

$U(N)$ instantons on $\mathcal{N}=\frac{1}{2}$ superspace: Exact solution and geometry of moduli space

Ruth Britto,^{1,*} Bo Feng,^{1,†} Oleg Lunin,^{1,‡} and Soo-Jong Rey^{1,2,§}

¹*Institute for Advanced Study, Einstein Drive, Princeton, New Jersey 08540, USA*

²*School of Physics & BK-21 Physics Division, Seoul National University, Seoul 151-747, Korea*

(Received 26 January 2004; published 28 June 2004)

We construct the exact solution of one (anti-)instanton in $\mathcal{N}=\frac{1}{2}$ super Yang-Mills theory defined on non(anti-)commutative superspace. We first identify $\mathcal{N}=\frac{1}{2}$ superconformal invariance as maximal spacetime symmetry. For the gauge group $U(2)$, the $SU(2)$ part of the solution is given by the standard (anti-)instanton, but the $U(1)$ field strength also turns out to be nonzero. The solution is $SO(4)$ rotationally symmetric. For the gauge group $U(N)$, in contrast with the $U(2)$ case, we show that the entire $U(N)$ part of the solution is deformed by non(anti-)commutativity and fermion zero modes. The solution is no longer rotationally symmetric; it is polarized into an axially symmetric configuration because of the underlying non(anti-)commutativity. We compute the “information metric” of one (anti-)instanton. We find that the moduli space geometry is deformed from the hyperbolic space \mathbb{H}_5 (Euclidean anti-de Sitter space) in a way anticipated from reduced spacetime symmetry. Remarkably, the volume measure of the moduli space turns out to be independent of the non(anti-)commutativity. Implications for D branes in the Ramond-Ramond flux background and the gauge-gravity correspondence are discussed.

DOI: 10.1103/PhysRevD.69.126004

PACS number(s): 11.25.-w, 11.25.Uv

I. INTRODUCTION

Recently, there has been considerable development in understanding superstrings and D branes in the background of the Ramond-Ramond flux. Take type-IIB superstring theory compactified on $X \times \mathbb{R}^4$, where X is a Calabi-Yau threefold. Turn on a Ramond-Ramond five-form G_5^+ on a holomorphic cycle of X ; the flux corresponds on \mathbb{R}^4 to a self-dual graviphoton flux. Introduce $D3$ branes whose world volume fills \mathbb{R}^4 . For closed strings, the graviphoton flux deforms the four-dimensional $\mathcal{N}=2$ supersymmetry algebra, in which half of the supersymmetry is realized nonlinearly. For open strings on the Euclidean $D3$ branes, the graviphoton flux deforms the $\mathcal{N}=1$ supersymmetry [1–5]. The deformation induces non(anti-)commutativity among the Grassmann-odd coordinates,

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}, \quad \{\bar{\theta}^\alpha, \bar{\theta}^\beta\} = 0, \quad \{\theta^\alpha, \bar{\theta}^\alpha\} = 0, \quad (1)$$

and breaks the underlying $\mathcal{N}=1$ supersymmetry to $\mathcal{N}=\frac{1}{2}$. Accordingly, the low-energy world-volume dynamics of a Euclidean $D3$ brane is governed by a non(anti-)commutative super-Yang-Mills theory with $\mathcal{N}=\frac{1}{2}$ supersymmetry.¹ The $\mathcal{N}=\frac{1}{2}$ super Yang-Mills theory is then defined by² the action functional [4]

$$S_{\text{YM}} = - \int_{\mathbb{R}^4} \text{Tr} \left[\frac{i\tau}{8\pi} W^\alpha \star W_\alpha \right]_{\theta^2} + \text{Tr} \left[\frac{i\bar{\tau}}{8\pi} \bar{W}_\alpha \star \bar{W}^\alpha \right]_{\bar{\theta}^2}, \quad (2)$$

where the non(anti-)commutativity (1) is realized in terms of the \star product:

$$A(\theta) \star B(\theta) \equiv A(\theta) \exp \left(-\frac{1}{2} C^{\alpha\beta} \frac{\bar{\partial}}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta} \right) B(\theta). \quad (3)$$

Though the non(anti-)commutativity parameter $C^{\alpha\beta}$ carries a nonzero scaling dimension, it turns out that, to all orders in perturbation theory, the non(anti-)commutative deformation of a renormalizable $\mathcal{N}=1$ supersymmetric field theory remains renormalizable [7,8]. Intuitively, in Wilson’s renormalization-group viewpoint, the renormalizability is explainable by chirally asymmetric assignments of scaling dimensions, a possibility made available by the non(anti-)commutative deformation [8].

We are primarily interested in the low-energy dynamics of $\mathcal{N}=\frac{1}{2}$ supersymmetric gauge theory. The motivation comes largely from two sides. First, the dynamics by itself is quite interesting and may provide a novel way of interpolating between gauge dynamics with $\mathcal{N}=1$ and $\mathcal{N}=0$ supersymmetries. Second, the dynamics may probe the Calabi-Yau geometry with the Ramond-Ramond flux G_5 turned on. It then becomes imperative to understand instantons in $\mathcal{N}=\frac{1}{2}$ supersymmetric gauge theories.

In exploring instantons in the $\mathcal{N}=\frac{1}{2}$ super-Yang-Mills theory, a variety of interesting questions arise. At the ultraviolet fixed point, the $\mathcal{N}=1$ theory is known to promote the Poincaré supersymmetry to superconformal symmetry. The superconformal symmetry algebra is $SU(4|1)$ and involves 16 Bosonic generators and 8 Fermionic ones. We will show that, once the non(anti-)commutativity is turned on, the sym-

*Electronic address: britto@ias.edu

†Electronic address: fengb@ias.edu

‡Electronic address: lunin@ias.edu

§Electronic address: sjrey@snu.ac.kr

¹For recent works dealing with various aspects of theories with $\mathcal{N}=\frac{1}{2}$ supersymmetry, see Ref. [6].

²Our conventions and notation are collected in Appendix A.

metry algebra is reduced to an $\mathcal{N}=\frac{1}{2}$ superconformal algebra. In this reduced symmetry algebra, the special conformal and the chiral $SU(2)_L$ generators (as well as their Fermionic partners) are removed, and the dilatation and the R -symmetry generators combine into a single generator of the form dictated precisely by the new scaling dimension assignment put forward in Ref. [8].

Despite being deformed by the non(anti-)commutativity, the instanton carries an integrally quantized topological charge,

$$Q_{\text{instanton}} = \int_{\mathbb{R}^4} \frac{1}{8\pi^2} \mathcal{F} = \mathbb{Z}, \quad \text{where} \quad \mathcal{F} := \text{Tr}_{U(N)} F \wedge F. \quad (4)$$

This is in full accord with the Atiyah-Singer index theorem and assures that the deformed anti-instantons are analytic in $C^{\alpha\beta}$. There is a good rationale behind this. The instanton supports Fermionic zero modes. What is nontrivial in the present context is that the instanton solution is corrected by the Fermionic zero modes. Accordingly, the topological charge *density* \mathcal{F} itself depends not only on the Bosonic zero modes but also on (even powers of) Fermionic zero modes. Moreover, since the non(anti-)commutative superspace is not invariant under the full $SO(4) = SU(2)_L \times SU(2)_R$ rotation group, the instanton would not be rotationally symmetric in general.

It turns out that the above two issues are intimately related. For the gauge group $G = U(2)$, we will find that the one-instanton solution exhibits trivial dependence on the non(anti-)commutativity—the $SU(2)$ part of the solution is the standard instanton, and the $U(1)$ part is a multipole configuration induced through Fermionic zero modes and non(anti-)commutativity. The $U(1)$ part cannot contribute to the topological charge; this is how the deformed instanton remains consistent with the Atiyah-Singer index theorem. The entire configuration is spherically symmetric, *viz.* the $U(2)$ instanton exhibits accidentally larger spacetime symmetries.

For a gauge group of higher rank, $G = U(N \geq 3)$, the story is far more interesting and intricate. Start with the standard $SU(2)$ instanton embedded in $U(N)$, and examine how the non(anti-)commutativity deforms the instanton configuration. In stark contrast to the $G = U(2)$ case, we find that the one-instanton solution is deformed not only in the $U(N-2)$ part but in the $SU(2)$ part as well. As the attentive reader will notice, this leads immediately to the possibility that the topological charge density, and hence the charge itself, depends on the Fermionic zero modes. We shall find that the topological charge density indeed depends on the Fermionic zero modes, but the charge itself is actually independent of them. The way this is made possible turns out to be nicely intertwined with the absence of rotational invariance in the problem. We will demonstrate that the deformation induced by the non(anti-)commutativity polarizes the topological charge density into a sort of dipole configuration. The deformation is axially symmetric but is fully compatible with the antichiral $SU(2)_R$ invariance. Thus, once integrated over \mathbb{R}^4 , the dipolar deformation is washed out, retaining only a

spherically symmetric contribution from the standard $SU(2)$ instanton. The latter yields integrally quantized topological charge.

One can learn more physics from the topological charge *density* \mathcal{F} , since it is a function of Bosonic and Fermionic zero modes in addition to being a function of coordinates on \mathbb{R}^4 . What one expects to be modified by the non(anti-)commutative deformation is the geometry of the one-instanton moduli space. To explore the issue, we compute the information metric of one instanton, first put forward by Hitchin [9]. For small instantons, we find that the information metric approaches that of a five-dimensional hyperbolic space H_5 (Euclidean anti-de Sitter space, AdS_5). The asymptotic isometry $SO(5,1)$ is much bigger than the $\mathcal{N}=\frac{1}{2}$ superconformal symmetry, so one expects the interior of the moduli space not to retain the H_5 geometry globally. Indeed, we find that the geometry of the moduli space is deformed for larger instantons—by non(anti-)commutativity, not only is each metric component deformed, but also off-diagonal components of the metric are induced. In fact, these corrections are fully compatible with the symmetries that underlie the theory: $\mathcal{N}=\frac{1}{2}$ supersymmetry, R -(pseudo)symmetry and dilatation symmetry. Remarkably, after a suitable change of zero-mode variables, the volume measure on the moduli space turns out to be *independent* of the non(anti-)commutative deformation. This observation bears implications for Maldacena’s gauge-gravity correspondence, on which we will elaborate in Sec. VII.

We have organized the present paper as follows. In Sec. II, we analyze the spacetime symmetry for theories defined on $\mathcal{N}=\frac{1}{2}$ superspace. We find that the underlying $\mathcal{N}=1$ superconformal symmetry is broken explicitly to “half” of it, yielding what we call $\mathcal{N}=\frac{1}{2}$ superconformal symmetry. This symmetry will provide a useful guideline for constructing $\mathcal{N}=\frac{1}{2}$ instantons in subsequent sections. In Sec. III, we derive self-duality and anti-self-duality equations by localizing the action on appropriate supersymmetric loci in field configuration space. In Sec. IV, we construct the instanton for the gauge group $G = U(2)$. This is a special situation where the instanton calculus becomes almost trivial, due in a major part to the trivial back reaction of the fermion quasi-zero modes to the undeformed instanton. In Sec. V, we construct the instanton for gauge groups of higher rank, namely $G = U(N)$ for $N \geq 3$. To illustrate the general strategy, we first set superconformal Fermionic zero modes to zero, and consider perturbations by supersymmetry Fermionic zero modes only. In Sec. VI, we include the superconformal Fermionic zero modes and find the exact instanton solution for gauge group $G = U(N)$. In both sections, we set out analytic strategy in a way adaptable for the Atiyah-Drinfeld-Hitchin-Manin (ADHM) method [10], relegating a direct ADHM construction for multi-instantons to future work. In Sec. VII, we study the profile on \mathbb{R}^4 of the topological charge density. We find that the density exhibits dipolar polarization, whose size is set by the non(anti-)commutative deformation and whose symmetry fits precisely with the underlying spacetime symmetries. We next study the density profile on the instanton moduli space by computing Hitchin’s information metric. We find that the geometry of the moduli space asymptotes to

that of the H_5 (Euclidean anti-de Sitter space) near the boundary. In the interior, the moduli space of the geometry is deformed by the non(anti-)commutativity, but again in a form fully compatible with the underlying spacetime symmetries. We discuss aspects of this observation in the context of Maldacena's gauge-gravity correspondence. In the appendixes, we collect conventions and notation, undeformed SU(2) instanton and anti-instanton solutions, and some essential steps of the computation for obtaining the exact U(N) solution presented in Sec. VI.

During the progress of this work, a paper by Imaanpur [11] appeared, overlapping with part of our Sec. IV. We find agreement (modulo errors and numerical factors) wherever both results overlap. Also, while this work was being written up, a paper by Grassi *et al.* [12] appeared, again overlapping with part of our Sec. IV. We believe that our motivation, results, and interpretation are in strong contrast to theirs.

II. $\mathcal{N}=\frac{1}{2}$ SUPERCONFORMAL ALGEBRA

We begin with observations regarding symmetry associated to the non(anti-)commutative $\mathcal{N}=\frac{1}{2}$ superspace. The underlying (anti-)commutative $\mathcal{N}=1$ superspace is parametrized by the coordinates $(x^{\alpha\dot{\alpha}}, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ —Bosonic, chiral, and antichiral Fermionic coordinates. The superspace displays $\mathcal{N}=1$ Poincaré supersymmetry. If dilatation invariance is additionally endowed, the symmetry is enlarged to $\mathcal{N}=1$ superconformal symmetry. This is the symmetry we are most interested in. For example, if a theory defined on the superspace has no mass scale, classically and/or quantum mechanically, then the operators and states of the theory are organized in irreducible representations of the superconformal group SU(2,2|1) or SU(4|1).

Once the non(anti-)commutativity deformation is turned on for the chiral Fermionic coordinates as in Eq. (1), the $\mathcal{N}=1$ supersymmetry is broken to $\mathcal{N}=\frac{1}{2}$ supersymmetry. This is seen by examining the deformation of the $\mathcal{N}=1$ supersymmetry algebra. Though the algebra among the $\mathcal{N}=1$ superspace derivatives

$$D_\alpha = +\partial_\alpha + 2i\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \quad \bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}}$$

remains unaffected:

$$\{D_\alpha, D_\beta\}_\star = 0,$$

$$\{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\}_\star = 0,$$

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\}_\star = -2P_{\alpha\dot{\alpha}},$$

the algebra among the $\mathcal{N}=1$ supersymmetry charges

$$Q_\alpha = +\partial_\alpha, \quad \bar{Q}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + 2i\theta^\alpha\partial_{\alpha\dot{\alpha}} \quad (5)$$

now obey deformed anticommutation relations:

$$\{Q_\alpha, Q_\beta\}_\star = 0,$$

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}_\star = 2P_{\alpha\dot{\alpha}},$$

$$\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\}_\star = 4C^{\alpha\beta}P_{\alpha\dot{\alpha}}P_{\beta\dot{\beta}}. \quad (6)$$

The last relation indicates that repeated action of the \bar{Q} supercharges is ill defined, violating the Leibnitz rule.³ As such, Eq. (6) does not form an algebra. The subalgebra generated by the Q_α 's is still preserved, and this defines precisely the chiral $\mathcal{N}=\frac{1}{2}$ supersymmetry algebra.

Implicit in the above route to the $\mathcal{N}=\frac{1}{2}$ supersymmetry is that the non(anti-)commutative superspace is parametrized in terms of so-called chiral coordinates (y, θ) , where y, \bar{y} refer to chiral and antichiral Grassmann-even coordinates:

$$y^{\alpha\dot{\alpha}} := (x^{\alpha\dot{\alpha}} - 2i\theta^\alpha\bar{\theta}^{\dot{\alpha}}) \quad \text{and} \quad \bar{y}^{\alpha\dot{\alpha}} := (x^{\alpha\dot{\alpha}} + 2i\theta^\alpha\bar{\theta}^{\dot{\alpha}}).$$

Various considerations point to this as the correct choice. First, in terms of the chiral coordinates, as observed by Seiberg [4], chiral and antichiral superfields are definable in a manner compatible with the non(anti-)commutative \star product (3). Second, the $\mathcal{N}=\frac{1}{2}$ superspace can be parametrized uniquely by (y, θ^α) , for which the $\mathcal{N}=\frac{1}{2}$ supersymmetry acts as a chiral Grassmann-odd translation:

$$(y^{\alpha\dot{\alpha}}, \theta^\alpha) \rightarrow (y^{\alpha\dot{\alpha}}, \theta^\alpha + \varepsilon^\alpha).$$

Having identified the canonical choice of coordinates on $\mathcal{N}=\frac{1}{2}$ superspace, we are now ready to analyze spacetime symmetries. In doing so, we will come across the idea [8] behind the intuitive proof of renormalizability of non(anti-)commutative field theories. In Ref. [8], it was argued that the most natural assignment of scaling dimensions is such that θ^α is dimensionless, and hence $C^{\alpha\beta}$ also is dimensionless. The new scaling dimension is now measured as a particular linear combination of the conventional scaling dimension and the R -symmetry charge. In other words, the new dilatation operator D_{new} is a linear combination of the conventional dilatation operator D and the R -symmetry charge R . We will now show that this is precisely what comes out of the analysis of spacetime symmetries associated with $\mathcal{N}=\frac{1}{2}$ superspace.

