S. Weinberg, Phys. Rev. Letters 27, 1688 (1971); For a survey of calculations carried out so far, see the report of J. Primack in the Proceedings of the XVI International Conference on High Energy Nuclear Physics, 1972 (to be published).

[6]G. 't Hooft, Nucl. Phys. B35, 167 (1971).

[7]B. W. Lee, Phys. Rev. D 5, 823 (1972).

[8]J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[9]A more complicated proof is given by Kibble, Ref. 4.

[10]For specific examples, see 't Hooft, Ref. 6, and Lee, Ref. 7. The general case has been worked out by R. Jackiw (unpublished). I will present the details in a forthcoming article on perturbative symmetry-breaking calculations.

[11]C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954); R. Utiyama, *ibid.* 101, 1597 (1956); M. Gell-Mann and S. Glashow, Ann. Phys. (N.Y.) 15, 437 (1961).

[12]It is at this point that we use our assumption that $\mathcal{G}$ is a *local* group. Kibble in Ref. 4 shows in detail how the gauge transformation transfers the Goldstone boson degrees of freedom to the gauge fields, but this is quite unnecessary in proving the existence of the unitarity gauge.

[13]It should be stressed that we did not have to assume in this proof that ϕ furnishes an irreducible representation of $\mathcal{G}$. Thus, we could add as many boson multiplets as we liked to theories like the model of leptons of Ref. 1, and still be confident of the existence of a gauge in which the linear combinations of these multiplets corresponding to Goldstone bosons were absent.

[14]S. Weinberg, Phys. Rev. D 5, 1962 (1972).

[15]This problem does not arise in the class of "simple" theories discussed here in Sec. VII, which includes most of the models considered in Refs. 1, 6, and 7.

[16]T. D. Lee and C. N. Yang, Phys. Rev. 128, 885 (1962); also see J. Honerkamp and K. Meetz, Phys. Rev. D 3, 1996 (1971); J. Charap, *ibid.* 3, 1998 (1971); I. S. Gerstein, R. Jackiw, B. W. Lee, and S. Weinberg, *ibid.* 3, 2486 (1971).

[17]The coefficient of the logarithm in the $SU(2)\times U(1)$ model of leptons was given as $6i\delta^4(0)$ in Ref. 5. This was a mistake, arising from neglect of the factor $\frac{1}{2}$ in Eq. (6.4). The number of broken generators in this model is three, and the correct coefficient of the logarithm is therefore $3i\delta^4(0)$.

PHYSICAL REVIEW D VOLUME 7, NUMBER 4 15 FEBRUARY 1973

Reduction Formulas for Charged Particles and Coherent States in Quantum Electrodynamics

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(Received 5 June 1972)

A weak asymptotic limit is proposed for a charged field as an operator on the space of asymptotic states. This leads to a modified Lehmann-Symanzik-Zimmermann reduction formula and a determination of the singularity near the mass shell of the Green's function of a charged particle in the presence of other charged particles. Coherent states of the electromagnetic field are also reduced out. The resultant expression for S-matrix elements in terms of vacuum expectation values of time-ordered fields yields a slight elaboration of the Feynman rules which allows a perturbative calculation that is free of infrared and Coulombic divergences order by order. As an application, the amplitude for scattering of a Dirac particle by an external Coulomb potential is calculated to second order in the external potential, with a finite result.

I. INTRODUCTION

The exact amplitude for scattering of a nonrelativistic particle by a Coulomb potential is given in textbooks on quantum mechanics:

$$f(p,\theta)=\frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)}\,\frac{-\gamma}{2p\sin^2(\frac{1}{2}\theta)}\,e^{-i\gamma\ln\sin^2(\theta/2)}, \tag{1.1}$$

where $\gamma=ee'(4\pi v)^{-1}$, $v=p/m$. This expression is analytic in $ee'(4\pi)^{-1}$ about the origin, and has a power series which converges for $|ee'(4\pi)^{-1}|<v$. On the other hand, because of the long range of the Coulomb field, the Born series for Coulomb scattering is known to be infrared-divergent term by term, so one is left with the question, "What perturbation theory yields term by term the series expansion of the exact amplitude?" We shall see that, by a correct formulation of the asymptotic condition, one obtains a slight elaboration of the Feynman rules for quantum electrodynamics which allows a perturbative calculation of all physical processes, and which is infrared-convergent order by order.

The correct asymptotic state of the electromagnetic field includes a coherent classical component which is present whenever charged particles are scattered. If this component is neglected, there results a divergence at zero photon momentum, which is what is usually referred to as the infrared divergence. Similarly, the correct asymptotic wave function of a charged particle is logarithmically distorted by the presence of other charged particles so that instead of a plane wave $e^{-ip\cdot x}$, it becomes $e^{-i(p\cdot x-\gamma\ln|p\cdot x|)}$, where γ depends on the charges and momenta of the particles in the asymptotic state. If this is neglected there also results a divergence at zero photon momentum which is sometimes called a Coulomb divergence or an "infinite Coulomb phase shift" and which manifests itself by the nonexistence of the terms of the Born series for Coulomb scattering. We apply the method of Lehmann, Symanzik, and Zimmermann[1] to the correct asymptotic states and obtain modified reduction formulas, namely modified formulas for S-matrix elements in terms of momentum-space Green's functions, i.e., Fourier transforms of expectation values of T products of Heisenberg fields. The modification is briefly described as follows.

Let the coherent radiated electromagnetic field be associated with a classical current $j_\mu(x)$ and the 4-vector potential $a_\mu^{\rm rad}(x)$ of the classical free field radiated by this current. The corresponding formula for reducing out the coherent radiation field, an initial photon of momentum k_i, and a final photon of momentum k_f, may be expressed as [see Eq. (5.38)]

$$\begin{aligned}\langle\cdots k_f,a(x)^{\rm out}|\cdots k_i{}^{\rm in}\rangle &= C^*[i(2\pi)^{-3/2}]^2\int d^4x_f\,e^{ik_f\cdot x_f}\int d^4x_i\,e^{-ik_i\cdot x_i}\\ &\quad\times\left\langle\cdots^{\rm out}\left|\,T\big([J(x_f)-j(x_f)]\exp\left(-i\int J\cdot a^{{\rm rad}(+)}d^4x\right)[J(x_i)-j(x_i)]\big)\right|\cdots^{\rm in}\right\rangle,\end{aligned} \tag{1.2}$$

where J_μ is the quantum-mechanical current, $a_\mu^{{\rm rad}(+)}(x)$ is the positive-frequency component of $a_\mu^{\rm rad}(x)$, and C is a constant. We see that because of the coherent field the external photons are coupled more weakly, namely to the difference of the quantum-mechanical current and the classical current. The factor of $\exp(-i\int J\cdot a^{{\rm rad}(+)}d^4x)$ makes all charged-particle propagators describe propagation in the external potential $a_\mu^{(+)}(x)$. This accounts for the radiation reaction of the coherent radiation on the charged particles. The real part of $a_\mu^{{\rm rad}(+)}(x)$, which is $\frac{1}{2}a_\mu^{\rm rad}(x)$, reduces classically to the familiar radiation reaction force $\frac{2}{3}e^2(\dddot{x}+\dot{x}\,\ddot{x}^2)$. The effect of the logarithmic distortion of the charged-particle wave function is that the usual reduction formula, which may be written as

$$\lim_{\eta\to 0}(-i)T(p(1+\eta))[\not p(1+\eta)-m]u(p) = m\lim_{\eta\to 0}T(p(1+\eta))(-i\eta)u(p),$$

becomes modified to [see Eq. (4.31) or (4.35)]

$$m\lim_{\eta\to 0}T(p(1+\eta))\,\frac{(\epsilon-i\eta)^{1-i\gamma}}{\Gamma(1-i\gamma)}u(p), \tag{1.3}$$

where $p^2=m^2$, $u(p)$ is the Dirac spinor with $\not p u=mu$, η is a measure of how the mass shell is approached, $\epsilon>0$ approaches zero before η, γ depends on the charges and momenta of the particles in the asymptotic state, and $T(P)$ is a momentum-space Green's function. This expression exhibits the singularity of the Green's function as a charged particle approaches its mass shell in the presence of other charged particles. These reduction formulas are nonperturbative. However, they may be used to provide a perturbative expansion according to the usual Feynman rules, Eq. (5.40). The modified reduction formulas assure infrared convergence. These formulas may also be useful in a systematic treatment of Coulombic and radiative corrections to weak decay processes, or of interference between Coulombic and strong forces. However, we have not proven convergence and unitarity order by order. Rather, by treating the asymptotic condition correctly, the cause of the difficulty is removed, but this should be verified in detail. Another open problem is that our formulas provide expected convergence for the unrenormalized perturbation series. However, conventional renormalization introduces new infrared divergences, since Z_1 and Z_2 are infrared-divergent, and a suitable formulation still remains to be done. The present formulas are a good starting point for this program.

Physically, the problem of infrared radiation has been well understood, at least since the classic work of Bloch and Nordsieck.[2] The formalism of Yennie, Frautschi, and Suura[3] yielded finite cross sections from divergent S-matrix elements. The modern description of infrared radiation using coherent states with finite S-matrix elements began with the work of Chung[4] and was pursued by Kibble.[5] The divergence of the terms in the Born series for Coulomb scattering was first analyzed by Dalitz.[6] He used a screened Coulomb potential, or equivalently a finite photon mass, and showed in low order that as the mass approaches zero the divergences contribute only to the phase of the amplitude. The modern treatment of Coulomb scattering originated with Dollard,[7] who was able to define a finite Coulomb scattering matrix by considering transitions between distorted plane-wave states, the distorted states being eigenstates of a modified H_0. Kulish and Faddeev[8] were able to eliminate both the infrared and Coulomb divergences in quantum electrodynamics by very cleverly separating out of the interaction Hamiltonian a part which contains both the infrared and Coulomb divergences and which they are able to solve exactly.

The present approach grew out of a study of the asymptotic electromagnetic fields in the theory of magnetic monopoles.[9] It differs from the Hamiltonian approach of Kulish and Faddeev[8] in that it deals directly with the Green's functions and leads immediately to a Feynman perturbation theory. Its starting point is a weak asymptotic limit for charged fields, Eq. (4.6), and representations for the asymptotic Maxwell fields, Eqs. (5.9) and (5.24), based on correspondence with classical results. Whereas Kulish and Faddeev associate one coherent field with the in-state and another with the out-state, in the present approach the in-state is pure Fock while the out-state contains a coherent classical radiation field, as in Eq. (1.2). This difference leads to S matrices which differ in the imaginary part of the complex potential appearing in Eq. (1.2). Both are finite and unitary. The present approach is adapted to the description of the actual physical scattering situation.

Section II is devoted to the asymptotic classical charged-particle trajectory. In Sec. III the corresponding asymptotic 1-particle wave function is found. The reduction formula for charged particles is derived in Sec. IV. It begins with a weak asymptotic limit for the charged field, and rests on the key assumption that S-matrix elements are finite, Eqs. (4.27) and (4.28), which leads to a determination of the singularity structure of the charged-particle Green's function in the neighborhood of the mass shell, Eq. (1.2) or (4.35). In Sec. V the coherent state is reduced out. Since perturbative calculations which are finite with respect to infrared emission have already been carried out by Chung,[4] we restrict ourselves to an application which shows the elimination of the divergence from the Coulomb Born series. In Sec. VI, the amplitude for scattering of a Dirac particle by an external Coulomb potential is calculated to second order in the potential, Eq. (6.16), and found to be finite. According to Eq. (1.2) the method involves regularization of the Feynman integrals by taking the electron off its mass shell, and multiplication by a known coefficient which is not a pure phase. In contrast the Dalitz method[6] involves a different regularization of the Feynman integrals by introducing a photon mass and then dropping an infinite phase. Surprisingly the two methods agree to second order, but it is not clear whether the agreement persists in higher order.

