Vector and Tensor Radiation from Schwarzschild Relativistic Circular Geodesics*

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For the case of high multipoles we give an analytic form of the spectrum of gravitational and electromagnetic radiation produced by a particle in a highly relativistic orbit $r_0 = (3+\delta)M$ around a Schwarzschild black hole of mass M. The general dependence of the power spectrum on the frequency in all three spin cases ($s=0$ for scalar, $s=1$ for vector, and $s=2$ for tensor fields) are summarized by power $P \propto \omega^{1-s}\exp(-2\omega/\omega_{\rm crit})$. Although they have the common feature of an exponential cutoff above a certain frequency $\omega_{\rm crit}=(4/\pi\delta)\omega_0$, where ω_0 is the frequency of the orbit, the tensor case has a much broader frequency spectrum than scalar or vector radiation.

I. INTRODUCTION

Regge and Wheeler[1] developed techniques to treat the most general small perturbation in a Schwarzschild geometry. These were applied to the stability problem for the Schwarzschild metric by Regge, Wheeler, and Vishveshwara.[2] First calculations of gravitational radiation from a highly relativistic source, based on an extended Regge-Wheeler formalism, were done by Thorne[3] and used for the analysis of gravitational radiation from neutron stars by Thorne and Campolattaro[4] and Price and Thorne.[5] Radiation from material falling into black holes has been given by Zerilli[6] and by Davis and Ruffini,[7] Davis, Ruffini, Press, and Price,[8] Davis, Ruffini, and Tiomno,[9] and Davis, Ruffini, Tiomno, and Zerilli.[10] In this paper the method is applied to treat (in the fixed Schwarzschild background geometry) the radiation emitted by a test particle moving at a velocity close to the local speed of light in the neighborhood of a black hole. As usual the particle as well as the radiation is described by a perturbation of the background metric, which can be expanded in terms of a set of scalar, vector, or tensor harmonics with both $(-1)^l$ electric (even) parity and $(-1)^{l+1}$ magnetic (odd) parity. This problem is of direct interest in relation to the possible enhancement of synchrotronlike gravitational radiation effects as recently suggested by Misner.[11]

As we are interested in the spectral distribution, we analyze only the Fourier transform of the perturbation. The radial part of the expansion satisfies a Schrödinger-type wave equation of the form

$$\frac{d^2u^{lm}}{dr^{*2}} + (\omega^2 - V^l_{\rm eff})u^{lm} = S^{lm}, \tag{1}$$

where $r^* = r - 3M + 2M\ln(r/M-2)$; M is the mass of the Schwarzschild black hole; l and m are the angular and azimuthal quantum numbers of the multipole expansion. $V^l_{\rm eff}$ depends on the particular spin of the field under examination. We can adapt the formalism to the study of scalar (spin-0), electromagnetic (spin-1), or gravitational (spin-2) radiation. In the limit $l \gg 1$ the potential $V^l_{\rm eff}$ acquires a standard form independent of the spin of the field[12]:

$$V^l_{\rm eff} = \left(1-\frac{2M}{r}\right)\frac{l(l+1)}{r^2} + O(l^0). \tag{2}$$

The source terms will be different in the three cases, even for $l \gg 1$. The explicit forms both for the potential and for the source in the case of vector fields have been derived by Ruffini and

Tiomno,[13] and those for the tensor field by Davis, Ruffini, and Tiomno.[14] We use their results as reproduced in Ref. 10. The scalar equations have been given by Misner *et al.*[12]

In this paper we look at the spectra of vector and tensor radiation produced by particles in extreme relativistic circular geodesic orbit close to the null orbit at $r=3M$, i.e., at $r_0=(3+\delta)M$. Then the radiated frequency ω is just a multiple m of the particle's orbit frequency $\omega_0=(M/r_0^3)^{1/2}$, namely $\omega=m\omega_0$.