We claim that, on the non(anti-)commutative superspace, the spacetime symmetry is realized on the following set of generators:

$$\bar{M}_{\dot{\alpha}\dot{\beta}}, \quad D_{\text{new}} \equiv D - \frac{1}{2}R, \quad P_{\alpha\dot{\alpha}}, \quad Q_\alpha, \quad \bar{S}^{\dot{\alpha}}, \quad (7)$$

which we refer to as the $\mathcal{N}=\frac{1}{2}$ superconformal symmetry generators. Notice that the special conformal transformation

³Note, however, that a single action of the \bar{Q} charge is meaningful. In particular, the second relation in Eq. (6) indicates that acting $\bar{Q}_{\dot{\alpha}}$ on the $\mathcal{N}=\frac{1}{2}$ supercharges Q_α generates a translation on \mathbb{R}^4 . In the next section, we will use this observation to derive instanton equations.

is no longer part of the symmetry, so the symmetry group does not encompass the conformal transformations. Rather, it should be viewed as supersymmetrization of the dilatation transformation. This implies that, at a renormalization-group fixed point, scale invariance of non(anti-)commutative field theories would not be enhanced to superconformal invariance, in stark contrast to the more familiar quantum field theories [13]. Implicit to the latter is the requirement of unitarity and Poincaré invariance, but these are precisely what we drop in non(anti-)commutative field theories.

The proof of Eq. (7) is elementary. Begin by realizing the $\mathcal{N}=1$ superconformal generators, again in the basis of chiral superspace coordinates $(y, \theta, \bar{\theta})$. They are

$$\begin{aligned}\bar{M}_{\dot{\alpha}\dot{\beta}} &= \frac{1}{2} y^\gamma (\dot{\alpha} \dot{\beta} \dot{\gamma}) - \bar{\theta}_{(\dot{\alpha}} \bar{\theta}_{\dot{\beta})}; & P_{\alpha\dot{\alpha}} &= i \partial_{\alpha\dot{\alpha}}; \\ M_{\alpha\beta} &= \frac{1}{2} y_{(\alpha} \dot{\gamma} \dot{\beta}) \dot{\gamma} - \theta_{(\alpha} \dot{\beta}), \\ K_{\alpha\dot{\alpha}} &= -i y_{\alpha} \dot{\beta} y^{\dot{\beta}} \dot{\alpha} \dot{\beta} + 2i y_{\alpha} \dot{\beta} \theta_{\dot{\beta}} \dot{\alpha} + 2i y_{\alpha\dot{\alpha}} \\ &\quad \times (\theta^{\dot{\beta}} \dot{\beta} + \bar{\theta}^{\dot{\beta}} \bar{\theta}_{\dot{\beta}}) + 4\theta_{\alpha} \bar{\theta}^{\dot{\beta}} \dot{\alpha} + 2i y_{\alpha} \dot{\beta} \bar{\theta}_{\dot{\beta}} \dot{\alpha}, \\ R &= i \theta^{\alpha} \partial_{\alpha} - i \bar{\theta}^{\dot{\alpha}} \dot{\alpha}; \\ D &= -\frac{i}{2} y^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} + \frac{i}{2} \theta^{\alpha} \partial_{\alpha} + \frac{i}{2} \bar{\theta}^{\dot{\alpha}} \dot{\alpha}, \\ Q_{\alpha} &= \partial_{\alpha}; & \bar{Q}_{\dot{\alpha}} &= -\bar{\theta}_{\dot{\alpha}} + 2i \theta^{\alpha} \partial_{\alpha\dot{\alpha}}, \\ \bar{S}_{\dot{\alpha}} &= y^{\alpha}_{\dot{\alpha}} Q_{\alpha} + 2i \bar{\theta}^{\dot{\beta}} \dot{\alpha}; \\ S_{\alpha} &= -(y_{\alpha} \dot{\beta} + 4i \theta_{\alpha} \bar{\theta}^{\dot{\beta}}) \bar{Q}_{\dot{\beta}} + 2i \theta^{\beta} D_{\alpha}.\end{aligned}$$

It is now straightforward to check \star (anti-)commutators among these generators. In doing so, we need to take into account the non(anti-)commutativity among the θ^{α} 's as in Eq. (1). As mentioned above, all other coordinates (anti-)commute *provided* one adopts the chiral superspace coordinates. One finds by straightforward computation that the algebra closes on the subset (7), whose nonvanishing \star commutators are

$$\begin{aligned}[\bar{M}_{\dot{\alpha}\dot{\beta}}, \bar{M}_{\dot{\gamma}\dot{\rho}}]_{\star} &= \epsilon_{\dot{\alpha}(\dot{\gamma}} \bar{M}_{\dot{\rho})\dot{\beta}} + \epsilon_{\dot{\beta}(\dot{\gamma}} \bar{M}_{\dot{\rho})\dot{\alpha}}, \\ [\bar{M}_{\dot{\alpha}\dot{\beta}}, P_{\dot{\gamma}\dot{\rho}}]_{\star} &= 4P_{\dot{\gamma}(\dot{\alpha}} \epsilon_{\dot{\rho})\dot{\beta}}, \\ [\bar{M}_{\dot{\alpha}\dot{\beta}}, \bar{S}_{\dot{\gamma}}]_{\star} &= \epsilon_{\dot{\alpha}(\dot{\gamma}} \bar{S}_{\dot{\beta})}, \\ [P_{\alpha\dot{\alpha}}, \bar{S}_{\dot{\beta}}]_{\star} &= 2i \epsilon_{\dot{\alpha}\dot{\beta}} Q_{\alpha}, \\ [D_{\text{new}}, P_{\alpha\dot{\alpha}}]_{\star} &= -i P_{\alpha\dot{\alpha}}, \\ [D_{\text{new}}, \bar{S}_{\dot{\alpha}}]_{\star} &= +i \bar{S}_{\dot{\alpha}},\end{aligned}$$

while the rest do not even form an algebra because the deformation induces terms violating the Leibniz rule, much as

the last relation in Eq. (6). We notice that only those generators whose expressions do not contain the coordinate θ are the ones preserved by the deformation.

The algebra (7) shows that translational invariance is retained, while (half of) the SO(4) rotational and special conformal invariance are lost. Therefore one expects that an instanton in $\mathcal{N}=\frac{1}{2}$ would produce only those zero modes associated with these generators, and span the coordinates of one-instanton moduli space. In the following sections, we shall see how this restricted symmetry plays out in adding deformation terms to the one-instanton solution and the metric on the moduli space.

III. (UN)DEFORMED INSTANTON EQUATIONS

In this section, we set up the problem of constructing instantons and anti-instantons in $\mathcal{N}=\frac{1}{2}$ super-Yang-Mills theory. First, and to help set up our notation, we recapitulate the definition of the theory. We then derive instanton and anti-instanton equations and argue that with the self-dual deformation by $C^{\alpha\beta}$, the anti-self-duality equations are deformed, while the self-duality equations are not.

Expanding in terms of the component fields, the action functional of the non(anti-)commutative Yang-Mills theory (2) is given by [4]

$$\begin{aligned}S_{\text{YM}} &= \frac{\text{Im } \tau}{4\pi} \int_{\mathbb{R}^4} \text{Tr} \left[-\frac{1}{2} F_{mn} F_{mn} - i \bar{\lambda} \lambda C_{mn} F_{mn} \right. \\ &\quad \left. + \frac{1}{4} (\bar{\lambda} \lambda)^2 C_{mn} C_{mn} - i \bar{\lambda} \bar{\sigma}^m \nabla_m \lambda + D^2 \right] \\ &\quad - i \frac{\text{Re } \tau}{8\pi} \int_{\mathbb{R}^4} \text{Tr} F_{mn} \tilde{F}_{mn}.\end{aligned}\quad (8)$$

Here we take the gauge group to be $G=U(N)$.⁴ We also denote the coupling parameters in the convention of Minkowski spacetime

$$\text{Re } \tau \equiv \frac{1}{2} (\tau + \bar{\tau}), \quad \text{Im } \tau \equiv \frac{1}{2i} (\tau - \bar{\tau}), \quad (9)$$

but, because the theory is defined on the Euclidean space \mathbb{R}^4 , we interpret them as referring to *two* independent complex coupling constants $\tau, \bar{\tau}$. In particular, by taking τ or $\bar{\tau}$ to infinity, one can localize the super Yang-Mills action functional to $D_{(\alpha} W_{\beta)} = 0$ or $\bar{D}_{(\dot{\alpha}} \bar{W}_{\dot{\beta})} = 0$ field configurations, viz. anti-self-duality and self-duality configurations. Decompose the gauge field strength into self-dual and anti-self-dual parts:

⁴Under the \star product (3), the enveloping algebra involving the Lie algebra $su(N)$ is $u(N)$. We adopt the conventions that the $u(N)$ generators \mathbf{T}^a ($a=0, 1, \dots, N^2-1$) are normalized as $\text{Tr } \mathbf{T}^a \mathbf{T}^b = \frac{1}{2} \delta^{ab}$, and the gauge covariant derivatives are $\nabla_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} + \frac{i}{2} [A_{\alpha\dot{\alpha}}, \cdot]$.

$$F_{mn}^{(+)} \equiv \frac{1}{2}(F + *F)_{mn} = \frac{1}{2}F_{\alpha\beta}\sigma_{mn}^{\alpha\beta},$$

$$F_{mn}^{(-)} \equiv \frac{1}{2}(F - *F)_{mn} = \frac{1}{2}F_{\dot{\alpha}\dot{\beta}}\bar{\sigma}_{mn}^{\dot{\alpha}\dot{\beta}}.$$

We will now derive the localization to self-duality or anti-self-duality configurations explicitly.

A. Antiholomorphic instanton from anti-self-duality

To derive the anti-self-dual equations, we arrange the action functional (8) into perfect squares involving $F^{(+)}$ as⁵

$$S_{\text{YM}} = \frac{\text{Im } \tau}{4\pi} \int_{\mathbb{R}^4} \text{Tr} \left[- \left(F_{mn}^{(+)} + \frac{i}{2} C_{mn} \bar{\lambda} \lambda \right)^2 - i \lambda \sigma^m \nabla_m \bar{\lambda} + D^2 \right] - \frac{i\bar{\tau}}{4\pi} \int_{\mathbb{R}^4} \text{Tr} F \wedge F. \quad (10)$$

The last term is a topological invariant, so the action functional has a critical point at which

$$F_{mn}^{(+)} + \frac{i}{2} C_{mn} \bar{\lambda} \lambda = 0, \quad \sigma^m \nabla_m \bar{\lambda} = 0, \quad \lambda = 0, \quad D = 0. \quad (11)$$

These equations define anti-self-duality conditions, whose solutions are anti-instantons. Notice that, compared to the $\mathcal{N}=1$ supersymmetric anti-self-dual equations, Eqs. (11) are deformed by the terms proportional to the self-dual non(anti)-commutativity parameter C_{mn} . Notice in Eq. (10) that, though expressed into a perfect square, the first term is not positive definite—the non(anti)-commutativity parameter C_{mn} is in general complex valued and the gaugino $\bar{\lambda}$ is no longer a Majorana fermion in Euclidean space. So, the critical point (11) should be understood as an enhanced symmetry point rather than a minimum action configuration. Closely related to this, in the first work of Ref. [7], it was shown that the supersymmetry state is not a configuration of minimum energy but of enhanced symmetry. In fact, owing to the non(anti)-commutativity, the energy (defined as eigenvalues of the Hamiltonian) is in general complex valued.

The anti-self-dual equations are also derivable by considering the $\mathcal{N}=\frac{1}{2}$ supersymmetry transformations. In the action functional (2), the chiral field strength superfield is given in the Wess-Zumino gauge as

$$W_\alpha(y, \theta) = -i\lambda_\alpha(y) + \left[\theta_\alpha D(y) - i \left(F_{\alpha\beta}(y) + \frac{i}{2} C_{\alpha\beta} \bar{\lambda} \lambda(y) \right) \theta^\beta \right] + \theta \theta \nabla_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(y).$$

Component fields transform under the $\mathcal{N}=\frac{1}{2}$ supersymmetry as

$$\delta \lambda^\alpha = i \epsilon^\alpha D + 2 \left(F^{\alpha\beta} + \frac{i}{2} C^{\alpha\beta} \bar{\lambda} \lambda \right) \epsilon_\beta,$$

$$\delta F_{\alpha\beta} = -i \epsilon_{(\alpha} \nabla_{\beta)} \bar{\lambda}^{\dot{\beta}},$$

$$\delta D = -\epsilon^\alpha \nabla_{\alpha\dot{\beta}} \bar{\lambda}^{\dot{\beta}},$$

$$\delta F_{\dot{\alpha}\dot{\beta}} = 0,$$

$$\delta \bar{\lambda}^{\dot{\alpha}} = 0. \quad (12)$$

Take in Eq. (2) the limit $\tau \rightarrow \infty$. In this limit, field configurations are localized to

$$0 = \text{Tr } \epsilon^{\alpha\beta} [W_\alpha \star W_\beta]_{\theta^2} = \text{Tr } \epsilon^{\alpha\beta} \left[-\frac{1}{2} (\delta_\alpha \lambda) (\delta_\beta \lambda) - i \lambda_\alpha \delta_\beta D \right].$$

We find that the configuration is localized where the $\mathcal{N}=\frac{1}{2}$ supersymmetry variations vanish. Moreover, inferring the supersymmetry transformation rules (12), the localization locus is precisely the critical point specified by Eq. (11). We will refer to a configuration satisfying the anti-self-duality conditions (11) as an *antiholomorphic instanton*, since its strength is proportional to multiple powers of $\exp(-2\pi i \bar{\tau})$.

Notice that each equation in Eq. (11) is preserved under the $\mathcal{N}=\frac{1}{2}$ supersymmetry transformations (12), but that does not mean that the functional form of the solution is preserved too. In fact, we shall find in the next section that the solution is corrected through the $C^{\alpha\beta}$ -dependent fermion bilinear term in Eq. (11). This correction has the following implications. Suppose we start with the ordinary instanton solving the anti-self-duality equation $F^{(+)}=0$. This instanton is an L^2 -normalizable solution of the $\bar{\lambda}$ equation in Eq. (11). As is evident from Eq. (11), this solution does not break the $\mathcal{N}=\frac{1}{2}$ supersymmetry; in particular, $\delta F_{\alpha\beta} = \delta D = 0$. It is illuminating to recast this from the underlying $\mathcal{N}=1$ supersymmetry viewpoint. The L^2 -normalizable $\bar{\lambda}$ zero-mode solution breaks “spontaneously” the antichiral supersymmetry (generated by \bar{Q}_α), but this is already broken “explicitly” as the non(anti)-commutativity deformation is turned on. As such, we will refer to the L^2 -normalizable $\bar{\lambda}$ solution solving Eq. (11) as *quasi-zero modes*. As discussed in the previous paragraph, the $\mathcal{N}=\frac{1}{2}$ supersymmetry does not preclude back reaction of these quasi-zero modes to the first equation in Eq. (11). It then modifies the vector potential one started with. Analogously, there will be quasi-superconformal zero modes, which will also react back to the Bosonic equations in Eq. (11).

B. Holomorphic instanton from self-duality

To derive the self-duality conditions, we arrange the action functional (8) into terms involving $F^{(-)}$ as

⁵This method was also considered independently in Ref. [11].

$$S_{\text{YM}} = \frac{\text{Im } \tau}{4\pi} \int_{\mathbb{R}^4} \text{Tr} [- (F_{mn}^{(-)})^2 + \bar{\lambda}^{\dot{\alpha}} [M, \bar{\lambda}_{\dot{\alpha}}] - i \bar{\lambda}^{\dot{\alpha}} \bar{\sigma}_{\dot{\alpha}\alpha}^m \nabla_m \lambda^\alpha + D^2] + \frac{i\tau}{4\pi} \int_{\mathbb{R}^4} \text{Tr} F \wedge F,$$

where the kernel M , which is an (anti-)commutator depending on $A_m, \bar{\lambda}$, is defined by

$$[M, \cdot] := -\frac{1}{2} C^{mn} \{F_{mn}, \cdot\} + C^{mn} \{A_m, \nabla_n \cdot\} + \frac{i}{4} C^{mn} \{A_m, [A_n, \cdot]\} - \frac{i}{16} C_{mn}^2 \{\bar{\lambda}\lambda, \cdot\}.$$

Again, the last term is a topological invariant, so the action functional has a critical point at which

$$F_{mn}^{(-)} = 0, \quad i \bar{\sigma}^m \nabla_m \lambda = 0, \quad \bar{\lambda} = 0, \quad D = 0. \quad (13)$$

These equations are the standard self-duality equations, and being independent of $C^{\alpha\beta}$, they are apparently unmodified by the non(anti-)commutativity deformation.