II. ASYMPTOTIC CLASSICAL SOLUTION FOR CHARGED PARTICLES

Let us first consider the asymptotic motion of classical relativistic particles under the mutual influence of their Coulomb fields. It is determined by the equations of motion[10]

$$m_i \frac{d^2 x_i^\mu}{d\tau_i^{\,2}} = e_i F^\mu{}_\nu(x_i)\frac{dx_i^\nu}{d\tau_i} \tag{2.1}$$

for the ith particle of mass m_i and charge e_i, and $F^{\mu\nu}(x_i)$ is the electromagnetic field produced by the other particles. Here x_i^μ is regarded as a function of the proper time τ_i, which may be eliminated from

$$\left(\frac{dx_i}{d\tau}\right)^2 = 1 .$$

It is convenient (and in fact necessary for massless particles) to parametrize the particle trajectories by

$$s_i = \tau_i / m_i . \tag{2.2}$$

The equations of motion become

$$\ddot{x}_i^\mu = e_i F^\mu{}_\nu(x_i)\dot{x}_i^{\,\nu}, \tag{2.3}$$

where dot means differentiation with respect to the curve parameter s_i, which may be eliminated at the end in favor of the mass m_i from

$$\dot{x}_i^{\,2} = m_i^{\,2} . \tag{2.4}$$

We are only interested in the asymptotic motion for early and late times $t\to\pm\infty$ or $s_i\to\pm\infty$, and for this it is sufficient to determine $F^{\mu\nu}$ in first approximation in the time. In zeroth approximation we put $F^{\mu\nu}=0$ because the particles are infinitely far apart, and the equation of motion becomes $\ddot{x}_i=0$, with solution

$$\lim_{t\to\pm\infty} x_i^\mu = p_i^\mu s_i + a_i^\mu . \tag{2.5}$$

Here p_i is a constant 4-vector, with $p_i^2=m_i^2$ by Eq. (2.3), which represents the asymptotic momenta p_i^{in} or p_i^{out} as $t\to-\infty$ or $t\to+\infty$, respectively. Each a_i^μ is a finite impact-parameter 4-vector, chosen orthogonal to p_i^μ, which fixes the origin of s_i. We further require

$$p_i \neq p_j\,, \quad i\neq j \tag{2.6}$$

to ensure that $\lim_{s_i\to\pm\infty} F(x_i(s_i))=0$. Here and in the following each p_i and a_i is understood to bear an index in or out for $t\to-\infty$ or $+\infty$.

From the trajectories in the zeroth approximation we find $F^{\mu\nu}$ in the first approximation. It is derived from the vector potential A^μ. For the straight-line trajectories (2.5), the retarded and advanced potentials coincide, and we have

$$\lim_{t\to\pm\infty} A^\mu(x_i) = \sum_{j\neq i} \frac{e_j}{4\pi} \frac{p_j^\mu}{\{[p_j\cdot(x_i-a_j)]^2 - p_j^2(x_i-a_j)^2\}^{1/2}} .$$

Here we sum only over $j\neq i$. The radiation-reaction term is proportional to $\dddot{x}$, which vanishes for the trajectory (2.5). We only require the vector potential for x_i at asymptotic times $\lim_{s_i\to\pm\infty} x_i = p_i s_i + a_i$, so the finite impact parameters a_j may be dropped, and we obtain

$$\lim_{t\to\pm\infty} A^\mu(x_i) = \sum_{j\neq i} \frac{e_j}{4\pi} \frac{p_j^\mu}{[(p_j\cdot x_i)^2 - p_j^2 x_i^2]^{1/2}}\,, \tag{2.7}$$

$$\lim_{t\to\pm\infty} F^{\mu\nu}(x_i) = \sum_{j\neq i} \frac{e_j}{4\pi} \frac{p_j^2(x_i^\mu p_j^\nu - x_i^\nu p_j^\mu)}{[(p_j\cdot x_i)^2 - p_j^2 x_i^2]^{3/2}} . \tag{2.8}$$

It is interesting that the asymptotic field depends only on the asymptotic momenta, not on the impact parameters. To obtain the first-order equation of motion we substitute for $x_i(s_i)$ on the right-hand side of Eq. (2.3) the zeroth-order trajectory (2.5). This yields for $F^{\mu\nu}$

$$\lim_{s_i\to\pm\infty} F^{\mu\nu}(x_i(s_i)) = \pm\frac{1}{s_i^2} \sum_{j\neq i} \frac{e_j}{4\pi} \frac{p_j^2(p_i^\mu p_j^\nu - p_i^\nu p_j^\mu)}{[(p_i\cdot p_j)^2 - p_i^2 p_j^2]^{3/2}}\,, \tag{2.9}$$

so the equation of motion (2.3) becomes in first order, with $\dot{x}_i = p_i$,

$$\lim_{s_i\to\pm\infty} \ddot{x}_i^\mu = \pm\frac{1}{s_i^2} \sum_{j\neq i} \frac{e_i e_j}{4\pi} \frac{p_j^2(p_i\cdot p_j\, p_i^\mu - p_i^2 p_j^\mu)}{[(p_i\cdot p_j)^2 - p_i^2 p_j^2]^{3/2}} . \tag{2.10}$$

On integrating once we obtain

$$\lim_{s_i\to\pm\infty} \dot{x}_i^\mu = p_i^\mu \mp \frac{1}{s_i} \sum_{j\neq i} \frac{e_i e_j}{4\pi} \frac{p_j^2(p_i\cdot p_j\, p_i^\mu - p_i^2 p_j^\mu)}{[(p_i\cdot p_j)^2 - p_i^2 p_j^2]^{3/2}}\,, \tag{2.11}$$

where we have put p_i^μ for the constant of integration in order to be consistent with the zeroth-order solutions. The 4-velocity approaches its asymptotic value like s_i^{-1}. On integrating again, we obtain the first-order asymptotic trajectory

$$\lim_{s_i\to\pm\infty} x_i^\mu = p_i^\mu s_i \mp \ln\left|\frac{s_i}{p_i^2}\right| \sum_{j\neq i} \frac{e_i e_j}{4\pi} \frac{p_j^2(p_i\cdot p_j\, p_i^\mu - p_i^2 p_j^\mu)}{[(p_i\cdot p_j)^2 - p_i^2 p_j^2]^{3/2}} + a_i^\mu . \tag{2.12}$$

The correction to uniform motion is of order $\ln|s_i|$, and thus we see that in the presence of Coulomb forces, the asymptotic trajectory of each particle is not characterized simply by an asymptotic momentum p_i^μ and an impact parameter a_i^μ but depends also on the charges and asymptotic momenta of the other particles.

III. ASYMPTOTIC WAVE FUNCTION FOR CHARGED PARTICLES

Preparatory to deriving the reduction formula, we seek wave functions which have the correct space-time behavior as $t\to\pm\infty$. We assume, as is appropriate for a scattering process, that at asymptotic times the particles occur in wave packets that are infinitely far apart, so it is correct to use single-particle wave functions. At asymptotic times the wave packets are at great distances from each other and follow the classical trajectories so the potentials are slowly varying over the dimensions of a wave packet and have their classical value.

Let the charged particles have spin $\frac{1}{2}$ and obey the Dirac equation. For the ith particle we seek the appropriate asymptotic solution of the equation[10]

$$(i\partial\!\!\!/ - e_i A\!\!\!/_i - m_i)\phi_i = 0, \tag{3.1}$$

as $t \to \pm\infty$, where A_i is the vector potential (2.7). We consider the iterated Dirac equation

$$[(i\partial - e_i A_i)^2 - \tfrac{1}{2} i e_i \gamma_\mu F_i^{\mu\nu}\gamma_\nu - m_i^2]\phi_i = 0 \tag{3.2}$$

and try the obvious positive- and negative-energy solutions in eikonal form,

$$\phi_{i+}(x) = e^{-iS_+(x)} u(p_i), \tag{3.3a}$$

$$\phi_{i-}(x) = e^{iS_-(x)} v(p_i), \tag{3.3b}$$

where $(p\!\!\!/_i - m_i)u(p_i) = 0$, $(-p\!\!\!/_i - m)v(p_i) = 0$, $p_i^2 = m_i^2$, and $p_i^0 > 0$. Including terms of order $1/t$, Eq. (3.2) becomes, as may be verified by substituting the explicit solutions obtained below,

$$(\pm\partial S_\pm - e_i A_i)^2 - m_i^2 = 0. \tag{3.4}$$

Putting

$$S_\pm(x) = p_i \cdot x + S_{1\pm}(x), \tag{3.5}$$

we obtain, to zeroth order in $1/t$, $p_i^2 = m_i^2$; and to first order

$$\pm p_i \cdot (\pm\partial S_{1\pm} - e_i A_i) = 0. \tag{3.6}$$

We choose the solution

$$S_{1\pm}(x) = \pm \int^x e_i A_i(x') \cdot dx', \tag{3.7}$$

where the integration extends along the curve $x' = p_i s$. This gives

$$S_{1\pm}(x) = \pm e_i \int^{(p_i \cdot x)/p_i^2} p_i \cdot A_i(p_i s) ds, \tag{3.8a}$$

which is required only for $t \to \pm\infty$. To avoid confusion of sign alternatives, we rewrite this equation as

$$S_{1\epsilon}(x) = \epsilon e_i \int^{p_i \cdot x/p_i^2} p_i \cdot A_i(p_i s) ds, \tag{3.8b}$$

where ϵ is the sign of the energy. From Eq. (2.7) we have

$$\lim_{s \to \pm\infty} A^\mu(p_i s) = \pm\frac{1}{s}\sum_{j\neq i}\frac{e_j}{4\pi}\frac{p_j^\mu}{[(p_i\cdot p_j)^2 - p_i^2 p_j^2]^{1/2}}, \tag{3.9}$$

which yields

$$\lim_{t \to \pm\infty} S_{1\epsilon}(x) = \pm\epsilon \ln|p_i \cdot x| \times \sum_{j\neq i}\frac{e_i e_j}{4\pi}\frac{p_i\cdot p_j}{[(p_i\cdot p_j)^2 - p_i^2 p_j^2]^{1/2}}. \tag{3.10}$$

It is convenient to define the dimensionless constant

$$\gamma_i \equiv \sum_{j\neq i}\frac{e_i e_j}{4\pi}\frac{p_i\cdot p_j}{[(p_i\cdot p_j)^2 - p_i^2 p_j^2]^{1/2}}. \tag{3.11}$$

Each term in the sum is the constant $e_i e_j (4\pi)^{-1}$ divided by the invariant relative rapidity of particles i and j, which is reminiscent of the relevant coupling parameter of the nonrelativistic Coulomb problem $ee'(4\pi v)^{-1}$. We thus obtain for the eikonal

$$\lim_{t\to\pm\infty} S_\epsilon(x, p_i) = p_i \cdot x \pm \epsilon\gamma_i \ln|p_i\cdot x|, \tag{3.12}$$

which gives for the wave functions (3.3)

$$\lim_{t\to\pm\infty} \phi_{i+}(x, p_i) = \exp[-i(p_i\cdot x \pm \gamma_i \ln|p_i\cdot x|)]u(p_i), \tag{3.13a}$$

$$\lim_{t\to\pm\infty} \phi_{i-}(x, p_i) = \exp[i(p_i\cdot x \mp \gamma_i \ln|p_i\cdot x|)]v(p_i). \tag{3.13b}$$

These equations are the principal result of this section. One may verify that a wave packet of these states

$$\psi_+(x) = \int \phi_+(x, p_i')\chi(p_i')\frac{d^3p_i'}{2E_i'}, \tag{3.14}$$

with $\chi(p_i')$ concentrated close to a particular momentum value p_i, follows precisely the classical trajectory (2.12), whereas the negative-energy wave packets follow a classical trajectory of opposite charge.

IV. REDUCTION FORMULA FOR CHARGED PARTICLES

We wish to relate S-matrix elements

$$\langle p_1' \cdots p_k', q_1' \cdots q_l'^{\,\text{out}} | p_1 \cdots p_m, q_1 \cdots q_n^{\text{in}}\rangle$$

for scattering of electrons and positrons with 4-momentum p and q respectively to vacuum expectation values of Heisenberg fields. The electromagnetic field variables are suppressed in this section. They will be dealt with in the following one.