Equation (1) has to be solved with the boundary conditions of purely outgoing waves for $r^*\sim r\to+\infty$,

$$u(r^*)=Ae^{i\omega r^*},\quad r^*\to+\infty$$

and purely ingoing waves at the surface of the black hole for $r^*=-\infty$ or $r=2M$,

$$v(r^*)=Be^{-i\omega r^*},\quad r^*\to-\infty$$

where u and v are solutions of Eq. (1) with $S^{lm}=0$. The Green's function is then given by

$$G(r^*,r_0^*)=\begin{cases}W^{-1}u(r^*)v(r_0^*), & r^*>r_0^*\\ W^{-1}u(r_0^*)v(r^*), & r^*<r_0^*\end{cases}$$

where the Wronskian $W=u(r_0^*)v'(r^*)-v(r_0^*)u'(r^*)$. In the solution of Eq. (1) for tensor and vector fields the Green's function and its derivative are needed as a consequence of the fact that in the source term both $\delta(r^*-r_0^*)$ and $\delta'(r^*-r_0^*)$ are present.[10] For the asymptotic regime ($l\gg1$) it has been shown that the following analytic forms can be obtained[15]:

$$G(r^*,r_0^*)=\tfrac{1}{4}i\omega_0^{-1/2}l^{-1/4}\frac{e^{-\pi\epsilon/8}}{\pi^{1/2}(m\omega_0)^{1/2}}\times e^{-i[\pi/8+(\epsilon/4)\ln(\epsilon/4)]}\Gamma(\tfrac{1}{4}+\tfrac{1}{4}i\epsilon)e^{i\omega r^*}\tag{3}$$

and

$$\frac{\partial G}{\partial r_0^*}(r^*,r_0^*)=\frac{1+i}{(8\pi)^{1/2}}\omega_0^{1/2}l^{1/4}\frac{e^{-\pi\epsilon/8}}{(m\omega_0)^{1/2}}\times e^{-i[\pi/8+(\epsilon/4)\ln(\epsilon/4)]}\Gamma(\tfrac{3}{4}+\tfrac{1}{4}i\epsilon)e^{i\omega r^*}.\tag{4}$$

Here ϵ has different values for the two different parities,

$$\begin{aligned}(-1)^l&:\ \epsilon_{\text{even}}=1+4p+|m|\delta,\quad 2p=l-|m|\\(-1)^{l+1}&:\ \epsilon_{\text{odd}}=3+4p+|m|\delta,\quad 2p=l-|m|-1.\end{aligned}\tag{5}$$

The quantity ϵ appears in the barrier penetration factor,[15] which is given by $e^{-\theta}$, where for a particle with energy E in the orbit r_0^*

$$\theta=\int_{r_0^*}^{r_+^*}[V(r^*)-E]^{1/2}dr^*=\tfrac{1}{4}\pi\epsilon$$

and r_+^* is the outer turning point defined by $V(r^*)=E$. From expressions (3) and (4) we see immediately that, both in the Green's function and in the derivative, terms with $l>|m|$ are negligible due to the exponential factor. The main contribution comes from the term $l=|m|$.

II. VECTOR RADIATION

A charge q moves circularly around a Schwarzschild black hole at a radius $r_0=(3+\delta)M$. The electromagnetic field it produces is expandable in multipole modes characterized by l,m.[16] The expression for the power emitted into a single mode, even and odd parity, is[10,13]

$$P_{\text{out}}(l,m)=\frac{\omega^2}{4\pi}(|R_{\text{even}}^{lm}|^2+|R_{\text{odd}}^{lm}|^2),\tag{6}$$

where

$$R_{\text{even}}^{lm}=\frac{4\pi q}{[l(l+1)]^{1/2}}Y_l^m(\tfrac{1}{2}\pi,0)\frac{\partial}{\partial r_0^*}G_{\text{even}}^{lm}(r^*\to\infty,r_0^*,\omega),\tag{7}$$

$$R_{\text{odd}}^{lm}=-\frac{4\pi q\omega_0}{[l(l+1)]^{1/2}}[l(l+1)-m(m+1)]^{1/2}Y_l^{m+1}(\tfrac{1}{2}\pi,0)G_{\text{odd}}^{lm}(r^*\to\infty,r_0^*,\omega).\tag{8}$$

