Cosmic gravitational-wave background in a cyclic universe

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(Received 18 July 2003; published 30 June 2004)

Inflation predicts a primordial gravitational wave spectrum that is slightly "red." In this Brief Report, we compute both the amplitude and spectral form of the primordial tensor spectrum predicted by cyclic or ekpyrotic models under the assumption that perturbations pass smoothly through the bounce. The spectrum is exponentially suppressed compared to inflation on long wavelengths. The strongest constraint is that the energy density in gravitational waves is less than 10% of the critical density at nucleosynthesis.

DOI: 10.1103/PhysRevD.69.127302 PACS number(s): 98.70.Vc, 98.80.Cq

The recently proposed cyclic model [1,2] differs radically from standard inflationary cosmology [3,4], while retaining the inflationary predictions of homogeneity, flatness, and nearly scale-invariant density perturbations. It has been suggested that the cosmic gravitational wave background provides the best experimental means for distinguishing the two models. Inflation predicts a nearly scale-invariant (slightly red) spectrum of primordial tensor perturbations, whereas the cyclic model predicts a blue spectrum [1]. The difference arises because inflation involves an early phase of hyper-rapid cosmic acceleration, whereas the cyclic model does not.

In this Brief Report, we compute the gravitational wave spectrum for cyclic models to obtain both the normalization and spectral shape as a function of model parameters, improving upon earlier heuristic estimates. We make the assumption that perturbations pass smoothly through the bounce. This assumption remains unproven and controversial [5], although recent results by several groups suggest it is plausible [6–8]. Under this assumption, we find that the spectrum is strongly blue. The amplitude is too small to be observed by currently proposed detectors on all scales. Hence, the discovery of a stochastic background of gravitational waves would be evidence in favor of inflation, and would rule out the cyclic model.

Readers unfamiliar with the cyclic model may consult [9] for an informal tour, and [10] for a recent analysis of phenomenological constraints. Cyclic cosmology draws strongly on earlier ideas associated with the "ekpyrotic universe" scenario [11–13]. Briefly, the scenario can be described in terms of the periodic collision of orbifold planes moving in an extra spatial dimension, or, equivalently, in terms of a four-dimensional theory with an evolving (modulus) field ϕ rolling back and forth in an effective potential $V(\phi)$. The potential (Fig. 1) is small and positive for large ϕ, falling steeply negative at intermediate ϕ, and increasing again for negative ϕ.

Each cycle consists of the following stages: (1) ϕ large and decreasing: the universe expands at an accelerated rate as $V(\phi)>0$ acts as dark energy; (2) ϕ intermediate and decreasing: the universe is dominated by a combination of scalar kinetic and potential energy, leading to slow contraction and to the generation of fluctuations; (3) ϕ negative and decreasing (beginning at conformal time $\tau_{end}<0$): the generation of fluctuations ends, ϕ rolls past ϕ_{end} and, in the four-dimensional description, the universe contracts rapidly, dominated by scalar field kinetic energy, to the bounce ($\tau=0$) at which matter and radiation are generated; (4) ϕ increasing from minus infinity: the universe remains dominated by scalar field kinetic energy, which decreases rapidly compared to the radiation energy; (5) ϕ large and increasing (beginning at $\tau_r>0$): the scalar field kinetic energy redshifts to a negligible value and the universe begins the radiation-dominated expanding phase; (6) ϕ large and nearly stationary: the universe undergoes the transitions to matter and dark energy domination, and the cycle begins anew.

During stage 2 ($\phi>\phi_{end}$), the potential $V(\phi)$ must be exponentially steep to produce acceptable scalar perturbations. But when $V(\phi)$ ceases to be exponentially steep (stage 3, $\phi<\phi_{end}$), the potential energy becomes negligible compared to the kinetic energy, which blueshifts and dominates the universe. Since ϕ acts like a free scalar field in this

