formulated the text in terms of the less customary point transformation, in order to facilitate the comparison with paper I (Ref. 3).

[3]G. Carmi, preceding paper, Phys. Rev. D 7, 1038 (1973), which is henceforth referred to as "paper I." Its equations will be quoted, as e.g., (I2.3).

[4]Professor Donald Newman, private communication.

[5]This is hindsight, of course. At the time paper I was written, the solution had to be found by trial and error.

[6]Thus the equilibrium-thermodynamics (i.e., $\sum_n \langle n|e^{-\rho H}|n\rangle$) cannot be evaluated exactly. However, if the system is first transformed by the two Bohm-Pines transformations [D. Bohm and D. Pines, Phys. Rev. 92, 609 (1953)] and the random-phase approximation (RPA) (and, possibly, also [G. Carmi and A. J. Lock, Phys. Rev. A 5, 1447 (1972)] the Bogoliubov approximation) is used, the residual interaction is small and the transformation (3.1) will (if it is chosen in such a way as to fulfill the other criteria of Ref. 1) describe also the thermodynamics of the system quite well (Ref. 1).

[7]This definition is obviously not the widest possible generalization of the corresponding definition in Sec. II, but it serves our purposes here.

[8]Again, this is not the most general system for which Lemma 1_n could be proved. More generally, one could write $f_i(x_1, \ldots, x_{i-1}, x_i+1, x_{i+1}, \ldots, x_n)$ on the left-hand side of (3.12).

[9]The main restriction is that the corresponding Lie element $pf + fp + g$ has a nonvanishing domain in L_2, as operator with range in L_2, and that this remains so for the Lie products of such elements.

[10]Robert Hermann, *Lie Groups and Physics* (Benjamin, New York, 1966), p. 139.

[11]L. Van Hove, Acad. Roy. Belg. Cl. Sci. Mém. (Series 8) 26, No. 6 (1951); Bull. Cl. Sci. Acad. Roy. Belg. (Series A) 37, 610 (1951).

[12]$(pf_1 + f_1p + g, \; pf_2 + f_2p + g_2) = pf_3 + f_3p + g_3$, with $f_3 = 2(f_1f_2' - f_1'f_2)$, etc.

[13]These results do not seem to be recorded in the literature but, judging from their simple nature, they must have occurred to almost everybody who has encountered this algebra.

[14]Within the underlying associative algebra (with respect to ordinary operator multiplication) the algebra can be spanned by a basis of *two* elements only, e.g., $A_{-2} = pe^{-2ix} + e^{-2ix}p$ and $B_1 = e^{ix}$.

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Spontaneously Broken Gauge Symmetries. IV. General Gauge Formulation

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The advent of the dimensional-regularization procedure allows the study of renormalizability of spontaneously broken gauge theories formulated in a wide class of gauges. We derive and study the Ward-Takahashi identities appropriate to such gauges. A consequence of the Ward-Takahashi identities is that the physical S matrix is invariant under a variation of the gauge condition. As remarked before, since the variation of a parameter in the R_ξ-gauge formulation shifts the masses of unphysical excitations, the above result, the ξ independence of the physical S matrix, implies that the unphysical excitations cannot contribute to the sum over intermediate states, establishing the unitarity of the S matrix. We also give the renormalization procedure of a model formulated in the R_ξ gauge.

I. INTRODUCTION

The advent of a very powerful regularization procedure for Feynman integrals – the so-called dimensional regularization[1] – permits us to discuss intelligently the renormalizability question of spontaneously broken gauge theories formulated in a fairly general class of gauges.[2] The present paper is dedicated to the derivation of the Ward-Takahashi (WT) identities in such gauges, which can be used to prove the renormalizability and unitarity of the theory in question. It has been observed[3] that in the so-called R_ξ-gauge formulation invariance of the physical S matrix under the variation of a gauge-specifying parameter (i.e., ξ) implies the unitarity of the S matrix, that is to say, that unphysical excitations do not contribute to sums over intermediate states. Thus, the ability to formulate quantum theory of spontaneously broken gauge symmetry in a general class of gauge conditions, in a way that reflects the gauge invariance of the action as expressed through the WT identities, is extremely useful in showing that the theory is in fact unitary and renormalizable. [Let us recall that the proof (in paper II) of the unitarity in the R gauge was extremely complicated.]

In our discussion we shall assume that all expressions are dimensionally regulated, so that formal manipulations of the kinds necessary in showing the WT identities for Feynman amplitudes are justified.[1] In the next section, we consider a general class of gauge conditions and derive the WT identities appropriate to the gauges being considered. An important lemma we use in the derivation, which is a generalization of the results of Fradkin and Tyutin, and Slavnov[4] for the Landau gauge formulation, is proved in the Appendix. In Sec. III we give a demonstration that the physical S matrix is invariant under an infinitesimal change in the gauge condition. This, together with the remark in Ref. 3, establishes the unitary of the physical S matrix. In Sec. IV we discuss renormalization conditions of a model formulated in the R_ξ gauge. Here we illustrate how renormalization constants may be chosen in accordance with the WT identities. Finally the Lagrangian of the model is written down in terms of renormalized fields and parameters.