Substituting the Green's functions (3) and (4) into (6), (7), and (8), and using a large-m approximation for the $Y_l^m(\tfrac{1}{2}\pi,0)$, we obtain

$$\begin{aligned}P_{\text{total}}^{\text{em}}(m\omega_0)&=\sum_{l\ge|m|}^{\infty}P_{\text{out}}(l,m)\equiv P_{\text{even}}+P_{\text{odd}}\\&\simeq\frac{q^2}{54\pi^{3/2}}[\exp(-\tfrac{1}{4}\pi\epsilon_{\text{even}})|\Gamma(\tfrac{3}{4}+\tfrac{1}{4}i\epsilon_{\text{even}})|^2+\tfrac{1}{2}\exp(-\tfrac{1}{4}\pi\epsilon_{\text{odd}})|\Gamma(\tfrac{1}{4}+\tfrac{1}{4}i\epsilon_{\text{odd}})|^2].\end{aligned}\tag{9}$$

It should be mentioned that in the above expression only the leading terms in the sum ($l=|m|$ for even and $l=|m|+1$ for odd parity) are included; both contributions to the power in (9) have basically the same exponentially decaying behavior in $(l-|m|)$ – as mentioned above for the Green's functions – and furthermore there are additional damping factors present if $|m|<l$ in the even case and $|m|+1<l$ in the odd case. Therefore we will set $p=0$ in (5) for all explicit evaluations.

In this case we can derive from expression (9) an approximate form, which holds in the region of the limit $\epsilon >> 1$ or $|m|\delta >> 1$. In that limit we can have a simplified form of the Γ functions[15] and get

$$P^{\text{em}}_{\text{total}}(\omega)=\frac{q^2}{54\pi^{1/2}}\left(e^{-\pi/2}\epsilon_{\text{even}}{}^{1/2}+2e^{-3\pi/2}\epsilon_{\text{odd}}{}^{-1/2}\right)e^{-2\omega/\omega_{\text{crit}}}\,, \tag{10}$$

where $|m|\delta=(4/\pi)\omega/\omega_{\text{crit}}$. From Eq. (10) we obtain for the ratio of the contributions of the odd- and even-parity terms in the asymptotic regime $|m|\delta >> 1$:

$$P^{\text{em}}_{\text{odd}}/P^{\text{em}}_{\text{even}}=2e^{-\pi}/m\delta\equiv\tfrac{1}{2}\pi e^{-\pi}\omega_{\text{crit}}/\omega\,. \tag{11}$$

In the case $|m|\delta >> 1$ the ratio tends instead to a constant of the order 10^{-2}. In Fig. 1 the total electromagnetic power, as given by expression (9), is presented. For comparison we also give the power spectrum for the emission of scalar waves, which was derived by Misner *et al.*,[12] for the modes $l=m$, which are the main contributors, as

$$P^S(\omega)\propto(\omega/\omega_{\text{crit}})\exp(-2\omega/\omega_{\text{crit}})$$

asymptotically for $|m|\delta >> 1$. Both cases are similar in that most of the radiated energy is emitted in a single decade of frequencies.

III. TENSOR RADIATION

Let us consider now the case of an uncharged particle with mass μ orbiting the Schwarzschild black hole and emitting gravitational radiation. In this case the power emitted is still given by expression (6). The functions analogous to (7) and (8) for even and odd parity are, however,[10] now given by

