Consistent coupling to Dirac fields in teleparallelism: Comment on "Metric-affine approach to teleparallel gravity"

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Contrary to the claim in a recent publication [Phys. Rev. D **67**, 044016 (2003)], I explicitly demonstrate the consistency of the coupling of Dirac fields to the teleparallelism equivalent of general relativity. Moreover, it is pointed out that, in a metric-affine framework, a $SL(4,R)$-covariant generalization of the Dirac equation needs to be considered.

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I. INTRODUCTION

Einstein's general relativity (GR) not only passes all observational tests but also permits a consistent coupling to Dirac spinors [1]. A very close rival is its *teleparallelism equivalent* ($\mathrm{GR}_\parallel$), suggested already by Einstein [2], which essentially differs from GR by a boundary term dC^*.

As has been well known since Hamilton, adding a boundary term to the action is the canonical method for generating new pairs of variables and momenta: In the transition from GR to teleparallelism, $C^* = \vartheta^\alpha \wedge {}^*D\vartheta_\alpha$ is the corresponding three-form, and the spacetime metric g_{ij} gets replaced by a local orthonormal coframe ϑ^α as independent variables, i.e., by the "tetrads." For a specific choice of the kinetic term in the Lagrangian, one arrives at the teleparallelism equivalent $\mathrm{GR}_\parallel$ of Einstein's theory.

The Poincaré gauge theory or its metric-affine generalization [3] encompasses the Einstein-Cartan theory and the teleparallelism equivalent ($\mathrm{GR}_\parallel$) of Einstein's theory as important subcases which are both *empirically* indistinguishable from classical general relativity.

Now and then the coupling of $\mathrm{GR}_\parallel$ to a Dirac field is debated, although this issue has essentially been answered already by Wigner [4].

II. DIRAC FIELDS IN RIEMANN-CARTAN SPACETIME

A Dirac field is a bispinor-valued zero-form ψ for which $\overline{\psi} := \psi^\dagger \gamma_0$ denotes the Dirac adjoint and $D\psi := d\psi + \Gamma \wedge \psi$ is the exterior covariant derivative with respect to the Riemann-Cartan (RC) connection $\Gamma := (i/4)\Gamma^{\alpha\beta}\sigma_{\alpha\beta}$, where $\sigma_{\alpha\beta} := (i/2)(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)$ are the Lorentz generators. The Dirac Lagrangian is given by the manifestly Hermitian four-form

$$L_{\mathrm{D}} = L(\gamma,\psi,D\psi) = \frac{i}{2}\{\overline{\psi}{}^*\gamma \wedge D\psi + \overline{D\psi} \wedge {}^*\gamma\psi\} + m\overline{\psi}\psi\eta, \tag{2.1}$$

where $\gamma := \gamma_\alpha \vartheta^\alpha$. Since $L_{\mathrm{D}} = \overline{L_{\mathrm{D}}} = L_{\mathrm{D}}^\dagger$ even in an unholonomic frame, it provides us automatically with the Hermitian charge current $j := \overline{\psi}{}^*\gamma\psi$ and axial current $j_5 := \overline{\psi}\gamma_5\ {}^*\gamma\psi$.

In order to separate out the purely Riemannian piece from torsion terms, we decompose the Riemann-Cartan connection $\Gamma = \Gamma^{\{\}} - K$ into the Riemannian (or Christoffel) connection $\Gamma^{\{\}}$ and the *contortion* one-form $K = (i/4)K^{\alpha\beta}\sigma_{\alpha\beta}$, obeying $\Theta := D\gamma = [\gamma, K] = \gamma_\alpha T^\alpha$. Accordingly, the Dirac Lagrangian (2.1) splits [5] into a Riemannian and a spin-contortion piece:

$$\begin{aligned} L_{\mathrm{D}} &= L(\gamma,\psi,D^{\{\}}\psi) - \frac{i}{2}\overline{\psi}({}^*\gamma \wedge K - K \wedge {}^*\gamma)\psi \\ &= L(\gamma,\psi,D^{\{\}}\psi) - \frac{1}{4}A \wedge \overline{\psi}\gamma_5\ {}^*\gamma\psi \\ &= L(\gamma,\psi,D^{\{\}}\psi) - T^\alpha \wedge \mu_\alpha. \end{aligned} \tag{2.2}$$

