Witten's ghost vertex made simple

D. M. Belov*

Department of Physics, Rutgers University, 136 Frelinghuysen Road, Piscataway, New Jersey 08854, USA

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First, we diagonalize the *bc*-ghost 3-string Neumann matrices using the technique described in Phys. Rev. D **68**, 066003 (2003). Their eigenvalues are in complete agreement with the previous authors. Second, we diagonalize the *N*-string gluing vertices for the bosonized ghost system. Third, we verify the descent and associativity relations for the combined bosonic matter+ghost gluing vertices. We find that in order for these relations to be true, the vertices must be normalized by the factor $\mathcal{Z}_N$. Here $\mathcal{Z}_N$ is the partition function of the bosonic matter+ghost CFT on the gluing surface, which is the unit disk with the Neumann boundary conditions and the midpoint conelike singularity specified by the angle excess $\pi(N-2)$.

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I. INTRODUCTION

Witten's open cubic string field theory [1] is usually formulated in terms of *N*-string gluing vertices [2,3]. The expressions for these vertices in the basis which diagonalizes L_0 lead to complicated calculations. Using the fact that $K_1 = L_1 + L_{-1}$ commutes with Witten's vertex, Rastelli *et al.* [4] transformed it to the basis with K_1 diagonal. They found that the Neumann matrices in the zero momentum vertex take a simple diagonal form in this basis. Further it was realized that many calculations in the string field theory which looks complicated in the L_0 basis can be done easily and analytically in the K_1 basis. Using their technique [4] the succeeding authors generalized their result to include momenta [5,6], ghosts [6–8], a background *B* field [9], and fermions [10]. However, all of these calculation were intrinsically indirect. Recently the authors of [11] formulated a simple and direct method of changing the basis. This method is so powerful that it allowed them to diagonalize at once *N*-string Neumann matrices for all scale dimensions in the matter sector, 3-string Neumann matrices for $\beta\gamma$ and *bc* ghosts, and to resolve the momentum difficulty. However, unlike all other cases, the 3-string *bc*-ghost vertex was diagonalized indirectly by relating it to the 6-string matter Neumann matrices [2].

The present paper has three aims. First, we diagonalize the 3-string *bc*-ghost vertex by directly changing the basis in its Neumann functions. Second, we consider the general bosonized ghost system [12] which is characterized by the background charge Q and the parity $\varepsilon = \pm 1$. We find an expression for the *N*-string gluing vertex of this system in the K_1 basis. Then we show that ghost numbers can be added to the vertex by a unitary transformation, and discuss the differences of this construction from the one for the matter sector [11].

Third, we test the descent relations

$$ {}_{1\cdots N,N+1}\langle V_{N+1}|V_1\rangle_{N+1} \stackrel{?}{=} {}_{1\cdots N}\langle V_N| \tag{1.1} $$

and other associativity relations for the combined bosonic matter+ghost gluing vertex $\langle V_N|$. The need to verify them arises from several inconsistencies in the calculations performed in the past two years. First, there is a strange anomaly in the multiplication of the wedge states $\langle h_N,0|$ [13] [Eq. (5.40) therein]. Second, the direct calculation of the inner product of two wedges $\langle h_3,3|h_3,0\rangle$ differs from the expected unity [8,14]. This result is in contradiction with the statement of [3] [Eq. (5.59) therein]. And third, assuming that the descent relations (1.1) are true for the vertices defined in [2] the authors of [15] find some contradictions in their calculations [compare Eqs. (3.34) and (3.38) therein]. In the present paper we show that for critical bosonic string there is a finite constant $\mathcal{Z}_{N+1,1;N}$ in the right-hand side (rhs) of the descent relation (1.1). This constant can be written as

$$ \mathcal{Z}_{N+1,1;N} = \frac{\mathcal{Z}_N}{\mathcal{Z}_1 \mathcal{Z}_{N+1}}, \tag{1.2} $$

where the function $\mathcal{Z}_N$ is

$$ \mathcal{Z}_N = \left[\frac{2}{N}\right]^{9/2} \exp\left\{\frac{27}{2}\left[D_0(N) - D_0(2)\right]\right\}. \tag{1.3} $$

The explicit expression for $D_0(N)$ is given in the text of this paper. Here one only needs to know that for $N \geqslant 1$ it monotonically increases, and goes to zero as $1/N$. Now it is obvious that in order to satisfy the descent relation one has to normalize the vertices by

$$ \langle V_N| \to \mathcal{Z}_N \langle V_N| \quad \text{for} \quad N \geqslant 1. \tag{1.4} $$

We also verify that the normalized vertices satisfy all other associativity conditions. Notice that for $N=2$ the normalization $\mathcal{Z}_N$ equals 1, therefore the string inner product is not affected by Eq. (1.4). The function $\mathcal{Z}_N$ is nothing else but the partition function of the matter+ghost CFT on the gluing surface, which is the unit disk with Neumann boundary conditions and midpoint cone-like singularity specified by the angle excess $\pi(N-2)$.

For $N=3$ the vertex normalized as in Eq. (1.4) coincides with Witten's original definition [1]. In that paper he defined it as the Polyakov integral over the gluing surface. It seems

*On leave from Steklov Mathematical Institute, Moscow, Russia; electronic address: belov@physics.rutgers.edu

that in most succeeding papers the Polyakov integral was changed into the correlation function on that surface, and the normalization factor $\mathcal{Z}_N$ was lost.

The paper is organized as follows. In Sec. II we review the notations used in [11]. In Sec. III we diagonalize the 3-string Neumann matrices for the *bc*-ghost system. In Sec. IV we consider the *N*-string gluing vertices for the general bosonized ghost system. We find its representation in the K_1 basis, and describe how to change the ghost number by a unitary transformation. In Sec. V we prove the associativity properties of the gluing vertices for the combined bosonic matter+ghost system. In Sec. VI we discuss the influence of the vertex normalization (1.4) on numeric calculations in SFT. The Appendix contains necessary technical information.

II. NOTATION

In this section we review the notations and main formulas from [11].

Consider the primary discrete series $\mathcal{D}_s^+$ of the SL(2,$\mathbb{R}$) representations. Here s is the scale dimension, $s=0,\frac{1}{2},1,\dots$. For example, $s=1$ corresponds to the zero momentum bosonic matter, $s=\frac{1}{2}$ to the fermions, and $s=0$ to the bosonic matter with the zero modes. An appropriate Hilbert space $\mathcal{H}_s$ consists of the functions $f(z)$ analytic inside the unit disk and square-integrable on the boundary. The inner product is [16]

$$\langle g|f\rangle=\frac{1}{\pi\Gamma(2s-1)}\int_{|z|\leqslant 1}d^2z[1-z\bar{z}]^{2s-2}\overline{g(z)}f(z). \tag{2.1}$$

The $s=\frac{1}{2}$ singularity is spurious [16], but there is a real one as s approaches zero. The algebra sl(2,$\mathbb{R}$) is generated by L_0, $L_{\pm 1}$ which are defined by

$$L_n=z^{n+1}\frac{d}{dz}+(n+1)sz^n. \tag{2.2}$$

This representation is unitary for $s>0$.

A. The discrete basis

The elliptic generator L_0 has discrete eigenvalues $(m+s)$, $m=0,1,2,\dots$. Its eigenfunctions normalized by Eq. (2.1) are

$$|m,s\rangle(z)=N_m^{(s)}z^m \quad \text{with} \quad N_m^{(s)}=\left[\frac{\Gamma(m+2s)}{\Gamma(m+1)}\right]^{1/2}. \tag{2.3}$$

Notice that for $s=0$ the only singular vector is $|0,0\rangle(z)$.