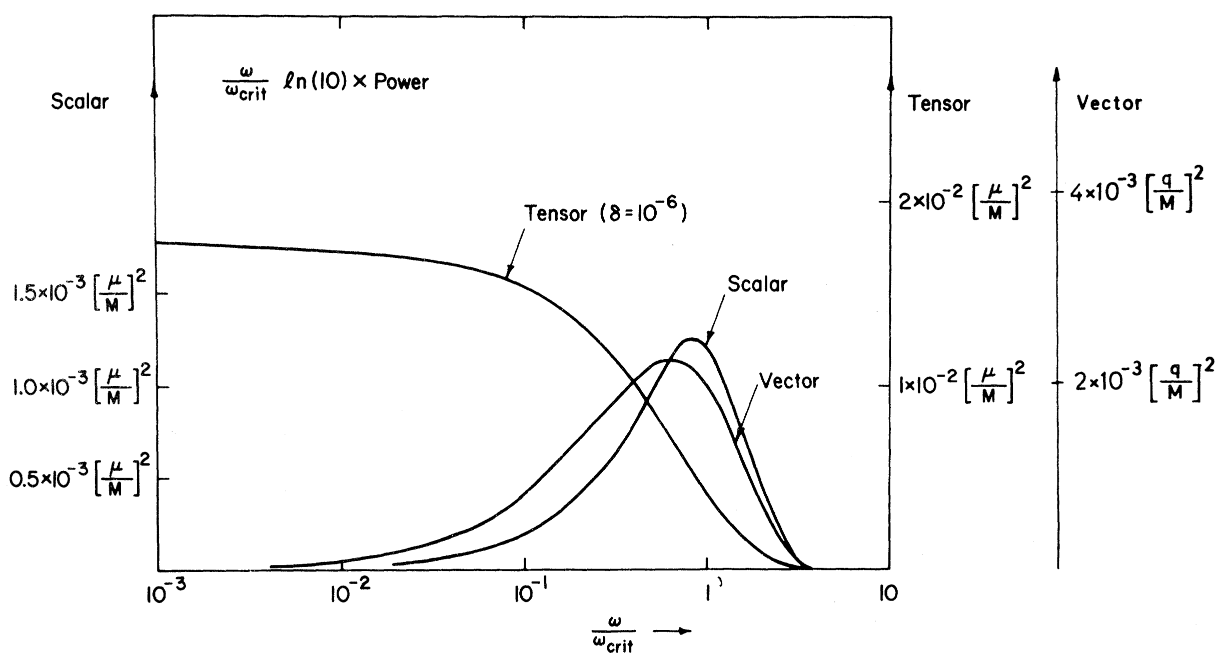

FIG. 1. Scalar, vector, and tensor synchrotron radiation. The quantity $(\omega/\omega_{\text{crit}})\ln(10)\times$power is plotted as the area under these curves and represents the total energy when plotted against $\log_{10}(\omega/\omega_{\text{crit}})$, where for the power the expressions in Eqs. (9) and (16) are used. Scalar power and vector power are functions of $\omega/\omega_{\text{crit}}$ only; hence their graphs are universal (independent of the choice of ω_{crit}). For the gravitational (tensor) power a radius $\delta=r_0/M-3$ of the orbiting particle has to be specified. For fixed $\delta=10^{-6}$ the tensor spectrum extends (to the left) till $m=2$ or $2\omega_0/\omega_{\text{crit}}=(\pi/2)\delta=10^{-5.805}$. μ stands for the scalar charge and the mass of the particle, and q is its electromagnetic charge. The scalar power is taken from Ref. 12.

$$R_{\text{even}}^{lm}=4\pi\mu u^0 Y_l^m(\tfrac{1}{2}\pi,0)\left(\frac{(l-1)(l+2)}{l(l+1)}\right)^{1/2}\left(\alpha(r_0)G^{\text{even}}+\frac{1}{\lambda}\frac{\partial}{\partial r_0^*}[\beta(r_0)G^{\text{odd}}]\right), \tag{12}$$

where

$$\lambda=\tfrac{1}{2}(l-1)(l+2),$$

$$u^0=(1-3M/r_0)^{-1/2}\simeq(3/\delta)^{1/2}$$

and

$$\alpha(r_0)=\frac{r_0-2M}{(r_0+3M/\lambda)^2}\left(1+\frac{1}{\lambda}+\frac{M}{\lambda r_0}-\frac{3M}{\lambda^2 r_0}+\frac{6M^2}{\lambda^2 r_0^2}\right)-\frac{2m^2-l(l+1)}{l(l+1)-2}\,r_0\omega_0^{\,2},$$

$$\beta(r_0)=\frac{1-2M/r_0}{1+3M/\lambda r_0};$$

and

$$R_{\text{odd}}^{lm}=4\pi\mu u^0 Y_l^{m+1}(\tfrac{1}{2}\pi,0)\left(\frac{l(l+1)-m(m+1)}{l(l-1)(l+1)(l+2)}\right)^{1/2}\frac{\partial}{\partial r_0^*}(r_0G^{\text{odd}}). \tag{13}$$