II. WT IDENTITIES

We shall discuss a theory consisting of Bose fields. Recently Bardeen[5] has given a discussion of renormalizability of gauge theories with fermions. He has shown that the problem associated with the anomalies[6-8] in fermion loops can be isolated from the general problem of renormalizability and gauge invariance, and if fermion loop anomalies are absent, or canceled among themselves *in lowest order,* the presence of fermions does not hinder the WT identities from being valid.

Let ϕ_i be the set of all fields including the gauge fields transforming as a linear (in general reducible) representation of a compact Lie group. The infinitesimal transformation is given by

$$\phi_i^g = \phi_i + (\Gamma_{ij}^\alpha \phi_j + \Lambda_i^\alpha) g_\alpha , \tag{1}$$

where our notation is such that the indices i and α stand for the space-time variables as well as the internal symmetry indices, and summation *and integration* over repeated indices will always be understood; g_α is the space-time-dependent parameter of the Lie group and Γ_{ij}^α is a reducible representation of the generator labeled by α, and Λ_i^α may involve space-time derivatives. The Lagrangian $\mathcal{L}$ is invariant under the transformation (1).

We choose as the gauge condition

$$F_\alpha(\phi) = a_\alpha , \tag{2}$$

and consider the integral

$$\Delta_F^{-1}(\phi) = \int [dg] \prod_\alpha \delta(F_\alpha(\phi^g) - a_\alpha) , \tag{3}$$

where $[dg] = \prod_x dg$ is the product of Hurwitz integrals at every space-time point and a_α is independent of ϕ_i and g. We need only compute, for our purpose, Δ_F with the restriction to the manifold $F_\alpha(\phi) = a_\alpha$. It is given by

$$\Delta_F(\phi) = \det M_{\alpha\beta} \tag{4}$$

for ϕ such that $F_\alpha(\phi) = a_\alpha$, where

$$M_{\alpha\beta} = \frac{\delta F_\alpha}{\delta \phi_i} (\Gamma_{ij}^\beta \phi_j + \Lambda_i^\beta) . \tag{5}$$

Popov and Faddeev[9] have shown that the vacuum-to-vacuum amplitude for a gauge theory should be written as

$$W(a_\alpha) = \int [d\phi] e^{iS[\phi]} \det M \prod \delta(F_\alpha(\phi) - a_\alpha) , \tag{6}$$

where $S[\phi]$ is the gauge-invariant action, $S[\phi] = \int d^4x \mathcal{L}(\phi)$. Consider the transformation (1) with g_α however restricted by the condition

$$g_\alpha = [M^{-1}(\phi)]_{\alpha\beta} \lambda_\beta , \tag{7}$$

where λ_α is an arbitrary infinitesimal number independent of ϕ. We show in the Appendix that $(\det M)[d\phi]$ is a measure invariant under the transformation (1) with the restriction (7). From Eq. (6), it follows then that W is invariant under an infinitesimal change of a_α, $\delta a_\alpha = \lambda_\alpha$,

$$\frac{dW(a_\alpha)}{da_\alpha} = 0 , \quad \text{for all } a_\alpha , \tag{8}$$

so W is independent of a_α. One can therefore integrate over a_α the right-hand side of Eq. (6) without changing the result,[10] up to some irrelevant normalization:

$$\begin{aligned} W &= \int [da] \prod_\alpha H(a_\alpha) \int [d\phi]\, e^{iS[\phi]} \\ &\qquad \times \det M \prod_\alpha \delta(F_\alpha(\phi) - a_\alpha) \\ &= \int [d\phi]\, e^{iS[\phi]} \det M \prod_\alpha H(F_\alpha(\phi)) . \end{aligned} \tag{9}$$

We shall specialize $H(a_\alpha)$ to a Gaussian function in the following:

$$\prod_\beta H(a_\beta) = \exp(-\tfrac{1}{2} i a_\beta^2) . \tag{10}$$

Equation (9) can be written as[10]

$$W = \int [d\phi] \exp(iS[\phi] - \tfrac{1}{2} i F_\alpha^2) \det M . \tag{11}$$

Furthermore $\det M$ can be described as loops generated by a set of complex scalar fields c_α and $\bar{c}_\alpha$ obeying Fermi-Dirac statistics.[9,11] Equation (11) can thus be written as[9,10]

$$W=\int[d\phi][dc][d\bar{c}]\exp(iS_{\rm eff}[\phi,c,\bar{c}])\,, \tag{12}$$

where

$$S_{\rm eff}[\phi,c,\bar{c}]=S[\phi]-\tfrac{1}{2}F_\alpha{}^2+\bar{c}_\alpha M_{\alpha\beta}c_\beta\,. \tag{13}$$

So far we have been silent about permissible choices of the gauge condition $F_\alpha=a_\alpha$. The gauge condition must be so chosen that the operator $M_{\alpha\beta}$ is nonsingular, i.e.,

$$\det M\neq 0\,, \tag{14}$$

and in order that the Green's functions of the theory be renormalizable the effective action, Eq. (13), must not contain terms of dimension higher than four. This requires in particular that the dimension of the function F not exceed two.