Actually, the self-duality equations (13) involve some highly nontrivial effects arising from the non(anti-)commutative deformation. This can be seen by resorting to the ‘‘broken’’ antichiral supersymmetry generated by \bar{Q} . The antichiral field strength superfield is given in the Wess-Zumino gauge by

$$\bar{W}_{\dot{\alpha}} = i \bar{\lambda}_{\dot{\alpha}}(\bar{y}) + [\bar{\theta}^{\dot{\alpha}} D(\bar{y}) - i F_{\dot{\alpha}\beta}(\bar{y}) \bar{\theta}^{\dot{\beta}}] + \bar{\theta}^2 [M, \bar{\lambda}_{\dot{\alpha}}].$$

Under the antichiral supersymmetry (generated by \bar{Q}), the component fields transform as

$$\begin{aligned} \bar{\delta} \bar{\lambda}^{\dot{\alpha}} &= -i \bar{\epsilon}^{\dot{\alpha}} D - 2 F^{\dot{\alpha}\beta} \bar{\epsilon}_{\dot{\beta}}, \\ \bar{\delta} A_{\alpha\dot{\alpha}} &= -2i \lambda_{\alpha} \bar{\epsilon}_{\dot{\alpha}}, \\ \bar{\delta} F_{\dot{\alpha}\beta} &= i \bar{\epsilon}_{(\dot{\alpha}} \nabla_{\alpha\dot{\beta})} \lambda^{\alpha}, \\ \bar{\delta} D &= -\bar{\epsilon}^{\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} \lambda^{\alpha} + i \bar{\epsilon}^{\dot{\alpha}} [M, \bar{\lambda}_{\dot{\alpha}}] - C^{\alpha\beta} \partial_{\alpha\dot{\alpha}} \partial_{\beta\dot{\beta}} \bar{\epsilon}^{\dot{\alpha}} \bar{\lambda}^{\dot{\beta}}, \\ \bar{\delta} \lambda_{\alpha} &= \bar{\epsilon}^{\dot{\beta}} C^{\gamma\beta} \partial_{\beta\dot{\beta}} \left[i \epsilon_{\alpha\gamma} D + 2 \left(F_{\alpha\gamma} + \frac{i}{2} C_{\alpha\gamma} \bar{\lambda} \bar{\lambda} \right) \right]. \end{aligned} \quad (14)$$

Take now the limit $\bar{\tau} \rightarrow \infty$. In this limit, the action localizes to the field configuration satisfying

$$\begin{aligned} 0 &= \text{Tr} \epsilon^{\dot{\alpha}\dot{\beta}} [\bar{W}_{\dot{\alpha}} \star \bar{W}_{\dot{\beta}}]_{\bar{\theta}\bar{\theta}} \\ &= \text{Tr} \epsilon^{\dot{\alpha}\dot{\beta}} \left[-\frac{1}{2} (\bar{\delta} \bar{\lambda}_{\dot{\alpha}}) (\bar{\delta} \bar{\lambda}_{\dot{\beta}}) + i \bar{\lambda}_{\dot{\alpha}} \bar{\delta}_{\dot{\beta}} D \right]. \end{aligned}$$

Here, we have used the cyclicity of color trace and the self-duality of the parameter $C^{\alpha\beta}$ to simplify the last term in the second line. Thus the partition function is localized at a place where variations under the broken antichiral supersymmetry

vanish. Recall that, though \bar{Q} 's are broken explicitly by the non(anti-)commutativity, *linear* transformations under the antichiral supersymmetry are well defined. Therefore for infinitesimal variations, the localization is a meaningful notion. We now see from Eq. (14) that the localization takes place precisely at the critical point (13). We will call the solutions of Eq. (13) *holomorphic instantons*, since their amplitude is proportional to $\exp(2\pi i \tau)$.

For holomorphic instantons, chiral fermion zero modes are protected. A nontrivial solution to the λ equation in Eq. (13) breaks the $\mathcal{N}=\frac{1}{2}$ supersymmetry spontaneously. Therefore these zero modes are true Goldstino modes, associated to the spontaneously broken $\mathcal{N}=\frac{1}{2}$ supersymmetry generated by Q_{α} . There will be also superconformal zero modes, since the theory is actually invariant under the $\mathcal{N}=\frac{1}{2}$ superconformal transformations, part of which includes the antichiral superconformal generators $\bar{S}^{\dot{\alpha}}$. Essentially, from the viewpoint of $\mathcal{N}=1$ super-Yang-Mills theory, the $\mathcal{N}=\frac{1}{2}$ supersymmetry coincides with the part spontaneously broken by the instantons.

Summarizing the above considerations, antiholomorphic instantons are solutions of the anti-self-duality equations

$$\begin{aligned} F_{mn}^{(+)} + \frac{i}{2} C_{mn} \bar{\lambda} \lambda &= 0, \quad i \sigma^m \nabla_m \bar{\lambda} = 0, \\ \lambda &= 0, \quad \text{Tr} \frac{1}{8\pi^2} \int_{\mathbb{R}^4} F \wedge F = Z_{-}, \end{aligned} \quad (15)$$

while holomorphic instantons are solutions of the self-duality equations

$$F_{mn}^{(-)} = 0, \quad i \bar{\sigma}^m \nabla_m \lambda = 0, \quad \bar{\lambda} = 0, \quad \text{Tr} \frac{1}{8\pi^2} \int_{\mathbb{R}^4} F \wedge F = Z_{+}. \quad (16)$$

IV. CONSTRUCTING INSTANTONS FOR $G=U(2)$

We will begin with the gauge group $G=U(2)$, as in this case the back-reaction of the fermion quasi-zero modes is rather trivial.⁶ We will always trade the $U(2)$ color indices for chiral or antichiral $SU(2) \times U(1)$ indices, so we express the gauge potential as

$$A_{\alpha\dot{\alpha}}^{\{ab\}} \equiv (2i \mathbf{T}_2 A_{\alpha\dot{\alpha}})_{ab}.$$

Of the Lie algebra $u(2)$, the symmetric part (ab) realizes the $su(2)$ subalgebra, while the antisymmetric part $[ab]$ realizes the $u(1)$ subalgebra.

As elaborated in the previous section, the self-duality equations (16) are exactly the same as that of $\mathcal{N}=1$ super-Yang-Mills theory, i.e., these equations are not deformed by turning on the non(anti-)commutativity. Hence the antiholomorphic instanton solutions are the same as those of $\mathcal{N}=1$

⁶This case of $G=U(2)$ was also considered in Refs. [11] and [12].

super-Yang-Mills theory. For a single antiholomorphic instanton of size ρ and center x_0 , the gauge potential and the field strength are

$$A_{\beta\dot{\beta}}^{\{ab\}} = -\frac{2i}{[(x-x_0)^2 + \rho^2]} \delta_{\beta}^{\{a} x_{\dot{\beta}}^{b\}},$$

$$F_{\alpha\dot{\beta}}^{\{ab\}} = \frac{8i\rho^2}{[(x-x_0)^2 + \rho^2]^2} \delta_{\alpha}^{\{a} \delta_{\dot{\beta}}^{b\}},$$

while the supersymmetry and the superconformal zero modes $\zeta, \bar{\eta}$ of the chiral fermion λ (associated with the spontaneously broken $\mathcal{N}=\frac{1}{2}$ supersymmetry) enter as

$$\lambda_{\alpha} = F_{\alpha\dot{\beta}} \xi^{\dot{\beta}}, \quad \text{where} \quad \xi^{\alpha} = \zeta^{\alpha} + x_{\alpha}^{\dot{\beta}} \bar{\eta}^{\dot{\beta}}.$$

Since the anti-instanton is unaffected by the non(anti-)commutativity and does not entail any new features, we shall not discuss it further.

The anti-self-duality equations (15) show that the gauge field strength is modified by quasi-zero modes of the fermion $\bar{\lambda}$. The coupled first-order equations (16) are solvable by formally treating the deformation parameter $C^{\alpha\beta}$ as a perturbation and iterating fermion back reactions. Because of the Grassmann nature of the fermion zero modes, the iterative procedure will terminate, and we will be able to construct the exact instanton solution.

So, begin with the solution at zeroth order in $C^{\alpha\beta}$. This is the standard instanton solution, solving the anti-self-duality equation, and is given by

$$A_{\beta\dot{\beta}}^{(0)\{ab\}} = -\frac{2i}{[(x-x_0)^2 + \rho^2]} \delta_{\beta}^{\{a} x_{\dot{\beta}}^{b\}},$$

$$F_{\alpha\dot{\beta}}^{(0)\{ab\}} = \frac{8i\rho^2}{[(x-x_0)^2 + \rho^2]^2} \delta_{\alpha}^{\{a} \delta_{\dot{\beta}}^{b\}}. \quad (17)$$

The zeroth-order solution for the quasi-zero modes of $\bar{\lambda}$ [transforming as an adjoint under the SU(2) subgroup] is also standard:

$$\lambda_{\alpha}^{- (0)} = F_{\alpha\dot{\beta}}^{(0)} \bar{\xi}^{\dot{\beta}}, \quad \text{where} \quad \bar{\xi}^{\dot{\beta}} \equiv \bar{\zeta}^{\dot{\beta}} + x_{\alpha}^{\dot{\beta}} \eta^{\alpha}. \quad (18)$$

In computing first-order corrections to the gauge potential, it is useful to keep track of the color indices. For the gauge group $G=U(2)$, the bilinear $(\bar{\lambda}\bar{\lambda})^{\{ab\}}$ is antisymmetric in the color indices a, b for an arbitrary spinor $\bar{\lambda}$, so the $\mathcal{O}(C)$ perturbation acts only on the diagonal U(1) subgroup, not on the SU(2) subgroup. In particular, one can express the perturbation as

$$\frac{i}{2} C_{mn} (\bar{\lambda}^{(0)} \bar{\lambda}^{(0)})^{\{ab\}} = -\epsilon^{ab} \frac{i}{4} C_{mn} \epsilon_{cd} (\bar{\lambda}^{(0)} \bar{\lambda}^{(0)})^{\{cd\}}. \quad (19)$$

This observation is elementary but simplifies the back-reaction computation considerably, and renders *the SU(2) part of the instanton solution unaffected by the non(anti-)*

commutativity. On the other hand, as we will see in the next section, this simplification no longer works for gauge groups $G=U(N \geq 3)$.

The anti-self-duality equation of the diagonal U(1) part now reads

$$(F + *F)_{mn} = -\frac{i}{2} C_{mn} (\bar{\lambda}\bar{\lambda})^{\{cd\}} \epsilon_{cd}.$$

To solve this equation, we first take the exterior derivative of the equation and obtain (after using the Bianchi identity)

$$d*F(x) = -\frac{i}{2} C \wedge d(\bar{\lambda}\bar{\lambda})^{\{cd\}}(x) \epsilon_{cd}.$$

This equation reduces in the Lorentz gauge to

$$\square *A(x) = \left[\frac{i}{2} C \wedge d(\bar{\lambda}\bar{\lambda})^{\{cd\}}(x) \epsilon_{cd} \right].$$

Introduce a prepotential $\Phi(x)$ for the gauge potential, and denote the fermion bilinear as a source J :

$$A(x) = *[C \wedge d\Phi(x)] \quad \text{and} \quad J(x) = \frac{i}{2} \epsilon_{cd} (\bar{\lambda}\bar{\lambda})^{\{cd\}}(x).$$

Notice that this prepotential ansatz for the gauge potential is consistent with the choice of the Lorentz gauge. We have thus reduced the first-order perturbation problem to solving a Poisson equation:

$$\square \Phi(x) = J(x) \quad \text{where} \quad J(x) = 3 \times 2^6 i \frac{\rho^4}{(x^2 + \rho^2)^4} \bar{\zeta}_{\alpha} \bar{\zeta}^{\dot{\alpha}}. \quad (20)$$

Three remarks are in order. First, it is worth emphasizing that the above procedure applies to the construction of multi-instantons as well. Second, concerning the field profile on \mathbb{R}^4 , not only is the zeroth-order solution (17), (18) SO(4) rotationally symmetric, but the deformed solution (20) is also. The SO(4) symmetry is certainly larger than the space-time symmetry identified in Sec. II. Third, the fermion zero modes are not deformed by the non(anti-)commutativity at all. In the latter two points, the gauge group $G=SU(2)$ is exceptional. In the next two sections, for higher-rank gauge groups, we will show that the instanton solution is only axisymmetric, retaining symmetries belonging to $SO(3) \subset SO(4)$ and that the fermion zero modes are deformed further.

V. DEFORMED INSTANTONS FOR $G=U(N \geq 3)$: HALF OF THE STORY

We next consider the gauge group $G=U(N)$ for $N \geq 3$ and find an exact solution for the antiholomorphic instanton. We do so by adopting the same iterative procedure, as it truncates at a finite order in the perturbative expansion. The procedure is, however, far more nontrivial than the $G=U(2)$ case, since there are extra fermion zero modes. To illustrate our strategy for constructing instantons exactly, we

consider in this section a special solution in which the superconformal quasi-zero modes are all set to zero.

Again, we start with the standard SU(2) instanton as the zeroth-order solution⁷ and then use perturbation theory in powers of $C^{\alpha\beta}$ to construct deformed solutions of Eq. (15). At zeroth order, the SU(2) instanton is embedded inside U(N), so we will decompose various U(N) fields into U(2) \times U($N-2$): an adjoint ($\mathbf{3} \oplus \mathbf{1}$), fundamentals ($\mathbf{2}, \overline{\mathbf{N}}-2$) and antifundamentals ($\overline{\mathbf{2}}, \mathbf{N}-2$), and singlets ($\mathbf{1}, \mathbf{N}-2 \otimes \mathbf{N}-2$) under U(2). We use the freedom of global gauge transformation under U($N-2$) to put the gaugino components transforming as fundamentals to some arbitrary unidirectional components in the U($N-2$) subspace. More precisely, we consider the zero mode

$$\bar{\lambda}_{\dot{\alpha}}^{(0)\{a\}i} = \frac{\chi^i}{(x^2 + \rho^2)^{3/2}} \delta_{\dot{\alpha}}^a,$$

and perform a U($N-2$) rotation to put χ^i in the form $\chi^3 \neq 0$, $\chi^4 = \chi^5 = \dots = \chi^N = 0$.⁸ In this way, we have reduced the effective number of gaugino equations to be solved in color space. Notice that the same gauge rotation does not in general put the gauge fields to the same unidirectional components in color space—they are generically nonzero and need to be solved through the anti-self-duality equations.

We expand the gauge field and Fermionic zero modes in powers of C_{mn} :

$$A_m = A_m^{(0)} + A_m^{(1)} + \dots \quad \text{and} \quad \bar{\lambda}_{\dot{\alpha}} = \bar{\lambda}_{\dot{\alpha}}^{(0)} + \bar{\lambda}_{\dot{\alpha}}^{(1)} + \dots, \quad (21)$$

where $A_m^{(k)}$ and $\bar{\lambda}_{\dot{\alpha}}^{(k)}$ are of order $\mathcal{O}(C^k)$, and $A_m^{(0)}$ and $\bar{\lambda}_{\dot{\alpha}}^{(0)}$ refer to the undeformed single instanton solution. Normally, such an iterative procedure would never yield an exact solution. In the present case, what saves us is the fact that the back reaction is generated by a finite number of fermion zero modes. As they are Grassmann valued, after some finitely many steps of the iteration, the back reaction terminates automatically. This is the motivation to first consider a special solution without superconformal zero modes, as the iteration there stops already at second order.

A. First-order back reaction

We now solve explicitly the first-order back reaction to the gauge and gaugino fields. Those residing in the U($N-2$) subgroup are not affected at all, so we only need to concentrate on the U(2) subgroup.

First, the self-dual gauge field equation becomes

⁷Our conventions and notations are summarized in Appendix A, and the explicit form of the undeformed instanton solution is given in Appendix B.