B. The continuous basis

The generator $K_1=L_1+L_{-1}$ commutes with Witten's star product [1], which therefore becomes simpler when it is diagonalized [4]. It is convenient to map

$$z=i\tanh w, \tag{2.4}$$

which takes the unit disk into the strip $|\mathrm{Im}\, w|\leqslant \pi/4$. We assume that under a map $z\mapsto w$ the vector $f(z)$ transforms in a trivial way

$$f(z)\mapsto f[z(w)]. \tag{2.4$'$}$$

Then

$$K_1=-i\frac{d}{dw}+2is\tanh w. \tag{2.5}$$

Since this is a hyperbolic generator, its eigenvalues are all real numbers κ. The normalized eigenfunctions of Eq. (2.5) are

$$|\kappa,s\rangle(z)=[A_s(\kappa)]^{1/2}(\cosh w)^{2s}e^{i\kappa w}, \tag{2.6}$$

where $A_s(\kappa)$ is the normalization constant:

$$A_s(\kappa)=\frac{2^{2s-2}}{\pi}\Gamma\left(s+\frac{i\kappa}{2}\right)\Gamma\left(s-\frac{i\kappa}{2}\right). \tag{2.7}$$

One sees that as $s\to 0$ the function (2.8) becomes ill-defined at $\kappa=0$. Nevertheless the $s=0$ K_1 eigenfunctions are well defined [11]:

$$|\kappa,0\rangle(z)=\mathcal{P}\frac{\sqrt{A_1(\kappa)}}{\kappa}e^{i\kappa w}=\mathcal{P}\frac{\sqrt{A_1(\kappa)}}{\kappa}+|\kappa,\Omega\rangle(z). \tag{2.8}$$

Here $\mathcal{P}$ means the principal value, and the function $|\kappa,\Omega\rangle(z)$ can also be written as the integral of the $s=1$ K_1 eigenfunction

$$|\kappa,\Omega\rangle(z)\equiv\int_0^z d\zeta|\kappa,1\rangle(\zeta). \tag{2.9}$$

The vector $|\kappa,\Omega\rangle(z)$ is that which was found by Rastelli *et al.* [4]. The important identity with $|\kappa,\Omega\rangle(z)$ is [11]

$$\frac{1}{2}\log(1+z^2)=\int_{-\infty}^{\infty}d\kappa\mathcal{P}\frac{\sqrt{A_1(\kappa)}}{\kappa}|\kappa,\Omega\rangle(z). \tag{2.10a}$$

Differentiating this with respect to z and using Eq. (2.9) one obtains another useful identity

$$\frac{z}{1+z^2}=\int_{-\infty}^{\infty}d\kappa\mathcal{P}\frac{\sqrt{A_1(\kappa)}}{\kappa}|\kappa,1\rangle(z). \tag{2.10b}$$

We will frequently use the notation $\langle\kappa,s|(\bar{z})\equiv\overline{|\kappa,s\rangle(z)}$.

C. The transition matrix

The transition matrix between the discrete and continuous bases is an orthogonal matrix with elements

$$\langle m,s|\kappa,s\rangle=V_m^{(s)}(\kappa)\frac{[A_s(\kappa)]^{1/2}}{N_m^{(s)}}. \tag{2.11}$$

Here the polynomials $V_m^{(s)}(\kappa)$ are given by the generating function

$$(\cosh w)^{2s} e^{i\kappa w} = \sum_{m=0}^{\infty} V_m^{(s)}(\kappa) z^m. \tag{2.12}$$

Due to Eq. (2.9) the transition matrices for $s=0$ and $s=1$ are related as [11]

$$\langle m+1,0|\kappa,0\rangle = \langle m,1|\kappa,1\rangle \quad (m \geqslant 0), \tag{2.13}$$

so $s=1$ is just $s=0$ with $m=0$ omitted.

III. 3-STRING VERTEX FOR *bc* SYSTEM

A. Overview

The *bc*-ghost system has a background charge $Q=-3$. Due to conservation of this charge it is convenient to write a 3-string vertex over the vacuum $\langle +|$, which is the conjugate of the ghost number 1 vacuum $|-\rangle$ (i.e., $\langle +|-\rangle = 1$). These vacua are defined by

$$b_n|-\rangle = 0 \ (n > -1); \quad \langle +|b_n = 0 \ (n<0); \tag{3.1a}$$

$$c_m|-\rangle = 0 \ \ (m \geqslant 1); \quad \langle +|c_m = 0 \ \ (m \leqslant 0); \tag{3.1b}$$

so $\{b_0, b_1, \ldots\}$ and $\{c_1, c_2, \ldots\}$ are the annihilation operators. The vacuum $|-\rangle$ is related to the SL(2,$\mathbb{R}$) invariant vacuum $|0\rangle$ by $|-\rangle = c(0)|0\rangle$. The 3-string vertex over these vacua was constructed in [2, paper II], and in our notations it reads

$$\langle V_3| = N_{bc\;123}\langle +| \times \exp\left[-\sum_{m=0,n=1}^{\infty} b_m^{(I)} (\mathcal{M}_{bc}^{IJ} C)_{mn} c_n^{(J)} \right]. \tag{3.2}$$

Here $_{123}\langle +|$ means the tensor product of three Fock vacua $\langle +|$, $(\mathcal{M}_{bc}^{IJ})_{mn}$ are the 3-string ghost Neumann matrices and $C_{mn} = (-1)^n \delta_{mn}$.

To obtain an expression for the matrix elements $(\mathcal{M}_{bc}^{IJ})_{mn}$ one can calculate the function

$$\langle V_3| c^{(I)}(z) b^{(J)}(z')|-\rangle_{123} \tag{3.3}$$

in two different ways: using expression (3.2) and using the conformal definition of the vertex. The details of this calculation can be found, for example, in Sec. 4 of [17] or in [18]. In our notations the result is

$$\begin{aligned}\mathcal{M}_{bc}^{IJ}(z,\overline{z'}) &\equiv \sum_{m,n=0}^{\infty} (\mathcal{M}_{bc}^{IJ})_{n,m+1} z^n \overline{z'}^m \\ &= \frac{[h_I'(z)]^{-1}[h_J'(-\overline{z'})]^2}{h_I(z) - h_J(-\overline{z'})} \frac{[h_I(z)]^3 - 1}{[h_J(-\overline{z'})]^3 - 1} \\ &\quad \times \left[\frac{\overline{z'}}{z}\right] + \frac{\delta^{IJ}}{z+\overline{z'}},\end{aligned} \tag{3.4}$$

and $N_{bc} = [3\sqrt{3}/4]^3$. Here the maps $h_I(z)$ are

$$h_I(z) = e^{i\varphi_I}\left(\frac{1-iz}{1+iz}\right)^{2/3} = e^{i\varphi_I} e^{4w/3}, \tag{3.5a}$$

$$h_I'(z) = -\frac{4i}{3}(\cosh w)^2 h_I(z), \tag{3.5b}$$

where $z = i \tanh w$, and $\varphi_I = (2\pi/3)(2-I)$ for $I=1,2,3$.

B. Diagonalizing Witten's 3-string ghost vertex

The aim of this section is to rewrite the operator $\mathcal{M}_{bc}^{IJ}(z,\overline{z'})$ in the K_1 basis. It is known that the vertices commute with the operator K_1 [2]. So one expects that $\mathcal{M}_{bc}^{IJ}(z,\overline{z'})$ takes a simple form in the K_1 basis.

To diagonalize the Neumann matrix (3.4) we first notice that the strange factors in Eq. (3.4) have a very simple expression in the variable w

$$\frac{[h_I(z)]^3 - 1}{z} = -4ie^{2w}(\cosh w)^2 \tag{3.6a}$$

and

$$\frac{[h_J(-\overline{z'})]^3 - 1}{\overline{z'}} = +4ie^{2\overline{w'}}(\cosh \overline{w'})^2. \tag{3.6b}$$

Now we proceed as in [11]—first do a binomial expansion of Eq. (3.4), then rewrite it as a contour integral, and finally do a Watson-Sommerfeld transformation. Assuming $\mathrm{Re}(\overline{w'} - w) < 0$ we obtain

$$\begin{aligned}\mathcal{M}_{bc}^{IJ}(z,\overline{z'}) &= \frac{4i}{3}(\cosh \overline{w'})^2 (-1)^{I-J} \\ &\quad \times \oint_C \frac{dj}{2i\sin \pi j} e^{\pm i\pi j}[e^{i(\varphi_J - \varphi_I)} e^{4/3(\overline{w'} - w)}]^{j+1/2} \\ &\quad + \frac{\delta^{IJ}}{z+\overline{z'}},\end{aligned} \tag{3.7}$$

where the contour C encircles the positive real axis counterclockwise (see Fig. 1). Notice that we have a sign ambiguity in the exponential, which comes from the analytic continuation of $(-1)^j$.

Before deforming the contour as in Fig. 1 we must worry about the convergence at infinity. Starting from here we will consider the cases $I \neq J$ and $I = J$ separately.