We will now introduce source terms J_i for the fields ϕ_i in order to generate Green's functions:

$$W[J]=\int[d\phi]\det M\exp i(S[\phi]-\tfrac{1}{2}F_\alpha{}^2+\phi_i J_i)\,. \tag{15}$$

Performing the nonlinear gauge transformations previously defined by Eq. (7) and remembering that a change of variables does not change the value of an integral, we obtain the identity

$$0=\int[d\phi]\det M\exp\{i(S[\phi]-\tfrac{1}{2}F_\alpha{}^2+J_i\phi_i)\}\frac{\delta}{\delta\lambda_\beta}(S[\phi]-\tfrac{1}{2}F_\alpha{}^2+J_i\phi_i)\,,$$

or

$$\int[d\phi]\det M\exp\{i(S[\phi]-\tfrac{1}{2}F_\gamma{}^2+J_i\phi_i)\}[-F_\alpha+J_i(\Gamma^\beta_{ij}\phi_j+\Lambda^\beta_i)(M^{-1})_{\beta\alpha}]=0\,. \tag{16}$$

Equation (16) can be translated into an equation for $W[J]$:

$$\left\{-F_\alpha\left(\frac{1}{i}\frac{\delta}{\delta J}\right)+J_i\left(\Gamma^\beta_{ij}\frac{1}{i}\frac{\delta}{\delta J}+\Lambda^\beta_i\right)\left[M^{-1}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right]_{\beta\alpha}\right\}W[J]=0\,. \tag{17}$$

This is the WT identity for generating functional $W[J]$ in the gauge defined by $F_\alpha(\phi)=a_\alpha$.

In the special case where F_α is a linear function of fields ϕ_i,

$$F_\alpha(\phi)=F_{\alpha i}\phi_i\,, \tag{18}$$

a set of solutions of Eq. (17) can be obtained in the following way. Let us compute $W[J]$ with $J_i=K_\alpha F_{\alpha i}$,

$$W[K_\alpha F_{\alpha i}]=\int[d\phi]\det M\times\exp(i\{S[\phi]-\tfrac{1}{2}(F_\alpha-K_\alpha)^2+\tfrac{1}{2}K_\alpha{}^2\})\,. \tag{19}$$

Using the fact that one can add to F_α an arbitrary function of space-time by a succession of gauge transformations which satisfy Eq. (7) and leave the metric $\det M[d\phi]$ and the action $S[\phi]$ invariant, we can perform the integral and obtain

$$W[K_\alpha F_{\alpha i}]=W[0]\,e^{iK_\alpha{}^2/2}$$

or

$$Z[K_\alpha F_{\alpha i}]\equiv -i\ln W[K_\alpha F_{\alpha i}]=\tfrac{1}{2}K_\alpha{}^2+\text{const}\,. \tag{20}$$

Equation (17) requires knowing the quantity

$$\left[M^{-1}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right]_{\beta\alpha}W[J]\,.$$

To this end we consider

$$W_{\alpha\beta}[J]=\int[d\phi][dc][d\bar{c}]c_\alpha\bar{c}_\beta\times\exp\{i(S[\phi]-\tfrac{1}{2}F_\alpha{}^2+\bar{c}_\alpha M_{\alpha\beta}c_\beta+J_i\phi_i)\}\,. \tag{21}$$

The functional $W_{\alpha\beta}[J]$, the Green's function for c in the presence of external sources, satisfies the equation

$$M_{\alpha\beta}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)W_{\beta\gamma}[J]=\delta_{\alpha\gamma}W[J]$$

or

$$\left[M^{-1}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right]_{\alpha\beta}W[J]=W_{\beta\gamma}[J]\,. \tag{22}$$

We believe that the discussion here of the WT identities for a general class of gauges parallels the combinatoric discussion of 't Hooft and Veltman[12] on the same subject.

III. SOME CONSEQUENCES OF WT IDENTITIES

The WT identities implied by Eq. (17) are satisfied by the dimensionally regulated Green's functions of the theory for general n, the number of space-time dimensions.[1] If the effective action $S_{\rm eff}$ in Eq. (12) is that of a renormalizable theory, then the singularities of the Green's functions at

$n=4$ are removed by the renormalization of fields and parameters of the Lagrangian and $F_\alpha(\phi)$. The renormalized Green's functions then satisfy the renormalized WT identities obtained by rescaling parameters and sources J_i in Eq. (17), as we have shown for the R-gauge formulation in paper II.

In this section we shall demonstrate that, for general n, a small variation in the unrenormalized expression for F_α leaves the physical S matrix invariant. It follows from the discussion in the last paragraph that when the theory is renormalized as we discussed, the physical S matrix is invariant under a variation of renormalized parameters appearing in the gauge condition.

As alluded to in the Introduction, a variation of the parameter ξ in the R_ξ-gauge formulation[3] causes the masses of unphysical particles to vary; in fact, in the limit $\xi\to 0$, these become infinite. Thus it is seen that the ξ-independence of the S matrix implies the unitarity of the S matrix; that is to say, the unphysical particles decouple from physical ones on mass shell.