In theories without massless particles one assumes that the renormalized spinor fields satisfy the weak limits[1]

$$\lim_{t\to\mp\infty}\psi(x)=\psi^{\mathrm{in(out)}}(x)=(2\pi)^{-3/2}\int\frac{d^3p}{2E}\sum_s[a_s(p)e^{-ip\cdot x}u_s(p)+b_s^\dagger(p)e^{ip\cdot x}v_s(p)], \quad (4.1a)$$

$$\lim_{t\to\mp\infty}\bar\psi(x)=\bar\psi^{\mathrm{in(out)}}(x)=(2\pi)^{-3/2}\int\frac{d^3p}{2E}\sum_s[a_s(p)e^{ip\cdot x}\bar u_s(p)+b_s(p)e^{-ip\cdot x}\bar v_s(p)], \quad (4.1b)$$

where the creation and annihilation operators for electrons, $a_s(p)$, and for positrons, $b_s(p)$, bear "in" or "out" subscripts for $t\to-\infty$ or $t\to+\infty$. For this limit to have a meaning, it must be contracted with a c-number solution of the free equation.[1] However, in the last section we saw that in the Coulomb field of other particles, the Dirac single-particle wave functions do not approach plane-wave limits, but depend on the momenta of the other particles present. Hence the limit (4.1) cannot be correct for charged particles. Nevertheless we may suppose that the asymptotic fields have only one-particle matrix elements, with the wave functions given by Eqs. (3.13). Thus the nonzero matrix elements of the asymptotic fields are (suppressing spin indices)

$$\lim_{t\to-\infty}\langle p_1'\cdots p_{n-1}',q_1'\cdots q_m'{}^{\mathrm{in}}|\psi(x)|p_1\cdots p_n,q_1\cdots q_m{}^{\mathrm{in}}\rangle=\sum_p(-1)^{-P}(2\pi)^{-3/2}\exp[-iS^{\mathrm{in}}(x,p_1;p_2\cdots p_n,q_1\cdots q_m)]u(p_1)$$
$$\times\frac{\delta(\vec p_1'-\vec p_2)}{2E(\vec p_2)}\cdots\frac{\delta(\vec p_{n-1}'-\vec p_n)}{2E(\vec p_n)}\frac{\delta(\vec q_1'-\vec q_1)}{2E(\vec q_1)}\cdots\frac{\delta(\vec q_m'-\vec q_m)}{2E(\vec q_m)} \quad (4.2a)$$

and

$$\lim_{t\to-\infty}\langle p_1'\cdots p_n',q_1'\cdots q_{m+1}'{}^{\mathrm{in}}|\psi(x)|p_1\cdots p_n,q_1\cdots q_m{}^{\mathrm{in}}\rangle=\sum_p(-1)^{P}(2\pi)^{-3/2}\exp[iS^{\mathrm{in}}(x,q_1';p_1\cdots p_n,q_1\cdots q_m)]v(q_1')$$
$$\times\frac{\delta(\vec p_1'-\vec p_1)}{2E(\vec p_1)}\cdots\frac{\delta(\vec p_n'-\vec p_n)}{2E(\vec p_n)}\frac{\delta(\vec q_2'-\vec q_1)}{2E(\vec q_1)}\cdots\frac{\delta(\vec q_{m+1}'-\vec q_m)}{2E(\vec q_m)}, \quad (4.2b)$$

where P represents each possible permutation. There is a corresponding limit for $t\to+\infty$ with "in" replaced by "out." Here the eikonals are given by

$$S(x,p;p_1\cdots p_n,q_1\cdots q_m)=p\cdot x\mp\gamma(p;p_1\cdots p_n,q_1\cdots q_m)\ln|p\cdot x|, \quad (4.3a)$$

$$S(x,q;p_1\cdots p_n,q_1\cdots q_m)=q\cdot x\mp\gamma(q;p_1\cdots p_n,q_1\cdots q_m)\ln|q\cdot x|, \quad (4.3b)$$

with upper and lower signs corresponding to S^{in} and S^{out}, respectively, where

$$\gamma(p;p_1\cdots p_n,q_1\cdots q_m)\equiv\frac{e^2}{4\pi}\left(\sum_{i=1}^n\coth(p,p_i)-\sum_{i=1}^m\coth(p,q_i)\right), \quad (4.4a)$$

$$\gamma(q;p_1\cdots p_n,q_1\cdots q_m)\equiv\frac{e^2}{4\pi}\left(-\sum_{i-1}^n\coth(p,q_i)+\sum_{i=1}^m\coth(q,q_i)\right), \quad (4.4b)$$

with

$$\coth(p_1,p_2)\equiv\frac{p_1\cdot p_2}{[(p_1\cdot p_2)^2-p_1^2p_2^2]^{3/2}}. \quad (4.5)$$

To have a meaning, Eqs. (4.2) and (4.6) must be contracted with a suitable wave function, as is done in Eq. (4.11) in the derivation of the reduction formula.

This asymptotic form for the matrix elements of the charged field may be expressed as the weak operator limit

$$\lim_{t\to-\infty}\psi(x)=\sum_{s=1}^2\left[\frac{d\vec p}{2E(\vec p)}\frac{1}{(2\pi)^{3/2}}e^{-iS^{\mathrm{in}}(x,p)}u_s(p)a_s^{\mathrm{in}}(p)+\frac{d\vec p}{2E(\vec q)}\frac{1}{(2\pi)^{3/2}}b_s^{\mathrm{in}\dagger}(q)e^{iS^{\mathrm{in}}(x,q)}v_s(q)\right], \quad (4.6)$$

and correspondingly for $t\to+\infty$ with "out" instead of "in." Here $a_s^{\mathrm{in}}(p)$ and $b_s^{\mathrm{in}}(q)$ are annihilation operators on the Fock space of electrons and positrons, respectively. The order of the eikonals and the creation and annihilation operators is significant here because the eikonals,

$$S^{\text{in}}(x,p)=p\cdot x-\gamma^{\text{in}}(p)\ln|p\cdot x|, \tag{4.7a}$$

$$S^{\text{in}}(x,q)=q\cdot x-\gamma^{\text{in}}(q)\ln|q\cdot x|, \tag{4.7b}$$

$$S^{\text{out}}(x,p)=p\cdot x+\gamma^{\text{out}}(p)\ln|p\cdot x|, \tag{4.7c}$$

$$S^{\text{out}}(x,q)=q\cdot x+\gamma^{\text{out}}(q)\ln|q\cdot x|, \tag{4.7d}$$

are operators on the scattering states through their dependence on $\gamma^{\text{in}}(p)$ and $\gamma^{\text{in}}(q)$, given by

$$\gamma^{\text{in}}(p)\equiv\frac{e^2}{4\pi}\int\frac{d\vec{p}\,'}{2E(\vec{p}\,')}\rho^{\text{in}}(p')\coth(p,p'), \tag{4.8a}$$

$$\gamma^{\text{in}}(q)\equiv-\frac{e^2}{4\pi}\int\frac{d\vec{p}\,'}{2E(\vec{p}\,')}\rho^{\text{in}}(p')\coth(q,p'), \tag{4.8b}$$

and similarly for $\gamma^{\text{out}}(p)$ and $\gamma^{\text{out}}(q)$. Here $[2E(\vec{p})]^{-1}\rho^{\text{in}}(\vec{p})]d\vec{p}$ is the operator representing the number of charges in $d\vec{p}$:

$$\rho^{\text{in}}(p)=\sum_{s=1}^{2}[a_s^{\text{in}\dagger}(p)a_s^{\text{in}}(p)-b_s^{\text{in}\dagger}(p)b_s^{\text{in}}(p)], \tag{4.9}$$

and similarly for $\rho^{\text{out}}(p)$.

Let us now reduce out an initial electron from the generic matrix element using Eq. (4.6) and suppressing spin indices:

$$\langle\cdots^{\text{out}}|p_1\cdots p_m,q_1\cdots q_n{}^{\text{in}}\rangle=\langle\cdots^{\text{out}}|a^{\text{in}\dagger}(p_1)|p_2\cdots p_m,q_1\cdots q_n{}^{\text{in}}\rangle, \tag{4.10}$$

$$=\lim_{x^0\to-\infty}(2\pi)^{-3/2}\int d^3x\langle\cdots^{\text{out}}|\bar{\psi}(x)|p_2\cdots p_m,q_1\cdots q_n{}^{\text{in}}\rangle\gamma^0u(p)\times\exp[-iS^{\text{in}}(x,p_1;p_2\cdots p_m,q_1\cdots q_n)]. \tag{4.11}$$

One may verify by a calculation like the one in the Appendix that the positron annihilation operator is eliminated by this projection. Proceeding as usual we rewrite this as

$$\begin{aligned}\langle\cdots^{\text{out}}|p_1\cdots p_m,q_1\cdots q_n\rangle&=-(2\pi)^{-3/2}\int d^4x\frac{\partial}{\partial x^0}\{\langle\cdots^{\text{out}}|\bar{\psi}(x)|p_2\cdots p_m,q_1\cdots q_n\rangle\gamma^0\exp[-iS^{\text{in}}(x,p_1;p_2\cdots p_m,q_1\cdots q_n)]u(p_1)\}\\&+\lim_{x^0\to\infty}(2\pi)^{-3/2}\int d^3x\langle\cdots^{\text{out}}|\bar{\psi}(x)|p_2\cdots p_m,q_1\cdots q_n{}^{\text{in}}\rangle\gamma^0\exp[-iS^{\text{in}}(x,p_1;p_2\cdots p_m,q_1\cdots q_n)]u(p_1).\end{aligned} \tag{4.12}$$

In this identity we have extended to all times the expression (4.3) for the eikonal which represents the wave function at early times only. The second term is studied in the Appendix. It is found there to consist of disconnected parts multiplied by a phase which diverges logarithmically with time, Eq. (A3). When smeared with smooth wave functions, so as to describe the scattering of wave packets, the contribution from this term vanishes due to the rapid oscillations of the phase factor for sufficiently large times. Hence there is no disconnected part in Coulomb scattering, corresponding to the fact that the long-range force causes all particles to be deviated. Thus we retain only the first term of Eq. (4.12). By adding a spatial divergence we obtain

$$\langle\cdots^{\text{out}}|p_1\cdots p_m,q_1\cdots q_n{}^{\text{in}}\rangle=i(2\pi)^{-3/2}\int d^4x\{\langle\cdots^{\text{out}}|\bar{\psi}(x)|p_2\cdots p_m,q_1\cdots q_n{}^{\text{in}}\rangle(i\overleftarrow{\partial\!\!\!/}+i\overrightarrow{\partial\!\!\!/})\times\exp[-iS^{\text{in}}(x,p_1;p_2\cdots p_m,q_1\cdots q_n)]u(p_1)\}. \tag{4.13}$$

To proceed further we go over to momentum space. We rewrite the wave function, given by Eq. (4.3), as

$$\exp[-iS^{\text{in}}(x,p_1,p_2\cdots p_m,q_1\cdots q_m)]=e^{-ip_1\cdot x}|p_1\cdot x|^{i\gamma_1}, \tag{4.14a}$$

where

$$\gamma_1\equiv\gamma(p_1;p_2\cdots p_m,q_1\cdots q_m), \tag{4.14b}$$

using the 1-dimensional Fourier transform

$$|s|^{i\gamma}=\int_{-\infty}^{\infty}d\lambda\, f(\lambda,\gamma)e^{is\lambda}=\int_{-\infty}^{\infty}d\lambda\, f(\lambda,\gamma)e^{-is\lambda}, \tag{4.15}$$

where

$$f(\lambda,\gamma)=\lim_{\epsilon\to 0}\frac{\Gamma(1+i\gamma)}{2\pi}\left[\frac{1}{(\epsilon-i\lambda)^{1+i\gamma}}+\frac{1}{(\epsilon+i\lambda)^{1+i\gamma}}\right]. \tag{4.16}$$

Then Eq. (4.13) becomes

$$\langle\cdots{}^{\text{out}}|p_1\cdots p_m, q_1\cdots q_n{}^{\text{in}}\rangle=i(2\pi)^{-3/2}\int_{-\infty}^{\infty}d\lambda\, f(\lambda,\gamma_1)\int d^4x\langle\cdots{}^{\text{out}}|\bar\psi(x)|p_2\cdots p_m, q_1\cdots q_n{}^{\text{in}}\rangle$$
$$\times(i\overleftarrow{\not\partial}+i\overrightarrow{\not\partial})\,e^{-ip_1\cdot x(1+\lambda)}u(p_1). \tag{4.17}$$

We introduce the Fourier transform $T(P)$ by

$$T(\ldots;P;p_2\cdots p_m, q_1\cdots q_n)\equiv\int\langle\cdots{}^{\text{out}}|\bar\psi(x)|p_2\cdots p_m, q_1\cdots q_n{}^{\text{in}}\rangle e^{-iP\cdot x}d^4x, \tag{4.18}$$

so Eq. (4.17) becomes

$$\langle\cdots{}^{\text{out}}|p_1\cdots p_m, q_1\cdots q_n{}^{\text{in}}\rangle=(-i)(2\pi)^{-3/2}\int_{-\infty}^{\infty}d\lambda\, f(\lambda,\gamma_1)(1+\lambda)T(\ldots;p_1(1+\lambda);p_2\cdots p_m, q_1\cdots q_n)(\overleftarrow{\not p}_1-m)u(p_1). \tag{4.19}$$

If the $\not p_1$ were multiplied to the right, we would get zero, just as in the usual reduction formula. However, we expect this zero to be eliminated by a factor of $(\not p_1-m)^{-1}$ on the left. This inverse does not exist, of course, unless $p_1{}^2\neq m^2$, so we must understand this expression to mean

$$\langle\cdots{}^{\text{out}}|p_1\cdots{}^{\text{in}}\rangle=(-i)(2\pi)^{-3/2}\lim_{\eta\to 0}\left\{\int_{-\infty}^{\infty}d\lambda\, f(\lambda,\gamma_1)(1+\lambda)T(\ldots;p_1(1+\eta)(1+\lambda);\ldots)[\not p_1(1+\eta)-m]\right\}u(p_1), \tag{4.20}$$

where $p_1{}^2=m^2$ and η is a parameter which measures approach to the mass shell.