The covariant derivative with respect to the Riemannian connection satisfies $D^{\{\}}\gamma = 0$. Hence, in a RC spacetime a Dirac spinor feels only the axial torsion one-form $A := {}^*(\vartheta^\alpha \wedge T_\alpha)$, or, equivalently, torsion merely couples to the spin-energy potential $\mu_\alpha = \frac{1}{4}\ {}^*j_5 \wedge \vartheta_\alpha$ (cf. Ref. [6]).

Since $L_{\mathrm{D}} \cong 0$ "on shell," the canonical energy-momentum three-form of the Dirac field reads

$$\Sigma_\alpha := \frac{\partial L_{\mathrm{D}}}{\partial \vartheta^\alpha} \cong \frac{i}{2}\{\overline{\psi}\ {}^*\gamma \wedge D_\alpha\psi - \overline{D_\alpha\psi} \wedge\ {}^*\gamma\psi\}, \tag{2.3}$$

where $D_\alpha := e_\alpha \rfloor D$. The spin current of the Dirac field is given by the Hermitian three-form

$$\tau_{\alpha\beta} := \frac{\partial L_{\mathrm{D}}}{\partial \Gamma^{\alpha\beta}} = \frac{1}{8}\overline{\psi}(\ {}^*\gamma\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\ {}^*\gamma)\psi, \tag{2.4}$$

with totally antisymmetric components $\tau_{\alpha\beta\gamma} := e_\gamma \rfloor {}^*\tau_{\alpha\beta} = \tau_{[\alpha\beta\gamma]}$.

In general, from local Poincaré invariance one obtains the "on shell" *Noether identities*

$$D\Sigma_\alpha \cong (e_\alpha \rfloor T^\gamma) \wedge \Sigma_\gamma + (e_\alpha \rfloor R^{\gamma\delta}) \wedge \tau_{\gamma\delta} \tag{2.5}$$

and

$$D\tau_{\alpha\beta} + \vartheta_{[\alpha} \wedge \Sigma_{\beta]} \cong 0, \tag{2.6}$$

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provided the matter field equation $\delta L/\delta\psi=0$ is satisfied. In the case of Dirac fields this can be proven directly by inserting the energy-momentum (2.3) and spin current (2.4), respectively, into the Noether identities (cf. [7]).

It is a distinguishing feature of a Weitzenböck spacetime [8] with vanishing Riemann-Cartan curvature, i.e., $R^{\alpha\beta}=0$, that the energy-momentum current Σ_α is conserved

$$\hat{D}\Sigma_\alpha \cong 0 \tag{2.7}$$

with respect to the transposed connection $\hat{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta + e_\alpha \rfloor T^\beta$ (cf. [9]).

III. TELEPARALLELISM EQUIVALENT

Let us recall two classically viable gravitational Lagrangians. (1) Hilbert's original choice

$$V_{\rm HE} = -\frac{1}{2\ell^2} R^{\{\}}_{\alpha\beta} \wedge {}^*(\vartheta^\alpha \wedge \vartheta^\beta), \tag{3.1}$$

where $R^{\{\}}_{\alpha\beta}$ denotes the Riemannian curvature and vanishing torsion as in GR (cf. [10]). (2) The torsion-square Lagrangian [11,12]

$$V_\| := \frac{1}{2\ell^2} T^\alpha \wedge {}^*\left(- {}^{(1)}T_\alpha + 2\, {}^{(2)}T_\alpha + \frac{1}{2}\, {}^{(3)}T_\alpha \right) \tag{3.2}$$