We will give a small variation ΔF_α to F_α and compute the variation ΔW of $W[J]$:

$$\Delta W[J]=\int[d\phi]\det M\exp\{i(S[\phi]-\tfrac{1}{2}F_\alpha{}^2+J_i\phi_i)\}\left(-iF_\alpha\Delta F_\alpha+\frac{\delta\Delta F_\alpha}{\delta\phi_i}(\Gamma^\beta_{ij}\phi_j+\Lambda^\beta_i)(M^{-1})_{\beta\alpha}\right). \tag{23}$$

We will now use Eq. (16). Applying $i\Delta F_\alpha(\delta/i\delta J)$ on it, we obtain

$$0=\int\left[-iF_\alpha\Delta F_\alpha+\left(J_i\Delta F_\alpha+\frac{\delta\Delta F_\alpha}{\delta\phi_i}\right)(\Gamma^\beta_{ij}\phi_j+\Lambda^\beta_i)(M^{-1})_{\beta\alpha}\right]\exp\{i(S[\phi]-\tfrac{1}{2}F_\alpha{}^2+J_i\phi_i)\}\det M[d\phi]. \tag{24}$$

Combining Eqs. (23) and (24), we finally obtain

$$\Delta W[J]=\int[d\phi]\exp\{i(S[\phi]-\tfrac{1}{2}F_\alpha{}^2+J_i\phi_i)\}\det M J_i\Delta F_\alpha(\phi)(\Gamma^\beta_{ij}\phi_j+\Lambda^\beta_i)(M^{-1})_{\beta\alpha}. \tag{25}$$

The S-matrix element, not necessarily connected, is obtained from W by differentiating with respect to J's around $J=0$, truncating external lines, and setting every external momentum on its mass shell. Let S_F be the quantity so obtained in the gauge specified by F_α. Then the structure of ΔW in Eq. (25) implies

$$S_F+\Delta S_F=\left[1+\tfrac{1}{2}\sum_e\left(\frac{\delta Z_F}{Z_F}\right)_e\right]S_F$$

or (26)

$$S_{F+\Delta F}=S_F\prod_e\left(\frac{Z_{F+\Delta F}{}^{1/2}}{Z_F{}^{1/2}}\right)_e,$$

where Z_F is the wave-function renormalization constant in the gauge specified by F_α, and the summation $\sum_e$ and the product $\prod_e$ run over all external lines e. The change in the renormalization constant, δZ_F, is obtained from the change in the propagator:

$$i\frac{\delta\Delta W[J]}{\delta J_i\delta J_k}\bigg|_{J=0}=-\int[d\phi]\det M\exp\{i(S[\phi]-\tfrac{1}{2}F_\alpha{}^2)\}[\phi_i\Delta F_\alpha(\Gamma^\beta_{kj}\phi_j+\Lambda^\beta_k)(M^{-1})_{\beta\alpha}+(i\leftrightarrow k)]. \tag{27}$$

The quantity $(\delta Z_F)_e$ is the residue of the pole of the Fourier transform of the above expression when i and k take the internal quantum numbers of e. Equation (26) shows that the renormalized, physical S-matrix element,

$$S=S_F\prod_e(Z_F{}^{-1/2})_e, \tag{28}$$

is independent of the gauge chosen, F_α, to compute it.

IV. RENORMALIZATION

In this section we apply the considerations of previous sections to a model, partly to illustrate the general arguments presented abstractly there, and partly to illustrate the renormalization procedure based on the WT identities is the R_ξ gauge.

We choose as our model the system of a quartet of scalar mesons. They form a representation of $[SU(2)]_G\times SU(2)$ where the first factor is a gauge symmetry. The $SU(2)\times SU(2)$ symmetry is spontaneously broken, leaving the diagonal SU(2) as invariance of the vacuum. In this model the triplet of gauge bosons all become massive.[13] In a gauge theory of weak and electromagnetic interactions, one of the gauge bosons remains massless and causes infrared divergences. In this model we avoid the infrared problem completely. In any case the infrared problem in the former is no worse than in quantum electrodynamics when formulated in the R_ξ gauge and can be resolved in the usual way.

The Lagrangian of the model is written as

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g\vec{A}_\mu \times \vec{A}_\nu)^2 + \tfrac{1}{2}[(\partial_\mu \psi)^2 + (\partial_\mu \vec{\chi})^2] + \tfrac{1}{2} g \vec{A}^\mu \cdot (\psi \partial_\mu \vec{\chi} - \chi\, \partial_\mu \psi + \vec{\chi} \times \partial_\mu \vec{\chi})$$
$$+ \tfrac{1}{8} g^2 \vec{A}_\mu{}^2 (\psi^2 + \vec{\chi}^2) - \tfrac{1}{2}\beta(\psi^2 + \vec{\chi}^2) - \tfrac{1}{4}\alpha(\psi^2 + \vec{\chi}^2)^2 . \quad (29)$$

The infinitesimal form of the local $[\mathrm{SU}(2)]_G$ transformations is

$$\begin{aligned} &\delta\phi_i = (\Gamma_{ij}^\alpha \phi_j + \Lambda_i^\alpha)\omega_\alpha ,\\ &\delta\vec{A}_\mu = -\vec{\omega}\times\vec{A}_\mu + \frac{1}{g}\partial_\mu \vec{\omega} ,\\ &\delta\chi = -\vec{\omega}\cdot\vec{\psi} ,\\ &\delta\vec{\chi} = \vec{\omega}\psi - \vec{\omega}\times\vec{\chi} . \end{aligned} \quad (30)$$