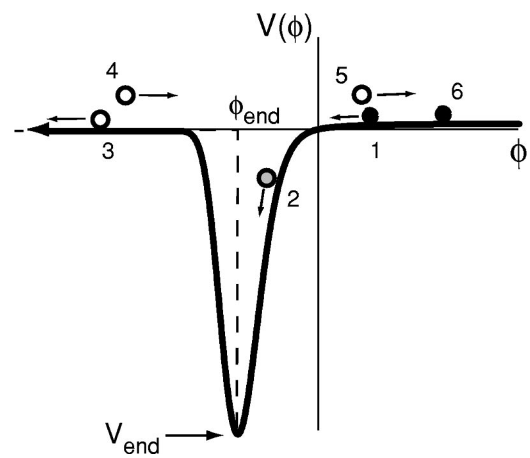

FIG. 1. Schematic of cyclic potential with numbers representing the stages described in the text. To the left of ϕ_{end}, where the scalar kinetic energy dominates, we approximate V with a Heaviside function, jumping to zero as shown by the dashed line.

regime, we may simplify calculations by setting $V(\phi)$ to zero without any loss of generality, as discussed in [2,10]. We therefore model the potential as

$$V(\phi)=V_0(1-e^{-c\phi/M_{pl}})\Theta(\phi-\phi_{end}) \tag{1}$$

where M_{pl} is the reduced Planck mass and the Heaviside step function $\Theta(\phi-\phi_{end})$ sets $V(\phi)$ to zero when $\phi<\phi_{end}$. Choosing $c=10$, for example, results in a scalar spectral index $n_s=.96$ which is compatible with current constraints.

Our calculation begins in the ''ekpyrotic phase,'' stage (2), with the Einstein-frame scale factor contracting

$$a(\tau)=a_{end}\left(\frac{\tau-\tau_{ek}}{\tau_{end}-\tau_{ek}}\right)^{\alpha}, \quad \tau<\tau_{end}, \tag{2}$$

where $\alpha\equiv 2/(c^2-2)\ll 1$ and $\tau_{ek}\equiv(1-2\alpha)\tau_{end}$, being the conformal time the potential would have diverged to minus infinity had the exponential form continued. At $\tau=\tau_{end}$, the ekpyrotic phase ends and the ''contracting kinetic phase,'' stage (3), begins:

$$a(\tau)=\left(\frac{-\tau}{(1+\chi)\tau_r}\right)^{1/2}, \quad \tau_{end}<\tau<0. \tag{3}$$

At $\tau=0$, the universe bounces and the ''expanding kinetic phase,'' stage (4), begins:

$$a(\tau)=\left(\frac{\tau}{\tau_r}\right)^{1/2}, \quad 0<\tau<\tau_r. \tag{4}$$

Radiation is produced at the bounce, but is less than the scalar kinetic energy until, at $\tau=\tau_r$, the expanding kinetic phase ends, and standard radiation-dominated, matter-dominated, and dark-energy-dominated epochs ensue. The transition times, τ_r and τ_{end}, are given by

$$\tau_r=(\sqrt{2}H_r)^{-1}, \quad \tau_{end}=-\tau_r/\Gamma, \tag{5}$$

and

$$\Gamma\equiv\left|\frac{\tau_r}{\tau_{end}}\right|=\left[\frac{1}{1+\chi}\left(\frac{2\alpha}{1-2\alpha}\right)\left(\frac{V_{end}}{H_r^2M_{pl}^2}\right)\right]^{1/3}, \tag{6}$$

where $H_r\equiv H(\tau_r)$ is the Hubble constant at τ_r, $V_{end}=-V(\phi_{end})$ is the depth of the potential at its minimum, and $\chi\ll 1$ is a small positive constant that measures the amount of radiation created at the bounce. Note that $a(\tau)$ and $a'(\tau)$ are both continuous at the transition time $\tau=\tau_{end}$, and we have chosen to normalize $a(\tau)$ to unity at the start of radiation domination ($a(\tau_r)=1$).