⁸From this point onward, all equations should be interpreted as equations in this particular frame, where $\chi_i = 0$ for $i > 3$. However, for presentational purposes, we will still keep the index i for $\chi_i = 0$. Notice that we do not impose any restriction on the conjugate representation spinor $\bar{\chi}_i$.

$$(\nabla_m A_n^{(1)} - \nabla_n A_m^{(1)})^{(+)} = -\frac{i}{2} C_{mn} \bar{\lambda}^{(0)} \bar{\lambda}^{(0)}.$$

Notice that in this paper we always use ∇_m to denote a covariant derivative with respect to the background gauge potential $A_m^{(0)}$. A more proper notation would be $\nabla_m^{(0)}$, but we hope that using ∇_m does not lead to confusion. An equation of this sort can be reduced to a Laplace equation (see the discussion in Appendix C) by taking an ansatz expressing the first-order gauge field in terms of a matrix-valued prepotential $\Phi^{(1)}$:

$$A_m^{(1)}(x) = C_{mn} \nabla_n \Phi^{(1)}(x).$$

The resulting Laplace equation for the ($N \times N$) matrix prepotential $\Phi(x)$ can be easily solved:

$$\Phi_a^{(1)b}(x) = \delta_a^b \left(\phi_1(x) \bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} + \phi_2(x) \frac{\bar{\chi}_i \chi^i}{\rho^2} \right)$$

and

$$\Phi_i^{(1)j} = \phi_3(x) \frac{\bar{\chi}_i \chi^j}{\rho^2},$$

where

$$\phi_1 = -8i \left[\frac{1}{(r^2 + \rho^2)} + \frac{\rho^2}{(r^2 + \rho^2)^2} \right],$$

$$\phi_2 = \frac{i}{8} \frac{1}{\rho^2 (r^2 + \rho^2)},$$

$$\phi_3 = \frac{i}{4} \frac{1}{\rho^2 (r^2 + \rho^2)}.$$

Next, we simplify the Weyl equation for $\bar{\lambda}^{(1)}$. Substituting the value of $A_m^{(1)}$ into the equation for $\bar{\lambda}$, we get in the first order in C_{mn} :

$$\begin{aligned} \sigma_{\alpha\dot{\alpha}}^m \cdot \nabla_m \bar{\lambda}^{(1)\dot{\alpha}} &= -\frac{i}{2} [A_m^{(1)}, \sigma_{\alpha\dot{\alpha}}^m \bar{\lambda}^{(0)\dot{\alpha}}] \\ &= -\frac{i}{2} C_{mn} [(\nabla_n \Phi^{(1)}), \sigma_{\alpha\dot{\alpha}}^m \bar{\lambda}^{(0)\dot{\alpha}}]. \end{aligned}$$

Using the Fierz identity:

$$C_{mn} \sigma_{\alpha\dot{\alpha}}^m = \sigma_{n\beta\dot{\alpha}} C^{\beta\gamma} \epsilon_{\gamma\alpha},$$

and the Weyl equation for $\bar{\lambda}^{(0)}$, one can show that

$$C_{mn} \sigma_{\alpha\dot{\alpha}}^m \cdot \nabla^n \bar{\lambda}^{(0)\dot{\alpha}} = C^{\beta\gamma} \epsilon_{\gamma\alpha} \sigma_{n\beta\dot{\alpha}} \nabla^n \bar{\lambda}^{(0)\dot{\alpha}} = 0.$$

This simplifies the first-order Weyl equation for $\bar{\lambda}^{(1)}$ as

$$\sigma_{\alpha\dot{\alpha}}^m \cdot \nabla_m \bar{\lambda}^{(1)\dot{\alpha}} = \frac{i}{2} C_{\alpha}^{\beta} \sigma_{\beta\dot{\alpha}}^m \nabla_m [\Phi^{(1)}, \bar{\lambda}^{(0)\dot{\alpha}}].$$

We take an ansatz for $\bar{\lambda}^{(1)}$ in terms of a spinor prepotential $\hat{\Psi}^{(1)}$ as

$$\bar{\lambda}^{(1)\beta} = \bar{\sigma}^{m\dot{\beta}\beta} \nabla_m \hat{\Psi}_\beta^{(1)}.$$

Using the anti-self-duality condition $F_{\alpha\beta}^{(0)}=0$, we then get

$$\nabla^2 \hat{\Psi}_\alpha^{(1)} = -\frac{i}{2} C_\alpha^\beta \sigma_{\beta\dot{\alpha}}^m \nabla_m [\Phi^{(1)}, \bar{\lambda}^{(0)\dot{\alpha}}]. \quad (22)$$

Here $\nabla^2 \equiv \nabla_m \nabla^m$ is the covariant Laplacian with respect to the background gauge potential $A_m^{(0)}$. We look for the solution in a form which factorizes out the C dependence from the prepotential:⁹

$$\hat{\Psi}_\alpha^{(1)} = C_\alpha^\beta \Psi_\beta^{(1)}.$$

Then one can re-express Eq. (22) as

$$i C_\alpha^\beta \sigma_{\beta\dot{\alpha}}^m \nabla_m \left[-i \bar{\sigma}^{n\dot{\alpha}\gamma} \nabla_n \Psi_\gamma^{(1)} - \frac{1}{2} [\Phi^{(1)}, \bar{\lambda}^{(0)\dot{\alpha}}] \right] = 0.$$

A particular solution to this equation obeys

$$\bar{\sigma}^{n\dot{\alpha}\gamma} \nabla_n \Psi_\gamma^{(1)} = \frac{i}{2} [\Phi^{(1)}, \bar{\lambda}^{(0)\dot{\alpha}}].$$

One can solve this equation again by using Green's functions for the Dirac operator. We obtain the solution

$$\begin{aligned} (\Psi_\alpha^{(1)})^b &= -\frac{i}{4} \delta_a^b \frac{x_{\alpha\dot{\alpha}}}{\rho^2} \bar{\xi}^{\dot{\alpha}} \left[\frac{1}{(r^2 + \rho^2)^2} \right] \bar{\chi}_k \chi^k, \\ (\Psi_\alpha^{(1)})^j &= +\frac{i}{2} \frac{x_{\alpha\dot{\alpha}}}{\rho^2} \bar{\xi}^{\dot{\alpha}} \left[\frac{1}{(r^2 + \rho^2)^2} \right] \bar{\chi}_i \chi^j, \\ (\Psi_\alpha^{(1)})^a &= -\frac{x_{\alpha\dot{\alpha}}}{\rho^4} \bar{\chi}_i \left[\frac{1}{(r^2 + \rho^2)^{1/2}} + \frac{\rho^4}{(r^2 + \rho^2)^{5/2}} \right] \bar{\xi}_\beta \bar{\xi}^{\dot{\beta}}, \\ (\Psi_\alpha^{(1)})_a &= +\frac{x_{\alpha\dot{\alpha}}}{\rho^4} \chi^j \left[\frac{1}{(r^2 + \rho^2)^{1/2}} + \frac{\rho^4}{(r^2 + \rho^2)^{5/2}} \right] \bar{\xi}_\beta \bar{\xi}^{\dot{\beta}}. \end{aligned} \quad (23)$$

This completes the first-order computation for back reaction of the fermion quasi-zero modes.

B. Second-order back reaction

Next we compute the second-order back reaction. The second-order perturbation for the gauge field $A_m^{(2)}$ satisfies the equation

$$\begin{aligned} (\nabla_m A_n^{(2)} - \nabla_n A_m^{(2)})^{(+)} + \frac{i}{2} [A_m^{(1)}, A_n^{(1)}]^{(+)} \\ + \frac{i}{2} C_{mn} (\bar{\lambda}_\alpha^{(1)} \bar{\lambda}^{(0)\dot{\alpha}} + \bar{\lambda}_\alpha^{(0)} \bar{\lambda}^{(1)\dot{\alpha}}) = 0. \end{aligned} \quad (24)$$

Again, we remind the readers that the superscript (+) denotes projection onto self-dual components of the antisymmetric tensor. Begin in Eq. (24) with the term

$$[A_m^{(1)}, A_n^{(1)}]^{(+)} = \left(C_{mk} C_{nl} + \frac{1}{2} \epsilon_{mnpq} C_{pk} C_{ql} \right) \nabla_{[k} \Phi^{(1)} \nabla_{l]} \Phi^{(1)}.$$

For an arbitrary antisymmetric tensor T_{kl} , by straightforward computation, one finds an identity:

$$\begin{aligned} \left(C_{mk} C_{nl} + \frac{1}{2} \epsilon_{mnpq} C_{pk} C_{ql} \right) T_{kl} \\ = -\frac{1}{2} C_{kl} C_{kl} T_{mn}^{(+)} + C_{mn} C_{kl} T_{kl}. \end{aligned}$$

This allows us to simplify the commutator:

$$\begin{aligned} [A_m^{(1)}, A_n^{(1)}]^{(+)} = -\frac{1}{2} C_{kl} C_{kl} (\nabla_{[m} (\Phi^{(1)} \nabla_{n]} \Phi^{(1)}))^{(+)} \\ + C_{mn} C_{kl} \nabla_{[k} \Phi^{(1)} \nabla_{l]} \Phi^{(1)}. \end{aligned}$$

Next, express the fermion contribution in Eq. (24) in terms of the prepotential $\Psi_\alpha^{(1)}$ in Eq. (23):

$$\bar{\lambda}_\alpha^{(0)} \bar{\lambda}^{(1)\dot{\alpha}} = \frac{1}{2} C_{kl} \bar{\lambda}_\alpha^{(0)} \bar{\sigma}^{m\dot{\alpha}\alpha} \sigma_{kl\beta}^m \nabla_m \Psi_\beta^{(1)}.$$

Using the identity

$$\bar{\sigma}^m \sigma^{kl} = \frac{1}{2} (\eta^{ml} \bar{\sigma}^k - \eta^{mk} \bar{\sigma}^l - \epsilon^{mkl n} \bar{\sigma}^n),$$

and the self-duality of the non(anti)-commutativity tensor C_{kl} , the fermion contribution in Eq. (24) can be simplified as

$$\bar{\lambda}_\alpha^{(0)} \bar{\lambda}^{(1)\dot{\alpha}} = C_{kl} \bar{\lambda}_\alpha^{(0)} \bar{\sigma}^{k\dot{\alpha}\alpha} \nabla_l \Psi_\alpha^{(1)}.$$

Substituting these two expressions into Eq. (24) for $A_m^{(2)}$, one finds

$$\begin{aligned} (\nabla_{[m} A_n^{(2)})^{(+)} - \frac{i}{8} C_{kl} C_{kl} (\nabla_{[m} (\Phi^{(1)} \nabla_{n]} \Phi^{(1)}))^{(+)} \\ + \frac{i}{4} C_{mn} C_{kl} (\nabla_{[k} \Phi^{(1)} \nabla_{l]} \Phi^{(1)} + \bar{\sigma}^{k\dot{\alpha}\alpha} [\bar{\lambda}_\alpha^{(0)}, \nabla_l \Psi_\alpha^{(1)}]) = 0. \end{aligned}$$

This equation is solvable by taking again a prepotential ansatz of the form

$$A_m^{(2)} = \frac{i}{8} C_{kl} C_{kl} \Phi \nabla_m \Phi + C_{mn} \nabla_n \Phi^{(2)},$$

and reducing it to the Poisson equation for the $(N \times N)$ matrix-valued prepotential $\Phi^{(2)}$:

⁹This step need not work in general, but will be justified by our explicit solution.

$$\nabla^2 \Phi^{(2)} = i C_{kl} (\nabla_{[k} \Phi^{(1)} \nabla_{l]} \Phi^{(1)} + \bar{\sigma}^{k\dot{\alpha}\alpha} [\bar{\lambda}_{\dot{\alpha}}^{(0)}, \nabla_l \Psi_{\alpha}^{(1)}]).$$

Again, the solution is obtained by convolving the scalar Green function on the right-hand side. We find that $\Phi^{(2)}$ has nonzero components on the SU(2) subspace only:

$$\begin{aligned} (\Phi^{(2)})_a^b &= -2i C_{mk} (\bar{\sigma}^{kn})^b{}_a \\ &\times \left[\frac{x_m x_n}{\rho^4} \left(\frac{1}{(r^2 + \rho^2)^2} + \frac{\rho^2}{(r^2 + \rho^2)^3} \right) \right] \bar{\chi}_i \chi^i \bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}}. \end{aligned} \quad (25)$$

To complete the iteration, one would next substitute the solution found above into the $\bar{\lambda}_{\dot{\alpha}}^{(2)}$ field equations and solve the second-order back reaction to the fermion quasi-zero modes. It is readily counted that the source term in the corresponding Weyl equation contains fifth powers of the fermion quasi-zero modes. Now that there are only four zero modes $\bar{\xi}_{\dot{\alpha}}$, $\bar{\chi}_i$, and χ^i , the source term vanishes identically. We thus find that second-order back reaction to the fermions is absent, i.e., $\bar{\lambda}_{\dot{\alpha}}^{(2)} = 0$. By the same reasoning, all higher-order back reactions $A_m^{(k)}$ and $\bar{\lambda}_{\dot{\alpha}}^{(k)}$ vanish identically for $k > 2$.

In summary, in the special situation where the superconformal zero mode is set to zero, we have succeeded in obtaining the exact solution for the anti-instanton as

$$\begin{aligned} A_m &= A_m^{(0)} + C_{mn} \nabla_n \Phi^{(1)} + \frac{i}{8} C_{kl} C_{kl} \Phi^{(1)} \nabla_m \Phi^{(1)} + C_{mn} \nabla_n \Phi^{(2)}, \\ \bar{\lambda}^{\dot{\alpha}} &= \bar{\lambda}^{(0)\dot{\alpha}} + \bar{\sigma}^{m\dot{\alpha}\alpha} C_{\alpha}^{\beta} \nabla_m \Psi_{\beta}^{(1)}, \end{aligned} \quad (26)$$

where the Bosonic prepotentials $\Phi^{(1)}, \Phi^{(2)}$ are given in Eqs. (22) and (25), while the Fermionic prepotential $\Psi_{\alpha}^{(1)}$ is given in Eq. (23).

VI. DEFORMED INSTANTONS FOR $G = U(N \geq 3)$: FULL STORY

We now extend the result of the previous section and obtain an exact solution for the antiholomorphic instanton in which all quasi-zero modes of the antichiral gaugino are turned on. Compared to the previous section, the iterative steps do not truncate at the second order because the superconformal zero modes of the fermions render the source terms in the Poisson equation far more complicated. Nevertheless, as there are only a finite number of fermion zero modes, the iteration is truncated beyond some higher order. One can thus follow the Green-function method illustrated in the previous section to obtain the instanton solution.

We performed the computation in the way indicated and obtained the exact anti-instanton solution in the following form:

$$\begin{aligned} A_m &= A_m^{(0)} + C_{mn} \nabla_n (\Phi^{(1)} + \Phi^{(2)} + \Phi^{(3)}) \\ &+ \frac{i}{16} C_{kl} C_{kl} [\Phi^{(1)}, \nabla_n \Phi^{(1)}], \\ \bar{\lambda}^{\dot{\alpha}} &= \bar{\lambda}^{(0)\dot{\alpha}} + \bar{\sigma}^{m\dot{\alpha}\alpha} C_{\alpha}^{\beta} \nabla_m (\Psi_{\beta}^{(1)} + \Psi_{\beta}^{(2)}) \\ &- \frac{C_{kl} C^{kl}}{32} [\Phi^{(1)}, [\Phi^{(1)}, \bar{\lambda}^{(0)\dot{\alpha}}]]. \end{aligned} \quad (27)$$