We assume that the parameters α and β are so arranged that the field ψ develops a vacuum expectation value,

$$\langle\psi\rangle = v . \quad (31)$$

We choose the gauge condition

$$\begin{aligned} &F_\alpha(\phi) = 0 ,\\ &\sqrt{\xi}\left(\partial_\mu \vec{A}^\mu + \frac{\lambda}{\xi}\vec{\chi}\right) = 0 , \end{aligned} \quad (32)$$

where λ and ξ are parameters to be specified. The operator M is given by

$$\begin{aligned} &M_{\alpha\beta} = \frac{\delta F_\alpha}{\delta\phi_i}(\Gamma_{ij}^\beta \phi_j + \Lambda_i^\beta) ,\\ &\frac{g}{\sqrt{\xi}} M = \partial^2 + g\,\partial_\mu \vec{A}\times + \frac{\lambda g}{2\xi}(\psi + \vec{\chi}\times) . \end{aligned} \quad (33)$$

We shall write the generating functional of Green's functions as

$$W[\vec{\eta}_\mu, \vec{J}, K] = \int [d\phi]\det M \exp\{i(S[\phi] - \tfrac{1}{2}F_\alpha{}^2)\} \exp\left(i\int d^4x(-\vec{\eta}_\mu\cdot\vec{A}^\mu + \vec{J}\cdot\vec{\chi} + K\psi)\right). \quad (34)$$

The identity (20) translates into

$$-i\ln W[\partial_\mu \vec{\Lambda}, (\lambda/\xi)\vec{\Lambda}, 0] = \frac{1}{2\xi}\vec{\Lambda}^2 + \mathrm{const} , \quad (35)$$

which yields

$$\left(i\partial_\mu \frac{\delta}{\delta\eta_\mu^a(x)} + \frac{\lambda}{\xi}\frac{1}{i}\frac{\delta}{\delta J^a(x)}\right)\left(i\partial_\nu \frac{\delta}{\delta\eta_\nu^b(y)} + \frac{\lambda}{\xi}\frac{1}{i}\frac{\delta}{\delta J^b(y)}\right) Z = -\frac{1}{\xi}\delta^4(x-y)\delta^{ab} , \quad (36)$$

where $Z = -i\ln W$, and a and b are isospin indices. Equation (36) yields a set of relations useful to renormalization of propagators and the parameters ξ and λ.

It is convenient to parametrize the $\vec{A}_\mu$ and $\vec{\chi}$ propagators as

$$\begin{aligned} &-i\langle (A_\mu^a(x) A_\nu^b(y))_+\rangle_0 \sim \left(g_{\mu\nu}\frac{1}{a} + p_\mu p_\nu \frac{c^2 - bd}{a[(a+bp^2)d - c^2p^2]}\right)\delta^{ab} ,\\ &-i\langle (A_\mu^a(x)\chi^b(y))_+\rangle_0 \sim ip\delta^{ab}\frac{c}{(a+bp^2)d - p^2c^2} ,\\ &-i\langle (\chi^a(x)\chi^b(y))_+\rangle_0 \sim \delta^{ab}\frac{a+bp^2}{(a+bp^2)d - p^2c^2} , \end{aligned} \quad (37)$$

where $a(p^2)$, $b(p^2)$, $c(p^2)$, and $d(p^2)$ are free of poles, at least in perturbation theory. Equation (36) gives

$$p^2\left(d + 2\frac{\lambda}{\xi}c\right) + \left(\frac{\lambda}{\xi}\right)^2 (a + bp^2) = -\frac{1}{\xi}[(a+bp^2)d - c^2p^2] . \quad (38)$$

We shall adjust the value of λ so that

$$c(0)=0\,. \tag{39}$$

Then we must have

$$d(0)=-\frac{\lambda^2}{\xi} \tag{40}$$

and

$$a(0)d'(0)=\lambda^2\,. \tag{41}$$

We can renormalize fields and parameters so that

$$a(p^2)\underset{p^2\to 0}{\sim} -Z_3{}^{-1}(p^2-m^2)\,, \qquad d(p^2)\underset{p^2\to 0}{\sim} Z_\chi{}^{-1}(p^2-\mu^2)\,. \tag{42}$$

Then from Eqs. (40) and (41) we find that

$$\lambda=m(Z_3Z_\chi)^{-1/2} \tag{43}$$

and

$$\mu^2=\frac{1}{\xi_r}m^2\,, \tag{44}$$

where

$$\xi_r=\xi Z_3\,. \tag{45}$$

How is the constant m related to the fundamental parameters of the theory? In lowest order, we have $m=\frac{1}{2}(gv)$. The vacuum expectation value v may be used as a fundamental parameter of theory instead of β in Eq. (29). What is more convenient, we may use m as a fundamental parameter instead of v. Thus the renormalized theory can be specified completely in terms of m and ξ_r, in addition to g_r and α_r to be defined shortly.