C. Matrices $\mathcal{M}_{bc}^{IJ}$ for $I \neq J$

For $I \neq J$ Eq. (3.7) can be rewritten as

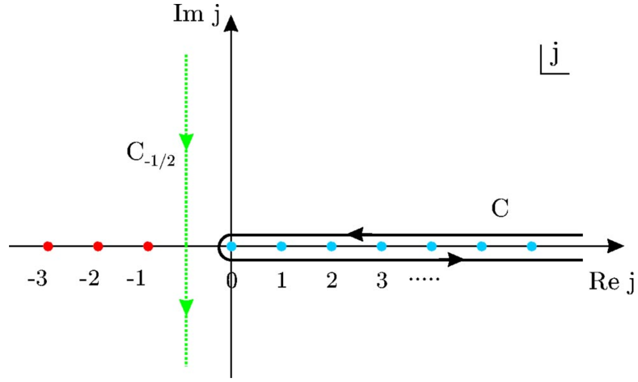

FIG. 1. The dots represent the poles of the integrand in Eq. (3.7). Contour C encircles the positive real axis counterclockwise. Then we deform it to the contour $C_{-1/2}$, which lies parallel to the imaginary axis at $\mathrm{Re}\, j = -\frac{1}{2}$.

$$\mathcal{M}^{IJ}_{bc}(z,\overline{z'}) = (\mp 1)\frac{4i}{3}(\cosh\overline{w'})^2(-1)^{I-J} \times \oint_C \frac{dj}{2\sin(\pi j)}\left[e^{i(\varphi_J-\varphi_I\pm\pi)}e^{4/3(\overline{w'}-w)}\right]^{j+1/2}. \tag{3.8}$$

To deform the contour as it is shown in Fig. 1 we must worry about convergence as $|\mathrm{Im}\, j| \to \infty$. To this end for $I<J$ we choose the upper "+" sign in the exponential. This guarantees that for $1 \leqslant I < J \leqslant 3$

$$|\varphi_J - \varphi_I + \pi| \leqslant \frac{\pi}{3}.$$

Taking into account the following asymptotics as $|\mathrm{Im}\, j| \to \infty$:

$$\frac{1}{\sin(\pi j)} \propto e^{-\pi|\mathrm{Im}\, j|}$$

and

$$|e^{4/3(\overline{w'}-w)j}| \leqslant e^{4/3|\mathrm{Im}(\overline{w'}-w)||\mathrm{Im}\, j|},$$

one concludes that for arbitrary small $\delta>0$ and $|\mathrm{Im}(\overline{w'}-w)| \leqslant (\pi/2) - \delta$ the integrand has at least the exponential falloff $e^{-\delta|\mathrm{Im}\, j|}$. (Oppositely, for $J<I$ one has to choose the lower "−" sign in the exponential. This guarantees the same falloff as $|\mathrm{Im}\, j| \to \infty$.) Now we can shift the contour C to $\mathrm{Re}\, j = -\frac{1}{2}$ by writing

$$j = -\frac{1}{2} - \frac{3i\kappa}{4} \tag{3.9}$$

to get (for $I<J$)

$$\mathcal{M}^{IJ}_{bc}(z,\overline{z'}) = (-1)^{I-J}(\cosh\overline{w'})^2 \times \int_{-\infty}^{\infty} d\kappa \frac{e^{(\pi\kappa/4)(2I-2J+3)}}{2\cosh(3\pi\kappa/4)} e^{i\kappa(w-\overline{w'})}. \tag{3.10}$$

Notice that the integral here converges now for all $w,\overline{w'}$ in the strip $|\mathrm{Im}\, w| < (\pi/4)$. Therefore (by the standard analytic continuation arguments) the right-hand side represents the operator (3.4) for all z and $\overline{z'}$ in the unit disk.

Comparing the cosh-factors in Eq. (3.10) with the ones in Eq. (2.6) one concludes that the continuum K_1 eigenfunctions correspond to $s=0$ and $s=1$. From Eqs. (2.8) and (2.6) it follows that the normalization factor of their tensor product must be $\mathcal{P}[A_1(\kappa)/\kappa]$. Insertion of unity $1 = \kappa/A_1(\kappa)\mathcal{P}[A_1(\kappa)/\kappa]$ into Eq. (3.10) yields

$$\mathcal{M}^{IJ}_{bc}(z,\overline{z'}) = -\int_{-\infty}^{\infty} d\kappa\, \mu^{IJ}_{bc}(\kappa)|\kappa,0\rangle(z) \otimes \langle\kappa,1|(\overline{z'}), \tag{3.11a}$$

where

$$\mu^{12}_{bc}(\kappa) = +e^{+x}\frac{\sinh 2x}{\cosh 3x}, \tag{3.11b}$$

$$\mu^{13}_{bc}(\kappa) = -e^{-x}\frac{\sinh 2x}{\cosh 3x} \tag{3.11c}$$

with $x \equiv (\pi\kappa/4)$.

D. Matrix $\mathcal{M}^{II}_{bc}$

For $I=J$ expression (3.4) becomes

$$\mathcal{M}^{II}_{bc}(z,\overline{z'}) = \frac{1}{z+\overline{z'}} + \frac{4i}{3}(\cosh\overline{w'})^2 \times \oint_C \frac{dj}{2i\sin(\pi j)} e^{\pm i\pi j}\left[e^{4/3(\overline{w'}-w)}\right]^{j+1/2}. \tag{3.12}$$

This expression contains two terms, and we will consider them separately. In the second term we want to deform the contour C as in Fig. 1. To this end we must first worry about convergence at infinity. For general w and $\overline{w'}$ the integrand does not go to zero as $|\mathrm{Im}\, j| \to \infty$. However for $0 < \mp\mathrm{Im}(\overline{w'}-w) \leqslant (\pi/2)$ the integrand has an exponential falloff. In this case we deform the contour C as in Fig. 1 by writing Eq. (3.9) to get

$$\text{second term} = \pm(\cosh\overline{w'})^2 \times \int_{-\infty}^{\infty} d\kappa \frac{e^{\mp 3\pi\kappa/4}}{2\cosh(3\pi\kappa/4)} e^{i\kappa(w-\overline{w'})}.$$

Comparing the cosh-factors here with the ones in Eq. (2.6) one concludes that the continuum eigenfunctions correspond to $s=0$ and $s=1$. Therefore we can write

$$\text{second term} = \pm \int_{-\infty}^{\infty} d\kappa e^{\pm 3\pi\kappa/4} \frac{\sinh \frac{\pi\kappa}{2}}{\cosh \frac{3\pi\kappa}{4}} \times |\kappa,0\rangle(z) \otimes \langle\kappa,1|(\overline{z'}), \tag{3.13}$$

where the eigenfunctions $|\kappa,0\rangle$ and $\langle\kappa,1|$ are defined in Eqs. (2.8) and (2.6), respectively. The representation (3.13) is valid only for $0 < \mp \mathrm{Im}(\overline{w'} - w) \leqslant (\pi/2)$.

Now we need to represent the first term in Eq. (3.12) through the tensor product of $s=0$ and $s=1$ eigenfunctions. The details of this calculation can be found in the Appendix. The result is

$$\frac{1}{z+\overline{z'}} = -\int_{-\infty}^{\infty} d\kappa e^{\pm \pi\kappa/2} |\kappa,0\rangle(z) \otimes \langle\kappa,1|(\overline{z'}) + \int_{-\infty}^{\infty} d\kappa \mathcal{P} \frac{\sqrt{A_1(\kappa)}}{\kappa} \otimes \langle\kappa,1|(\overline{z'}). \tag{3.14}$$

The first integral here converges for $0 < \mp \mathrm{Im}(\overline{w'} - w) \leqslant (\pi/2)$, while the second integral converges for all $\overline{z'}$ in the unit disk.

Substitution of Eqs. (3.13) and (3.14) into Eq. (3.12) yields

$$\mathcal{M}^{II}_{bc}(z,\overline{z'}) = \int_{-\infty}^{\infty} d\kappa \mathcal{P} \frac{\sqrt{A_1(\kappa)}}{\kappa} \otimes \langle\kappa,1|(\overline{z'}) + \int_{-\infty}^{\infty} d\kappa \left[\pm e^{\pm 3\pi/4} \frac{\sinh \frac{\pi\kappa}{2}}{\cosh \frac{3\pi\kappa}{4}} - e^{\pm \pi\kappa/2} \right] \times |\kappa,0\rangle(z) \otimes \langle\kappa,1|(\overline{z'}). \tag{3.15}$$

This expression has a sign ambiguity in the second term. However, if we want the first integral to be convergent in the region $|\mathrm{Im}(w-\overline{w'})| \leqslant (\pi/2)$ we must choose the upper "+" sign in this expression. Finally, we get

$$\mathcal{M}^{II}_{bc}(z,\overline{z'}) = -\int_{-\infty}^{\infty} d\kappa \mu^{II}_{bc}(\kappa) |\kappa,0\rangle(z) \otimes \langle\kappa,1|(\overline{z'}) + \int_{-\infty}^{\infty} d\kappa \mathcal{P} \frac{\sqrt{A_1(\kappa)}}{\kappa} \otimes \langle\kappa,1|(\overline{z'}), \tag{3.16a}$$

where

$$\mu^{II}_{bc}(\kappa) = \frac{\cosh x}{\cosh 3x} \quad \text{with} \quad x \equiv \frac{\pi\kappa}{4}. \tag{3.16b}$$