Contributions to this integral from finite values of λ, compared to η which approaches zero, are annihilated by

$$[\not p_1(1+\eta)-m]u(p_1)=m\eta u(p_1), \tag{4.21}$$

for when λ is finite $T(\ldots;p_1(1+\eta)(1+\lambda);\ldots)$ is regular in η, as $\eta\to 0$, and cannot contribute to the residue of $[\not p_1(1+\eta)-m]^{-1}$. The antiparticle singularity of $T(\ldots;p_1(1+\eta)(1+\lambda);\ldots)$ at $\lambda=-2$ is also annihilated because $f(\lambda)$ is regular at $\lambda=-2$. Hence all the contribution to Eq. (4.20) comes from the neighborhood of the mass shell of $T(\ldots;p_1(1+\eta)(1+\lambda);\ldots)$, namely

$$p_1{}^2(1+\eta)^2(1+\lambda)^2-m^2\approx m^2(2\eta+2\lambda)\approx 0, \tag{4.22}$$

or from

$$\lambda=O(\eta). \tag{4.23}$$

Therefore let us evaluate the integral (4.20) for $|\eta|^{3/4}\geq\lambda\geq-|\eta|^{3/4}$, where the limits are chosen to be much greater than η in absolute value, and then verify that the remaining contribution vanishes as $\eta\to 0$. Temporarily assuming this last point, we write

$$\langle\cdots{}^{\text{out}}|p_1\cdots{}^{\text{in}}\rangle=(-i)(2\pi)^{-3/2}m\lim_{\eta\to 0}\int_{-|\eta|^{3/4}}^{|\eta|^{3/4}}d\alpha\, f(\alpha,\gamma_1)(1+\lambda)\eta T(\ldots;p_1(1+\eta)(1+\lambda);\ldots)u(p_1), \tag{4.24}$$

where the domain of integration is now entirely in the neighborhood of the mass shell. Setting $\lambda=|\eta|\alpha$, and noting from Eq. (4.16) that $f(\lambda,\gamma)$ is homogeneous in λ of degree $(1+i\gamma)^{-1}$, we obtain

$$\langle\cdots{}^{\text{out}}|p_1\cdots{}^{\text{in}}\rangle=(-i)(2\pi)^{-3/2}m\lim_{\eta\to 0}\int_{-|\eta|^{-1/4}}^{|\eta|^{-1/4}}d\lambda\, f(\lambda,\gamma_1)(1+\alpha|\eta|)$$
$$\times\operatorname{sgn}\eta\,|\eta|^{1-i\gamma_1}T(\ldots;p_1(1+\eta)(1+|\eta|\alpha);\ldots)u(p_1). \tag{4.25}$$

General considerations tell us that $T(\ldots;P;\ldots)$ has a cut in P^2 beginning at $P^2 - m^2$. Hence we may write

$$\lim_{\eta\to 0} T(\ldots;P;\ldots)u(p_1) = \lim_{\eta\to 0} S\left(\frac{P^2-m^2+i\epsilon}{2m^2}\right)R(\ldots;p_1;\ldots)u(p_1), \tag{4.26}$$

where $P = p_1(1+\eta)(1+|\eta|\alpha)$, $R(\ldots;p_1;\ldots)u(p_1)$ is regular, and all the mass-shell singularity is in $S((P^2-m^2+i\epsilon)/2m^2)$. The $i\epsilon$ indicates the Feynman boundary condition. Here and in the following whenever the symbol ϵ appears it approaches zero before η (or any other variable) does. Equation (4.25) becomes

$$\langle\cdots{}^{\rm out}|p_1\cdots{}^{\rm in}\rangle = -im(2\pi)^{-3/2}\lim_{\eta\to 0}\left\{\int_{-|\eta|^{-1/4}}^{|\eta|^{-1/4}} d\alpha\, f(\alpha,\gamma_1)\,{\rm sgn}\eta|\eta|^{1-i\gamma_1}S(|\eta|({\rm sgn}\eta+\alpha+i\epsilon))\right\}$$
$$\times R(\ldots;p_1;\ldots)u(p_1), \tag{4.27}$$

where we have used

$$\lim_{\eta\to 0} S\left(\frac{P^2-m^2+i\epsilon}{2m^2}\right) = \lim_{\eta\to 0} S(|\eta|({\rm sgn}\eta+\alpha+i\epsilon)),$$

for $P = p_1(1+\eta)(1+\alpha|\eta|)$. The only way for this integral to have a finite limit as $|\eta|$ approaches zero is for $\lim_{\eta\to 0}S[|\eta|({\rm sgn}\eta+\alpha+i\epsilon)]$ to be homogeneous in $|\eta|$ of degree $(-1+i\gamma_1)$, or

$$\lim_{\eta\to 0} S[|\eta|({\rm sgn}\eta+\alpha+i\epsilon)] = \frac{1}{[|\eta|({\rm sgn}\eta+\alpha+i\epsilon)]^{1-i\gamma_1}}. \tag{4.28}$$

The integral over α may now be effected:

$$\int_{-\infty}^{\infty} d\alpha\, f(\alpha,\gamma_1)\frac{{\rm sgn}\eta}{({\rm sgn}\eta+\alpha+i\epsilon)^{1-i\gamma_1}} = \frac{e^{-\pi\gamma_1/2}}{\Gamma(1-i\gamma_1)}, \tag{4.29}$$

with $f(\alpha,\gamma_1)$ defined as in Eq. (4.16), so Eq. (4.27) becomes

$$\langle\cdots{}^{\rm out}|p_1\cdots{}^{\rm in}\rangle = -im(2\pi)^{-3/2}\,\Gamma^{-1}(1-i\gamma_1)e^{-\pi\gamma_1/2}R(\ldots;p_1;\ldots)u(p_1). \tag{4.30}$$

Let us express this in terms of the original Fourier transform $T(P)$, using Eq. (4.26) and (4.28) with $P = p_1(1+\eta)$. This gives, restoring the in-particle labels,

$$\langle\cdots{}^{\rm out}|p_1\cdots p_m, q_1\cdots q_n{}^{\rm in}\rangle = m(2\pi)^{-3/2}\lim_{\eta\to 0}\left[T(\ldots;p_1(1+\eta);p_2\cdots p_m,q_1\cdots q_n)\frac{(\epsilon-i\eta)^{1-i\gamma_1}}{\Gamma(1-i\gamma_1)}u(p_1)\right]. \tag{4.31}$$

For $\gamma_1 = 0$ the usual reduction formula is regained since $m\eta$ may be replaced by $[\not{p}_1(1+\eta)-m]$ acting on $u(p_1)$. There remains only to verify that the contributions to the integral (4.20) from $\lambda > |\eta|^{3/4}$ and $\lambda < -|\eta|^{3/4}$ do indeed vanish. This is easily done, recalling that the integral vanishes for λ finite as $\eta\to 0$, and making use of the singularity structure given by (4.26) and (4.28).

Formula (4.31) achieves our purpose of reducing out one electron and expressing the S-matrix element in terms of the Fourier transform, (4.18), of the matrix element of the renormalized field operator. It works like a truncation formula for removing a leg by multiplying by an inverse propagator S^{-1}. Here, however, $S(p_1(1+\eta),\gamma_1)$ is not the free propagator $i[\not{p}_1(1+\eta)-m+i\epsilon]^{-1}\to(\epsilon-i\eta)^{-1}m^{-1}$, but describes instead propagation of particle 1 in the Coulomb field of the other particles. The expression for $S^{-1}[p_1(1+\eta),\gamma_1]$ is however greatly simplified by the limit $\eta\to 0$ and right multiplication by the positive-energy spinor $u(p_1)$. The explicit appearance of $\Gamma(1-i\gamma_1)$ is characteristic of Coulomb scattering.

We may reduce out a second electron and obtain analogously

$$\langle\cdots{}^{\rm out}|p_1\cdots p_n, q_1\cdots q_n{}^{\rm in}\rangle = [m(2\pi)^{-3/2}]^2\lim_{\eta_1\to 0}\lim_{\eta_2\to 0}\left[T(\ldots;p_1(1+\eta_1),p_2(1+\eta_2);p_3\cdots p_m,q_1\cdots q_n)\right.$$
$$\left.\times\frac{(\epsilon-i\eta_1)^{1-i\gamma_1}}{\Gamma(1-i\gamma_1)}\frac{(\epsilon-i\eta_2)^{1-i\gamma_2}}{\Gamma(1-i\gamma_2)}\right]u(p_1)u(p_2), \tag{4.32}$$

where γ_1 is given in Eq. (4.14b),

$$\gamma_2 = \gamma_2(p_2;p_3\cdots p_m, q_1\cdots q_n), \tag{4.33}$$

$$T(\ldots; P_1, P_2; p_3 \cdots p_m, q_1 \cdots q_n) = \int d^4x_1 d^4x_2 \langle \cdots {}^{\text{out}} | T(\overline{\psi}(x_1)\overline{\psi}(x_2)) p_3 \cdots p_m, q_1 \cdots q_n{}^{\text{in}} \rangle e^{-iP_1 \cdot x_1} e^{-iP_2 \cdot x_2}, \tag{4.34}$$

and spinor indices are summed separately for each particle. Comparing γ_1 and γ_2 we see that the order of limits cannot be interchanged in Eq. (4.32) because the singularity associated with particle 2 is independent of particle 1, which is off its mass shell when particle 2 goes on, whereas the singularity associated with particle 1 depends on particle 2, which is already on its mass shell as particle 1 goes on. This apparent asymmetry arises because we only have a weak limit (4.6) for the field, which is used successively for each particle.