Certain useful information is obtained from the WT identity which follows from differentiating Eq. (17) with respect to J_i and then putting all external sources equal to zero:

$$\left\{-F_\alpha\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\frac{\delta}{\delta J_i}+\left(\Gamma^\beta_{ij}\frac{1}{i}\frac{\delta}{\delta J_i}+\Lambda^\beta_i\right)\left[M^{-1}\left(\frac{1}{i}\frac{\delta}{\delta J_i}\right)\right]_{\beta\alpha}\right\}W[J]\bigg|_{J=0}=0\,. \tag{46}$$

Specializing to $J_i=J^a(y)$, we learn that the inverse ghost propagator $G^{-1}(p^2)$,

$$-i\langle(c^a(x)\bar{c}^b(y))_+\rangle\sim G(p^2)\delta^{ab}\,, \tag{47}$$

is proportional to

$$G^{-1}(p^2)\sim[(a+p^2b)d-c^2p^2]\,, \tag{48}$$

so that $G(p^2)$ has a pole where the χ propagator does. Moreover, the low-energy limit of $G^{-1}(p^2)$ is given by

$$g^{-1}(p^2)\underset{p^2\to 0}{\sim}-\tilde{Z}_3{}^{-1}\left(p^2-\frac{1}{\xi_r}m^2\right), \tag{49}$$

where $\tilde{Z}_3$ is cutoff-dependent.

Differentiating Eq. (17) with respect to J_k and J_l and letting all external sources vanish, we obtain

$$\left\{-F_\alpha\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\frac{\delta}{\delta J_k}\frac{\delta}{\delta J_l}+\left[\left(\Gamma^\beta_{kj}\frac{1}{i}\frac{\delta}{\delta J_j}+\Lambda^\beta_k\right)\right]\left[M^{-1}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right]_{\beta\alpha}\frac{\delta}{\delta J_l}+(k\leftrightarrow l)\right\}W[J]\bigg|_{J=0}=0\,, \tag{50}$$

specializing to the case $J_i=\eta^b_\mu$, $J_l=\eta^c_\nu$, where η^b_μ and η^c_ν are transverse, $\partial^\mu\eta^b_\mu=0$, we obtain a relation between the $(A_\mu)^3$ and $A_\mu\bar{c}c$ vertices. Denoting the former by $i\Gamma^{abc}_{\lambda\mu\nu}(p,q,r)$ and the latter by $i\gamma^{abc}_\lambda(p,q;r)$ with $p+q+r=0$ [see Fig. 1 of paper I (Ref.14)], and expressing the low-energy limits of these quantities by

$$\begin{aligned} i\Gamma^{abc}_{\lambda\mu\nu}(p,q,r)&\underset{p,q,r\to 0}{\sim}\epsilon^{abc}\left(g\frac{1}{Z_1}[(p-q)_\nu g_{\lambda\mu}+(q-r)_\lambda g_{\mu\nu}+(r-p)_\mu g_{\nu\lambda}]+\text{cutoff-independent terms}\right),\\ i\gamma^{abc}_\lambda(p,q;r)&\underset{p,q,r\to 0}{\sim}\epsilon^{abc}p_\lambda\left(g\frac{1}{\tilde{Z}_1}+\text{cutoff-independent terms}\right), \end{aligned} \tag{51}$$

we find from Eq. (50) that[15]

$$Z_1/Z_3 = \tilde{Z}_1/\tilde{Z}_3 . \tag{52}$$

The renormalized coupling constant g_r is defined as

$$g = g_r \frac{Z_1}{Z_3^{3/2}} = g_r \frac{\tilde{Z}_1}{\tilde{Z}_3 Z_3^{1/2}} . \tag{53}$$

[Caution: Actually the quantity appearing in Eq. (50) is not γ_χ^{abc}, but a related quantity whose cutoff-dependent part is the same as γ_χ^{abc}.]

We may also define α_r to be the value of the renormalized $(\chi)^4$ coupling when all external momenta vanish. By the use of Eq. (17) it is possible to show that all remaining renormalization parts can be expressed finitely in terms of g_r, α_r, and m. We have done this. However, the process is much too arduous to reproduce here. We shall be content to give a renormalization procedure based on this study by specifying renormalization constants.

The effective Lagrangian of the theory is

$$\begin{aligned}\mathcal{L}_{\text{eff}} = &-\tfrac{1}{4}(\partial_\mu \vec{A} - \partial_\nu \vec{A}_\mu + g\vec{A}_\mu \times \vec{A}_\nu)^2 + \tfrac{1}{2}[(\partial_\nu S)^2 + (\partial_\mu \vec{\chi})^2] + \tfrac{1}{2} g \vec{A}^\mu \cdot (S\partial_\mu \vec{\chi} - \vec{\chi}\partial_\mu S + \vec{\chi}\times\partial_\mu\vec{\chi}) + \tfrac{1}{2} g v \vec{A}^\mu \cdot \partial_\mu \vec{\chi} \\ &+ \tfrac{1}{8} g^2 v^2 \vec{A}_\mu^2 + \tfrac{1}{4} g^2 v \vec{A}_\mu^2 S + \tfrac{1}{8} g^2 \vec{A}_\mu^2 (S^2 + \vec{\chi}^2) - \tfrac{1}{4}\alpha(S^2+\chi^2) - \alpha v S(S^2+\vec{\chi}^2) - \alpha v^2 S^2 - \delta\mu^2(S^2+\vec{\chi}^2) - v\delta\mu^2 S \\ &+ \vec{\bar{c}}\left(\partial^2 + \frac{\lambda g v}{2\xi}\right)\vec{c} + g\vec{\bar{c}}\cdot\partial^\mu[\vec{A}_\mu \times c] + \frac{\lambda g}{2\xi}(\vec{\bar{c}}\cdot\vec{c} S + \vec{\bar{c}}\cdot\vec{\chi}\times\vec{c}) - \tfrac{1}{2}\xi\left(\partial_\mu A^\mu + \frac{\lambda}{\xi}\chi\right)^2 ,\end{aligned} \tag{54}$$