E. The Neumann matrices in the discrete basis

Collecting Eqs. (3.16) and (3.11), one concludes that the ghost 3-string Neumann functions (3.4) have the following diagonal representation in the K_1 basis:

$$\mathcal{M}^{IJ}_{bc}(z,\overline{z'}) = -\int_{-\infty}^{\infty} d\kappa \mu^{IJ}_{bc}(\kappa) |\kappa,0\rangle(z) \otimes \langle\kappa,1|(\overline{z'}) + \delta^{IJ} \int_{-\infty}^{\infty} d\kappa \mathcal{P} \frac{\sqrt{A_1(\kappa)}}{\kappa} \otimes \langle\kappa,1|(\overline{z'}), \tag{3.17a}$$

where $1 \leqslant I,J \leqslant 3$,

$$\mu^{II}_{bc}(\kappa) = \frac{\cosh \mathrm{x}}{\cosh 3\mathrm{x}}, \tag{3.17b}$$

$$\mu^{I,I+1}_{bc}(\kappa) = +e^{\mathrm{x}} \frac{\sinh 2\mathrm{x}}{\cosh 3\mathrm{x}}, \tag{3.17c}$$

$$\mu^{I+1,I}_{bc}(\kappa) = -e^{-\mathrm{x}} \frac{\sinh 2\mathrm{x}}{\cosh 3\mathrm{x}} \tag{3.17d}$$

and $\mathrm{x} \equiv (\pi\kappa/4)$.

To find the Neumann matrices $(\mathcal{M}^{IJ}_{bc})_{n,m+1}$ we can calculate the matrix elements of the operator $\mathcal{M}^{IJ}_{bc}(z,\overline{z'})$ between the vectors $\langle n,0|$ and $|m,1\rangle$. In this calculation one obviously gets a divergence for $n=0$ [see Eq. (2.3)]. We can handle this divergence by considering instead of the operator $\mathcal{M}^{IJ}_{bc}(z,\overline{z'})$ the vector $\mathcal{M}^{IJ}_{bc}(0,\overline{z'})$ and calculate its inner product with $|m,1\rangle$. Proceeding in this way one obtains

$$(\mathcal{M}^{IJ}_{bc})_{n+1,m+1} = -\frac{\sqrt{m+1}}{\sqrt{n+1}} \int_{-\infty}^{\infty} d\kappa \mu^{IJ}_{bc}(\kappa) \times \langle n+1,0|\kappa,0\rangle\langle\kappa,1|m,1\rangle \tag{3.18a}$$

for $n,m \geqslant 0$ and

$$(\mathcal{M}^{IJ}_{bc})_{0,m+1} = -\sqrt{m+1} \int_{-\infty}^{\infty} d\kappa [\mu^{IJ}_{bc}(\kappa) - \delta^{IJ}] \times \frac{\sqrt{A_1(\kappa)}}{\kappa} \langle\kappa,1|m,1\rangle. \tag{3.18b}$$

Here the square roots come from the calculation of the inner products of z^n or $\overline{z'}^m$, appearing in the definition (3.4) of $\mathcal{M}^{IJ}_{bc}$, with the vectors $\langle n,0|$ or $|m,1\rangle$.

Taking into account Eq. (2.13) and $\mu^{IJ}_{bc}(0) = \delta^{IJ}$ we find that the representation (3.18) and eigenvalues (3.17d) completely agree with the ones obtained in [11] [Eqs. (6.7) and (6.8) therein].

IV. *N*-STRING VERTEX FOR BOSONIZED GHOSTS

A. Overview

The aim of this section is to review a construction of LeClair *et al.* [3] of the *N*-string gluing vertex for the bosonized ghosts.

Here we consider CFT for the general bosonized ghost system [12], which is characterized by the background charge Q and the parity $\varepsilon=\pm 1$. The ghost number current $j(z)=\varepsilon\partial\phi(z)$ is an anomalous primary operator of dimension 1 and transforms under a conformal map $h(z)$ as

$$j(z)\mapsto(h\circ j)(z)=h'(z)j[h(z)]+\frac{Q}{2}\frac{h''(z)}{h'(z)}. \tag{4.1}$$

This current has the following mode expansion:

$$j(z)=\frac{j_0}{z}+\sum_{n=1}^{\infty}\sqrt{n}\{a_n^+z^{n-1}-a_n^-z^{-n-1}\}$$

with

$$[a_n^-,a_m^+]=-\varepsilon\,\delta_{mn}. \tag{4.2}$$

Here $a_m^\pm$ are creation/annihilation operators over the vacuum $|q\rangle$, which is an eigenvector of the operator j_0 with eigenvalue q. Due to the anomalous transformation law (4.1) the conjugate vacuum to $|q\rangle$ is $\langle -q-Q|$. From this it follows that $j_0^\dagger=-j_0-Q$.

The OPEs of the fields $\phi(z)$ and $j(z)$ are

$$\phi(z)\phi(z')\sim\varepsilon\log(z-z'), \tag{4.3a}$$

$$j(z)e^{q\phi(z)}\sim\frac{q}{z-z'}e^{q\phi(z')}. \tag{4.3b}$$

The matter field X^μ can be obtained from the expression above by identifying ε with $-\eta^{\mu\nu}$, $j_0=-i\sqrt{2\alpha'}p^\mu$ and $a_m^\pm=\mp(i/\sqrt{m})\,\alpha_{\mp m}^\mu$.

The gluing vertex for the bosonized ghosts differs in two ways from that for the X field. First, it has nonzero background charge Q, and second the momentum eigenvalues are no longer continuous but form a discrete set. The vertex reads [3] [Eq. (5.1) therein]

$$\begin{aligned}\langle V_{N,Q}^{(0)}|=&\sum_{\{q^I\}}\delta_{q^1+\dots+q^N+Q,0}\bigotimes_{I=1}^{N}{}_I\langle -q^I-Q|\\&\times\exp\Bigg[\frac{\varepsilon}{2}\sum_{I=1}^{N}q^I(Q+q^I)+\frac{\varepsilon}{2}\sum_{I\neq J}q^IN_{00}^{IJ}q^J\\&-\frac{\varepsilon}{2}\sum_{I=1}^{N}\sum_{n=0}^{\infty}QK_n^Ia_n^{-(I)}-\varepsilon\sum_{I,J=1}^{N}\sum_{n=1}^{\infty}q^IN_{0n}^{IJ}a_n^{-(J)}\\&+\frac{\varepsilon}{2}\sum_{I,J=1}^{N}\sum_{m,n=1}^{\infty}a_m^{-(I)}N_{mn}^{IJ}a_n^{-(J)}\Bigg],\end{aligned} \tag{4.4}$$

where the Neumann function coefficients N_{nm}^{IJ} are defined by the same formula as the ones for the matter part of the vertex [3] and

$$K^I(z)\equiv\sum_{n=1}^{\infty}K_n^I\sqrt{n}z^n=\frac{h_I''(z)}{h_I'(z)}. \tag{4.5}$$

The fact that the terms in the exponent contain the coefficient Q is a direct consequence of the transformation law (4.1). Using the *anomalous* momentum conservation law and $\langle -q-Q|j_0=\langle -q-Q|q$ we can rewrite the exponential as [3]

$$\begin{aligned}&\exp\Bigg[\frac{\varepsilon}{2}\sum_{I,J}j_0^I\mathcal{N}_{00}^{IJ}j_0^J-\varepsilon\sum_{I,J}\sum_{n=1}^{\infty}j_0^I\mathcal{N}_{0n}^{IJ}a_n^{-(J)}\\&+\frac{\varepsilon}{2}\sum_{I,J}\sum_{m,n=1}^{\infty}a_m^{-(I)}\mathcal{N}_{mn}^{IJ}a_n^{-(J)}\Bigg],\end{aligned} \tag{4.6}$$

where the new Neumann-function coefficients are related to the old ones by

$$\mathcal{N}_{00}^{IJ}=N_{00}^{IJ}-\frac{1}{2}N_{00}^{II}-\frac{1}{2}N_{00}^{JJ}, \tag{4.7a}$$

$$\mathcal{N}_{0m}^{IJ}=N_{0m}^{IJ}-\frac{1}{2}K_m^J, \tag{4.7b}$$

$$\mathcal{N}_{mn}^{IJ}=N_{mn}^{IJ}. \tag{4.7c}$$

The new coefficients can be expressed in terms of the gluing maps $\{h_I(z)\}$ as follows [3]:

$$\mathcal{N}_{00}^{IJ}=(1-\delta^{IJ})\log\left[\frac{|h_I(0)-h_J(0)|}{|h_I'(0)h_J'(0)|^{1/2}}\right], \tag{4.8a}$$

$$\begin{aligned}\mathcal{N}_0^{IJ}(z)&\equiv\sum_{n=1}^{\infty}\mathcal{N}_{0n}^{IJ}\sqrt{n}z^{n-1}\\&=-\frac{h_J'(z)}{h_I(0)-h_J(z)}-\frac{\delta^{IJ}}{z}-\frac{h_J''(z)}{2h_J'(z)},\end{aligned} \tag{4.8b}$$

$$\begin{aligned}\mathcal{N}^{IJ}(z,z')&\equiv\sum_{m,n=1}^{\infty}\mathcal{N}_{mn}^{IJ}\sqrt{mn}z^{m-1}z'^{n-1}\\&=\frac{h_I'(z)h_J'(z')}{[h_I(z)-h_J(z')]^2}-\frac{\delta^{IJ}}{(z-z')^2}.\end{aligned} \tag{4.8c}$$