Here the superscripts denote the order of C expansion they contribute. The zeroth-order, undeformed solution $A_m^{(0)}$ is summarized in Appendix B. Now that we have $2N$ fermion zero modes, reduced effectively via $U(N-2)$ gauge rotation to six zero modes ($\bar{\xi}_{\dot{\alpha}}$, η^{α} , $\bar{\chi}_i$, χ^i), the perturbation expansions continue to the third order for the gauge field and to the second order for the fermion zero modes. We relegate computational details to Appendix D, and collect below the final result, order by order, using the shorthand notation $\bar{\xi}^{\dot{\alpha}} \equiv \bar{\xi}^{\dot{\alpha}} + x_{\alpha}^{\dot{\alpha}} \eta^{\alpha}$. The Bosonic prepotentials are

$$\begin{aligned} (\Phi^{(1)})_a^b &= -8i \left[\frac{\rho^2}{(r^2 + \rho^2)^2} \bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} + \frac{1}{r^2 + \rho^2} (\bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} + \rho^2 \eta^{\alpha} \eta_{\alpha}) \right. \\ &\quad \left. - \frac{1}{r^2 + \rho^2} \frac{\bar{\chi}_i \chi^i}{64\rho^2} \right] \delta_a^b, \\ (\Phi^{(1)})_a^i &= -\frac{2\bar{\xi}_{\dot{\alpha}} \chi^i}{(r^2 + \rho^2)^{3/2}}, \quad (\Phi^{(1)})_i^a = -\frac{2\bar{\chi}_i \bar{\xi}^{\dot{\alpha}}}{(r^2 + \rho^2)^{3/2}}, \\ (\Phi^{(1)})_i^j &= i \frac{1}{r^2 + \rho^2} \frac{\bar{\chi}_i \chi^j}{4\rho^2} \end{aligned} \quad (28)$$

at $\mathcal{O}(C)$,

$$\begin{aligned} (\Phi^{(2)})_a^b &= -2i C_{mk} \frac{1}{(\rho^2 + r^2)^3} (\bar{\sigma}^{kn})^b{}_a \frac{x_m x_n \bar{\chi}_i \chi^i}{\rho^2} \\ &\times \left[\frac{\bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}}}{\rho^2} (r^2 + 2\rho^2) - \rho^2 \eta^{\alpha} \eta_{\alpha} - \eta^{\alpha} x_{\alpha\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} \right] \\ &+ 2i \frac{1}{(\rho^2 + r^2)^2} \frac{\bar{\chi}_i \chi^i}{\rho^2} [\bar{\xi}_{\dot{\alpha}} (\bar{x}C)^{b\alpha} \eta_{\alpha} + \bar{\xi}^b (\bar{x}C)_{\dot{\alpha}}^{\alpha} \eta_{\alpha}], \\ (\Phi^{(2)})_i^a &= -8 \frac{\bar{\chi}_i}{(r^2 + \rho^2)^{5/2}} \left[(r^2 + 2\rho^2) \frac{\bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}}}{\rho^2} (\bar{x}C)^{a\alpha} \eta_{\alpha} \right. \\ &\quad \left. + \eta^{\alpha} \eta_{\alpha} (\bar{x}C)^a{}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} \right], \end{aligned} \quad (29)$$

$$(\Phi^{(2)})_a^i = -8 \frac{\chi^i}{(r^2 + \rho^2)^{5/2}} \left[(r^2 + 2\rho^2) \frac{\bar{\xi}_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\alpha}}}{\rho^2} (\bar{x}C)_a^\alpha \eta_\alpha + \eta^\alpha \eta_\alpha (\bar{x}Cx)_{a\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} \right]$$

at $\mathcal{O}(C^2)$, and

$$\Phi^{(3)} = i \frac{C_{kl} C_{kl}}{2} \frac{\eta^\alpha \eta_\alpha \bar{\xi}_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\alpha}} \bar{\chi}_i \chi^i}{\rho^4 (r^2 + \rho^2)^3} \text{diag}[r^4 + 6r^2 \rho^2 + 3\rho^4, r^4 + 6r^2 \rho^2 + 3\rho^4, 2(r^4 + 4r^2 \rho^2 + \rho^4)] \quad (30)$$

at $\mathcal{O}(C^3)$, respectively. Here, x denotes the matrix $x^{\alpha\dot{\alpha}}$, \bar{x} denotes $x^{\alpha\alpha}$, and C is a matrix with components

$$C_\alpha^\beta = \epsilon_{\alpha\gamma} C^{\gamma\beta} = \frac{1}{2} C_{mn} (\sigma^{mn})_\alpha^\beta.$$

The Fermionic prepotentials are

$$(\Psi_\alpha^{(1)})_a^b = -\frac{i}{4} \delta_a^b \left[\frac{x_{\alpha\dot{\alpha}} \bar{\xi}^{\dot{\alpha}}}{\rho^2} \frac{1}{(r^2 + \rho^2)^2} - \frac{r^2 \eta_\alpha}{\rho^4} \frac{1}{(r^2 + \rho^2)} \right] \bar{\chi}_k \chi^k,$$

$$(\Psi_\alpha^{(1)})_i^a = -\frac{x_\alpha^a}{\rho^4} \bar{\chi}_i \left[\frac{(r^2 + 3\rho^2)}{(r^2 + \rho^2)^{3/2}} \bar{\xi}_{\dot{\beta}} \bar{\xi}^{\dot{\beta}} + \frac{2\rho^4}{(r^2 + \rho^2)^{5/2}} \bar{\xi}_{\dot{\beta}} \bar{\xi}^{\dot{\beta}} \right] - \frac{4\bar{\xi}^a \eta_\alpha \bar{\chi}_i}{(r^2 + \rho^2)^{3/2}}, \quad (31)$$

$$(\Psi_\alpha^{(1)})_a^i = \frac{x_{\alpha a}}{\rho^4} \chi^i \left[\frac{(r^2 + 3\rho^2)}{(r^2 + \rho^2)^{3/2}} \bar{\xi}_{\dot{\beta}} \bar{\xi}^{\dot{\beta}} + \frac{2\rho^4}{(r^2 + \rho^2)^{5/2}} \bar{\xi}_{\dot{\beta}} \bar{\xi}^{\dot{\beta}} \right] + \frac{4\bar{\xi}_a \eta_\alpha \chi^i}{(r^2 + \rho^2)^{3/2}},$$

$$(\Psi_\alpha^{(1)})_i^j = \frac{i}{2} \frac{\bar{\chi}_i \chi^j}{\rho^2 (r^2 + \rho^2)} \left[x_{\alpha\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} - \frac{r^2}{\rho^2} \eta_\alpha \right]$$

at $\mathcal{O}(C)$ order, and

$$(\Psi_\alpha^{(2)})_a^b = \frac{\bar{\chi}_i \chi^i}{\rho^2 (r^2 + \rho^2)^3} \left[4C_{mn} x_m (\bar{\sigma}^{nk})^b_{\alpha} x_k x_{\alpha\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} \eta^\beta \eta_\beta - 2(x_{\alpha\dot{\alpha}} \bar{x}^{b\dot{\beta}} C_{\dot{\beta}\gamma} \eta_\gamma + \bar{x}^{b\dot{\beta}} C_{\beta\dot{\alpha}} \eta^\gamma x_{\gamma\dot{\alpha}}) \bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} - \frac{r^2 + \rho^2}{\rho^2} (x_{\alpha\dot{\alpha}} \bar{x}^{b\dot{\beta}} C_{\dot{\beta}\gamma} \eta_\gamma + x^b_{\alpha} x_{\sigma\dot{\alpha}} \eta^\gamma C_{\gamma}^\sigma) \bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} \right],$$

$$(\Psi_\alpha^{(2)})_a^i = -8i \frac{C_{\alpha\beta}^{\beta a} \chi^i}{(r^2 + \rho^2)^{5/2}} \eta^\gamma \eta_\gamma \bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}};$$

$$(\Psi_\alpha^{(2)})_i^a = -8i \frac{C_{\alpha\beta}^{\beta a} \bar{\chi}_i}{(r^2 + \rho^2)^{5/2}} \eta^\gamma \eta_\gamma \bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} \quad (32)$$

at $\mathcal{O}(C^2)$ order.

We emphasize again the reasoning behind truncation of the iterative process at the third order, not N th order. Though there are $2N$ fermion zero modes, by making use of the $U(N-2)$ gauge rotation, we have brought the bi-fundamental fermions $\chi, \bar{\chi}$ to a unidirection. Therefore, with the aid of the gauge freedom, we have effectively reduced the independent components of the fermion zero modes to 6: two (would-be) supersymmetry $\bar{\xi}$'s, two (would-be) superconformal η 's, and two gauge zero modes $\chi, \bar{\chi}$. As such, with $U(N-2)$ gauge orientation chosen to be unidirectional, the iterative procedure terminates at third order. We stress that for presentational purposes we kept an index i for the zero modes χ_i , even though the results are relevant only in the particular frame where the only nonvanishing component is χ_3 .

VII. INSTANTON TOMOGRAPHY: POLARIZATION AND GEOMETRY OF MODULI SPACE

Having obtained the exact solution for one antiholomorphic instanton, we are now ready to learn aspects of semiclassical or nonperturbative physics in $\mathcal{N}=\frac{1}{2}$ super Yang-Mills theory. The simplest gauge-invariant quantity we would like to study is the action functional density, which in the present case is simply the topological charge density \mathcal{F} . Notice that we are interested in the *density* of the topological charge, since the latter is nothing but the zeroth moment of the former. The zero modes supported by the instanton span the moduli space, which we denote as \mathcal{M} .

With particular attention to the fate of spacetime symmetries discussed in Sec. II, we are primarily interested in the five-dimensional subspace in \mathcal{M} spanned by the instanton center $X^{\alpha\dot{\alpha}}$ and size ρ . The instanton density \mathcal{F} then depends not only on the coordinates $x^{\alpha\dot{\alpha}}$ of \mathbb{R}^4 but also on the coordinates of \mathcal{M} . Therefore one needs to examine moments of the instanton density \mathcal{F} separately on \mathbb{R}^4 and on \mathcal{M} , respectively. This is precisely what we will do in this section. First, we will examine the profile of \mathcal{F} on \mathbb{R}^4 for a fixed position on \mathcal{M} . We will then find that the instanton charge density \mathcal{F} contains a dipole-moment component [in addition to the $O(4)$ -symmetric monopole-moment component]. The dipole-moment component refers to axially symmetric polarization of the instanton and is invariant only under $O(3) \subset O(4)$. Second, we will examine the profile of \mathcal{F} on \mathcal{M} (after integrating it over \mathbb{R}^4). We will compute Hitchin's information metric and study the deformation of the geometry of \mathcal{M} . Remarkably, we will discover that, though the metric is deformed, the volume measure is independent of the non(anti)-commutative deformation.

A. Instanton density

The topological charge density (which is the same as the action functional density for instantons) is defined by

$$\mathcal{F}[x; Z^A] = \text{Tr}_{\text{U(N)}}(F \wedge F)_{\text{instanton}}.$$

The field configurations of the instanton are functions both of the coordinates on \mathbb{R}^4 and of the Bosonic and Fermionic quasi-zero modes. The quasi-zero modes span the moduli space \mathcal{M} , so we will denote coordinates on \mathcal{M} as $Z^A = (X^{\alpha\dot{\alpha}}, \rho, \eta_\alpha, \bar{\zeta}^{\dot{\alpha}}, \chi^i, \bar{\chi}_i)$. Therefore the instanton density \mathcal{F} could depend not only on coordinates $x^{\alpha\dot{\alpha}}$ of \mathbb{R}^4 but also on coordinates Z^A on \mathcal{M} .

Substituting the exact one-instanton solution constructed in the previous section, after a straightforward algebra, we obtain the action functional density as

$$\begin{aligned} \mathcal{F} = & -\frac{96\rho^4}{(r^2 + \rho^2)^4} \left[1 - \frac{C_{kl}C_{kl}}{\rho^2} \frac{(r^4 - 6r^2\rho^2 + 3\rho^4)}{(r^2 + \rho^2)^2} \bar{\chi}_i \chi^i \bar{\zeta}_{\dot{\alpha}} \zeta^{\dot{\alpha}} \right. \\ & + 2 \frac{C_{kl}C_{kl}}{\rho^2} \frac{(2r^2\rho^2 - 3\rho^4)}{(r^2 + \rho^2)^2} (\eta^\alpha x_{\alpha\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}} \bar{\chi}_i \chi^i \\ & \left. + 16\rho^2 r^2 \bar{\zeta}_{\dot{\alpha}} \zeta^{\dot{\alpha}} \eta^\alpha \eta_\alpha) \right]. \end{aligned} \quad (33)$$

We have shown that SO(4) Lorentz symmetry is broken explicitly on non(anti-)commutative superspace. Still, as is evident from the spinor index contractions, the instanton density is invariant under SO(4) rotations, *provided*, in addition to $x^{\alpha\dot{\alpha}}$, all Fermionic zero modes are rotated simultaneously. Notice that, under this SO(4) symmetry transformation, C_{kl} transforms nontrivially but $C_{kl}C^{kl}$ is invariant. We will refer to this invariance as SO(4) (pseudo)symmetry and make further use of it in the following subsections.

One learns from the result (33) that, with non(anti-)commutativity turned on, the instanton density is deformed from the standard one by $\mathcal{O}(C^2)$ contributions. Notice that, though the antiholomorphic instanton solution itself is modified up to cubic order in the non(anti-)commutativity parameter $C^{\alpha\beta}$, the instanton density terminates at quadratic order. Notice also that there is no $\mathcal{O}(C)$ deformation in the instanton density. These features are not due to any delicate cancellations, but originate from $\text{SU}(2)_L$ symmetry and the Grassmann-odd nature of the fermion zero modes.

As it stands, the result (33) is quite complicated, primarily because of the last two terms involving various combinations of fermion zero modes. To expose further puzzles, recall that the topological charge of the antiholomorphic instanton, which is defined by the integral of the action functional density \mathcal{F} over the Euclidean space \mathbb{R}^4 , equals

$$\mathcal{Q}_{\text{instanton}} \equiv \int_{\mathbb{R}^4} d^4x \frac{\mathcal{F}}{8\pi^2} = -1.$$

It takes an integer value, though in general the integral de-

pends on the fermion zero modes. On the other hand, the topological charge ought to be integer valued, and hence *independent* of the fermion zero modes whatsoever. It is also independent of the instanton size ρ , but this is a well-known result for the ordinary instanton, again from a topological argument. What is *a priori* not so obvious in the present case is that the result is also independent of the fermion zero modes.¹⁰ The way this independence on fermion zero modes comes about is highly nontrivial: integrals over x of the second and the third terms vanish individually. We are thus led to examine tomographically the instanton density and understand how precisely the Fermionic zero-mode dependence is distributed.

It would also be illuminating to recast the instanton density \mathcal{F} in the context of Maldacena's gauge-gravity correspondence [14]. As is well known in the context of five-dimensional anti-de Sitter spacetime as holographic dual of $\mathcal{N}=4$ super-Yang-Mills theory, the instanton density \mathcal{F} defines the bulk-to-boundary propagator (as introduced in Ref. [15]) of a massless bulk scalar field that couples to the topological charge density of the super-Yang-Mills theory residing at the boundary [16]. This can be understood from the elementary observation that

$$\Delta_{\text{AdS}}^{(5)} \frac{1}{8\pi^2} \mathcal{F}(Z;x) \Big|_{C=0} = 0 \quad \text{for } \rho \neq 0,$$

$$\lim_{\rho \rightarrow 0} \frac{1}{8\pi^2} \mathcal{F}(Z;x) \Big|_{C=0} = -\delta^{(4)}(x)$$

$$\text{obeying} \quad \int_{\mathbb{R}^4} \frac{\mathcal{F}}{8\pi^2} = -1. \quad (34)$$

In this context, the coordinates $(X^{\alpha\dot{\alpha}}, \rho)$ are interpreted as the bulk location, while the coordinate x refers to the boundary location. Once the non(anti-)commutativity is turned on, neither of the two relations would hold. Therefore one expects that both the geometry of the five-dimensional gravity background and the instanton density would be modified. In the following subsections, we will explore aspects of these modifications in detail.

B. Instanton polarization by non(anti-)commutativity

With a fair amount of guesswork based on underlying symmetries, we were able to show that the action functional density can be packaged into the following form:

¹⁰Evidently, dependence of the result on the fermion zero mode would lead to Grassmann-valued c -number contribution to the topological charge. This is unphysical.

$$\begin{aligned}
 \mathcal{F} = & 96(\rho^4 - C^2 \rho^{-2} \bar{\chi}_i \chi^i \bar{\zeta}_{\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}} + 64 C^2 \rho^2 \bar{\zeta}_{\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}} \eta^\alpha \eta_\alpha) \\
 & \times \left[\left(x + \sqrt{10} C w - C^2 \frac{\bar{\chi}_i \chi^i w}{2\rho^4} \right)^2 \right. \\
 & + \rho^2 + 2\sqrt{10} C \left(\bar{\zeta}_{\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}} - \frac{\bar{\chi}_i \chi^i}{16} \right) \\
 & + 2 C^2 \bar{\zeta}_{\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}} \left(18 \eta^\alpha \eta_\alpha \right. \\
 & \left. \left. - \frac{\bar{\chi}_i \chi^i}{\rho^4} \right) \right]^{-2} \left[\left(x - \sqrt{10} C w - C^2 \frac{\bar{\chi}_i \chi^i w}{2\rho^4} \right)^2 + \rho^2 \right. \\
 & \left. - 2\sqrt{10} C \left(\bar{\zeta}_{\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}} - \frac{\bar{\chi}_i \chi^i}{16} \right) \right. \\
 & \left. + 2 C^2 \bar{\zeta}_{\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}} \left(18 \eta^\alpha \eta_\alpha - \frac{\bar{\chi}_i \chi^i}{\rho^4} \right) \right]^{-2}. \tag{35}
 \end{aligned}$$

Here we have introduced the following shorthand notation:

$$w^{\alpha\dot{\alpha}} \equiv \eta^\alpha \bar{\zeta}^{\dot{\alpha}}, \quad C^2 \equiv \frac{1}{4} \det C^{\alpha\beta}, \quad C \equiv \sqrt{C^2}.$$

Notice that, in Eq. (35), the two square brackets are exchanged by the inversion Π in \mathbb{R}^4 (which is the Euclidean version of the combined operation of parity P and time reversal T); since $\bar{\chi}$ is an antichiral fermion, Π essentially rotates all the fermion zero modes by $e^{i\pi/2} = +i$. Therefore the new expression (35) of the instanton density exhibits the \mathbb{Z}_2 antipodal reflection symmetry manifestly. This \mathbb{Z}_2 reflection is nothing but a subgroup of the (pseudo) $\text{SO}(4)$ symmetry discussed below Eq. (33).