The primordial spectrum, $\Delta h(k,\tau_r)$. A quasi-stationary stochastic background of gravitational waves is characterized by the quantity $\Delta h(k,\tau)$, the rms dimensionless strain per unit logarithmic wave number at time τ. Accounting for both polarizations, it is given by $\Delta h(k,\tau)=k^{3/2}|h_k(\tau)|/\pi$, where the Fourier amplitude $h_k(\tau)$ satisfies

$$h_k''+2\frac{a'}{a}h_k'+k^2h_k=0. \tag{7}$$

In the cyclic model, all modes inside the horizon today exited the horizon during the contracting phase, and re-entered during the expanding phase. Early in the ekpyrotic phase [that is, in stage (2), with $\tau\to-\infty$], these modes were far inside the Hubble volume, which had been smoothed, flattened and cleaned of debris by the dark energy epoch, stage (1). All modes of interest are therefore expected to be in their usual Minkowski vacuum at the start of stage (2), implying the boundary condition

$$h_k(\tau)\to\frac{e^{-ik\tau}}{a(\tau)M_{pl}\sqrt{2k}} \quad \text{as} \quad \tau\to-\infty. \tag{8}$$

To solve Eq. (7), it is useful to define $f_k(\tau)\equiv a(\tau)h_k(\tau)$ and rewrite Eq. (7) as

$$f_k''+\left(k^2-\frac{a''}{a}\right)f_k=0. \tag{9}$$

During the ekpyrotic phase, $a(\tau)$ is given by Eq. (2), and the general solution of (9) is

$$f_k(\tau)=\sqrt{y}[A_1(k)H_n^{(1)}(y)+A_2(k)H_n^{(2)}(y)], \tag{10}$$

where $A_{1,2}(k)$ are arbitrary constants, $n\equiv\frac{1}{2}-\alpha$, $y\equiv-k(\tau-\tau_{ek})$, and $H_n^{(1,2)}$ are the Hankel functions. Asymptotically, $H_n^{(1,2)}(y)\to\sqrt{2/\pi y}\,e^{\pm iy}$, so (8), (10) imply

$$A_1(k)=\frac{1}{2}\sqrt{\frac{\pi}{k}}, \quad A_2(k)=0, \tag{11}$$

where we have dropped a physically irrelevant phase. In the contracting kinetic phase, stage (4), $a(\tau)$ is given by Eq. (3), and the general solution of Eq. (9) is

$$f_k(\tau)=\sqrt{-k\tau}[B_1(k)H_0^{(1)}(-k\tau)+B_2(k)H_0^{(2)}(-k\tau)] \tag{12}$$

where $B_{1,2}(k)$ are arbitrary constants. Then, continuity of h_k and h_k' at $\tau=\tau_{end}$ implies

$$B_{1,2}(k)=\mp\frac{i\pi}{4}\sqrt{\frac{\pi\alpha}{2k}}x_e[H_1^{(2,1)}(x_e)H_n^{(1)}(2\alpha x_e)+H_0^{(2,1)}(x_e)H_{n-1}^{(1)}(2\alpha x_e)] \tag{13}$$

where $x_e\equiv k|\tau_{end}|$. Finally, in the expanding kinetic phase, $a(\tau)$ is given by Eq. (4), and the general solution of Eq. (9) is

$$f_k(\tau)=\sqrt{k\tau}[C_1(k)H_0^{(1)}(k\tau)+C_2(k)H_0^{(2)}(k\tau)]. \tag{14}$$

From Eq. (7), each gravity wave polarization acts just like a massless scalar, and quantum field theory for a massless scalar in the contracting/expanding Milne geometry near the cyclic model's bounce has already been treated in [8]. (Our problem is even simpler: tensor perturbations are gaugeinvariant.) They find a unique sensible matching condition. One can think of continuing the fields from the contracting Milne wedge to the expanding one through the Minkowski space in which they are both naturally embedded. Equivalently, one may analytically continue the positive (negative) frequency parts of $h_k \equiv f_k/a$ around the origin in the lower (upper) half of the complex τ plane, so $H_0^{(1,2)}(-k\tau) \to -H_0^{(2,1)}(k\tau)$ [8]. This yields

$$C_{1,2}(k) = -\sqrt{1+\chi}B_{2,1}(k). \tag{15}$$

The pre-factor arises because $a(\tau)$ differs by a factor of $\sqrt{1+\chi}$ between the kinetic contracting and expanding phases; see Eqs. (3) and (4). Combining our results, we arrive at the "primordial" dimensionless strain spectrum at the beginning of the radiation dominated epoch:

$$\Delta h(k,\tau_r) = (k^2/\pi M_{pl})\sqrt{2(1+\chi)\tau_r}\left|B_2(k)H_0^{(1)}(x_r) + B_1(k)H_0^{(2)}(x_r)\right| \tag{16}$$

where $x_r \equiv k\tau_r$ and $k < k_{end}$. For $k > k_{end}$, the spectrum is cut off because these modes are not amplified and, instead, Eq. (16) converges to the result for a static Minkowski background.