Notice that all functions here are manifestly PSL(2) invariant [3, pp. 487–488], i.e., they do not change under $\{h_I\}\to\{T\circ h_I\}$ with $T(z)\in$ PSL(2). Therefore all Neumann coefficients $\mathcal{N}_{mn}$ are SL(2,$\mathbb{R}$) invariant. Let us remember that, for example, the generating function for the coefficients N_{0n}^{IJ} *does* depend on a choice of the SL(2,$\mathbb{R}$) frame. However, this dependence is cancelled by the nonanomalous momentum conservation, and therefore the X vertex is SL(2) invariant.

Writing the vertex in the form (4.6) eliminates the explicit dependence on the background charge Q. In this notation the X^μ vertex can be obtained simply by changing ε to $-\eta^{\mu\nu}$, $j_0=-i\sqrt{2\alpha'}p^\mu$ and $a_n^{\pm}\to\mp i(\alpha^\mu_{\mp n}/\sqrt{n})$. Of course some of the terms in Eq. (4.8b) will drop out due to the normal (nonanomalous) momentum conservation [3].

For Witten's N-string vertex the maps $h_I(z)$ are

$$h_I(z)=\left(\frac{1-iz}{1+iz}\right)^{2/N}=e^{i\varphi_I}e^{4w/N}, \tag{4.9a}$$

$$h_I'(z)=-\frac{4i}{N}(\cosh w)^2 h_I(z), \tag{4.9b}$$

where $z=i\tanh w$ and $\varphi_I=(2\pi/N)(\alpha_N-I)$. Here α_N is a real number which is chosen in such a way that all angles φ_I lie in the range $(-\pi,\pi]$.

B. Diagonalization of the Neumann coefficients

1. Operator $\mathcal{N}^{IJ}$

The operator $\mathcal{N}^{IJ}(z,z')$ was diagonalized in Sec. III of [11]:

$$\mathcal{N}^{IJ}(z,-\overline{z'})=\int_{-\infty}^{\infty}d\kappa\,\mu^{IJ}_{1,N}(\kappa)|\kappa,1\rangle(z)\otimes\langle\kappa,1|(\overline{z'}), \tag{4.10}$$

where $|\kappa,1\rangle(z)$ is defined in Eq. (2.6), $\langle\kappa,1|(\overline{z'})\equiv\overline{|\kappa,1\rangle(z')}$ and

$$\mu^{II}_{1,N}(\kappa)=-\frac{\sinh(N-2)x}{\sinh Nx}, \tag{4.11a}$$

$$\mu^{IJ}_{1,N}(\kappa)=e^{+x(N+2I-2J)}\frac{\sinh 2x}{\sinh Nx}\quad (I<J), \tag{4.11b}$$

$$\mu^{IJ}_{1,N}(\kappa)=e^{-x(N-2I+2J)}\frac{\sinh 2x}{\sinh Nx}\quad (I>J). \tag{4.11c}$$

Here $x\equiv(\pi\kappa/4)$.

2. Matrix $\mathcal{N}^{IJ}_{00}$

Substitution of the maps (4.9) into Eq. (4.8a) yields

$$\mathcal{N}^{IJ}_{00}=(1-\delta^{IJ})\log\left[\frac{N}{2}\sin\left(\frac{\pi}{N}|I-J|\right)\right]. \tag{4.12}$$

This expression coincides with the matrix $M'^{IJ}_{N,\,00}$ which is defined by Eq. (5.8) in [11]. The numbers $\mathcal{N}^{IJ}_{00}$ can also be represented by the following integral:

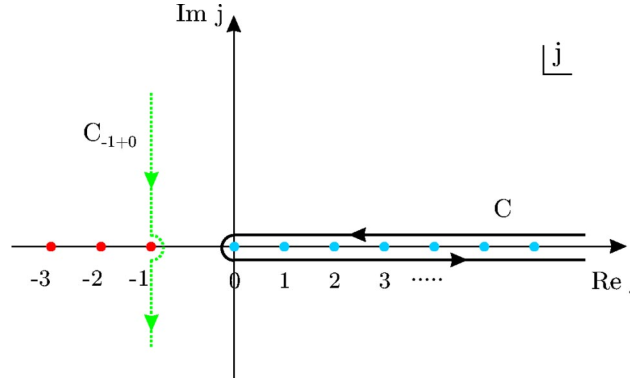

FIG. 2. The dots represent the poles of the integrand in Eq. (4.15). Contour C encircles the positive real axis counterclockwise. Then we deform it to the contour C_{-1+0}, which lies parallel to the imaginary axis at Re $j=-1$ and passes the pole at $j=-1$ on the right.

$$\mathcal{N}^{IJ}_{00}=-\int_{-\infty}^{\infty}d\kappa\frac{A_1(\kappa)}{\kappa^2}\left\{\frac{1}{2}[\mu^{IJ}_{1,N}(\kappa)+\mu^{JI}_{1,N}(\kappa)]-\mu^{II}_{1,N}(\kappa)+\delta^{IJ}-1\right\}. \tag{4.13}$$

The simplest way to obtain this expression is to notice that it can also be written as the following limit:

$$-\lim_{s\to+0}\int_{-\infty}^{\infty}d\kappa A_s(\kappa)\left\{\frac{1}{2}[\mu^{IJ}_{s,N}(\kappa)+\mu^{JI}_{s,N}(\kappa)]-\mu^{II}_{s,N}(\kappa)+\delta^{IJ}-1\right\},$$

which was calculated in Eqs. (5.7)–(5.9) in [11] with the result $M'^{IJ}_{N,\,00}$.

3. Vector $\mathcal{N}^{IJ}_0$

Substitution of the maps (4.9) into Eq. (4.8b) yields

$$\mathcal{N}^{IJ}_0(-\overline{z'})=\frac{4i}{N}(\cosh\overline{w'})^2\frac{e^{i(\varphi_J-\varphi_I)}e^{4\overline{w'}/N}}{1-e^{i(\varphi_J-\varphi_I)}e^{4\overline{w'}/N}}+\frac{\delta^{IJ}}{\overline{z'}}-\frac{\overline{z'}}{1+\overline{z'}^2}+\frac{2i}{N}(\cosh\overline{w'})^2. \tag{4.14}$$

Assuming Re $\overline{w'}<0$ we can expand the first term in a binomial series, and then rewrite it as a contour integral. All other terms we leave as they are for a while. So the first term becomes

$$\text{first term}=-\frac{4i}{N}(\cosh\overline{w'})^2\oint_C\frac{dj}{2i\sin(\pi j)}\times e^{i(\varphi_J-\varphi_I\pm\pi)(j+1)}e^{4\overline{w'}/N(j+1)}, \tag{4.15}$$

where the contour C encircles the positive real axis in the counterclockwise direction (see Fig. 2). Notice the sign ambiguity in the exponential, which comes from the analytic continuation of $(-1)^j$.