Finally we write down the complete formula for reducing out in succession all electrons followed by all positrons:

$$\langle p_1' \cdots p_k', q_1' \cdots q_l'{}^{\text{out}} | p_1 \cdots p_m, q_1 \cdots q_n{}^{\text{in}} \rangle$$

$$= [m(2\pi)^{-3/2}]^{k+l+m+n}$$

$$\times \lim_{\eta_1' \to 0} \cdots \lim_{\eta_{k+1}' \to 0} \cdots \lim_{\eta_1 \to 0} \cdots \lim_{\eta_{m+1} \to 0} \left[\frac{\overline{u}(p_1')(\epsilon - i\eta_1')^{1-i\gamma_1'}}{\Gamma(1-i\gamma_1')} \cdots \frac{\overline{v}(q_1)(\epsilon - i\eta_{m+1})^{1-i\gamma_{m+1}}}{\Gamma(1-i\gamma_{m+1})} \cdots \right.$$
$$\times T(p_1'(1+\eta_1')\ldots, -q_1(1+\eta_{m+1})\ldots; p_1(1+\eta_1)\ldots, -q_1'(1+\eta_{k+1}')\ldots)$$
$$\left. \times \frac{(\epsilon - i\eta_1)^{1-i\gamma_1}}{\Gamma(1-i\gamma_1)} u(p_1) \cdots \frac{(\epsilon - i\eta_{k+1}')^{1-i\gamma_{k+1}'}}{\Gamma(1-i\gamma_{k+1}')} v(q_1') \cdots \right], \tag{4.35}$$

where

$$\gamma_1 = \gamma(p_1; p_2 \cdots p_m, q_1 \cdots q_n),$$
$$\gamma_{m+1} = \gamma(q_1, q_2 \cdots q_n),$$
$$\gamma_1' = \gamma(p_1'; p_2' \cdots p_n', q_1' \cdots q_l'),$$
$$\gamma_{k+1}' = \gamma(q_1'; q_2' \cdots q_l'),$$

etc.

[see Eq. (4.4)], and

$$T(P_1'\ldots, Q_1\ldots; P_1\ldots, Q_1'\ldots) = \int d^4x_1' e^{iP_1' \cdot x_1'} \cdots d^4x_{k+1}' e^{iQ_1' \cdot x_{k+1}'} \cdots d^4x_1 e^{-iP_1 \cdot x_1} \cdots d^4x_{m+1} e^{-iQ_1 \cdot x_{m+1}}$$
$$\times \langle {}^{\text{out}} | T(\psi(x_1') \cdots \psi(x_{m+1}) \cdots \overline{\psi}(x_1) \cdots \overline{\psi}(x_{k+1}') \cdots) | {}^{\text{in}} \rangle. \tag{4.36}$$

Here the order of limits is again important, and the in- and out-states are asymptotic states of the electromagnetic field. These will be reduced out in the following section.

V. REDUCTION FORMULA FOR COHERENT STATES

The vector potential of the electromagnetic field satisfies the equation of motion in the Gupta-Bleuler gauge,

$$\partial^2 A^\nu = J^\nu. \tag{5.1}$$

The in- and out-fields are, as usual, introduced by

$$A^\nu(x) = A^{\nu\,\text{in}}(x) + \int \Delta^{\text{ret}}(x-y) J^\nu(y) d^4y, \tag{5.2a}$$

$$A^\nu(x) = A^{\nu\,\text{out}}(x) + \int \Delta^{\text{adv}}(x-y) J^\nu(y) d^4y, \tag{5.2b}$$

with Δ^{ret} and Δ^{adv} the retarded and advanced solutions of

$$\partial^2 \Delta^{\text{ret}}(x) = \partial^2 \Delta^{\text{adv}}(x) = \delta^4(x). \tag{5.3}$$

The radiation field is a free field defined for all space-time by[11]

$$A^{\nu\,\text{rad}}(x) \equiv A^{\nu\,\text{out}}(x) - A^{\nu\,\text{in}}(x)$$
$$= \int \Delta(x-y) J^\nu(y) d^4y, \tag{5.4}$$

where

$$\Delta(x) = \Delta^{\text{ret}}(x) - \Delta^{\text{adv}}(x), \tag{5.5a}$$

$$\Delta(x) = \frac{i}{(2\pi)^3} \int (e^{-ik\cdot x} - e^{ik\cdot x}) \frac{d^3k}{2\omega}, \tag{5.5b}$$

with $k^\mu = (\omega, \vec{k}) = (|\vec{k}|, \vec{k})$. The in- and out-fields obey the canonical commutation relations

$$[A^{\mu\,\text{in}}(x), A^{\nu\,\text{in}}(y)] = [A^{\mu\,\text{out}}(x), A^{\nu\,\text{out}}(y)] = ig^{\mu\nu}\Delta(x-y). \quad (5.6)$$

The fact that the same Δ appears in the definition of the radiation field and in the commutation relations is exploited below, for example, in showing that Eq. (5.15b) follows from (5.18). The asymptotic fields have the usual expansions

$$A^{\nu\,\text{in}}(x) = (2\pi)^{-3/2}\int \frac{d^3k}{2\omega}[a^{\nu\,\text{in}}(k)e^{-ik\cdot x} + a^{\nu\,\text{in}}(k)e^{ik\cdot x}], \quad (5.7)$$

with

$$[a^{\nu\,\text{in}}(k), a^{\mu\,\text{in}\dagger}(k')] = -g^{\mu\nu}\,2\omega\,\delta(\vec{k}-\vec{k}'), \quad (5.8a)$$

$$[a^{\nu\,\text{in}}(k), a^{\nu\,\text{in}}(k')] = 0, \quad (5.8b)$$

and similarly for in $\to$ out.

We will be concerned in this section with the usual scattering situation in which the initial state contains a finite number of charged particles and photons, so $A^{\nu\,\text{in}}(x)$ will have the Fock representation defined by the vacuum state

$$A^{\nu\,\text{in}\,(-)}(x)\,|\,0\rangle = a^{\nu\,\text{in}}(k)\,|\,0\rangle = 0. \quad (5.9)$$

Whether or not A^{out} has a Fock representation depends entirely on the operator (5.4). The usual infrared divergences are symptoms of a contradiction between Eq. (5.4) and a Fock representation for A^{out} also. The cure is well known; namely, to account correctly for the infinite number of photons that are emitted coherently in every scattering process and which add up in the infrared limit to the classical radiation field of the corresponding classical scattering process.

Let us now construct the classical radiation field that corresponds to the given scattering process and that has the correct infrared behavior. For this purpose we need consider only the initial and final charges and momenta

$$e_i, p_i \to e_f, p_f, \quad \sum e_i = \sum e_f$$

and may take as the classical current

$$j_c^\nu(x) = \sum_i e_i p_i^\nu \int_{-\infty}^{0} \delta^4(x - p_i s)ds + \sum_f e_f p_f^\nu \int_0^\infty \delta^4(x - p_f s)ds, \quad (5.10)$$

corresponding to the initial and final particles approaching and leaving the origin uniformly. Any other current with the same low-frequency behavior would do as well. In fact we will retain only the low-frequency components of (5.10) by introducing a cutoff $f(k)$ in momentum space and choosing as our classical current

$$j^\nu(x) = \int \tilde{j}^\nu(k)e^{-ik\cdot x}d^4k, \quad (5.11)$$

with

$$\tilde{j}^\nu(k) = f(k)\tilde{j}_c^\nu(k),$$

where $\tilde{j}_c^\nu(k)$ is the Fourier transform of $j_c^\nu(x)$,

$$\tilde{j}_c^\nu(k) = (2\pi)^{-4}\left(\sum_i e_i p_i^\nu \int_{-\infty}^0 e^{ik\cdot p_i\tau}d\tau + \sum_f e_f p_f^\nu \int_0^\infty e^{ik\cdot p_f\tau}d\tau\right).$$

This gives

$$\tilde{j}^\nu(k) = \frac{f(k)}{(2\pi)^4}\sum_i \frac{e_i p_i^\nu}{\epsilon + ik\cdot p_i} + \sum_f \frac{e_f p_f^\nu}{\epsilon - ik\cdot p_f}, \quad (5.12a)$$

where $f(k) = f^*(-k)$ cuts off high frequencies,

$$\lim_{k\to 0} f(k) = 1, \quad \lim_{k\to\infty} f(k) = 0. \quad (5.12b)$$

The corresponding classical radiation field $a^{\nu\,\text{rad}}(x)$ is given by Eq. (5.4), with the quantum-mechanical current J replaced by the classical current j,

$$a^{\nu\,\text{rad}}(x) = \int \Delta(x-y)j^\nu(y)d^4y, \quad (5.13a)$$

or, from Eqs. (5.5b) amd (5.11),

$$a^{\nu\,\text{rad}}(x) = 2\pi i\int \frac{d^3k}{2\omega}[\tilde{j}^\nu(k)e^{-ik\cdot x} - \tilde{j}^\nu(-k)e^{ik\cdot x}], \quad (5.13b)$$

where $k^\mu = (\omega, \vec{k})$. A suitable representation for the out-field $A^{\nu\,\text{out}}(x)$ is given by

$$A^{\nu\,\text{out}}(x) = A_f^{\nu\,\text{out}}(x) + a^{\nu\,\text{rad}}(x), \quad (5.14)$$

where $A_f^{\nu\,\text{out}}(x)$ is a canonical free field in the Fock representation.

Whether or not $A^{\nu\,\text{out}}(x)$ has a Fock representation now depends on whether there exists a unitary operator s in the Fock space satisfying

$$A_f^{\nu\,\text{out}}(x) = sA^{\nu\,\text{out}}(x)s^{-1} = A^{\nu\,\text{out}}(x) - a^{\nu\,\text{rad}}(x) \quad (5.15a)$$

or

$$sA^{\nu\,\text{out}}(x)s^{-1} = A^{\nu\,\text{out}}(x) - \int \Delta(x-y)j^\nu(y)d^4y. \quad (5.15b)$$

We see by comparison with Eq. (5.4) that s is the S matrix for the coupling of a quantum field $A^\nu(x)$ to the classical current source

$$\partial^2 A^\nu(x) = j^\nu(x). \quad (5.16)$$

From time-dependent perturbation theory we know that if s exists it is given by

$$s = T\exp\left[-i\int j(x)\cdot A^{\text{out}}(x)d^4x\right]. \tag{5.17}$$

It will turn out, of course, that s does not exist. However, the product $s^{-1}S$, defined by a suitable limiting process, exists, where S is the Dyson S matrix for the scattering produced by the quantum current $J^\nu(x)$, and the physical scattering amplitude is a matrix element of $s^{-1}S$. To proceed further we let $f(k)$ in Eq. (5.12) also contain an infrared cutoff, so that s exists. In our final expression the infrared cutoff in $f(k)$ is dropped. From Eq. (5.17) we may rewrite s in normal form as

$$s = C:\exp\left(-i\int j\cdot A^{\text{out}}d^4x\right):, \tag{5.18}$$

where : : means normal ordering, the constant C is given by

$$C = \exp\left[\tfrac{1}{2}\int j_\mu(x)\Delta_F(x-y)j^\mu(y)d^4x d^4y\right], \tag{5.19}$$

and $\Delta_F(x)$ is the zero-mass Feynman propagator,

$$\Delta_F(x) = \frac{i}{(2\pi)^4}\int\frac{e^{-ik\cdot x}}{k^2+i\epsilon}d^4k. \tag{5.20}$$

Using Eq. (5.18) one may verify (5.15b) directly. With this expression for the Feynman propagator, we find

$$C = \exp\left[\tfrac{1}{2}i(2\pi)^4\int\tilde{j}_\mu(-k)\frac{1}{k^2+i\epsilon}\tilde{j}^\mu(k)d^4k\right],$$

or from Eq. (5.12a)

$$C = \exp\left[\tfrac{1}{2}i\frac{1}{(2\pi)^4}\int|f(k)|^2\left(\sum_i\frac{e_ip_i}{p_i\cdot k} - \sum_f\frac{e_fp_f}{p_f\cdot k}\right)^2\frac{d^4k}{k^2+i\epsilon}\right]. \tag{5.21}$$

This gives for $|C|$

$$|C| = \exp\left[\frac{\pi}{(2\pi)^4}\int|f(k)|^2\left(\sum_i\frac{e_ip_i}{p_i\cdot k} - \sum_f\frac{e_fp_f}{p_f\cdot k}\right)^2\frac{d^3k}{2\omega}\right], \tag{5.22}$$

where $k=(\omega,\vec{k})=(|\vec{k}|,\vec{k})$. We see that if $\lim_{k\to 0}f(k)\to 1$, the integral is divergent at low frequency. More precisely, because of current conservation, $k\cdot\tilde{j}(k)=0$ with $k^2=0$, $\tilde{j}(k)$ is spacelike, so such a divergence makes $|C|$ vanish. In this case s does not exist as an operator on Fock space and $A^{\mu\,\text{out}}$ does not have a Fock representation. To avoid this we implicitly assume that $f(k)$ also cuts off low frequencies, and it is understood that this low-frequency cutoff is removed in our final formula.