where $\delta\mu^2 = \alpha v^2 + \beta$, and $\psi = v + S$. We shall renormalize fields and parameters according to

$$\begin{aligned}\vec{A}_\mu &= Z_3^{1/2}(\vec{A}_\mu)_r , \\ \vec{\chi} &= Z_\chi^{1/2}\vec{\chi}_r , \\ (S, v) &= Z_\psi^{1/2}(S, v)_r , \\ \vec{c} &= Z_3^{1/2}\vec{c}_r , \\ g &= g_r Z_1/Z_3^{3/2} , \\ \alpha &= \alpha_r Z_4/Z_\chi^2 ,\end{aligned} \tag{55}$$

and define

$$m = \tfrac{1}{2} g_r v_r . \tag{56}$$

We recall

$$\xi = \frac{1}{Z_3}\xi_r , \tag{57}$$

$$\lambda = m(Z_3 Z_\chi)^{-1/2} . \tag{58}$$

We determine Z_3, Z_χ, and $\tilde{Z}_3$ from Eqs. (42) and (49). We choose Z_ψ in such a way that m^2 determined by Eq. (42) is the same as $\frac{1}{4}(g_r v_r)^2$ by rescaling v_r and S_r. [This does not make the S field normalized to the unit amplitude asymptotically, but no matter. The renormalized S propagator is finite.] The constants Z_1 and Z_4 are determined by requiring that the renormalized coupling constants associated with the $(A_3)^3$ and $(\chi)^4$ vertices at zero external momenta are g_r and α_r, respectively. The counterterm $\delta\mu^2$ is determined by the condition that the S field does not have a vacuum expectation value (see paper II).

Once they are renormalized according to Eqs. (55)-(58), Green's functions of the theory are finite in terms of m, g_r, α_r, and ξ_r. The S matrix is independent of ξ_r. Since the poles of the χ propagator, the longitudinal part of the A_μ propagator, and the ghost propagator $G(p^2)$ at

$$p^2 = \frac{1}{\xi_r} m^2 + \text{a finite correction}$$

arising from the zero of $[(a+bp^2)d - c^2p^2]$ vary as ξ_r is varied, while the S matrix is independent of ξ_r, we see that the contributions from these poles must cancel in all S-matrix elements. In fact, in the limit $\xi_r \to 0$, we find that the poles in question recede to infinity and the corresponding particles do not contribute to sum over intermediate states at any finite energy. The limit $\xi_r \to 0$ defines the U-gauge formulation discussed in paper III (see also Ref. 16) as an analytic limit of an infinite set of renormalizable gauge formulations.

APPENDIX

We wish to show the invariance of the metric

$$dV = [d\phi]\Delta_F, \quad \Delta_F = \det M , \tag{A1}$$

under the gauge transformation (1) constrained by Eq. (7). We shall first compute the Jacobian J of the transformation (1) to first order in g,

$$\begin{aligned}J &= \exp[\Gamma_{ii}^\alpha g_\alpha + (\Gamma_{ij}^\alpha \phi_j + \Lambda_i^\alpha)\delta g_\alpha/\delta\phi_i] , \\ \delta g_\alpha/\delta\phi_i &= -(M^{-1})_{\alpha\beta}(\delta M_{\beta\gamma}/\delta\phi_i) g_\gamma .\end{aligned} \tag{A2}$$

Since the measure $[d\phi]$ is invariant under the group of linear transformations, $\exp(\Gamma_{ii}^\alpha g_\alpha) = 1$, and we have

$$J=\exp[-(\Gamma^{\alpha}_{ij}\phi_j+\Lambda^{\alpha}_i)(M^{-1})_{\alpha\beta}(\delta M_{\beta\gamma}/\delta\phi^i)g_\gamma]\,. \tag{A3}$$

We shall now compute the change in Δ_F to this order,

$$\Delta^g_F=\det(M+\delta M)=\Delta_F\exp \mathrm{Tr}M^{-1}\delta M\,, \tag{A4}$$

where

$$\delta M_{\beta\gamma}=\frac{\delta M_{\beta\gamma}}{\delta\phi_i}(\Gamma^{\delta}_{ij}\phi_j+\Lambda^{\delta}_i)g_\delta\,. \tag{A5}$$

So we have

$$\delta[\ln\Delta_F]=\frac{\delta M_{\beta\gamma}}{\delta\phi_i}(M^{-1})_{\gamma\beta}(\Gamma^{\delta}_{ij}\phi_j+\Lambda^{\delta}_i)g_\delta\,. \tag{A6}$$

The effect of the transformation on the measure dV is then

$$\delta[\ln\Delta_F]+\ln J=\left(\frac{\delta M_{\beta\gamma}}{\delta\phi_i}(\Gamma^{\delta}_{ij}\phi_j+\Gamma^{\delta}_i)(M^{-1})_{\gamma\beta}-(\Gamma^{\gamma}_{ij}\phi_j+\Lambda^{\gamma}_i)(M^{-1})_{\gamma\beta}\frac{\delta M_{\beta\delta}}{\delta\phi_i}\right)g_\delta\,. \tag{A7}$$