Now we are going to deform the contour C to lie parallel to the imaginary axis as in Fig. 2. To this end we need to worry about the convergence as $|\mathrm{Im}\, j|\to\infty$. Let us assume $I\leqslant J$. In this case the integral will have the exponential fall-off if we choose the upper "+" sign and temporarily assume $0<-\mathrm{Im}\,\overline{w'}<(\pi/4)$. With these assumptions we can safely deform the contour C as in Fig. 2 by writing

$$j=-1+0-\frac{iN\kappa}{4}$$

to get

$$\text{first term}=(\cosh\overline{w'})^2\int_{-\infty}^{\infty}\frac{d\kappa}{2\sinh\left(\frac{\pi N\kappa}{4}+i0\right)}\times e^{\pi\kappa/4(N+2I-2J)}e^{-i\kappa\overline{w'}}. \quad (4.16)$$

Using the fact that

$$\frac{1}{\sinh\left(\frac{\pi N\kappa}{4}+i0\right)}=\mathcal{P}\frac{1}{\sinh\frac{\pi N\kappa}{4}}-\frac{4i}{N}\delta(\kappa)$$

one finds that the term in Eq. (4.16) coming from the δ function cancels the last term in Eq. (4.14). Now we represent $1/\overline{z'}$ in Eq. (4.14) as in Eq. (3.14) with the "+" sign, and write $\overline{z'}/(1+\overline{z'}^2)$ as in Eq. (2.10b) to get

$$\mathcal{N}_0^{IJ}(-\overline{z'})=\int_{-\infty}^{\infty}d\kappa\mathcal{P}\frac{\sqrt{A_1(\kappa)}}{\kappa}\times(\mu_{1,N}^{IJ}(\kappa)-1+\delta^{IJ})\langle\kappa,1|(\overline{z'}), \quad (4.17)$$

where $\mu_{1,N}^{IJ}(\kappa)$ is defined in Eq. (4.11). Notice now that the integral in the rhs converges for all $\overline{w'}$ in the strip $|\mathrm{Im}\,\overline{w'}|<(\pi/4)$. Therefore (by the standard analytic continuation arguments) this expression represents the vector $\mathcal{N}_0^{IJ}(z)$ for all z in the unit disk. One can easily check that Eq. (4.17) remains the same if we assume $I\geqslant J$ and choose the "−" sign in Eqs. (4.14) and (3.14).

C. The *N*-string vertex in the continuous basis

1. Vertex in the continuous basis

We introduce the $s=1$ continuum oscillators

$$a^{\pm}(\kappa)=\sum_{n=0}^{\infty}a_{n+1}^{\pm}\langle\kappa,1|n,1\rangle$$

with

$$[a^-(\kappa),a^+(\kappa')]=-\varepsilon\,\delta(\kappa-\kappa'). \quad (4.18)$$

In order to write the vertex we also need a twist operator C, which roughly speaking corresponds to the substitution $\overline{z'}\to-\overline{z'}$. It is defined [2] in the discrete basis by $(Ca^{\pm})_n=(-1)^n a_n^{\pm}$. Hence in the continuous basis it becomes

$$(Ca^{\pm})(\kappa)=-a^{\pm}(-\kappa). \quad (4.19)$$

Finally, substituting Eqs. (4.10), (4.17) into Eq. (4.6) and using Eq. (4.18) one obtains

$$\begin{aligned}\langle V_{N,\,Q}^{(0)}|=&\sum_{\{q^I\}}\bigotimes_{I=1}^{N}{}_I\langle -q^I-Q|\,\delta_{j_0^1+\cdots+j_0^N+Q,0}\\&\times\exp\Bigg[-\frac{\varepsilon}{2}\sum_{I,J=1}^{N}\int_{-\infty}^{\infty}d\kappa a^{-(I)}(\kappa)\mu_{1,N}^{IJ}(\kappa)\\&\times(Ca^{-(J)})(\kappa)+\varepsilon\sum_{I,J=1}^{N}j_0^I\int_{-\infty}^{\infty}d\kappa\mathcal{P}\frac{\sqrt{A_1(\kappa)}}{\kappa}\\&\times\{\mu_{1,N}^{IJ}(\kappa)-1+\delta^{IJ}\}(Ca^{-(J)})(\kappa)\\&+\frac{\varepsilon}{2}\sum_{I,J=1}^{N}j_0^I\mathcal{N}_{00}^{IJ}j_0^J\Bigg],\end{aligned} \quad (4.20)$$

where j_0 is the ghost number operator, $\langle -q-Q|j_0=q\langle -q-Q|$, the continuous oscillators $a^-(\kappa)$ are defined in Eq. (4.18), the N-string Neumann matrix eigenvalues $\mu_{1,N}^{IJ}$ are defined in Eq. (4.11), and the coefficients $\mathcal{N}_{00}^{IJ}$ are listed in Eq. (4.12). Notice that the gluing vertex for the matter field X^μ can be obtained from this by putting $Q=0$ and replacing ε with $-\eta^{\mu\nu}$, $j_0=-i\sqrt{2\alpha'}p_\mu$ and $a^{-(I)}\to a_\mu^{-(I)}$ [compare Eq. (4.20) with Eq. (5.12) from [11]].

2. The unitary transformation

In [11] it was shown that the $s=0$ N-string matter vertex can be obtained from the zero momentum (i.e., $s=1$) vertex by the unitary transformation

$$U_p=\exp\left\{i\sqrt{2\alpha'}\hat{p}\int_{-\infty}^{\infty}d\kappa\mathcal{P}\frac{\sqrt{A_1(\kappa)}}{\kappa}\times[a^+(\kappa)+a^-(\kappa)]\right\}. \quad (4.21)$$

In the case of nonzero background charge Q the appropriate unitary operator is

$$U_{j_0}=\exp\left\{\varepsilon\left(j_0+\frac{Q}{2}\right)\int_{-\infty}^{\infty}d\kappa\mathcal{P}\frac{\sqrt{A_1(\kappa)}}{\kappa}\times[a^+(\kappa)+a^-(\kappa)]\right\}. \quad (4.22)$$

Here the linear combination of j_0 and Q was fixed from the conjugation property [12] of j_0: $j_0^\dagger=-j_0-Q$. Notice that in contrast to Eq. (4.21) the unitary transformation (4.22) remains nontrivial even for $j_0=0$. Under this unitary transformation the $s=1$ continuum oscillator $a^{\pm}(\kappa)$ transforms to the $s=0$ oscillator

$$a^{\pm}(\kappa,j_0)\equiv U_{j_0}^{-1}a^{\pm}(\kappa)U_{j_0} = \pm\left(j_0+\frac{Q}{2}\right)\mathcal{P}\frac{\sqrt{A_1(\kappa)}}{\kappa}+a^{\pm}(\kappa). \tag{4.23}$$

Now let us try to proceed as in [11] and add the ghost numbers by applying N copies of the unitary transformation (4.22). To this end we have to regularize the principal value in Eq. (4.22). It does not matter what regularization we choose, the final result should be regularization independent. For definite we will assume the following regularization [11]:

$$U_{j_0,s}=\exp\left\{\varepsilon\left(j_0+\frac{Q}{2}\right)\int_{-\infty}^{\infty}d\kappa\xi_s(\kappa) \times[a^+(\kappa)+a^-(\kappa)]\right\}, \tag{4.24}$$

where

$$\xi_s(\kappa)=\sqrt{A_s(\kappa)}=\frac{\mathrm{sgn}(\kappa)}{\sqrt{\kappa^2+4s^2}}\sqrt{A_{1+s}(\kappa)}. \tag{4.25}$$

Now U_{j_0} can be normal ordered

$$U_{j_0,s}=\exp\left\{-\frac{\varepsilon}{2}\left(j_0+\frac{Q}{2}\right)^2\langle\xi_s,\xi_s\rangle\right\}:U_{j_0,s}:, \tag{4.26}$$

where

$$\langle\xi_s,\xi_s\rangle=\int_{-\infty}^{\infty}d\kappa A_s(\kappa)=\Gamma(2s).$$

Whenever possible we will write $\langle\xi_s,\xi_s\rangle$ instead of its value $\Gamma(2s)$. We will do this in order to be able to choose another regularization without extra problems.

So we want to calculate

$$\lim_{s\to+0}\langle V_N^{(1)}|\bigotimes_{I=1}^{N}U_{j_0^I,s}\delta_{j_0^1+\cdots+j_0^N+Q,0}, \tag{4.27}$$

where the vertex $\langle V_N^{(1)}|$ is defined by the first line in Eq. (4.20). Substitution of Eq. (4.26) yields

$$\lim_{s\to+0}\sum_{q^1,\ldots,q^N}\bigotimes_{I=1}^{N}{}_I\langle -q^I-Q|\delta_{q^1+\cdots+q^N+Q,0} \times\exp\left\{-\frac{\varepsilon}{2}\sum_{I,J=1}^{N}\int_{-\infty}^{\infty}d\kappa a^{-(I)}(\kappa)\mu_{1,N}^{IJ}(\kappa)(Ca^{-(J)})(\kappa) +\varepsilon\sum_{I,J=1}^{N}\left(q^I+\frac{Q}{2}\right)\int_{-\infty}^{\infty}d\kappa\xi_s(\kappa)[\mu_{1,N}^{IJ}(\kappa)+\delta^{IJ}] \times(Ca^{-(J)})(\kappa)-\frac{\varepsilon}{2}\sum_{I,J=1}^{N}\left(q^I+\frac{Q}{2}\right) \times\left(q^J+\frac{Q}{2}\right)\int_{-\infty}^{\infty}d\kappa\,\xi_s(\kappa)[\mu_{1,N}^{IJ}(\kappa)+\delta^{IJ}]\xi_s(\kappa)\right\}.$$

One sees that the integrals in the first and second term in the exponential are well defined as $s\to 0$, but there is a problem with the $s\to 0$ limit in the third term.