of $\mathrm{GR}_\|$, where $H^\|_\alpha := -\partial V_\| / \partial T^\alpha = (1/2\ell^2)\, \eta_{\alpha\beta\gamma} \wedge K^{\beta\gamma}$ is dual to the contortion one-form $K_{\alpha\beta}$ which features in the decomposition $\Gamma_{\alpha\beta} = -\Gamma_{\beta\alpha} = \Gamma^{\{\}}_{\alpha\beta} - K_{\alpha\beta} = \Gamma^{\{\}}_{\alpha\beta} + e_\alpha \rfloor T_\beta + (e_\alpha \rfloor e_\beta \rfloor T_\gamma) \wedge \vartheta^\gamma$ of the RC connection with $T^\alpha = K_\beta{}^\alpha \wedge \vartheta^\beta$.

Because of the geometric identity

$$V_\| \equiv V_{\rm HE} + \frac{1}{2\ell^2} R_{\alpha\beta} \wedge {}^*(\vartheta^\alpha \wedge \vartheta^\beta) + \frac{1}{2\ell^2} d(\vartheta^\alpha \wedge {}^*T_\alpha), \tag{3.3}$$

in a Weitzenböck spacetime $\mathrm{GR}_\|$ is classically equivalent to GR up to a boundary term dC^* where $C^* := \vartheta^\alpha \wedge {}^*D\vartheta_\alpha$ is a Chern-Simons type term for the dual torsion.

IV. PROPER TELEPARALLELISM VIA CONSTRAINTS

In a consistent Lagrangian formulation and in order to avoid particular gauges, the constraint $R^{\alpha\beta}=0$ on the RC connection Γ has to be imposed by subtracting $R^{\alpha\beta} \wedge \lambda_{\alpha\beta}$ from Eq. (5.2) below, where the two-form $\lambda_{\alpha\beta} = -\lambda_{\beta\alpha}$ is a Lagrange multiplier (cf. [13]). Then the proper teleparallelism Lagrangian reads

$$\tilde{V}_\| = V_\| - R^{\alpha\beta} \wedge \lambda_{\alpha\beta}. \tag{4.1}$$

By varying this Lagrangian independently with respect to ϑ^α, $\Gamma^{\alpha\beta}$, and the multiplier $\lambda_{\alpha\beta}$, one obtains [14] as field equations

$$DH^\|_\alpha - E^\|_\alpha = \Sigma_\alpha, \tag{4.2}$$

$$D\lambda_{\alpha\beta} + \vartheta_{[\alpha} \wedge H^\|_{\beta]} = \tau_{\alpha\beta}, \tag{4.3}$$

and

$$R^{\alpha\beta} = 0. \tag{4.4}$$

Since the multiplier term in Eq. (4.1) does not depend on the coframe, the resulting first field equation (4.2) is the same as that of the naive Lagrangian $V_\|$. As to the second field equation (4.3), it satisfies identically the integrability condition as an equation for $\lambda_{\alpha\beta}$. Indeed, in a Weitzenböck spacetime

$$DD\lambda_{\alpha\beta} = -2R_{[\alpha|}{}^\gamma \wedge \lambda_{\gamma|\beta]} = 0. \tag{4.5}$$

Hence the condition for local solvability of Eq. (4.3) with respect to $\lambda_{\alpha\beta}$ is

$$D(\tau_{\alpha\beta} - \vartheta_{[\alpha} \wedge H^\|_{\beta]}) = D\tau_{\alpha\beta} + \vartheta_{[\alpha} \wedge \Sigma_{\beta]} + \vartheta_{[\alpha} \wedge E^\|_{\beta]} - T_{[\alpha} \wedge H^\|_{\beta]} = 0, \tag{4.6}$$

where the right-hand side follows after inserting the first field equation (4.2). Since the metrical gauge energy-momentum current (5.4.15) of Ref. [3] satisfies $m_{[\alpha\beta]} := \vartheta_{[\alpha} \wedge E^\|_{\beta]} - T_{[\alpha} \wedge H^\|_{\beta]} = 0$ in a Weitzenböck spacetime, the second Noether identity (2.6) for matter is recovered.