The present-day spectrum, $\Delta h(k,\tau_0)$. To convert from the primordial spectrum to the present-day spectrum $\Delta h(k,\tau_0) \equiv T_h(k)\Delta h(k,\tau_r)$ we need to know the transfer function, $T_h(k)$. To approximate $T_h(k)$, note that $\Delta h(k,\tau)$ is roughly time-independent outside the horizon, and decays as a^{-1} once a mode re-enters the horizon. Therefore, the transfer function is $\sim 1/(1+z_r)$ for modes already inside the horizon at the onset of radiation domination τ_r, and $\sim 1/(1+z_k)$ for modes that entered at redshift z_k between τ_r and τ_0. Using the fact that $H \propto a^{-2}$ during radiation domination and $H \propto a^{-3/2}$ during matter domination (neglecting the change in g_*), we find

$$T(k) \approx \left(\frac{k_0}{k}\right)^2\left[1 + \frac{k}{k_{eq}} + \frac{k^2}{k_{eq}k_r}\right] \tag{17}$$

where $k_0 \equiv a_0 H_0$, $k_{eq} \equiv a_{eq}H_{eq}$, $k_r \equiv a_r H_r$, and $k_{end} \equiv a_{end}|H_{end}|$ denote the modes on the horizon today (τ_0), at matter-radiation equality (τ_{eq}), at the start of radiation domination (τ_r), and at the end of the ekpyrotic phase (τ_{end}), respectively.

The gravitational wave spectrum can be divided into three regimes. There is a low frequency (LF) regime corresponding to long wavelength modes that re-enter after matter-radiation equality ($k < k_{eq}$), and a medium frequency (MF) regime consisting of modes which re-enter between equality and the onset of radiation domination ($k_{eq} < k < k_r$). (We ignore the recent dark energy dominated phase, which has negligible effect.) The spectrum for these two regimes is

$$\Delta h \approx \frac{\Gamma^{1/2}k_0^2}{\pi M_{pl}H_r^\alpha}\begin{cases} k^{-1+\alpha} & \text{(LF)} \\ k^\alpha/k_{eq} & \text{(MF)}. \end{cases} \tag{18}$$

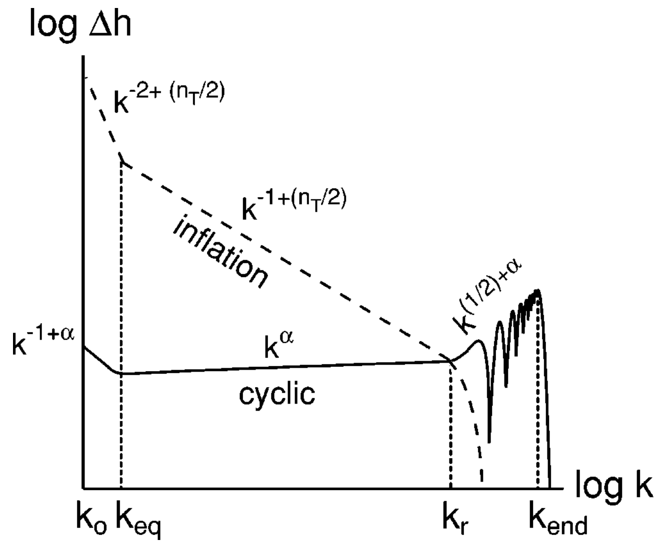

FIG. 2. A schematic comparison of the dimensionless strain observed today $\Delta h(k,\tau_0)$, as predicted by inflation and the cyclic model. Here n_T is the inflationary tensor spectral index (a small negative number), and $\alpha \ll 1$ in the cyclic model is a small positive number. k_r denotes the mode on the horizon at the start of radiation domination.