In passing, we would like to emphasize that, though each square bracket in Eq. (35) seems to contain a nonanalytic expression of $C^{\alpha\beta}$, the instanton density \mathcal{F} is actually analytic—Eq. (35) is merely rewriting Eq. (33), whose expression is manifestly analytic in $C^{\alpha\beta}$.

The alternative expression (35) for the instanton density now offers an intuitive understanding of the effect of non(anti-)commutativity. From the two square brackets in Eq. (35), one readily finds a variety of deformations. A class of deformation of the most interest to our discussion is the one arising from the last term in the round brackets, proportional to $C^2 \bar{\chi}_i \chi^i w$. We will now argue that this term corresponds to polarizing \mathcal{F} so that the dipole moment is induced.¹¹

Begin by noting that the term proportional to $C^2 \bar{\chi}_i \chi^i w^{\alpha\dot{\alpha}} x_{\alpha\dot{\alpha}}$ flips sign under the aforementioned antipodal \mathbb{Z}_2 reflection. Bearing in mind that \mathbb{Z}_2 is nothing but a subgroup of $\text{SO}(4)$ (pseudo)symmetry, one finds that the instanton density \mathcal{F} is polarized along the direction set by

$$\Delta_p^{\alpha\dot{\alpha}} = \frac{C^2 \bar{\chi}_i \chi^i}{\rho^4} w^{\alpha\dot{\alpha}}. \tag{36}$$

As the dipole moment Δ_p is proportional to C^2 , we discover that the first moment of \mathcal{F} is induced as a consequence of turning on non(anti-)commutativity—an important indication that the Grassmann-even space \mathbb{R}^4 and the Grassmann-odd space spanned by (θ^1, θ^2) are not merely in a direct product, but rather intertwined. This is as expected. We have shown in Sec. II that the conformal extension of the non(anti-)commutative superspace is nothing but $\mathcal{N}=\frac{1}{2}$ conformal superspace. We will make this more concrete in the next two subsections.

Notice that the dipole moment Δ_p is also set by the product of supersymmetry and superconformal zero modes *and* to the off-diagonal zero modes $\chi^i, \bar{\chi}_i$. In particular, dependence on the latter is quite interesting since $\chi^i, \bar{\chi}_i$ zero modes are the ones present only for higher-rank gauge groups $G = \text{U}(N \geq 3)$. It also follows that the instanton in the $\text{U}(2)$ gauge theory does not support enough Fermionic zero modes to exhibit the full intricacy of physics on the non(anti-)commutative superspace. In fact, for $G = \text{U}(2)$, the instanton density \mathcal{F} in Eq. (33) is considerably simplified and one observes the maximum at $x^{\alpha\dot{\alpha}} = 0$. It is also $\text{SO}(4)$ rotationally symmetric and thus carries no first moment.

After all, polarization of the antiholomorphic instanton is fully consistent with symmetries of the non(anti-)commutative superspace. As explained in Sec. II, the non(anti-)commutativity breaks the underlying $\text{SO}(4)$ Lorentz symmetry to the antichiral $\text{SU}(2)_R$ symmetry, acting on antichiral, dotted indices. This implies that the instanton configuration would no longer be a spherically symmetric configuration on \mathbb{R}^4 , but rather a configuration invariant only under $\text{SU}(2)_R$. Indeed, we have just observed that the instanton is polarized such that its topological charge density is axisymmetric, where the polarization direction is set by the product of supersymmetry and superconformal zero modes.

We should, however, emphasize that the induced polarization is set entirely by the fermion zero modes and hence Grassmann valued. Modulo this point, what underlies the polarization is precisely the same physics as the UV/IR mixing phenomenon discovered in Ref. [17] and by now well understood in terms of open Wilson lines [18,19] in the noncommutative spacetime. We recall that the UV/IR mixing phenomenon was also shown to take place in non(anti-)commutative superspace [20]. With such a caveat, we plot the induced dipole-moment component of the instanton density in Fig. 1, and contrast it with the monopole-moment component.

¹¹We also note that all deformation terms in Eq. (35) other than Δ_p contribute to deformation of the monopole-moment component in the instanton density \mathcal{F} .

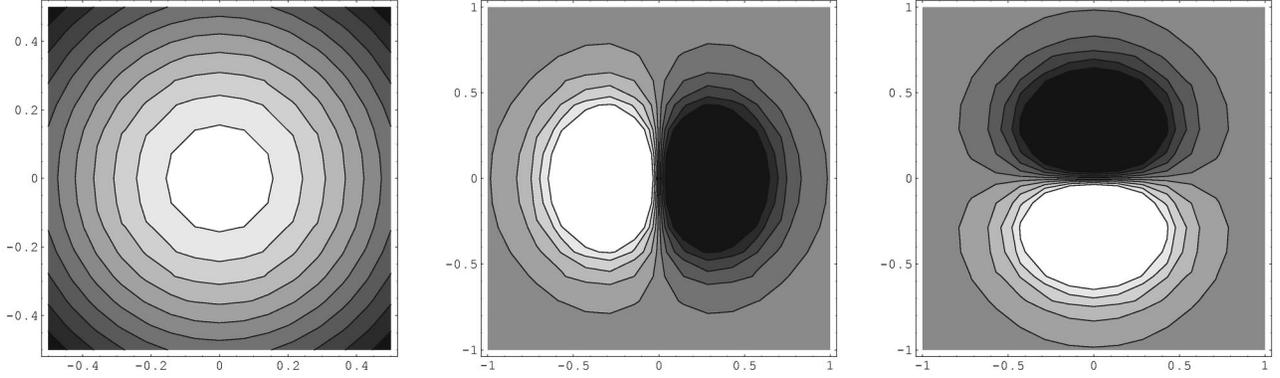


FIG. 1. Contour plots of the instanton density (33). We consider the simplest case where $\bar{\zeta}^2 = \eta^2 = 0$ and present the section in x_3 - x_4 plane: (a) section of the C -independent, monopole-moment component of Eq. (33); (b) dipole-moment component for real part of coefficient of $C^2 \eta^1 \bar{\zeta}^1 \bar{\chi}_i \chi^i$; (c) dipole-moment component for imaginary part of coefficient of $C^2 \eta^1 \bar{\zeta}^1 \bar{\chi}_i \chi^i$.

C. Geometry of moduli space: Information metric

We will now dwell on the other side of the instanton density (33), (35), viz. variation of the density over one-instanton moduli space \mathcal{M} . Traditionally, the instanton moduli space is defined in terms of the so-called L^2 metric—the induced metric on the space of zero modes obtained by choosing a conformal structure on \mathbb{R}^4 .¹² Practically, the L^2 metric is not so convenient, since it comes with various technical complications. For example, the formalism is not manifestly gauge invariant, the moduli space is typically afflicted with small instanton singularities at a finite distance in the moduli space,¹³ and the metric does not exhibit manifest conformal invariance though the (anti-) self-dual equation does. To remedy these shortcomings, Hitchin proposed an alternative definition of the moduli space metric based on the so-called information metric [9,21,22]. The idea is that one views the instanton action density as a family of probability distributions on \mathbb{R}^4 , parametrized by the Grassmann-even and Grassmann-odd zero modes of instantons. Implicit to the idea is an assumption that the moduli space is a submanifold of the infinite-dimensional affine space of all *smooth* volume forms with unit volume. Since we are more interested in incorporating all the spacetime symmetries inherent to the theory and studying differential geometry on the moduli space, we prefer studying the information metric over the L^2 metric.

As mentioned, Hitchin's information metric is defined entirely in terms of the instanton density \mathcal{F} , and is given by

$$\mathcal{G}_{ab} dZ^a dZ^b \equiv dZ^a dZ^b \int_{\mathbb{R}^4} d^4x \frac{\partial_a \mathcal{F} \partial_b \mathcal{F}}{\mathcal{F}}. \quad (37)$$

¹²For a compact hyper-Kähler manifold, it is well known that the L^2 metric on the moduli space is again hyper-Kähler. This assertion extends also to noncompact hyper-Kähler manifolds such as \mathbb{R}^4 .

¹³This is particularly a drawback for making contact with Maldacena's gauge-gravity correspondence [14], since differential geometry is ill defined at the singularity.

The information metric has many virtues compared to the L^2 metric. First, as the instanton density \mathcal{F} is gauge invariant, the metric defines the moduli space geometry with manifest gauge invariance. Second, the metric is geodesically complete. Third, elementary scaling analysis proves that the metric exhibits manifest conformal invariance.

In this section, we shall compute Hitchin's information metric explicitly for a single antiholomorphic instanton, and learn the geometry of the moduli space \mathcal{M} . Recall that the moduli space \mathcal{M} is five dimensional [apart from the trivial $SU(2) \subset U(2)$ gauge orientations], spanning the instanton's size and center. To introduce the instanton center position, we first shift the coordinates $x^{\alpha\dot{\alpha}}$ on \mathbb{R}^4 in Eq. (33) by $(x-X)^{\alpha\dot{\alpha}}$, where $X^{\alpha\dot{\alpha}}$ now refers to the center of the anti-instanton, and compute the integrals (37). After some algebra, we find that

$$\begin{aligned} \mathcal{G}_{AB} dZ^A dZ^B = & \frac{2}{5} \left[64 \frac{d\rho^2}{\rho^2} \left(1 - \frac{15}{7\rho^6} C^2 S_1 - \frac{320}{7\rho^2} C^2 S_2 \right) \right. \\ & + 64 \frac{dX^2}{\rho^2} \left(1 - \frac{5}{7\rho^6} C^2 S_1 - \frac{88}{7\rho^2} C^2 S_2 \right) \\ & + \frac{32C^2}{7\rho^7} T^m dX_m d\rho - \frac{16C^2}{\rho^6} dX_m dT^m \\ & \left. + \frac{32C^2}{\rho^7} d\rho dS_1 + \frac{1024C^2}{\rho^3} d\rho dS_2 \right]. \quad (38) \end{aligned}$$

Here, we introduced the shorthand notation S_1 , S_2 , and T_m for three independent products involving the fermion zero modes:

$$S_1 \equiv \bar{\xi}_\alpha \bar{\xi}^{\dot{\alpha}} \bar{\chi}_i \chi^i, \quad S_2 \equiv \bar{\xi}_\alpha \bar{\xi}^{\dot{\alpha}} \eta^\alpha \eta_\alpha, \quad T_m \equiv \bar{\chi}_i \chi^i \eta^\alpha \sigma_{\alpha\dot{\alpha}}^m \bar{\xi}^{\dot{\alpha}}. \quad (39)$$

As they are quartic in Grassmann-odd variables, the product of any two such objects vanishes identically.

Much as the instanton density itself, the information metric (38) is quite complicated because of C -dependent Fermionic zero-mode effects. This is again the manifestation that the moduli space \mathcal{M} is not the standard superconformal superspace, but rather its non(anti-)commutative counterpart with $\mathcal{N}=\frac{1}{2}$ supersymmetry explained in Sec. II. Nontrivial mixing between Grassmann-even and Grassmann-odd coordinates on \mathcal{M} as observed in Eq. (38) originates from the non(anti-)commutativity. All these mixings, however, are removable by a suitable change of variables. Introduce the following shift to the Grassmann-even coordinates X_m and ρ^2 :

$$\tilde{X}^m = X^m - \frac{C^2}{8\rho^4} T^m \quad \text{and} \quad \tilde{\rho}^2 = \rho^2 \left[1 + \frac{C^2}{2} \frac{S_1}{\rho^6} + 16C^2 \frac{S_2}{\rho^2} \right].$$

Notice that we still maintain the translational invariance on conformal slice of \mathbb{R}^4 in \tilde{X}^m . In terms of the shifted variables, the information metric becomes

$$\begin{aligned} \mathcal{G}_{AB} dZ^A dZ^B = & \frac{128}{5} \left[\frac{d\tilde{\rho}^2}{\tilde{\rho}^2} \left(1 + \frac{6}{7\tilde{\rho}^6} C^2 S_1 - \frac{96}{7\tilde{\rho}^2} C^2 S_2 \right) \right. \\ & + \frac{d\tilde{X}^2}{\tilde{\rho}^2} \left(1 - \frac{3}{14\tilde{\rho}^6} C^2 S_1 + \frac{24}{7\tilde{\rho}^2} C^2 S_2 \right) \\ & \left. - C^2 \frac{13}{14\tilde{\rho}^7} T^m d\tilde{\rho} dX_m \right]. \end{aligned} \quad (40)$$

One readily sees that, in the limit that the Grassmann-odd coordinates vanish, the information metric is precisely the metric of the hyperbolic space \mathbb{H}_5 , describing the five-dimensional Euclidean anti-de Sitter (AdS) space. Therefore, for nonzero $C^{\alpha\beta}$, one would still interpret the instanton density \mathcal{F} as the bulk-to-boundary propagator obeying

$$\begin{aligned} \tilde{\Delta}_{\mathbb{H}}^{(5)} \frac{1}{8\pi^2} \mathcal{F}(Z;x) &= 0 \quad \text{for } Z^A \neq 0, \\ \lim_{Z^A \rightarrow 0} \frac{1}{8\pi^2} \mathcal{F}(Z;x) &= -\delta^{(4)}(x) \quad \text{obeying } \int_{\mathbb{R}^4} \frac{\mathcal{F}}{8\pi^2} = -1, \end{aligned} \quad (41)$$

where $\tilde{\Delta}_{\mathbb{H}}^{(5)}$ refers to the scalar Laplacian operator defined by Hitchin's information metric (38), and the limit $Z^A \rightarrow 0$ is interpreted as taking all Grassmann-even and Grassmann-odd coordinates approaching zero while holding $C^{\alpha\beta}$ finite.

A quantity of interest is the volume measure of Hitchin's information metric, as it defines the sum over instanton effects in $\mathcal{N}=\frac{1}{2}$ super-Yang-Mills theory. From Eq. (40), one readily finds that

$$d \text{Vol} = \sqrt{\det \tilde{\mathcal{G}}} d\tilde{\rho} d^4 X \quad \text{where} \quad \det \tilde{\mathcal{G}} = \left(\frac{128}{5} \right)^5 \frac{1}{\tilde{\rho}^{10}}.$$

Remarkably, we find that the volume form is exactly the same as in ordinary Yang-Mills theories. It should be noted,

however, that the natural coordinate is not ρ but $\tilde{\rho}$. This may indicate that $\tilde{\rho}$ should be interpreted as the size modulus of the deformed instanton.

D. Symmetry considerations

Having examined the instanton density and the information metric, one might be able to infer the results from considerations of the spacetime conformal symmetry identified in Sec. II. Indeed, the functional form of various terms in the information metric (38) (but not the coefficients) is determinable by those symmetries. Let us briefly discuss these symmetries and their consequences for the metric (38).

- (i) Translational invariance on the conformal slice of \mathbb{R}^4 guarantees that the metric components \mathcal{G}_{ab} do not depend upon the translational Bosonic zero modes X_m 's.
- (ii) Global rotation in the $U(N-2)$ part of the gauge group (i.e., unitary rotations acting by $\chi^i \rightarrow U^i_j \chi^j$, $\bar{\chi}_i \rightarrow U^j_i \bar{\chi}_j$) restricts possible contributions of the bi-fundamental Fermionic zero modes to $\bar{\chi}_i \chi^i$ or $d(\bar{\chi}_i \chi^i)$.
- (iii) Chiral $SU(2)_L$ and antichiral $SU(2)_R$ symmetries (acting on dotted and undotted indices) restrict possible contractions of supersymmetric and superconformal Fermionic zero modes to $\bar{\zeta}_\alpha \bar{\zeta}^\alpha, \eta^\alpha \eta_\alpha, \eta^\alpha \sigma_{\alpha\dot{\alpha}}^m \bar{\zeta}^\alpha$, and

$$\eta_\alpha C^{\alpha\beta} \sigma_{\beta\dot{\alpha}}^m \bar{\zeta}^\alpha, \quad \eta^\alpha C_{\alpha\beta} C^{\beta\gamma} \sigma_{\gamma\dot{\alpha}}^m \bar{\zeta}^\alpha. \quad (42)$$

Here we also made use of the Fermionic statistics and of the fact that we can have at most two $C_{\alpha\beta}$ if only one η and one ζ are present. Notice that $\eta^\alpha \sigma_{\alpha\dot{\alpha}}^m \bar{\zeta}^\alpha$ can appear only in the combination $dX_m \eta^\alpha \sigma_{\alpha\dot{\alpha}}^m \bar{\zeta}^\alpha$ in order to be consistent with Lorentz (pseudo)symmetry and translation symmetry.