We choose as a basis for the out-states

$$|a^{\text{rad}}(x)^{\text{out}}\rangle = s|0\rangle,\quad |k,\,a^{\text{rad}}(x)^{\text{out}}\rangle = sa^{\dagger\,\text{out}}(k)|0\rangle, \tag{5.23}$$

$$|k_1,k_2,a^{\text{rad}}(x),^{\text{out}}\rangle = sa^{\text{out}\,\dagger}(k_1)a^{\text{out}\,\dagger}(k_2)|0\rangle,\ \text{etc.},$$

with $a^{\text{out}}(k)|0\rangle=0$, and where polarization indices are suppressed. Since s is unitary these states have the same inner products as the usual Fock-space basis vectors, and are interpreted as a superposition of photon fields and a classical coherent field $a^{\text{rad}}(x)$. The state $|a^{\text{rad}}(x),^{\text{out}}\rangle$ is coherent, since we have from Eq. (5.15a)

$$A^{\nu\,\text{out}(-)}(x)|a^{\text{rad}}(x),^{\text{out}}\rangle = a^{\nu\,\text{rad}(-)}(x)|a^{\text{rad}}(x),^{\text{out}}\rangle, \tag{5.24a}$$

or from Eq. (5.13b)

$$a^{\nu\,\text{out}}(k)|a^{\text{rad}}(x),^{\text{out}}\rangle = (2\pi)^{5/2}i\tilde{j}^\nu(k)|a^{\text{rad}}(x),^{\text{out}}\rangle, \tag{5.24b}$$

where, as usual superscripts (+) and (−) refer to positive- and negative-frequency parts of a free field. Let us denote an element of the basis (5.23) by

$$|k_1\cdots k_m,a^{\text{rad}}(x)^{\text{out}}\rangle = |\{k_i\}a^{\text{rad}}(x)^{\text{out}}\rangle = s\prod_i a^{\text{out}\dagger}(k_i)|0\rangle. \tag{5.25}$$

From Eqs. (5.15a) and (5.13b) we have

$$sa^{\nu\,\text{out}\,\dagger}(k)s^{-1} = a^{\nu\,\text{out}\dagger}(k) + (2\pi)^{5/2}i\tilde{j}^\nu(-k), \tag{5.26}$$

which gives

$$|\{k_i\}a^{\text{rad}}(x)^{\text{out}}\rangle = \prod_i[a^{\text{out}\,\dagger}(k_i) + (2\pi)^{5/2}i\tilde{j}^\nu(-k)]s|0\rangle,$$

or, from Eq. (5.18),

$$|\{k_i\}a^{\text{rad}}(x)^{\text{out}}\rangle = C\exp\left(-i\int jA^{\text{out}(+)}d^4x\right)\prod_i a_s^{\text{out}\,\dagger}(k_i)\Big|\,0\Big\rangle, \tag{5.27}$$

where for concision we have introduced the shifted creation and annihilation operators

$$a_s^{\text{out}\,\nu}(k) \equiv a^{\text{out}\,\nu}(k) - (2\pi)^{5/2}i\tilde{j}^\nu(k), \tag{5.28a}$$

$$a_s^{\text{out}\,\dagger\nu}(k) \equiv a^{\text{out}\,\dagger\nu}(k) + (2\pi)^{5/2}i\tilde{j}^\nu(-k), \tag{5.28b}$$

which are creation and annihilation operators for

$$A_s^{\text{out}}(x) = sA^{\text{out}}(x)s^{-1} = A^{\text{out}}(x) - a^{\text{rad}}(x). \tag{5.28c}$$

We note from Eqs. (5.7) and (5.11) that

$$\int j \cdot A^{\text{out}(+)} d^4x = (2\pi)^{5/2} \int a^{\text{out}\dagger}(k) \cdot j(k) \frac{d^3k}{2\omega}, \tag{5.29}$$

or, using also (5.13b),

$$\int j^\nu(x) A_\nu^{\text{out}(+)}(x) d^4x = \int A^{\nu\,\text{out}}(x) \left(\frac{\overrightarrow{\partial}}{\partial t} - \frac{\overleftarrow{\partial}}{\partial t} \right) a_\nu^{\text{rad}(-)}(x) d^3x. \tag{5.30}$$

The 3-dimensional integral on the right-hand side is evaluated at some fixed time $t = x^0$, and because both $A^{\nu\,\text{out}}(x)$ and $a^{\nu\,\text{rad}}(x)$ are solutions of the wave equation the result is time-independent. Hence from Eq. (5.27) we obtain finally

$$|\{k_i\} a^{\text{rad}}(x)^{\text{out}}\rangle = C \exp\left[(-i) \int A^{\nu\,\text{out}} (\overrightarrow{\partial}_t - \overleftarrow{\partial}_t) a_\nu^{\text{rad}(-)} d^3x \right] \prod_i a_s^{\text{out}\dagger}(k_i) \Big| 0 \Big\rangle. \tag{5.31}$$

Having specified the in- and out-states of the electromagnetic field, we are ready to derive reduction formulas for these states. According to Eq. (4.35) we require the matrix element

$$\langle k_1' \cdots k_m', a^{\text{rad}}(x)^{\text{out}} | T(\psi, \overline{\psi}) | k_1 \cdots k_m^{\text{in}} \rangle = \langle \{k_1'\} a^{\text{rad}}(x)^{\text{out}} | T(\psi, \overline{\psi}) | \{k_j\}^{\text{in}} \rangle,$$

where $T(\psi, \overline{\psi})$ is a time-ordered product of ψ's and $\overline{\psi}$'s. From Eq. (5.31) this may be written

$$\langle \{k_i'\} a^{\text{rad}}(x)^{\text{out}} | T(\psi, \overline{\psi}) | \{k_i\}^{\text{in}} \rangle = C^* \Big\langle 0 \Big| \prod_i a_s^{\text{out}}(k_i') \exp\left[(i) \int A^{\nu\,\text{out}} (\overrightarrow{\partial}_t - \overleftarrow{\partial}_t) a_\nu^{\text{rad}(+)} d^3x \right] T(\psi, \overline{\psi}) \prod_j a^{\text{in}\dagger}(k_j) \Big| 0 \Big\rangle. \tag{5.32}$$

We now state the identity which is fundamental to the reduction formula for coherent states:

$$\exp\left[i \int A^{\text{out}} (\overrightarrow{\partial}_t - \overleftarrow{\partial}_t) a^{\text{rad}(+)} d^3x \right] T(\psi, \overline{\psi}) \exp\left[-i \int A^{\text{in}} (\overrightarrow{\partial}_t - \overleftarrow{\partial}_t) a^{\text{rad}(+)} d^3x \right] = T\left(\psi, \overline{\psi} \exp\left(-i \int J \cdot a^{\text{rad}(+)} d^4x \right) \right). \tag{5.33}$$

This identity also holds if $a^{\text{rad}(+)}$ is replaced by a function which has positive and/or negative frequency components. It is established by expanding the exponentials, or more rapidly, by substituting $a^{\text{rad}(+)}(x) \to \lambda a^{\text{rad}(+)}(x)$ on left- and right-hand sides and verifying that for $\lambda = 0$ the identity holds and that left and right satisfy the same first-order differential equation in λ. One uses the equation of motion for the Heisenberg field $\partial^2 A = J$, and the vanishing equal-time commutators of A and $\dot{A}$ with ψ and $\overline{\psi}$ and also, naively, with J. Hence we obtain

$$\langle \{k_i'\} a^{\text{rad}}(x)^{\text{out}} | T(\psi, \overline{\psi}) | \{k_j\}^{\text{in}} \rangle$$
$$= C^* \Big\langle 0 \Big| \prod_i a_s^{\text{out}}(k_i') T\left(\psi, \overline{\psi} \exp\left(-i \int J \cdot a^{\text{rad}(+)} d^4x \right) \right) \exp\left[i \int A^{\text{in}} (\overrightarrow{\partial}_t - \overleftarrow{\partial}_t) a^{\text{rad}(+)} d^3x \right] \prod_i a^{\text{in}\dagger}(k_j) \Big| 0 \Big\rangle. \tag{5.34}$$

Using the Hermitian conjugate of Eq. (5.30) with out → in, and Eqs. (5.18) and (5.26) with out → in, we get

$$\langle \{k_i'\} a^{\text{rad}}(x)^{\text{out}} | T(\psi, \overline{\psi}) | \{k_j\}^{\text{in}} \rangle = C^* \Big\langle 0 \Big| \prod_i a_s^{\text{out}}(k_i') T\left(\psi, \overline{\psi} \exp\left(-i \int J \cdot a^{\text{rad}(+)} \right) d^4x \right) \prod_j a_{-s}^{\text{in}\dagger}(k_j) \Big| 0 \Big\rangle, \tag{5.35}$$

where the shifted creation operators for in-states are

$$a_{-s}^{\text{in}\dagger}(k) \equiv a^{\text{in}\dagger}(k) - (2\pi)^{5/2} i \tilde{j}(-k). \tag{5.36}$$

In Eq. (5.35) the exponential function of the annihilation operator has been reduced out and there remains only to reduce out the photon operators. Using Eqs. (5.28a), (5.36), and (5.11), one easily verifies the identities

$$a_s^{\nu\,\text{out}}(k) T(\psi, \overline{\psi}, J) = (2\pi)^{-3/2} i \int d^4x\, e^{ik\cdot x}\, T([J^\nu(x) - j^\nu(x)] \psi, \overline{\psi}, J) + T(\psi, \overline{\psi}, J) a^{\nu\,\text{in}}(k), \tag{5.37a}$$

$$T(\psi, \overline{\psi}, J)a_{-s}^{\nu\,\mathrm{in}\dagger}(k) = (2\pi)^{-3/2} i \int d^4x\, T([J^\nu(x) - j^\nu(x)]\psi, \overline{\psi}, J)\, e^{-ik\cdot x} + a^{\nu\,\mathrm{out}\dagger}(k) T(\psi, \overline{\psi}, J). \tag{5.37b}$$

Hence we have from Eq. (5.35), dropping disconnected parts in the photon momenta,

$$\langle \{k_i'\} a^{\mathrm{rad}}(x)^{\mathrm{out}} | T(\psi, \overline{\psi}) | \{k_j\}^{\mathrm{in}} \rangle$$

$$= C * \int \prod_i d^4x_i' (2\pi)^{-3/2} i e^{ik_i'\cdot x_i'} \prod_j d^4x_j\, (2\pi)^{-3/2} i e^{-ik_j\cdot x_j}$$

$$\times \left\langle 0 \left| T\left(\cdots [J(x_i') - j(x_i')] \cdots [J(x_j) - j(x_j)] \cdots \exp\left(-i \int J\cdot a^{\mathrm{rad}(+)} d^4x \right) \psi, \overline{\psi} \right) \right| 0 \right\rangle. \tag{5.38}$$

This formula completes the reduction of the coherent and photon states. It is interpreted as follows: (1) The final and initial photons are coupled to the difference of the quantum-mechanical source J and the classical source j. Because $j(x)$ has only low-frequency components, by Eq. (5.12) only the coupling of low-frequency photons $[f(k) \neq 0]$ is weakened. (2) The factor $\exp(-i\int J\cdot a^{\mathrm{rad}(+)} d^4x)$ means that the charged particles propagate in the complex external potential $a^{\mathrm{rad}(+)}(x)$, which describes the reaction of the radiated coherent field upon the charged particles. In fact its real part, $\frac{1}{2}a^{\mathrm{rad}}(x)$, is the vector potential of an electromagnetic field $\frac{1}{2} f_{\mu\nu}^{\mathrm{rad}}(x) = \frac{1}{2}[f_{\mu\nu}^{\mathrm{out}} - f_{\mu\nu}^{\mathrm{in}}]$, which when evaluated in the classical limit at the position of the particle gives the classical radiation-reaction force,[11]

$$e\tfrac{1}{2} f_{\mu\nu}^{\mathrm{rad}}[x(\tau)]u(\tau) = \frac{2}{3}\frac{e^2}{c^3}(\dddot{u} - \dot{u}^2 u). \tag{5.39}$$

The reduction formula (5.38) may be expanded order by order in e using

$$C * \left\langle 0 \left| T\left(\psi, \overline{\psi}, J,\ \exp\left(-i \int J\cdot a^{\mathrm{rad}(+)} d^4x \right) \right) \right| 0 \right\rangle = C * \frac{\langle 0 | T(\psi, \overline{\psi}_I, J_I \exp[-i\int J_I\cdot (A_I + a^{\mathrm{rad}(+)}) d^4x]) | 0 \rangle}{\langle 0 | T \exp[-i\int J_I \cdot A_I\, d^4x] | 0 \rangle}, \tag{5.40}$$

where all fields are the free interaction-representation fields. The resulting expression is infrared-finite order by order in e. Alternatively, the Green's functions (5.40) may be characterized nonperturbatively, by the familiar equation for the Green's functions of electrodynamics in an external potential $a^{\mathrm{rad}+}(x)$. When Eq. (5.38) is combined with Eq. (4.35), the resulting expression for the scattering amplitude is free of both Coulombic and infrared divergences.