Finally, modes which exit the horizon during the ekpyrotic phase (before τ_{end}), and re-enter during the expanding kinetic phase (after the bound but before τ_r) result in a high frequency (HF) band ($k_r < k < k_{end}$):

$$\Delta h \approx \left(\frac{\sqrt{2}}{\pi}\right)^{3/2}\frac{(\Gamma H_r)^{1/2-\alpha}k_0^2}{M_{pl}k_{eq}k_r}\left|\cos\left(k\tau_r - \frac{\pi}{4}\right)\right|k^{1/2+\alpha} \text{ (HF)}. \tag{19}$$

The HF band runs over a range $k_{end}/k_r = \Gamma$, and this quantity is strongly constrained by the requirement that the scalar field cross the negative region of the potential before radiation domination begins, which requires that [10]

$$H_r \lesssim \frac{V_{end}^{1/2}}{M_{Pl}}\left(\frac{V_0}{V_{end}}\right)^{\sqrt{3/2}/c}, \tag{20}$$

where V_0 is today's value of the dark energy density. This equation, combined with Eq. (6), gives a lower bound on Γ, $\Gamma \gtrsim (V_{end}/V_0)^{\sqrt{2/3c^2}}$. For example, for V_{end} around the grand unified theory (GUT) scale and $c=10$, we find $\Gamma \geqslant 10^8$. Figure 2 schematically depicts $\Delta h(k,\tau_0)$ in the cyclic scenario and compares it to the inflationary spectrum [15].

Another useful quantity is $\Omega_{gw}(k,\tau_0)$, the gravitational wave energy per unit logarithmic wave number, in units of the critical density [14,15]

$$\Omega_{gw}(k,\tau_0) \equiv \frac{k}{\rho_{cr}}\frac{d\rho_{gw}}{dk} = \frac{1}{6}\left(\frac{k}{k_0}\right)^2\Delta h(k,\tau_0)^2. \tag{21}$$

In the cyclic model, $\Omega_{gw}(k,\tau_0)$ is very blue, with nearly all the gravitational wave energy concentrated at the high-frequency end of the distribution.

Observational constraints and detectability. The strongest observational constraint on the gravitational spectrum in the cyclic model comes from the requirement that the successful predictions of big bang nucleosynthesis (BBN) not be affected, which requires

$$\int_{k_{BBN}}^{k_{end}} \Omega_{gw}(k,\tau_0)\frac{dk}{k} \lesssim \frac{0.1}{1+z_{eq}}. \tag{22}$$

From the above equations, (19) and (21), and using $1+z_{eq} \approx k_{eq}^2/k_0^2$, and $T_r \sim H_r^{1/2} M_{Pl}^{1/2}$ for the temperature at radiation domination, we obtain a total Ω in gravitational waves of $\sim (2\alpha V_{end}/T_r M_{Pl}^3)^{4/3} [36\pi^3(1+z_{eq})]^{-1}$, which from Eq. (22) implies

$$T_r \gtrsim \frac{\alpha}{20} V_{end} M_{Pl}^{-3}, \tag{23}$$

where, for simplicity, we have ignored the factor which depends on the number of thermal degrees of freedom, which further weaken this bound.

The other observational constraints are much weaker [16]. From the cosmic microwave background (CMB) anisotropy, one infers $\Delta h(f \sim 10^{-18}\ \mathrm{Hz}) \lesssim 10^{-5}$; from precision pulsar timing, $\Delta h(f \sim 10^{-8}\ \mathrm{Hz}) \lesssim 10^{-14}$. Optimistic goals for the Laser Interferometer Space Antenna and the advanced Laser Interferometer Gravitational Wave Observatory (LIGO) are strain sensitivities of $\Delta h(f \sim 10^{-4}\ \mathrm{Hz}) \sim 10^{-20.5}$ and $\Delta h(f \sim 10^{2}\ \mathrm{Hz}) \sim 10^{-24}$, respectively. Figure 3 shows results for values of T_r and V_{end} consistent with all constraints on the cyclic model [10].

Even if the parameters are chosen to saturate the BBN constraint, the spectrum is still orders of magnitude below the sensitivity of anticipated instruments. Hence, the detection of a scale-invariant, stochastic gravitational wave imprint in the CMB polarization would be consistent with inflation and rule out the cyclic model.