Again using Eqs. (3) and (4) for the asymptotic expressions of the Green's functions and their derivatives, we obtain for the total power of the gravitational radiation ($\delta \ll 1$)

$$P_{\text{even}}^{\text{GR}}(m\omega_0)=\mu^2\frac{\exp(-\frac{1}{4}\pi\epsilon_{\text{even}})}{54\pi^{3/2}m\delta}\left|(1+\tfrac{1}{2}m\delta)\Gamma(\tfrac{1}{4}+\tfrac{1}{4}i\epsilon_{\text{even}})+\frac{\sqrt{2}\,(1-i)}{\sqrt{3m}}\,\Gamma(\tfrac{3}{4}+\tfrac{1}{4}i\epsilon_{\text{even}})\right|^2, \tag{14}$$

$$P_{\text{odd}}^{\text{GR}}(m\omega_0)=\mu^2\frac{\exp(-\frac{1}{4}\pi\epsilon_{\text{odd}})}{54\pi^{3/2}m\delta}\left|\sqrt{2}\,\Gamma(\tfrac{3}{4}+\tfrac{1}{4}i\epsilon_{\text{odd}})+\frac{1+i}{2\sqrt{3m}}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{4}i\epsilon_{\text{odd}})\right|^2. \tag{15}$$

As in the case of the electromagnetic radiation, the only significant contributions are given for even parity by the term with $l=|m|$, and for odd parity by the term with $l=|m|+1$. Then

$$P_{\text{total}}^{\text{GR}}(m\omega_0)=P_{\text{even}}^{\text{GR}}(m\omega_0)+P_{\text{odd}}^{\text{GR}}(m\omega_0). \tag{16}$$

Notice that for large values of m the second terms in both $P_{\text{even}}^{\text{GR}}(m\omega_0)$ and $P_{\text{odd}}^{\text{GR}}(m\omega_0)$ are negligible compared with the first. We are now again interested in finding the asymptotic expression for the power in the region of the exponential cutoff, where $|m|\delta \gg 1$ or $\epsilon \gg 1$.

$$P^{\text{GR}}(m\omega_0)=\frac{\mu^2}{54\pi^{1/2}}\left(\frac{4}{\pi}\frac{\omega}{\omega_{\text{crit}}}\epsilon_{\text{even}}^{-1/2}e^{-\pi/2}+\tfrac{1}{2}\pi\frac{\omega_{\text{crit}}}{\omega}\epsilon_{\text{odd}}^{1/2}\epsilon^{-3\pi/2}\right)e^{-2\omega/\omega_{\text{crit}}}, \tag{17}$$

where as before $\omega/\omega_{\text{crit}}=\frac{1}{4}\pi|m|\delta$. Also the ratio of the even- and odd-mode parts in the power is similar to the electromagnetic case. It is almost constant for low values of m ($|m|\delta \ll 1$), and in the asymptotic region where $|m|\delta \gg 1$ we have

$$P_{\text{odd}}^{\text{GR}}/P_{\text{even}}^{\text{GR}}\simeq\tfrac{1}{2}\pi e^{-\pi}\,\omega_{\text{crit}}/\omega, \tag{18}$$

which is exactly the same as in (11).

The total power for gravitational radiation as given by Eq. (16) is plotted in Fig. 1. In contrast to the scalar and vector spectra, for the tensor radiation each decade from the fundamental frequency up to the cutoff contains roughly equal energy. In Fig. 2 the comparison of the power obtained in the present analysis with the one gained from numerical integration is made for $\delta=10^{-2}$. Only below an m value for which also the second terms in P_{odd} and P_{even} contribute significantly is the exact solution missed. At $m=10$ the analytic solution is off by ~1% in the gravitational case and by ~15% in the electromagnetic case.

Details on the behavior of the polarization of GSR in the gravitational case have been given by some of us.[17] For the electromagnetic case see the subsequent paper (this issue).