The first term in Eq. (42) is removable in the information metric.¹⁴ To show this, we notice that there are only two ways of incorporating the first term in Eq. (42) consistently with Lorentz (pseudo)symmetry:

$$dX_m \eta_\alpha C^{\alpha\beta} \sigma_{\beta\dot{\alpha}}^m \bar{\zeta}^\alpha, \quad C_{mn} dX_n \eta_\alpha C^{\alpha\beta} \sigma_{\beta\dot{\alpha}}^m \bar{\zeta}^\alpha. \quad (43)$$

The first term is linear in $C^{\alpha\beta}$ and thus cannot appear in the instanton density \mathcal{F} or in the information metric. Recall that the part of the solution that is linear in C is a singlet under $SU(2)$. Using the Fierz identity (22),

$$C_{mn} \sigma_{\beta\dot{\alpha}}^m = \sigma_{n\rho\dot{\alpha}} C^{\rho\gamma} \epsilon_{\gamma\beta},$$

we observe that the second term in Eq. (43) becomes

$$\begin{aligned} C_{mn} dX_n \eta_\alpha C^{\alpha\beta} \sigma_{\beta\dot{\alpha}}^m \bar{\zeta}^\alpha &= dX_n \eta_\alpha C^{\alpha\beta} \sigma_{n\rho\dot{\alpha}} C^{\rho\gamma} \epsilon_{\gamma\beta} \bar{\zeta}^\alpha \\ &= -\frac{1}{2} C^{\rho\beta} C_{\rho\beta} dX_n \eta^\alpha \sigma_{\alpha\dot{\alpha}}^n \bar{\zeta}^\alpha. \end{aligned}$$

¹⁴The second term can be written in terms of $\eta^\alpha \sigma_{\alpha\dot{\alpha}}^m \bar{\zeta}^\alpha$ due to relation $C_{\alpha\beta} C^{\beta\gamma} = \frac{1}{2} C_{\sigma\beta} C^{\beta\sigma} \delta_\alpha^\gamma$.

Putting these considerations together, we find that only the following combinations of zero modes are permitted by the spacetime symmetries:

$$S_1 \equiv \bar{\zeta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} \bar{\chi}_i \chi^i, \quad S_2 \equiv \bar{\zeta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} \eta^\alpha \eta_\alpha,$$

$$S_3 \equiv \eta^\alpha \eta^\alpha \bar{\chi}_i \chi^i, \quad dX_m T_m \equiv dX_m \bar{\chi}_i \chi^i \eta^\alpha \sigma_{\alpha\dot{\alpha}}^m \bar{\zeta}^{\dot{\alpha}},$$

and they should be multiplied by C^2 in the information metric to be consistent with U(1) (pseudo) R symmetry.

(i) The fermion zero modes transform under the $\mathcal{N}=\frac{1}{2}$ supersymmetry as follows:

$$(\delta X)_{\alpha\dot{\alpha}} = 4i\epsilon_{\alpha\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}},$$

$$\delta\rho^2 = 4i\rho^2(\epsilon\eta),$$

$$\delta\eta = 4i\eta(\epsilon\eta),$$

$$\delta\chi = 6i\chi(\epsilon\eta),$$

$$\delta\bar{\chi} = 6i\bar{\chi}(\epsilon\eta),$$

$$\delta\bar{\zeta} = 0.$$

From these rules, one readily finds that

$$\delta\left(\frac{S_1}{\rho^6}\right) = 0; \quad \delta\left(\frac{S_2}{\rho^2}\right) = 0; \quad \text{and} \quad \delta(\bar{\chi}\chi\eta^2) = 0.$$

Both the instanton density \mathcal{F} and the information metric \mathcal{G}_{AB} are invariant under these transformations.¹⁵

(ii) Powers of ρ in the instanton density \mathcal{F} and the information metric \mathcal{G}_{AB} are determinable by elementary dimensional analysis.

ACKNOWLEDGMENTS

We thank D. Berenstein, J. Maldacena, and N. Seiberg for helpful discussions. R.B. thanks the Aspen Center for Physics for hospitality during this work. S.J.R. was a member at the Institute for Advanced Study during this work. He thanks the School of Natural Sciences for hospitality and for the grant-in-aid from the Fund for Natural Sciences. This work was supported in part by NSF grant PHY-0070928 (R.B., B.F., O.L.) and by KRF-2003-013-C00023 (S.J.R.).

APPENDIX A: CONVENTIONS AND NOTATION

The conventions and notation we adopt are as follows. The signature of Lorentzian spacetime $\mathbb{R}^{3,1}$ is $\text{diag}(+---)$. Wick rotation to Euclidean \mathbb{R}^4 is achieved by $x^0 \rightarrow ix^4$ and $S_{\text{Lorentzian}} \rightarrow iS_{\text{Euclidean}}$. We continue adopting

¹⁵There is a possible combination of the form $C^2 S_3 = C^2 \bar{\chi}_i \chi^i \eta^\alpha \eta_\alpha$ that is consistent with all symmetries; thus it can in principle appear in the instanton density \mathcal{F} and in the information metric \mathcal{G}_{AB} . On the other hand, explicit computation indicates that the coefficient of this term is zero.

Lorentzian spinor notation. We freely change between SO(4) and $SU(2)_L \times SU(2)_R$ indices, viz.

$$\partial_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^m \partial_m; \quad \partial_{\alpha\dot{\alpha}} x^{\beta\dot{\beta}} = -2\delta_{\alpha}^{\beta} \delta_{\dot{\alpha}}^{\dot{\beta}},$$

$$x_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^m x_m, \quad x_m = -\frac{1}{2} \bar{\sigma}_m^{\dot{\alpha}\alpha} x_{\alpha\dot{\alpha}}.$$

We normalize the gauge covariant derivatives as

$$\nabla_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} + \frac{i}{2} [A_{\alpha\dot{\alpha}}],$$

For the U(2) gauge group, we also freely interchange color indices between adjoint and spinor indices [23] (suppressing spacetime indices):

$$A := A^a \mathbf{T}^a; \quad A^{\{ab\}} := (2i \mathbf{T}_2 A)^{ab}; \quad \text{tr}_{u(2)} \mathbf{T}^a \mathbf{T}^b = \frac{1}{2} \delta^{ab}.$$

In spinor notation, the traceless SU(2) subgroup is symmetric in the spinor indices a, b , while the diagonal U(1) subgroup is proportional to ϵ^{ab} . In explicit form,

$$A^{\{ab\}} = \epsilon^{ac} A_c{}^b, \quad A_a{}^b = \epsilon_{ac} A^{\{cb\}}.$$

In differential forms, the gauge field strengths are

$$F = \frac{1}{2} F_{mn} dx^m \wedge dx^n; \quad *F = \frac{1}{2} \tilde{F}_{mn} dx^m \wedge dx^n,$$

where $\tilde{F}_{mn} = \frac{1}{2} \epsilon_{mnpq} F_{pq}$. Thus $\frac{1}{2}(F_{mn} F_{mn}) = *(F \wedge *F)$, and $\frac{1}{2}(F_{mn} \tilde{F}_{mn}) = *(F \wedge F)$.

The gauge coupling constants $\tau, \bar{\tau}$ are customarily taken as per Lorentzian theory conventions:

$$\tau = \frac{\theta_{\text{YM}}}{2\pi} + i \frac{4\pi}{g_{\text{YM}}^2}; \quad \bar{\tau} = \frac{\theta_{\text{YM}}}{2\pi} - i \frac{4\pi}{g_{\text{YM}}^2}.$$

In Euclidean theory, we interpret τ and $\bar{\tau}$ as two independent, complex-valued coupling constants.

APPENDIX B: SINGLE UNDEFORMED ANTIHOLOMORPHIC INSTANTON

The zeroth-order (undeformed) solution for the gauge field is

$$A_{\beta\dot{\beta}}^{(0)\{ab\}} = -\frac{2i}{x^2 + \rho^2} (\delta_{\beta}^a x_{\dot{\beta}}^b + \delta_{\dot{\beta}}^b x_{\beta}^a),$$

$$F_{\alpha\dot{\beta}}^{(0)\{ab\}} = \frac{8i\rho^2}{(x^2 + \rho^2)^2} (\delta_{\alpha}^a \delta_{\dot{\beta}}^b + \delta_{\alpha}^b \delta_{\dot{\beta}}^a). \quad (\text{B1})$$

Consider a fermion $\bar{\alpha}$ transforming in the adjoint representation of U(2). The zeroth-order solution for its zero modes is

TABLE I. Scaling dimension and R charge of zero modes.

	dim	U(1) $_R$
X_m	-1	0
ρ	-1	0
η_α	$\frac{1}{2}$	-1
$\bar{\xi}^\alpha$	$-\frac{1}{2}$	-1
χ^i	$-\frac{3}{2}$	-1
$\bar{\chi}_i$	$-\frac{3}{2}$	-1

$$\bar{\lambda}^{(0)}_{\dot{\alpha}} \equiv F_{\alpha\dot{\beta}}^{(0)} \bar{\xi}^{\dot{\beta}} + F_{\alpha\dot{\beta}}^{(0)} \chi^{\dot{\beta}} \eta^\alpha \equiv F_{\alpha\dot{\beta}}^{(0)} \bar{\xi}^{\dot{\beta}}. \quad (\text{B2})$$

Consider next fermions $\chi, \bar{\chi}$ transforming in the fundamental and antifundamental representations of U(2), respectively. The zeroth-order solution for their zero modes is

$$\bar{\lambda}_{\dot{\alpha}a}^{(0)} \equiv \epsilon_{ac} \bar{\lambda}^{(0)\{c\}i}_{\dot{\alpha}}, \quad \bar{\lambda}^{(0)\{a\}i}_{\dot{\alpha}} = \frac{\chi^i}{(x^2 + \rho^2)^{3/2}} \delta_{\dot{\alpha}}^a,$$

$$\bar{\lambda}^{(0)}_{\dot{\alpha}i} \equiv \bar{\lambda}^{(0)\{a\}}_{\dot{\alpha}i} = \frac{\bar{\chi}_i}{(x^2 + \rho^2)^{3/2}} \delta_{\dot{\alpha}}^a.$$

See Table I for the scaling dimensions and R charges for the fermion zero modes.

In the instanton solution for gauge group $G=U(N)$, χ_i is the fermion component transforming as a bi-fundamental under $U(N-2)$, and $\bar{\chi}_i$ is its complex conjugate. Starting with an arbitrary zero mode, one can always perform a constant $U(N-2)$ rotation, so that there is only one nontrivial component χ_3 . In other words, one can reduce the general discussion of $G=U(N)$ with $N \geq 3$ effectively to $G=U(3)$.

APPENDIX C: SOLVING DIFFERENTIAL EQUATIONS

While constructing the instanton solution in perturbation theory, we repeatedly encounter equations of the following form:

$$\nabla_m A_n - \nabla_n A_m + \epsilon_{mnl} \nabla_l A_l = -C_{mn} J, \quad (\text{C1})$$

where the covariant derivative ∇_n is computed with instanton background field $A^{(0)}$ taken in the appropriate representation of SU(2). Applying a differential operator $\epsilon_{mnr} \nabla_s$ to both sides of the above equation, we find¹⁶

$$2iF_{nr} A_n + 2\nabla_r \nabla_l A_l - 2\nabla^2 A_r = -\epsilon_{mnr} \nabla_s C_{mn} J.$$

Here we used the definition of F_{mn} and its anti-self-duality:

$$\nabla_m \nabla_n - \nabla_n \nabla_m = \frac{i}{2} F_{mn}, \quad \epsilon_{mnr} F_{sm} = 2F_{nr}.$$

In the Lorentz gauge ($\nabla_l A_l = 0$), we have

¹⁶To simplify notation, we use F_{mn} instead of $F_{mn}^{(0)}$ in this appendix.

$$iF_{nm} A_n - \nabla^2 A_m = -C_{mn} \nabla_n J.$$

This equation is solvable by taking an ansatz

$$A_m = C_{mn} \nabla_n \Phi, \quad (\text{C2})$$

which automatically satisfies the gauge condition $\nabla_m A_m = (i/4) C_{mn} F_{mn} \Phi = 0$ because of the anti-self-duality of F_{mn} . Using the relation $F_{nm} C_{nk} = F_{nk} C_{nm}$, the left-hand side of Eq. (C2) can be rewritten as

$$iF_{nm} A_n - \nabla^2 A_m = -\frac{i}{2} [\nabla_k, F_{kl}] C_{ml} \Phi - C_{ml} \nabla_l \nabla^2 \Phi.$$

We thus demonstrated that there exists a natural ansatz (C2) for the gauge potential A_m , allowing us to reduce Eq. (C1) to a *single* differential equation of Poisson type:

$$\nabla^2 \Phi = J.$$

Here ∇ is a covariant derivative in an appropriate representation of the gauge group G .

APPENDIX D: DETAILS OF THE COMPUTATION

In this appendix we present some intermediate steps in computing the solution (27)–(32). We start with the conventional instanton solution and construct the deformed instanton in perturbation theory in the non(anti-)commutativity parameter $C^{\alpha\beta}$.

1. First order for the bosons

We begin by solving the equation for the first correction to the gauge field $A_m^{(1)}$:

$$(F_{\alpha\beta}^{(1)})^a{}_b = -\frac{i}{2} C_{\alpha\beta} \delta_b^a \left(\frac{3 \cdot 64 \rho^4}{(x^2 + \rho^2)^4} \bar{\xi}_\alpha \bar{\xi}^\alpha - \frac{\bar{\chi}_i \chi^i}{(x^2 + \rho^2)^3} \right),$$

$$(F_{\alpha\beta}^{(1)})^j{}_i = +i C_{\alpha\beta} \frac{\bar{\chi}_i \chi^j}{(x^2 + \rho^2)^3},$$

$$(F_{\alpha\beta}^{(1)})^i{}_a = -C_{\alpha\beta} \frac{12\rho^2}{(x^2 + \rho^2)^{7/2}} \bar{\xi}_a \bar{\xi}^i,$$

$$(F_{\alpha\beta}^{(1)})^a{}_i = -C_{\alpha\beta} \frac{12\rho^2}{(x^2 + \rho^2)^{7/2}} \bar{\chi}_i \bar{\xi}^a.$$

As discussed in Appendix C, an ansatz

$$A_m^{(1)} = C_{mn} \nabla_n \Phi^{(1)}$$

reduces the entire problem to solving a Poisson-type equation in the instanton background, and the solution of the Poisson equation is given by Eq. (28).

2. First order for the fermions

At the next step, we solve the equation for $\bar{\lambda}^{(1)}$:

$$\begin{aligned}\sigma_{\alpha\dot{\alpha}}^m \nabla_m \bar{\lambda}^{(1)\dot{\alpha}} &= -\frac{i}{2} [A_m^{(1)}, \sigma_{\alpha\dot{\alpha}}^m \bar{\lambda}^{(0)\dot{\alpha}}] \\ &= -\frac{i}{2} C_{mn} [(\nabla_n \Phi^{(1)}), \sigma_{\alpha\dot{\alpha}}^m \bar{\lambda}^{(0)\dot{\alpha}}].\end{aligned}$$

Using the Fierz identity and equation of motion for $\bar{\lambda}^{(0)}$, we reduce this equation to

$$\sigma_{\alpha\dot{\alpha}}^m \nabla_m \bar{\lambda}^{(1)\dot{\alpha}} = \frac{i}{2} C_{\alpha\dot{\alpha}}^{\beta} \sigma_{\beta\dot{\beta}}^m \nabla_m [\Phi^{(1)}, \bar{\lambda}^{(0)\dot{\alpha}}].$$

With an ansatz

$$\bar{\lambda}^{(1)\dot{\alpha}} = \bar{\sigma}^{m\dot{\alpha}\alpha} C_{\alpha\dot{\alpha}}^{\beta} \nabla_m \Psi_{\beta}^{(1)},$$

it is reduced further to

$$\sigma_{\beta\dot{\alpha}}^m \nabla_m \left[-i \bar{\sigma}^{n\dot{\alpha}\gamma} \nabla_n \Psi_{\gamma}^{(1)} - \frac{1}{2} [\Phi^{(1)}, \bar{\lambda}^{(0)\dot{\alpha}}] \right] = 0. \quad (D1)$$

Notice that this derivation does not rely on the specific form of $\bar{\lambda}^{(0)}$: the steps above are essentially the same as those described in more detail in Sec. V for the special situation $\eta_{\alpha} = 0$.