In the second term we can substitute $Q=-\Sigma_I q^I$ and use that $\Sigma_I[\mu^{IJ}(\kappa)+\delta^{IJ}]=2$ to obtain

$$\text{second term}=\varepsilon\sum_{I,J=1}^{N}q^I\int_{-\infty}^{\infty}d\kappa\mathcal{P}\frac{\sqrt{A_1(\kappa)}}{\kappa} \times[\mu_{1,N}^{IJ}(\kappa)+\delta^{IJ}-1](Ca^{-(J)})(\kappa).$$

In the third term, we can use Eq. (4.13) and the anomalous conservation law to get

$$\text{third term}=\frac{\varepsilon}{2}\sum_{I,J=1}^{N}q^I\mathcal{N}_{00}^{IJ}q^J -\frac{\varepsilon Q^2}{2}\int_{-\infty}^{\infty}d\kappa\frac{A_1(\kappa)}{\kappa^2} \times[\mu_{1,N}^{II}(\kappa)-\mu_{1,N}^{II}(0)]-\frac{\varepsilon Q^2}{2}\frac{(N-2)^2}{2N}\langle\xi_s,\xi_s\rangle.$$

The integral here is easy to calculate, and we finally get the relation

$$\langle V_{N,Q}^{(0)}|\{q^I\}\rangle=\lim_{s\to+0}e^{F_{N,s}}\langle V_{N,Q}^{(1)}| \times U_{q^1}\otimes\cdots\otimes U_{q^N}\delta_{q^1+\cdots+q^N+Q,0}, \tag{4.28}$$

where

$$F_{N,s}\equiv\frac{\varepsilon Q^2}{2}\frac{(N-2)^2}{2N}\langle\xi_s,\xi_s\rangle+\frac{\varepsilon Q^2}{2}\left[\log\frac{N}{2}-\frac{N-2}{N}\log 2\right]. \tag{4.29}$$

Notice that for general Q the function $F_{N,s}$ is zero only for $N=2$. This means that the string inner product $\langle V_2^{(0)}|$ is not affected by this singularity. For $Q=0$ $F_{N,s}$ is identically zero, and hence after replacing $q^I\to -i\sqrt{2\alpha'}p_\mu^I$ Eq. (4.28) reproduces the result of [11] for the matter sector.

V. ASSOCIATIVITY OF WITTEN'S STAR PRODUCT

A. Descent relation between the gluing vertices

The aim of this section is to verify the following descent relation:

$${}_{1\cdots N}\langle V_N|={}_{1\cdots N,N+1}\langle V_{N+1}|V_1\rangle_{N+1}, \tag{5.1}$$

where $\langle V_N|$ is the combined matter+ghost gluing N-string vertex. The combined vertex $\langle V_N|$ has the form (4.20) with the replacements: $a^-\to a^-_\mu$ ($\mu=-1,\ldots,D-1$) where $\mu=-1$ corresponds to the ghost oscillator and $\mu=0,\ldots,D-1$ to the matter ones; $\varepsilon\to-\eta^{\mu\nu}$ where $\eta^{\mu\nu}=\mathrm{diag}(-\varepsilon,-1,1,\ldots,1)$; and $j_0\to j_0^\mu=(j_0,-i\sqrt{2\alpha'}\vec{p})$.

We know that the vertex depending on the momenta and ghost numbers can be obtained from the $s=1$ vertex by the unitary transformation as in Eq. (4.28). Since adding the momentum does not produce any divergencies [11] we put it equal to zero. So we need to calculate the following product:

$$e^{F_{N+1,s}+F_{1,s}}{}_{1\cdots N,N+1}\langle V_{N+1}^{(1)}|V_1^{(1)}\rangle_{N+1}\times U_{q^1}\otimes\cdots\otimes U_{q^N}\delta_{q^1+\cdots+q^N+Q,0}. \tag{5.2}$$

The inner product in the $(N+1)$th tensor component is easy to calculate:

$$\begin{aligned}{}_{1\cdots N,N+1}&\langle V_{N+1}^{(1)}|V_1^{(1)}\rangle_{N+1}\\ &=\det(1-\mu_{N+1}^{11})^{-(D+1)/2}\\ &\times{}_{1\cdots N}\langle 0|\exp\left[\frac{1}{2}\sum_{\mu,\nu=-1}^{D-1}\sum_{I,J=1}^{N}\int_{-\infty}^{\infty}d\kappa\, a_\mu^{-(I)}(\kappa)\,\eta^{\mu\nu}\right.\\ &\left.\times\left\{\mu_{N+1}^{IJ}+\frac{\mu_{N+1}^{I,N+1}\mu_{N+1}^{N+1,J}}{1-\mu_{N+1}^{11}}\right\}(\kappa)(Ca_\nu^{-(J)})(\kappa)\right],\end{aligned}$$

where $\mu_N^{IJ}\equiv\mu_{1,N}^{IJ}$. It is a matter of simple algebra to show that the term in $\{\ \}$ equals $\mu_N^{IJ}(\kappa)$. Hence we see that the descent relation is actually satisfied up to a numeric coefficient

$${}_{1\cdots N,N+1}\langle V_{N+1}|V_1\rangle_{N+1}=(\mathcal{Z}_{N+1,1;N})\,{}_{1\cdots N}\langle V_N|, \tag{5.3}$$

where

$$\log\mathcal{Z}_{N+1,1;N}\equiv-\frac{D+1}{2}\log\det(1-\mu_{N+1}^{11})+F_{N+1,s}+F_{1,s}-F_{N,s} \tag{5.4}$$

and $F_{N,s}$ is defined in Eq. (4.29). In other words $\log\mathcal{Z}_{N+1,1;N}$ is

$$\begin{aligned}\log\mathcal{Z}_{N+1,1;N}\equiv&+\frac{\varepsilon Q^2}{2}\log\frac{N+1}{2N}\\ &-\frac{D+1}{2}\int_{-\infty}^{\infty}d\kappa\log(1-\mu_{N+1}^{11}(\kappa))\rho_1(\kappa)\\ &+\frac{\varepsilon Q^2}{2}\frac{(N-1)(N+2)}{N(N+1)}\{\langle\xi_s,\xi_s\rangle+\log 2\},\end{aligned} \tag{5.5}$$

where $\rho_1(\kappa)$ is the trace density which was calculated in [14]

$$\begin{aligned}\rho_1(\kappa)=&\frac{1}{\pi}[\langle\xi_s,\xi_s\rangle+\log 2]\\ &-\frac{1}{4\pi}\left[\psi\left(1+\frac{i\kappa}{2}\right)+\psi\left(1-\frac{i\kappa}{2}\right)+2\gamma_E\right].\end{aligned}$$

From this it follows that $\mathcal{Z}_{2,1;1}=1$. If the descent relation (5.1) were true, all $\mathcal{Z}_{N+1,1;N}$ with $N>1$ would be 1. But as we will see in a moment $\mathcal{Z}_{N+1,1;N}$ is a nontrivial function of N, and therefore the vertices must contain an additional normalization factor.

For the critical bosonic string ($D=26$, $\varepsilon=1$, and $Q=-3$) $\mathcal{Z}_{N+1,1;N}$ is a finite function of N

$$\log\mathcal{Z}_{N+1,1;N}=-\frac{9}{2}\sum_{\alpha\in A}s_\alpha\log\alpha+\frac{27}{2}\sum_{\alpha\in A}s_\alpha D_0(\alpha), \tag{5.6}$$

where $A=\{2,N,N+1,1\}$, $\{s_\alpha|\alpha\in A\}=\{1,1,-1,-1\}$, and

$$D_0(\alpha)=\int_0^\infty\frac{dt}{t}\left\{\frac{\coth\frac{t}{\alpha}-\frac{\alpha}{t}}{e^t-1}-\frac{1}{3\alpha}\frac{1}{1+t}\right\}. \tag{5.7}$$

The details of this calculation will be presented in [19]. From the representation (5.6) it follows that $\mathcal{Z}_{N+1,1;N}$ can be written as

$$\mathcal{Z}_{N+1,1;N}=\frac{\mathcal{Z}_N}{\mathcal{Z}_1\mathcal{Z}_{N+1}}. \tag{5.8}$$

Here the logarithm of $\mathcal{Z}_N$ is given by

$$\log\mathcal{Z}_N=-\frac{9}{2}\log\frac{N}{2}+\frac{27}{2}[D_0(N)-D_0(2)]. \tag{5.9}$$