IV. DISCUSSION

The main result of our analysis is the following: In the three cases the power spectrum $P(\omega)$ can be summarized by a formula of the type

$$P(\omega)\sim(\omega/\omega_{\text{crit}})^{1-s}e^{-2\omega/\omega_{\text{crit}}}. \tag{19}$$

s is here the spin of the field under consideration ($s=0$ scalar, $s=1$ electromagnetic, $s=2$ gravitational). Equation (19) is to be seen by inspection

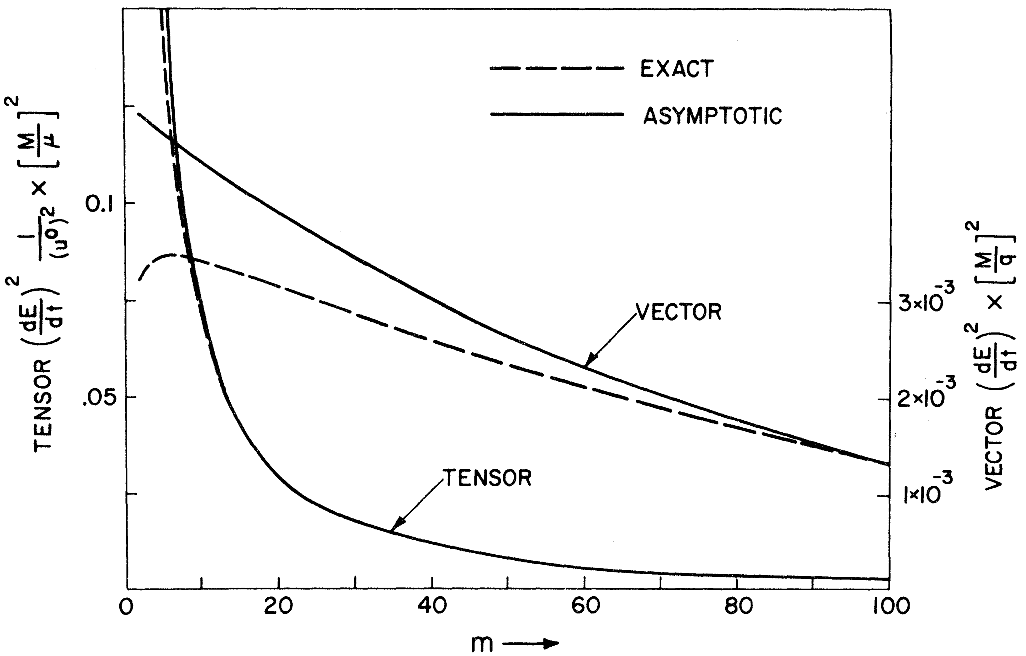

FIG. 2. Low-mode comparison of numerically integrated (exact) and analytic solutions (asymptotic) which generally hold for $m \gg 1$. The radius r_0 of the particle's circular orbit is given by $\delta = r_0/M - 3 = 10^{-2}$. μ and q are the mass and the charge of the particle, respectively.

of Eq. (9) – the expression tends to a constant for low ω as all $\epsilon \to 1$ in that limit – and Eqs. (14) and (15) – they show an over-all factor $(m\delta)^{-1}$ $\propto (\omega/\omega_{crit})^{-1}$. These properties are still visible in the asymptotic expressions (10) and (17), respectively, although other factors take over in that limit. From (19) it follows that at high frequencies ($\omega >> \omega_{crit}$) all the spectra are qualitatively the same as at low frequencies ($\omega \lesssim \omega_{crit}$); they differ by a factor ω^{1-s} (see also Fig. 1).

It is of physical interest to explore how these spectra might affect the energy requirement implied by a possible detection of gravitational radiation. Of course, the spectra as given in (19) which we are using here for that purpose are certainly not produced by events occurring daily over a long period of time, as no astrophysically plausible process of generation of these spectra has yet been conceived.

If we have a detector of bandwidth $\Delta\omega$ centered around a frequency ω_{exp} we have for the total amount of energy E_{tot} implied by the given spectral distribution as a function of the energy E_{obs} available in the fixed bandwidth

$$E_{tot} = E_{obs} \int_0^\infty P(\omega) d\omega \Big/ \int_{\omega_{exp} - \Delta\omega/2}^{\omega_{exp} + \Delta\omega/2} P(\omega) d\omega . \quad (20)$$