A typical scattering situation is one in which no photons with frequency less than a minimum frequency ω_0 are observed. In this case $f(k)$ should satisfy

$$f(k) = 0, \quad k^0 > \omega_0 \tag{5.41}$$

and cross sections should include a summation over all discrete final-state photons with frequency $\omega < \omega_0$. In any finite order of perturbation theory, there will only be at most a finite number of them.

VI. APPLICATION TO COULOMB SCATTERING

To illustrate the use of the reduction formula for charged particles, we apply it to the simplest case, namely scattering of a Dirac particle by an external Coulomb field[12]

$$a_\mu^c(x) = \frac{e'}{4\pi}\delta_{\mu 0}\frac{1}{r}. \tag{6.1}$$

Heretofore this problem has been treated by the method of Dalitz,[6] who replaces the pure Coulomb potential by the shielded potential

$$a_\mu^c(x, \lambda) = \frac{e'}{4\pi}\delta_{\mu 0}\frac{e^{-\lambda r}}{r}, \tag{6.2}$$

which corresponds to giving the photon a small mass λ. In this case each term in the Born series is convergent for finite λ. As λ approaches zero, a divergence develops which has been verified in low order to contribute only to the phase of the scattering amplitude, leaving cross sections finite.

Consider the amplitude $\langle p_f^{\mathrm{out}} | p_i^{\mathrm{in}} \rangle$ for the scattering of a Dirac particle by the potential (6.1). According to Eq. (4.35) it is given by

$$\langle p_f^{\text{out}} | p_i^{\text{in}} \rangle = [m(2\pi)^{-3/2}]^2 \lim_{\eta_f \to 0} \lim_{\eta_i \to 0} \left\{ \bar{u}(p_f) \frac{(\epsilon - i\eta_f)^{1-i\gamma_f}}{\Gamma(1-i\gamma_f)} T[p_f(1+\eta_f), p_i(1+\eta_i)] \frac{(\epsilon - i\eta_i)^{1-i\gamma_i}}{\Gamma(1-i\gamma_i)} u(p_i) \right\}, \tag{6.3}$$

where

$$\gamma_i = \frac{ee'}{4\pi} \frac{E_i}{|\vec{p}_i|}, \quad \gamma_f = \frac{ee'}{4\pi} \frac{E_f}{|\vec{p}_f|}, \tag{6.4}$$

$$T(p_f, p_i) = \int d^4x\, d^4y\, e^{i(p_f \cdot x - p_i \cdot y)} \langle 0 | T[\psi(x)\bar{\psi}(y)] | 0 \rangle . \tag{6.5}$$

Here $S_c(x, y) \equiv \langle 0 | T(\psi(x)\bar{\psi}(y)) | 0 \rangle$ is the Dirac propagator in the external Coulomb field and satisfies the equation

$$[i\not{\partial}_x - e\not{a}^c(x) - m] S_c(x, y) = i\delta^4(x-y)$$

with Feynman boundary conditions. We will evaluate (6.3) to second order in the potential, and thus write

$$S_c(x, y) = S(x-y) + \int S(x-z)(-ie)\not{a}^c(z) S(z-y) d^4z + \int S(x-z)(-ie)\not{a}^c(z) S(z-u)(-ie)\not{a}^c(u) S(u-y) d^4z d^4u, \tag{6.6}$$

where $S(x-y)$ is the free propagator,

$$S(x) = \frac{i}{(2\pi)^4} \int \frac{e^{-ip\cdot x}}{\not{p} - m + i\epsilon} d^4x .$$

When

$$\int e^{i(p_2 - p_1)\cdot z} (-ie)\not{a}^c(z) d^4z = -2\pi i \frac{ee'\delta(p_2^0 - p_1^0)\gamma^0}{|\vec{p}_2 - \vec{p}_1|^2} \tag{6.7}$$

is inserted into Eqs. (6.5) and (6.6), one finds

$$T(p_f, p_i) = (2\pi)^4 i \left\{ \frac{\delta^4(p_f - p_i)}{\not{p}_i - m} + \frac{i}{(2\pi)^4} \frac{1}{\not{p}_f - m} \frac{(-2\pi i) ee'\delta(p_2^0 - p_1^0)\gamma^0}{(\vec{p}_f - \vec{p}_i)^2} \frac{1}{\not{p}_i - m} \right.$$
$$\left. + \left[\frac{i}{(2\pi)^4}\right]^2 \frac{1}{\not{p}_f - m} \int \frac{(-2\pi i) ee'\delta(p_f^0 - q^0)\gamma^0}{(\vec{p}_f - \vec{q})^2} \frac{1}{\not{q} - m} \frac{(-2\pi i) ee'\delta(q^0 - p_i^0)\gamma^0}{(\vec{q} - \vec{p}_i)^2} d^4q \right\} \frac{1}{\not{p}_i - m} .$$

We drop the first term, since it will be annihilated in Eq. (6.3), and find

$$T(P_f, P_i) = (2\pi)^4 i\delta(P_f^0 - P_i^0) \frac{1}{\not{P}_f - m} \left\{ \frac{ee'\gamma^0}{(2\pi)^3} \frac{1}{|\vec{P}_f - \vec{P}_i|^2} + \left[\frac{ee'}{(2\pi)^3}\right]^2 \int \frac{\gamma^0}{|\vec{P}_f - \vec{q}|^2} \frac{\delta(q^0 - P_i^0)}{\not{q} - m + i\epsilon} \frac{\gamma^0}{|\vec{q} - \vec{P}_i|^2} d^4q \right\} \frac{1}{\not{P}_i - m} . \tag{6.8}$$

When this is substituted into Eq. (6.3) with $P_f = p_f(1+\eta_f)$ and $P_i = p_i(1+\eta_i)$, the integral over q diverges as $\eta_i \to 0$ or $\eta_f \to 0$, but in such a way that the product which appears in Eq. (6.3) should have a finite limit to the order considered. The calculation is simplified by setting $\eta_f = \eta_i = \eta$. In this case, because of the energy-conserving δ function in T, we have

$$|\vec{p}_f| = |\vec{p}_i| = p, \quad E_f = E_i = (\vec{p}^2 + m^2)^{1/2}, \tag{6.9}$$

$$\gamma_f = \gamma_i = \gamma = \frac{ee'}{4\pi} \frac{E}{|\vec{p}|}, \tag{6.10}$$

and Eq. (6.3) becomes

$$\langle p_f^{\text{out}} | p_i^{\text{in}} \rangle = \frac{-(2\pi)}{\Gamma^2(1-i\gamma)} i\delta(E_f - E_i) \lim_{\eta \to 0} (\epsilon - i\eta)^{-2i\gamma} \bar{u}(p_f)$$
$$\times \left\{ \frac{ee'}{(2\pi)^3} \frac{\gamma^0}{|\vec{p}_f - \vec{p}_i|} + \left[\frac{ee'}{(2\pi)^3}\right]^2 \int \frac{\gamma^0}{|\vec{p}_f(1+\eta) - \vec{q}|^2} \frac{\delta(q^0 - p_i^0(1+\eta))}{\not{q} - m + i\epsilon} \frac{\gamma^0}{|\vec{q} - \vec{p}_i(1+\eta)|^2} d^4q \right\} u(p_i) . \tag{6.11}$$

On making the change of variable $q \to q(1+\eta)$, one finds

$$\langle p_f^{\text{out}}|p_i^{\text{in}}\rangle = \frac{-2\pi i\delta(E_f-E_i)}{\Gamma^2(1-i\gamma)} \lim_{\eta\to 0}(\epsilon - i\eta)^{-2i\gamma}\bar{u}(p_f)$$
$$\times\left\{\frac{ee'}{(2\pi)^3}\frac{\gamma^0}{|\vec{p}_f-\vec{p}|^2}+\left[\frac{ee'}{(2\pi)^3}\right]^2\int\frac{\gamma^0}{|\vec{p}_f-\vec{q}|}\frac{E\gamma^0-\vec{\gamma}\cdot\vec{q}+m}{\vec{p}^2-\vec{q}^2+2m^2\eta+i\epsilon}\frac{\gamma^0}{|\vec{q}-\vec{p}_i|^2}d^3q\right\}u(p_i). \tag{6.12}$$

As always, ϵ approaches zero before η does, and we have systematically neglected η compared to 1. The Feynman denominator is obtained from

$$\lim_{\eta\to 0} q^2(1+\eta)^2 - m^2 + i\epsilon \approx (1+\eta)^2[E^2 - \vec{q}^2 - m^2(1-2\eta)+i\epsilon] \approx \vec{p}^2-\vec{q}^2+2m^2\eta+i\epsilon\,.$$

We see that η, which measures distance from the mass shell, appears only in the coefficient and in the Feynman denominator. As long as it is different from zero, the vanishing of $\vec{p}^2-\vec{q}^2+2m^2\eta$ does not coincide with the vanishing of $(\vec{p}_f-\vec{q})^2$ or $(\vec{q}-\vec{p}_i)^2$, and the integral is finite. The integral over $\vec{q}$ may be effected, with the result

$$\lim_{\eta\to 0}\int\frac{\gamma^0}{|\vec{p}_f-\vec{q}|^2}\frac{E\gamma^0-\vec{\gamma}\cdot\vec{q}+m}{\vec{p}^2-\vec{q}^2+2m^2\eta+i\epsilon}\frac{\gamma^0}{|\vec{q}-\vec{p}_i|^2}d^3\vec{q}$$
$$=\frac{i\pi^2}{p|\vec{p}_f-\vec{p}_i|^2}(E\gamma^0+m)\left\{2\ln\left[\frac{m^2}{2p^2}(\eta+i\epsilon)\right]+\left(\ln\frac{4p^2}{|\vec{p}_f-\vec{p}_i|^2}-i\pi\right)\right\}$$
$$+\tfrac{1}{2}\vec{\gamma}\cdot(\vec{p}_f+\vec{p}_i)\left\{2\ln\left[\frac{m^2}{2p^2}(\eta+i\epsilon)\right]+\frac{p^2}{p^2-\frac{1}{4}|\vec{p}_f-\vec{p}_i|^2}\left(\ln\frac{4p^2}{|\vec{p}_f-\vec{p}_i|^2}-i\pi\frac{p-\frac{1}{2}|\vec{p}_f-\vec{p}_i|}{p}\right)\right\}, \tag{6.13}$$

where $p=|\vec{p}_f|=|\vec{p}_i|$.