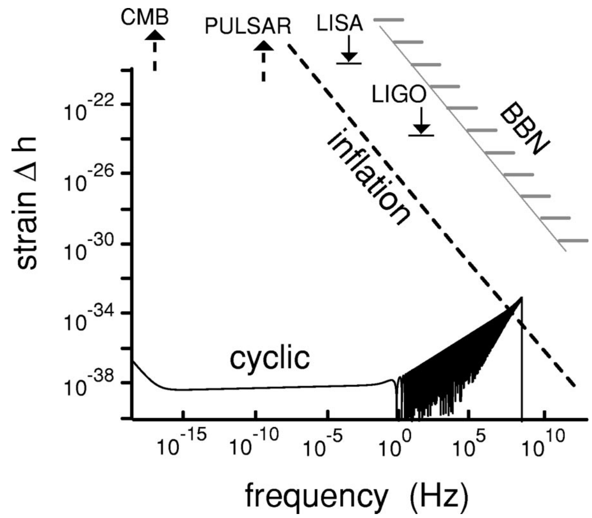

FIG. 3. The present-day dimensionless strain $\Delta h(k,\tau_0)$ predicted by the cyclic model with $T_r = 10^7$ GeV and $V_{end}^{1/4} = 10^{14}$ GeV. These parameters yield a gravity wave density four orders of magnitude below the BBN bound. Some observational bounds and (optimistic) future strain sensitivities are indicated. The dashed arrows mean the empirical bounds lie well above range of Δh displayed here.

We thank A. Tolley and J. Khoury for helpful conversations. L.B. is supported by the NSF. This work was also supported in part by U.S. Department of Energy grant DE-FG02-91ER40671 (P.J.S.) and by PPARC-UK (N.T.).

[1] P.J. Steinhardt and N. Turok, Science **296**, 1436 (2002).
[2] P.J. Steinhardt and N. Turok, Phys. Rev. D **65**, 126003 (2002).
[3] A.H. Guth, Phys. Rev. D **23**, 347 (1981).
[4] A.D. Linde, Phys. Lett. **108B**, 389 (1982); A. Albrecht and P.J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).
[5] G.T. Horowitz and J. Polchinski, Phys. Rev. D **66**, 103512 (2002); H. Liu, G. Moore, and N. Seiberg, J. High Energy Phys. **06**, 045 (2002); D. Lyth, Phys. Lett. B **524**, 1 (2002); R. Brandenberger and F. Finelli, J. High Energy Phys. **11**, 056 (2001); J. Martin, P. Peter, N. Pinto-Neto, and D.J. Schwarz, Phys. Rev. D **65**, 123513 (2002); **67**, 028301 (2003); J. Martin and P. Peter, Phys. Rev. Lett. **92**, 061301 (2004).
[6] R. Durrer and F. Vernizzi, Phys. Rev. D **66**, 083503 (2002); C. Cartier, R. Durrer, and E.J. Copeland, *ibid.* **67**, 103517 (2003); B. Craps and B.A. Ovrut, *ibid.* **69**, 066001 (2004).
[7] T.J. Battefield, S.P. Patil, and R. Brandenberger, hep-th/0401010.
[8] A.J. Tolley and N. Turok, Phys. Rev. D **66**, 106005 (2002); A.J. Tolley, N. Turok, and P.J. Steinhardt, *ibid.* **69**, 106005 (2004).
[9] P.J. Steinhardt and N. Turok, Nucl. Phys. B (Proc. Suppl.) **124**, 38 (2003).
[10] J. Khoury, P.J. Steinhardt, and N. Turok, Phys. Rev. Lett. **92**, 031302 (2004).
[11] J. Khoury, B.A. Ovrut, P.J. Steinhardt, and N. Turok, Phys. Rev. D **64**, 123522 (2002).
[12] J. Khoury, B.A. Ovrut, N. Seiberg, P.J. Steinhardt, and N. Turok, Phys. Rev. D **65**, 086007 (2002).
[13] J. Khoury, B.A. Ovrut, P.J. Steinhardt, and N. Turok, Phys. Rev. D **66**, 046005 (2002).
[14] K.S. Thorne, in *300 Years of Gravitation*, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1987), pp. 330–458.
[15] M.S. Turner, Phys. Rev. D **55**, 435 (1997).
[16] B. Allen, gr-qc/9604033; K.S. Thorne, gr-qc/9506086.