Due to the presence of high multipoles in the spectra, the effect of beaming has to be taken also into proper account in the estimate of E_{obs}. If the radiation is confined to a disk of half-width $\Delta\vartheta$, then

$$E_{obs} = \Delta\vartheta \times E_{obs}^{isot} , \quad (21)$$

where E_{obs}^{isot} is the amount of radiation to be expected by the detector in the absence of beaming (isotropic emission). As a property of the spherical harmonics, radiation into a multipole m shows a θ dependence $\propto \sin^{2|m|}\theta \simeq e^{-\vartheta^2 |m|/2}$. The half-width of the beam is determined by $\Delta\vartheta \sim |m|^{-1/2}$.[11] We have, therefore, for the final total energy implied by the observations,

$$E_{tot} = \Delta\vartheta \left(\int_0^\infty P(\omega) d\omega \Big/ \int_{\omega_{exp} - \Delta\omega/2}^{\omega_{exp} + \Delta\omega/2} P(\omega) d\omega \right) E_{obs}^{isot}$$

$$\equiv \alpha_s E_{obs}^{isot} , \quad (22)$$

where E_{obs}^{isot} is the amount of energy extrapolated from the energy observed by the antenna within its bandwidth under the assumption of isotropic radiation. α_s is a conversion factor between this energy and E_{tot}, the total energy requirement due to the entire spectrum and with the beaming effect included. Since for circular orbits the radiated frequency ω is related to the orbital frequency ω_0 by

$\omega = m\omega_0$, and for the orbit close to $r = 3M$ the particle orbits with the frequency $\omega_0 = (M/r^3)^{1/2} = 3.91\times 10^4 (M_\odot/M)\times \sec^{-1}$, we can set a lower limit for the mass of the black hole by the minimal condition $\omega_{\exp} \geq 2\omega_0$ or

$$M/M_\odot \geq 2(3\sqrt{3}\, M_\odot)^{-1}\times \omega_{\exp}^{-1}. \tag{23}$$

From expression (19) for the spectra we can now give an explicit evaluation of E_{tot} involved in the cases $s=0$, $s=1$, and $s=2$. However, before calculating the explicit integral we have to fix a particular cutoff frequency of the spectrum. For the scalar and vector cases we assume that for a given detector frequency $\omega_{\exp}$ the cutoff frequency ω_{crit} is chosen in a way so that $\omega_{\exp}$ coincides with the peak of the spectrum (see Fig. 1), i.e., $\omega_{\exp} = \omega_{peak} = (\frac{1}{4}\pi)\omega_{crit}$ or $\omega_{crit} = (4/\pi)\omega_{\exp}$. Furthermore we assume the earth to be located within the half-width of the beam, so that we have the relation

$$m_{\exp} = \frac{\omega_{\exp}}{\omega_0} = 2.55\times 10^{-5}(M/M_\odot), \tag{24}$$

$$\Delta\vartheta = m_{\exp}^{-1/2}.$$

Now we are able to express all quantities involved in terms of the detector frequency and the bandwidth of the experiment. We insert the spectrum as given by (19) into (22) to obtain the conversion factor α_s. For the scalar and vector cases we get

$$\alpha_0 = \frac{1}{\pi}\,\alpha_1 = \frac{2}{\pi^2} e^{\pi/2}\,\omega_0^{1/2}\,\frac{\omega_{\exp}^{1/2}}{\Delta\omega} = 3.86\times 10^2 \sec^{-1/2}\times \frac{\omega_{\exp}^{1/2}}{\Delta\omega}\left(\frac{M_\odot}{M}\right)^{1/2}, \quad (\Delta\omega \ll \omega_{\exp}). \tag{25}$$

To get a rough estimate for α_2 in the tensor case we choose ω_{crit} in a way that locates the detector frequency in the flat part of the spectrum, say $\omega_{crit} = 10\omega_{\exp}$. A crude evaluation of (22) then yields

$$\alpha_2 = e^{2/5}\,\omega_0^{1/2}\,\frac{\omega_{\exp}^{1/2}}{\Delta\omega}\ln\left(10\,\frac{\omega_{\exp}}{\omega_0}\right) = 2.95\times 10^2 \sec^{-1/2}\times \ln(\delta^{-1})\,\frac{\omega_{\exp}^{1/2}}{\Delta\omega}\left(\frac{M_\odot}{M}\right)^{1/2}. \tag{26}$$

Equation (27) shows that even for $\delta = 10^{-6}$ the enhancement of the radiation for tensor waves is less than that for scalar waves only by a factor of 10.

As we are aware of several gravity-wave experiments operating already or in the near future at different frequencies, (26) can be used to calculate the energy requirement employed by this model.

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