The coefficient in Eq. (6.12) may be rewritten

$$\frac{(\epsilon-i\eta)^{-2i\gamma}}{\Gamma^2(1-i\gamma)}=\frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)}\frac{\sinh\pi\gamma}{\pi\gamma}e^{-\gamma[2i\ln(\eta+i\epsilon)+\pi]}, \tag{6.14}$$

which, to the order required, we express as

$$\frac{(\epsilon-i\eta)^{-2i\gamma}}{\Gamma^2(1-i\gamma)}=\frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)}\left\{1-\frac{ee'}{4\pi}\frac{E}{p}[2i\ln(\eta+i\epsilon)+\pi]\right\}. \tag{6.15}$$

When Eqs. (6.13) and (6.15) are substituted into (6.12), one obtains

$$\langle p_f^{\text{out}}|p_i^{\text{in}}\rangle = -(\pi)^{-1}i\delta(E_f-E_i)\frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)}$$
$$\times\bar{u}(p_f)\left[\left(\frac{ee'}{4\pi}\right)\frac{\gamma^0}{4p^2\sin^2(\frac{1}{2}\theta)}\left(\frac{ee'}{4\pi}\right)^2\frac{\gamma^0}{4p^2\sin^2(\frac{1}{2}\theta)}\frac{E}{p}i\ln\frac{4p^4\sin^2(\frac{1}{2}\theta)}{m^4}\right.$$
$$\left.-\left(\frac{ee'}{4\pi}\right)^2\frac{\gamma^0E-m}{2p}\left(\frac{\pi}{4p^2\sin\frac{1}{2}\theta}\frac{1}{1+\sin\frac{1}{2}\theta}+\frac{i\ln\sin^2(\frac{1}{2}\theta)}{4p^2\cos^2(\frac{1}{2}\theta)}\right)\right]u(p_i). \tag{6.16}$$

This is the desired finite amplitude for scattering of a Dirac particle by a Coulomb potential. In the nonrelativistic limit it agrees with Eq. (1.1) to second order, apart from a θ-independent phase. After substituting $ee'(4\pi)^{-1}\to -Ze^2$, it agrees with Dalitz expression,[6] apart from a θ-independent infinite phase which is present in the latter work (and a change of sign in the second term). This agreement is remarkable because the coefficient (6.15) is not a pure phase, but it is nevertheless replaced by unity in the Dalitz method. A compensation occurs, because the regularization of the integral (6.13) by the η in the electron propagator is not equivalent to regularization in the manner of Dalitz in which a small photon mass is introduced, which miraculously makes up for the different coefficient (6.15). Since different regularizations of the same Feynman integral yield different values, some doubt may be cast on the Dalitz method, which is essentially an *ad hoc* regularization. However, it is based on the physical idea of a screened Coulomb potential, and it would therefore be interesting to know whether the Dalitz method and the present one agree to all orders.

As a final remark we wish to place the present work within the larger context of the structure of quantum electrodynamics. We have tacitly assumed that the Green's functions of quantum electrodynamics exist as 4-dimensional distributions in each variable, as discussed by Blanchard and Seneor.[13] The infrared and Coulombic divergences arise only when the 4-momenta of the charged particles approach their mass shell. We have been concerned here with the strength of the singularity of the Green's functions at the mass shell, and more particularly with the prescription for removing the singular factors to obtain the finite S-matrix elements.

ACKNOWLEDGMENTS

It is a pleasure to thank Dr. Gianfausto Dell'Antonio for stimulating conversations, and Mr. Nikolas Papanikolaou for many enjoyable sessions of detailed discussion.

APPENDIX

A typical contribution to the second term of Eq. (4.12) is the integral

$$\begin{aligned}&\lim_{x^0\to\infty}(2\pi)^{-3}\int d^3x\exp[iS^{\text{out}}(x,p_1';p_2'\cdots p_n',q_1'\cdots q')]\bar{u}(p_1')\gamma^0u(p_1)\exp[-iS^{\text{in}}(x,p_1;p_2\cdots p_m,q_1\cdots q_n)],\\&\lim_{x^0\to\infty}(2\pi)^{-3}\int d^3x\,\bar{u}(p_1')\gamma^0u(p_1)\exp\{+i[p_1'\cdot x+\gamma'(p_1')\ln|p_1'\cdot x|]\}\exp\{-i[p_1\cdot x-\gamma(p_1)\ln|p_1\cdot x|]\}.\end{aligned}\tag{A1}$$

This is a distribution in $\vec{p}_1$ and $\vec{p}_1'$. To evaluate it we smooth it with well-behaved test functions $\phi(\vec{p}_1)$, $\psi^*(\vec{p}_1')$ and consider the integral

$$\begin{aligned}I=\lim_{x^0\to\infty}(2\pi)^{-3}\int d^3x\int d^3p'\psi^*(\vec{p}\,')\bar{u}(\vec{p}\,')\exp\{i[p'\cdot x+\gamma'(p')\ln|p'\cdot x|]\}\\\times\gamma^0\int d^3p\exp\{-i[p\cdot x-\gamma(p)\ln|p\cdot x|]\}u(\vec{p})\phi(\vec{p}).\end{aligned}\tag{A2}$$

Using the representation (4.15) we rewrite this as

$$\begin{aligned}I&=\lim_{x^0\to\infty}(2\pi)^{-3}\int d^3p'\,d^3p\,\psi^*(\vec{p}\,')\bar{u}(\vec{p}\,')\gamma^0u(\vec{p})\phi(\vec{p})d\lambda\,d\mu\,f(\lambda,\gamma'(\vec{p}\,'))f(\mu,\gamma(\vec{p}))\int d^3x e^{ip'\cdot x(1+\lambda)}e^{-ip\cdot x(1+\mu)}\\&=\lim_{t\to\infty}\int d^3p'\,d^3p\,\psi^*(\vec{p}\,')\bar{u}(\vec{p}\,')\gamma^0u(\vec{p})\phi(\vec{p})d\lambda\,d\mu\,f(\lambda,\gamma'(p'))f(\mu,\gamma(p))\\&\qquad\times\exp\{i[E_{p'}(1+\lambda)-E_p(1+\mu)]t\}\,\delta^3(\vec{p}\,'(1+\lambda)-\vec{p}(1+\mu))\\&=\lim_{t\to\infty}\int\frac{d^3p\,d\lambda\,d\mu}{(1+\lambda)^3}\psi^*\left(\frac{\vec{p}(1+\mu)}{(1+\lambda)}\right)\bar{u}\left(\frac{\vec{p}(1+\mu)}{1+\lambda}\right)\gamma^0u(\vec{p})\phi(\vec{p})f\left(\lambda,\gamma'\left[\frac{\vec{p}(1+\mu)}{(1+\lambda)}\right]\right)f(\mu,\gamma(p))\\&\qquad\times\exp\left\{i\left[\left(\frac{\vec{p}^2(1+\mu)^2}{(1+\lambda)^2}+m^2\right)^{1/2}(1+\lambda)-(\vec{p}^2+m)^{1/2}(1+\mu)\right]t\right\}.\end{aligned}$$

Upon introducing as new variable of integration $\vec{p}\,'=\vec{p}(1+\lambda)^{-1}$, we find, after dropping the prime,

$$\begin{aligned}I=\lim_{t\to\infty}\int d^3p\,d\lambda\,d\mu\,\psi^*(\vec{p}(1+\mu))\bar{u}(\vec{p}(1+\mu))\gamma^0u(\vec{p}(1+\lambda))\phi(\vec{p}(1+\lambda))f(\lambda,\gamma'(\vec{p}(1+\lambda)))f(\mu,\gamma(\vec{p}(1+\mu)))\\\times\exp(i\{[\vec{p}^2(1+\mu)^2+m^2]^{1/2}(1+\lambda)-[\vec{p}^2(1+\lambda)^2+m^2]^{1/2}(1+\mu)\}t).\end{aligned}$$

We now introduce the new variables $\alpha=\lambda t$ and $\beta=\mu t$, and letting t approach infinity within the integral, we obtain

$$I=\lim_{t\to\infty}\int d^3p\,d\alpha\,d\beta\,\psi^*(\vec{p})\bar{u}(\vec{p})\gamma^0u(\vec{p})\phi(\vec{p})t^{-1}f(t^{-1}\alpha,\gamma'(\vec{p}))t^{-1}f(t^{-1}\beta,\gamma(\vec{p}))\exp\left[i\frac{m^2}{(\vec{p}^2+m^2)^{1/2}}(\alpha-\beta)\right].$$

From Eq. (4.16) we have $t^{-1}f(t^{-1}\alpha,\gamma'(p))=t^{i\gamma'}f(\alpha,\gamma'(p))$, and using (4.15) we get finally

$$I=\lim_{t\to\infty}\int d^3p\,\psi^*(\vec{p})\bar{u}(\vec{p})\gamma^0u(\vec{p})\phi(\vec{p})\exp\left\{i[\gamma(p)+\gamma'(p)]\ln\left[\frac{m^2t}{(\vec{p}^2+m^2)^{1/2}}\right]\right\}.\tag{A3}$$

This result is the same as we would obtain if in Eq. (A2) we evaluated $\ln|p\cdot x|$ along the classical trajectory $x=\vec{p}(\vec{p}^2+m^2)^{-1/2}t$, so

$$\ln|p\cdot x| = \ln\left[(\vec{p}^2+m^2)^{1/2}t - \frac{\vec{p}^2}{(\vec{p}^2+m^2)^{1/2}}t\right] = \ln\left[\frac{m^2}{(\vec{p}^2+m^2)^{1/2}}t\right],$$

and similarly for $\ln|p'\cdot x|$.

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*The research reported in this paper was supported in part by funds from the National Science Foundation.

[1]H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento 1, 205 (1955); 6, 319 (1957).

[2]F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937).

[3]D. Yennie, S. Frautschi, and H. Suura, Ann. Phys. (N.Y.) 13, 379 (1961).

[4]V. Chung, Phys. Rev. 140, B1110 (1965).

[5]T. Kibble, Phys. Rev. 173, 1527 (1968); 174, 1882 (1968); 175, 1624 (1968); J. Math. Phys. 9, 315 (1968).

[6]R. Dalitz, Proc. Roy. Soc. (London) A206, 509 (1951); C. Kacser, Nuovo Cimento 13, 303 (1959).

[7]J. D. Dollard, J. Math. Phys. 5, 729 (1964); D. Muhlerin and I. Zinnes, *ibid.* 11, 1402 (1970).

[8]P. Kulish and L. Faddeev, Teoret. i. Mat. Fiz. 4, 153 (1970) [Theoret. and Math. Phys. 4, 745 (1970)].

[9]D. Zwanziger, Phys. Rev. D 6, 458 (1972).

[10]The metric is $g^{\mu\nu}=\text{diag}(1,-1,-1,-1)$ and $\hbar=c=1$. Our convention for Dirac matrices is $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}$, and $\not{a}\equiv a\gamma^\mu$, with $\bar{\psi}=\psi^\dagger\gamma^0$.

[11]P. A. M. Dirac, Proc. Roy. Soc. (London) A167, 148 (1938).

[12]The calculation reported here was effected in collaboration with Nikolas Papanikolaou.

[13]P. Blanchard and R. Seneor, CERN Report No. TH 1420 (unpublished).

PHYSICAL REVIEW D VOLUME 7, NUMBER 4 15 FEBRUARY 1973

Approximate Solutions of Predictive Relativistic Mechanics for the Electromagnetic Interaction

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(Received 2 October 1972)

We solve the equations of predictive relativistic mechanics for the electromagnetic interaction of two structureless point charges, up to second order in the coupling constant $g=e_1e_2$, using as a subsidiary condition the Liénard-Wiechert formulas, for both the advanced and the retarded potentials, separately or in the time-reversal-invariant combination. Our general results reduce in the case of one-dimensional rectilinear motion to those obtained previously by Hill, which, as shown recently by Andersen and von Baeyer, are reliable in the low energy regime. In the time-reversal-invariant combination, if $g<0$, concentric circular motion is possible; and assuming that both charges have equal masses we compare the speed-vs-radius relation obtained in this theory to that obtained in the Breit-Darwin approximation and in Wheeler-Feynman electrodynamics.

I. INTRODUCTION

The equations of predictive relativistic mechanics (P.R.M.) can be written in a time-symmetric formalism,[1-4] or in a manifestly covariant one.[5,4] Up to now no physically meaningful exact solution of the equations of P.R.M. has been obtained in either formalism. In this paper we develop a perturbation technique which permits the recurrent calculation of the four-accelerations of the manifestly covariant formalism in the case of two pointlike structureless particles, by assuming that these functions can be expanded into power series of a coupling constant. From them we obtain very easily the corresponding three-accelerations of the time-symmetric formalism.

We had to solve at each order a very simple linear partial differential equation for each unknown function, whose solutions, after a suitable change of variables, can be obtained by quadratures. This leaves the problem still undetermined because a choice of a subsidiary condition is still possible.

We have considered the interaction of two point charges and used as subsidiary conditions the Liénard-Wiechert formulas for both the advanced and retarded potentials, separately or in the time-reversal-invariant combination.