The expression (A7) vanishes for every g if we have

$$\left(\frac{\delta M_{\beta\gamma}}{\delta\phi_i}(\Gamma^{\delta}_{ij}\phi_j+\Lambda^{\delta}_i)-\frac{\delta M_{\beta\delta}}{\delta\phi_i}(\Gamma^{\gamma}_{ij}\phi_j+\Lambda^{\gamma}_i)\right)(M^{-1})_{\gamma\beta}=0\,. \tag{A8}$$

We must now compute $\delta M/\delta\phi$:

$$\frac{\delta M_{\beta\gamma}}{\delta\phi_i}=\frac{\delta^2F^\beta}{\delta\phi_i\delta\phi_j}(\Gamma^{\gamma}_{jk}\phi_k+\Lambda^{\gamma}_k)+\frac{\delta F^\beta}{\delta\phi_j}\Gamma^{\gamma}_{ji}\,. \tag{A9}$$

When Eq. (A9) is substituted in Eq. (A8), the term $\delta^2F/\delta\phi_i\delta\phi_j$ drops out. So Eq. (A8) becomes

$$\frac{\delta F^\beta}{\delta\phi_k}[\Gamma^{\gamma}_{ki}(\Gamma^{\delta}_{ij}\phi_j+\Lambda^{\delta}_i)-\Gamma^{\delta}_{ki}(\Gamma^{\delta}_{ij}\phi_j+\Lambda^{\gamma}_i)](M^{-1})_{\gamma\beta}=0\,. \tag{A10}$$

Let $f_{\alpha\beta\gamma}$ be the structure constant of the gauge group. Then

$$\Gamma^{\gamma}_{ki}(\Gamma^{\delta}_{ij}\phi_j+\Lambda^{\delta}_j)-\Gamma^{\delta}_{ki}(\Gamma^{\gamma}_{ij}\phi_j+\Lambda^{\gamma}_i)=f_{\gamma\delta\alpha}(\Gamma^{\alpha}_{kj}\phi_j+\Lambda^{\alpha}_k)\,. \tag{A11}$$

Therefore Eq. (A10) can be written as

$$f_{\gamma\delta\alpha}\frac{\delta F^\beta}{\delta\phi_k}(\Gamma^{\alpha}_{kj}\phi_j+\Lambda^{\alpha}_k)(M^{-1})_{\gamma\beta}=0\,,$$

or (A12)

$$f_{\gamma\delta\alpha}(M^{-1})_{\gamma\beta}M_{\beta\alpha}=0\,.$$

But Eq. (A12) is true because $f_{\alpha\delta\alpha}=0$ for any compact Lie group.

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[1]G. 't Hooft and M. T. Veltman, Nucl. Phys. B44, 189 (1972); C. G. Bollini, J. J. Giambiagi, and A. Gonzales-Dominguez, Nuovo Cimento 31, 550 (1964); G. Cicuta and Montaldi, Lett. Nuovo Cimento 4, 329 (1972).

[2]B. W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972); 5, 3137 (1972); 5, 3155 (1972). These papers will be referred to as papers I, II, and III, respectively.

[3]K. Fujikawa, B. W. Lee, and A. I. Sanda, Phys. Rev. D 6, 2923 (1972). See also Y. P. Yao, Phys. Rev. D 6, 3129 (1972).

[4]E. S. Fradkin and I. V. Tyutin, Phys. Rev. D 2, 2841 (1970); A. Slavnov, Kiev Report No. ITP-71-83E (unpublished).

[5]W. A. Bardeen, in Proceedings of the Sixteenth International Conference on High Energy Physics, National Accelerator Laboratory, Batavia, Ill., 1972 (unpublished).

[6]J. S. Bell and R. Jackiw, Nuovo Cimento 51, 47 (1969).

[7]S. L. Adler, Phys. Rev. 177, 2426 (1969).

[8]In the context of spontaneously broken gauge theories, see C. Bouchiat, J. Iliopoulos, and Ph. Meyer, Phys. Letters 38B, 519 (1972); D. Gross and R. Jackiw, Phys. Rev. D 6, 477 (1972); H. Georgi and S. L. Glashow, *ibid.* 6, 429 (1972).

[9]V. Popov and L. D. Faddeev, Kiev Report No. ITP-67-36 (unpublished) [English translation available as NAL Report No. NAL-ThY-57 (unpublished)]; L. D. Faddeev and V. Popov, Phys. Letters 25B, 29 (1967).

[10]G. 't Hooft, Nucl. Phys. B33, 173 (1971).

[11]R. P. Feynman, Acta Phys. Polon. 26, 697 (1963).

[12]G. 't Hooft and M. T. Veltman, Utrecht report, 1972 (unpublished).

[13]For constructions of this kind, see Ref. 3, Appendix B; K. Bardakci and M. Halpern, Phys. Rev. D 6, 696 (1972).

[14]There is an error in Fig. 1 of paper I: The bare expression for $G(p^2)$ should be $(-p^2)^{-1}$.

[15]J. C. Taylor, Nucl. Phys. B33, 436 (1971); Slavnov, Ref. 4.

[16]S. Weinberg, Phys. Rev. Letters 27, 1688 (1971).