Notice that the function $\mathcal{Z}_N$ *cannot* be uniquely determined from the relation (5.8). It is defined up to a rescaling

$$\mathcal{Z}_N\mapsto(\mathrm{const})^{N-2}\mathcal{Z}_N. \tag{5.10}$$

The function $\mathcal{Z}_N$ defined in Eq. (5.9) monotonically goes to zero on the interval $[1,\infty)$, and its asymptotic at infinity is

$$\mathcal{Z}_N\propto\left[\frac{2}{N}\right]^{9/2}\exp\left\{-\frac{27}{2}D_0(2)+O\left(\frac{1}{N}\right)\right\}.$$

Now it is obvious that in order to have the descent relation (5.1) one has to introduce the normalized gluing vertices $\langle\!\langle V_N|$, which are defined by

$$\langle\!\langle V_N|=\mathcal{Z}_N\langle V_N|\quad\text{for}\quad N\geqslant 1. \tag{5.11}$$

The normalization $\mathcal{Z}_N$ is given by Eq. (5.9) or any of Eq. (5.10). Notice that $\mathcal{Z}_2=1$ independently on the choice of the scaling factor in Eq. (5.10). The ambiguity (5.10) is closely related to the string field redefinition. Indeed the factor $(\mathrm{const})^{N-2}$ in the vertex can be cancelled by simultaneous rescaling of the string field $\mathcal{A}\mapsto(\text{const})^{-1}\mathcal{A}$ and the coupling constant $g_o\mapsto(\text{const})^{-1}g_o$. The natural choice of the factor in Eq. (5.10) is that where $\mathcal{Z}_N$ coincides with the partition function of the bosonic matter+ghost CFT on the gluing surface, which is the unit disk with Neumann boundary conditions and the angle excess $\pi(N-2)$. This choice basically follows from the relation

$$\mathcal{Z}_N=\langle\langle V_N|(|0\rangle_1\otimes\cdots\otimes|0\rangle_N),$$

where $\langle\langle V_N|$ is supposed to be a surface (multi)state, and therefore the rhs must be the partition function of this surface (see Fig. 5 in [20]).

For $N=3$ the normalized as in Eq. (5.11) 3-string vertex coincides with Witten's original definition [1]. In that paper he defined it as the Polyakov integral over the gluing surface which, of course, includes the partition function in its definition.

Notice that in Moyal formulation of SFT (MSFT) [21] the star product is associative by construction and there is a way to obtain the Neumann matrix elements from it [21]. Therefore it should be also possible to extract the normalization of the gluing vertices and compare them with $\mathcal{Z}_N$.

B. Associativity

The associativity requires many relations between the gluing vertices. For example,

$$\begin{aligned}({}_{123}\langle V_3|\otimes{}_{456}\langle V_3|)|V_2\rangle_{34}&={}_{1256}\langle V_4|,\\ ({}_{123}\langle V_3|\otimes{}_{456}\langle V_3|\otimes{}_{78}\langle V_2|)|V_3\rangle_{368}&={}_{12457}\langle V_5|,\end{aligned}\tag{5.12}$$

and many more. The question is if these relations are satisfied for the normalized vertices (5.11). Actually the question is only about the normalization factors, since the exponentials were worked out in [3,22].

We claim that the normalized vertices (5.11) indeed satisfy the relations like Eq. (5.12). The proof is simple and does not require complicated calculations. Let us prove, for example, the first relation in Eq. (5.12). Suppose that it is false, and there is a constant $A\neq1$ in the rhs:

$$({}_{123}\langle\langle V_3|\otimes{}_{456}\langle\langle V_3|)|V_2\rangle\rangle_{34}=A\quad{}_{1256}\langle\langle V_4|.$$

Now we contract this equation with the identity state $|V_1\rangle\rangle_6$ in the sixth tensorial space. Using the descent relations (5.1) we obtain

$$({}_{123}\langle\langle V_3|\otimes{}_{45}\langle\langle V_2|)|V_2\rangle\rangle_{34}=A\quad{}_{125}\langle\langle V_3|.$$

Noticing that ${}_{45}\langle\langle V_2|V_2\rangle\rangle_{34}=\delta_{53}$ one finds that the constant A equals 1. So the first relation in Eq. (5.12) is true. Actually all relations like Eq. (5.12) can be proved in this manner. Therefore the normalized vertices (5.11) *do* satisfy the associativity relations.

VI. DISCUSSION

Here I want to discuss some consequences of the normalization (5.11) for the numeric calculations of the tachyon condensation [23]. First, the calculations in which one uses only vertices $\langle V_2|$ and $\langle V_3|$ (see, for example, [24]) are *not* affected by the normalization (5.11). One can simply cancel the factor $\mathcal{Z}_3$ in the cubic vertex by simultaneous rescaling of the string field and the coupling constant as $\mathcal{A}\mapsto\mathcal{Z}_3^{-1}\mathcal{A}$ and $g_o\mapsto\mathcal{Z}_3^{-1}g_o$ correspondingly. However, the calculations in the bosonic string which involve the higher vertices (if any were done) have to be revised.

Second, the fact that $\mathcal{Z}_N\neq1$ ($N=1$ and $N\geqslant3$) for the bosonic string may potentially affect the numeric calculations in the nonpolynomial fermionic string field theory (see, for example, [25]). To check this one has to calculate the contribution of the matter fermions and superghosts into the partition function $\mathcal{Z}_N$. For the same reason as for the bosonic cubic SFT the calculations in the cubic fermionic string field theory (see, for example, [26]) are not affected.

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APPENDIX: REPRESENTATION OF $(Z+\overline{z'})^{-1}$ THROUGH $s=0\times s=1$ K_1 EIGENFUNCTIONS

In this Appendix we derive a representation of the first term in Eq. (3.12) through the tensor product of $s=0$ and $s=1$ K_1 eigenfunctions (2.8) and (2.6).

We start from the following equation for $0<\mp\text{Im}(\overline{w'}-w)\leqslant(\pi/2)$ and $s>0$:

$$\frac{\Gamma(2s)}{(z+\overline{z'})^{2s}}=(\cosh w\cosh\overline{w'})^{2s}\times\int_{-\infty}^{\infty}d\kappa A_s(\kappa)e^{\pm\pi\kappa/2}e^{i\kappa(w-\overline{w'})}.\tag{A1}$$

This expression follows from Eqs. (3.14) and (3.23) in [11]. Obviously differentiating the left-hand side with respect to $\overline{z'}$ and taking the limit $s\to0$ one obtains $-(z+\overline{z'})^{-1}$. Hence the problem is to perform these operations on the rhs.

Differentiation by $\overline{z'}=-i\tanh\overline{w'}$ of Eq. (A1)'s rhs yields

$$(\cosh w)^{2s}(\cosh\overline{w'})^{2s+2}\int_{-\infty}^{\infty}d\kappa e^{\pm\pi\kappa/2}\times A_s(\kappa)e^{i\kappa(w-\overline{w'})}\{-2s\overline{z'}+\kappa\}.\tag{A2}$$

Using the following relations

$$A_s(\kappa)=\frac{A_{1+s}(\kappa)}{\kappa^2+4s^2}, \quad \lim_{s\to 0}\frac{\kappa}{\kappa^2+4s^2}=\mathcal{P}\frac{1}{\kappa},$$

and

$$\lim_{s\to 0}\frac{2s}{\kappa^2+4s^2}=\pi\delta(\kappa),$$

one can take the $s\to 0$ limit in Eq. (A2):

$$(\cosh \overline{w'})^2\int_{-\infty}^{\infty}d\kappa e^{\pm\pi\kappa/2}\mathcal{P}\frac{A_1(\kappa)}{\kappa}e^{i\kappa(w-\overline{w'})}-\frac{\overline{z'}}{1+\overline{z'}^2}.$$

The last term in this expression comes from the midpoint and therefore can be written as in Eq. (2.10b). Using Eqs. (2.8) and (2.6) we finally obtain

$$-\frac{1}{z+\overline{z'}}=\int_{-\infty}^{\infty}d\kappa e^{\pm\pi\kappa/2}|\kappa,0\rangle(z)\otimes\langle\kappa,1|(\overline{z'})$$
$$-\int_{-\infty}^{\infty}d\kappa\mathcal{P}\frac{\sqrt{A_1(\kappa)}}{\kappa}\langle\kappa,1|(\overline{z'}). \tag{A3}$$

Here the first integral converges for $0<\mp\mathrm{Im}(\overline{w'}-w)\leqslant(\pi/2)$, while the second integral converges for all $\overline{z'}$ in the unit disk.

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