Using an identity for sigma matrices: $\sigma_m \bar{\sigma}_n = -\eta_{mn} + 2\sigma^{mn}$ and anti-self-duality of the undeformed solution (which leads to $\sigma^{mn} \nabla_m \nabla_n = 0$), we can express Eq. (D1) as a Poisson equation:

$$\nabla^2 \Psi_{\alpha}^{(1)} = -\sigma_{\alpha\dot{\alpha}}^n \nabla_n J^{(1)\dot{\alpha}}, \quad J^{(1)\dot{\alpha}} \equiv \frac{i}{2} [\Phi^{(1)}, \bar{\lambda}^{(0)\dot{\alpha}}].$$

Explicit evaluation of the current gives

$$\begin{aligned}(J^{\dot{\alpha}})_{a^b} &= -\delta_a^b \frac{i \bar{\xi}^{\dot{\alpha}} \bar{\chi}_i \chi^i}{(r^2 + \rho^2)^3}; \quad (J^{\dot{\alpha}})_i{}^j = 2 \frac{i \bar{\xi}^{\dot{\alpha}} \bar{\chi}_i \chi^j}{(r^2 + \rho^2)^3}, \\ (J_{\dot{\alpha}})_a{}^i &= -\epsilon_{\dot{\alpha}a} \frac{4\chi^i}{(r^2 + \rho^2)^{7/2}} [(r^2 + \rho^2)(\bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} + \rho^2 \eta^{\alpha} \eta_{\alpha}) \\ &\quad + 4\rho^2 \bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}}], \quad (D2) \\ (J_{\dot{\alpha}})_i{}^a &= -\delta_{\dot{\alpha}}^a \frac{4\bar{\chi}_i}{(r^2 + \rho^2)^{7/2}} [(r^2 + \rho^2)(\bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} + \rho^2 \eta^{\alpha} \eta_{\alpha}) \\ &\quad + 4\rho^2 \bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}}].\end{aligned}$$

We used the following relation: $2x_m \bar{\xi}^m \eta = \bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} - \bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} - x^2 \eta^{\alpha} \eta_{\alpha}$. The solution of Eq. (D2) is given by Eq. (31).

3. Second order for the bosons

Repeating the arguments of Sec. V, we find the equation

$$\begin{aligned}2(\nabla_m A_n^{(2)})^+ - \frac{i}{4} C_{kl} C_{kl} [\nabla_m (\Phi^{(1)} \nabla_n \Phi^{(1)})]^+ \\ + \frac{i}{2} C_{mn} C_{kl} [\nabla_k \Phi^{(1)} \nabla_l \Phi^{(1)} + \bar{\sigma}^{k\dot{\alpha}\alpha} [\bar{\lambda}_{\dot{\alpha}}^{(0)}, \nabla_l \Psi_{\alpha}^{(1)}]] = 0.\end{aligned} \quad (D3)$$

In Sec. V, we put forward the following ansatz for $A^{(2)}$:

$$A_m^{(2)} = \frac{i}{8} C_{kl} C_{kl} \Phi^{(1)} \nabla_m \Phi^{(1)} + C_{mn} \nabla_n \Phi^{(2)}. \quad (D4)$$

With the ansatz, Eq. (D3) was reduced to an Poisson equation for $\Phi^{(2)}$. Notice that, in special situations, the superconformal modes were set to zero, the solution $\Phi^{(1)}$ was such that $\text{Tr} \Phi^{(1)} \nabla_m \Phi^{(1)} = 0$. As such, $A_m^{(2)}$ did not have a component proportional to the identity matrix. In the presence of superconformal modes, $\text{Tr} \Phi^{(1)} \nabla_m \Phi^{(1)}$ is no longer zero, and we found it convenient to modify the ansatz (D4) so that the trace component in $A_m^{(2)}$ is avoided. The simplest such modification is

$$A_m^{(2)} = \frac{i}{16} C_{kl} C_{kl} \Phi^{(1)} \bar{\nabla}_m \Phi^{(1)} + C_{mn} \nabla_n \Phi^{(2)}, \quad (D5)$$

and it does not spoil the Poisson equation for $\Psi^{(2)}$, since the difference between Eqs. (D5) and (D4), $\delta A_m^{(2)} = -(i/16) C_{kl} C_{kl} \nabla_m (\Phi^{(1)} \Phi^{(1)})$, satisfies $(\nabla_m \delta A_n^{(2)})^+ = 0$. With the ansatz (D5), we get the equation for $\Phi^{(2)}$:

$$\nabla^2 \Phi^{(2)} = i C_{kl} \{ \nabla_k \Phi^{(1)} \nabla_l \Phi^{(1)} + \bar{\sigma}^{k\dot{\alpha}\alpha} [\bar{\lambda}_{\dot{\alpha}}^{(0)}, \nabla_l \Psi_{\alpha}^{(1)}] \} \equiv J^{(2)}.$$

Explicit evaluation of the current $J^{(2)}$ yields

$$\begin{aligned}(J^{(2)})_a{}^b &= \frac{16i C_{mk}}{(\rho^2 + r^2)^5} (\bar{\sigma}^{kn})^b{}_a x_m x_n \bar{\chi}_i \chi^i \left[\frac{\bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}}}{\rho^2} (r^2 + 6\rho^2) \right. \\ &\quad \left. + (r^2 - 4\rho^2) \eta^{\alpha} \eta_{\alpha} - 10 \eta^{\alpha} x_{\alpha\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} \right] \\ &\quad - \frac{40i \bar{\chi}_i \chi^i}{(\rho^2 + r^2)^4} \epsilon_{a\dot{\alpha}} [\bar{\xi}^{\dot{\alpha}} (\bar{x}C)^{b\alpha} \eta_{\alpha} + \bar{\xi}^b (\bar{x}C)^{\dot{\alpha}\alpha} \eta_{\alpha}], \\ (J^{(2)})_i{}^a &= \frac{32\bar{\chi}_i}{(r^2 + \rho^2)^{9/2}} [(r^2 + 9\rho^2) \bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} (\bar{x}C)^{a\alpha} \eta_{\alpha} \\ &\quad + 8\rho^2 \eta^{\alpha} \eta_{\alpha} (\bar{x}C x)^a{}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}}], \\ (J^{(2)})_a{}^i &= \frac{32A^i \epsilon_{a\dot{\beta}}}{(r^2 + \rho^2)^{9/2}} [(r^2 + 9\rho^2) \bar{\xi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} (\bar{x}C)^{\dot{\beta}\alpha} \eta_{\alpha} \\ &\quad + 8\rho^2 \eta^{\alpha} \eta_{\alpha} (\bar{x}C x)^{\dot{\beta}}{}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}}].\end{aligned}$$

Solving the Poisson equation, we get Eq. (29).

4. Second order for the fermions

In the second order in C , we get the following equation for the fermions:

$$\begin{aligned} \sigma_{\alpha\dot{\alpha}}^m \nabla_m \bar{\lambda}^{(2)\dot{\alpha}} &= -\frac{i}{2} [A_m^{(1)}, \sigma_{\alpha\dot{\alpha}}^m \bar{\lambda}^{(1)\dot{\alpha}}] - \frac{i}{2} [A_m^{(2)}, \sigma_{\alpha\dot{\alpha}}^m \bar{\lambda}^{(0)\dot{\alpha}}] \\ &= -\frac{i}{2} C_{mn} \nabla_n ([\Phi^{(1)}, \sigma_{\alpha\dot{\alpha}}^m \bar{\lambda}^{(1)\dot{\alpha}}] \\ &\quad + [\Phi^{(2)}, \sigma_{\alpha\dot{\alpha}}^m \bar{\lambda}^{(0)\dot{\alpha}}]) \\ &\quad + \frac{i}{2} C_{mn} [\Phi^{(1)}, \sigma_{\alpha\dot{\alpha}}^m \nabla_n \bar{\lambda}^{(1)\dot{\alpha}}] \\ &\quad + \frac{1}{32} C_{kl} C_{kl} [\Phi^{(1)} \nabla_m \Phi^{(1)}, \sigma_{\alpha\dot{\alpha}}^m \bar{\lambda}^{(0)\dot{\alpha}}]. \end{aligned}$$

After straightforward algebra, this equation is simplified as

$$\begin{aligned} \sigma_{\alpha\dot{\alpha}}^m \nabla_m \bar{\lambda}^{(2)\dot{\alpha}} &= -\frac{i}{2} C_{mn} \nabla_n ([\Phi^{(1)}, \sigma_{\alpha\dot{\alpha}}^m \bar{\lambda}^{(1)\dot{\alpha}}] \\ &\quad + [\Phi^{(2)}, \sigma_{\alpha\dot{\alpha}}^m \bar{\lambda}^{(0)\dot{\alpha}}]) \\ &\quad - \frac{C_{kl} C^{kl}}{32} \sigma_{\alpha\dot{\alpha}}^m \nabla_m [\Phi^{(1)}, [\Phi^{(1)}, \bar{\lambda}^{(0)\dot{\alpha}}]]. \end{aligned}$$

This suggests the following ansatz:

$$\bar{\lambda}^{(2)\dot{\alpha}} = -\frac{C_{kl} C^{kl}}{32} [\Phi^{(1)}, [\Phi^{(1)}, \bar{\lambda}^{(0)\dot{\alpha}}]] + \bar{\sigma}^{m\dot{\alpha}\alpha} C_{\alpha\beta}^m \nabla_m \Psi_{\beta}^{(2)},$$

which leads to the following Poisson-type equation:

$$-\nabla^2 \Psi_{\alpha}^{(2)} = \frac{i}{2} \sigma_{\alpha\dot{\alpha}}^m \nabla_m ([\Phi^{(1)}, \bar{\lambda}^{(1)\dot{\alpha}}] + [\Phi^{(2)}, \bar{\lambda}^{(0)\dot{\alpha}}]).$$

The solution is given in Eq. (32).

5. Third order for the bosons

Finally, in the third order in C , we need to solve for the gauge fields only. The equations are

$$\begin{aligned} 2(\nabla_{[m} A_{n]}^{(3)})^+ + \frac{i}{2} 2[A_{[m}^{(1)}, A_{n]}^{(2)}]^+ + \frac{i}{2} C_{mn} (\bar{\lambda}_{\dot{\alpha}}^{(2)} \bar{\lambda}^{(0)\dot{\alpha}} \\ + \bar{\lambda}_{\dot{\alpha}}^{(0)} \bar{\lambda}^{(2)\dot{\alpha}} + \bar{\lambda}_{\dot{\alpha}}^{(1)} \bar{\lambda}^{(1)\dot{\alpha}}) = 0. \end{aligned}$$

Direct computation of the current yields

$$\begin{aligned} (\nabla_m A_n^{(3)} - \nabla_n A_m^{(3)})^+ &= -\frac{i}{2} C_{mn} \frac{C_{kl} C_{kl}}{4} \frac{32 \eta^{\alpha} \eta_{\alpha} \bar{\zeta}_{\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}} \bar{\chi}_i \bar{\chi}^i}{\rho^2 (r^2 + \rho^2)^5} \\ &\quad \times \text{diag}(3(r^4 - 4r^2 \rho^2 - \rho^4), 3(r^4 \\ &\quad - 4r^2 \rho^2 - \rho^4), 2(r^4 - 10r^2 \rho^2 + \rho^4)), \end{aligned}$$

where the nonzero entries reside in the U(3) block. This equation is soluble by taking

$$A_m^{(3)} = C_{mn} \partial_n \Phi^{(3)},$$

where

$$\begin{aligned} \Phi^{(3)} &= 2i \frac{C_{kl} C_{kl}}{4} \frac{\eta^{\alpha} \eta_{\alpha} \bar{\zeta}_{\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}} \bar{\chi}_i \bar{\chi}^i}{\rho^4 (r^2 + \rho^2)^3} \text{diag}(r^4 + 6r^2 \rho^2 + 3\rho^4, r^4 \\ &\quad + 6r^2 \rho^2 + 3\rho^4, 2(r^4 + 4r^2 \rho^2 + \rho^4)). \end{aligned}$$

-
- [1] R. Dijkgraaf and C. Vafa, Nucl. Phys. **B644**, 3 (2002); **B644**, 21 (2002); ‘‘A perturbative window into non-perturbative physics,’’ hep-th/0208048; R. Dijkgraaf, M.T. Grisaru, C.S. Lam, C. Vafa, and D. Zanon, Phys. Lett. B **573**, 138 (2003).
- [2] H. Ooguri and C. Vafa, Adv. Theor. Math. Phys. **7**, 53 (2003); **7**, 407 (2003).
- [3] J. de Boer, P.A. Grassi, and P. van Nieuwenhuizen, Phys. Lett. B **574**, 981 (2003).
- [4] N. Seiberg, J. High Energy Phys. **06**, 010 (2003).
- [5] N. Berkovits and N. Seiberg, J. High Energy Phys. **07**, 010 (2003).
- [6] M. Hatsuda, S. Iso, and H. Umetsu, Nucl. Phys. **B671**, 217 (2003); T. Araki, K. Ito, and A. Ohtsuka, Phys. Lett. B **573**, 209 (2003); Y. Shibusa and T. Tada, *ibid.* **579**, 211 (2004); E. Ivanov, O. Lechtenfeld, and B. Zupnik, J. High Energy Phys. **02**, 012 (2004); S. Ferrara and E. Sokatchev, Phys. Lett. B **579**, 226 (2004); R. Abbaspur, ‘‘Scalar solitons in non(anti-)commutative superspace,’’ hep-th/0308050; A. Sako and T. Suzuki, Phys. Lett. B **582**, 127 (2004); B. Chandrasekhar and A. Kumar, J. High Energy Phys. **03**, 013 (2004); S. Iso and H. Umetsu, Phys. Rev. D **69**, 105003 (2004); A. Imaanpur, J. High Energy Phys. **12**, 009 (2003).
- [7] R. Britto, B. Feng, and S.J. Rey, J. High Energy Phys. **07**, 067 (2003); S. Terashima and J.T. Yee, *ibid.* **12**, 053 (2003); M.T. Grisaru, S. Penati, and A. Romagnoni, *ibid.* **08**, 003 (2003); R. Britto and B. Feng, Phys. Rev. Lett. **91**, 201601 (2003); A. Romagnoni, J. High Energy Phys. **10**, 016 (2003); O. Lunin and S.J. Rey, *ibid.* **09**, 045 (2003); M. Alishahiha, A. Ghodsi, and N. Sadooghi, ‘‘One-loop perturbative corrections to non(anti-)commutativity parameter of $N=1/2$ supersymmetric U(N) gauge theory,’’ hep-th/0309037.
- [8] D. Berenstein and S.J. Rey, Phys. Rev. D **68**, 121701(R) (2003).
- [9] N.J. Hitchin, in *Global Geometry and Mathematical Physics*, Lecture Notes in Mathematics No. 1451 (Springer, Heidelberg, 1988), pp. 1–48.
- [10] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld, and Y.I. Manin, Phys. Lett. **65A**, 185 (1978).
- [11] A. Imaanpur, J. High Energy Phys. **09**, 077 (2003).
- [12] P.A. Grassi, R. Ricci, and D. Robles-Llana, ‘‘Instanton calculations for $N=1/2$ super Yang-Mills theory,’’ hep-th/0311155.
- [13] J. Polchinski, Nucl. Phys. **B303**, 226 (1988).
- [14] J.M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theory Phys. **38**, 1113 (1999)].

- [15] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, *Phys. Lett. B* **428**, 105 (1998); E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998).
- [16] S.-J. Rey (unpublished); V. Balasubramanian, P. Kraus, A.E. Lawrence, and S.P. Trivedi, *Phys. Rev. D* **59**, 104021 (1999) (see also Ref. [22]).
- [17] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, *J. High Energy Phys.* **02**, 020 (2000).
- [18] S.R. Das and S.J. Rey, *Nucl. Phys.* **B590**, 453 (2000); D.J. Gross, A. Hashimoto, and N. Itzhaki, *Adv. Theor. Math. Phys.* **4**, 893 (2000); Y. Kiem *et al.*, *Phys. Rev. D* **65**, 026002 (2002); *Eur. Phys. J. C* **22**, 757 (2002); *Nucl. Phys.* **B641**, 256 (2002); *Phys. Rev. D* **65**, 046003 (2002); *Eur. Phys. J. C* **22**, 781 (2002).
- [19] S. J. Rey, in *Gravity, Gauge Theories and Strings*, Les Houches 2001, edited by C. Bachas *et al.* (Springer, Berlin, 2002), pp. 587–609; *J. Korean Phys. Soc.* **39**, S527 (2001).
- [20] R. Britto, B. Feng, and S.J. Rey, *J. High Energy Phys.* **08**, 001 (2003).
- [21] M.K. Murray, *Exp. Math.* **2**, 271 (1994); D. Groisser and M.K. Murray, “Instantons and the information metric,” [dg-ga/9611008](https://arxiv.org/abs/dg-ga/9611008); S. Parvizi, *Mod. Phys. Lett. A* **17**, 341 (2002).
- [22] M. Blau, K.S. Narain, and G. Thompson, “Instantons, the information metric, and the AdS/CFT correspondence,” [hep-th/0108122](https://arxiv.org/abs/hep-th/0108122).
- [23] M. A. Shifman and A. I. Vainshtein, in *ITEP Lectures in Particle Physics and Field Theory*, edited by M. Shifman (World Scientific, Singapore, 1999), Vol. 2, pp. 485–648.