In particular, this holds for Dirac fields: one should not overlook that the transition from GR to $\mathrm{GR}_\|$ generated by C^* is, in general, accompanied by the related change $L_{\rm D} \to L_{\rm D} + dU$ of the Dirac Lagrangian. Even for a trivial three-form $U = \vartheta^\alpha \wedge \mu_\alpha = 0$, the corresponding boundary term $dU = T^\alpha \wedge \mu_\alpha - \vartheta^\alpha \wedge D\mu_\alpha = \vartheta^\alpha \wedge [e_\beta \rfloor (T^\beta \wedge \mu_\alpha) - D\mu_\alpha]$ compensates the torsion coupling in Eq. (2.2) and thereby induces the relocalization

$$\Sigma_\alpha \to \sigma_\alpha := \Sigma_\alpha - D\mu_\alpha + e_\beta \rfloor (T^\beta \wedge \mu_\alpha),$$

$$\tau_{\alpha\beta} \to \hat{\tau}_{\alpha\beta} := \tau_{\alpha\beta} - \vartheta_{[\alpha} \wedge \mu_{\beta]} = 0 \tag{4.7}$$

of the Noether currents [cf. (R1) and (R2) of Ref. [9]], such that the relocalized spin $\hat{\tau}_{\alpha\beta}$ vanishes. Equivalently, the correspondence $H^\|_\alpha \to \mu_\alpha$ emerging from Eq. (2.2) is sufficient for a consistently relocalized Dirac spin on the left-hand side of Eq. (4.6) even for $\mathrm{GR}_\|$.

Thus, the only role of the second field equation is to determine the Lagrangian multiplier $\lambda_{\alpha\beta}$ nonuniquely, i.e., only up to a covariant divergence $D\Phi_{\alpha\beta}$. The Cauchy problem for $\mathrm{GR}_\|$, however, is not completely settled (cf. Refs. [10,15,16]).

V. DISCUSSION

Recently, it has been claimed [17] that there is an inconsistency in the coupling of spinors to metric-affine generalizations of teleparallelism. However, this is erroneous for at least three reasons.

(1) Dirac fields satisfy the Noether identities (2.5), (2.6) and thereby couple consistently to $\mathrm{GR}_\|$ in the Poincaré framework with "spontaneously" broken local translations

[18], as has been shown here. (A previous comment [19] cannot be regarded as conclusive, since there the teleparallel gauge $\Gamma \stackrel{*}{=} 0$ is assumed and then $GR_{\parallel}$ is tacitly remapped to Einstein's GR about which there were no doubts in the first place.)

(2) For $GR_{\parallel}$ extensions with nonmetricity based on gauging $SL(4,R)$ and the identity (5.9.16) of Ref. [3], a metric-affine generalization [20] of the Dirac equation needs to be considered.

(3) Moreover, in spaces without nonmetricity, the Lagrangian (4.1) of Ref. [15] is not that of GR but $V_{\mathrm{HE}} + (1/2\ell^2) dC^*$, the boundary term of which signals a canonical transformation of variables. Then, in a consistent formulation [3], the Belinfante-Rosenfeld symmetrized energy-momentum tensor $\sigma_\alpha := \Sigma_\alpha - D\mu_\alpha + e_\beta \rfloor (T^\beta \wedge \mu_\alpha)$ arises, which, by construction, satisfies $\vartheta_{[\alpha} \wedge \sigma_{\beta]} = 0$.

On the other hand, the teleparallelism equivalent of GR merits further investigation because of several attractive features: the apparent absence [21] of the chiral anomaly, in contradistinction [22] to Einstein-Cartan theory, a complete formal solvability [23] of its Ashtekar type constraints by loop type Cartan circuits, the possible implementation of torsional instantons into a quadratic Weyl model [24,25] amended by the Euler invariant and dC^* as boundary terms, and consistent noncommutative [26,27] and superspace extensions [28].

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