Perfect fluid models in noncomoving observational spherical coordinates

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We use null spherical (observational) coordinates to describe a class of inhomogeneous cosmological models. The proposed cosmological construction is based on the observer past null cone. A known difficulty in using inhomogeneous models is that the null geodesic equation is not integrable in general. Our choice of null coordinates solves the radial ingoing null geodesic by construction. Furthermore, we use an approach where the velocity field is uniquely calculated from the metric rather than put in by hand. Conveniently, this allows us to explore models in a noncomoving frame of reference. In this frame, we find that the velocity field has shear, acceleration, and expansion rate in general. We show that a comoving frame is not compatible with expanding perfect fluid models in the coordinates proposed and dust models are simply not possible. We describe the models in a noncomoving frame. We use the dust models in a noncomoving frame to outline a fitting procedure.

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I. INTRODUCTION

The study of inhomogeneous cosmological models is a well motivated and justified endeavor (see [1,2] for reviews). These models provide more freedom in discussing very early or very late evolution of the irregularities in the Universe. Their study also complements perturbation approaches. It is worth mentioning that there are a few hundreds of inhomogeneous cosmological models that reproduce a metric of the Friedmann-Lemaitre-Robertson-Walker (FLRW) class of solutions when their arbitrary constants or functions take certain limiting values [1]. They become then, in that limit, compatible with the almost homogeneous and almost isotropic observed Universe. This shows the richness of these studies.

A difficulty that is encountered in these models is that the null geodesic equation is not integrable in general. In this paper, we explore the alternative of using null (observational) spherical coordinates in which the radial null geodesic equation of interest is solved by construction. However, when considering null coordinates and a given metric for the spacetime some subtleties arise regarding the frame of reference used. In order to explore this point we will use in this paper the approach described by Ishak and Lake [4] where the velocity field is calculated from the metric and not put in by hand. Conveniently, this approach allows one to explore noncomoving frames of reference, an important point for this paper.

Surprisingly, little work has been done in noncomoving coordinates [3,6–11] despite some interesting features particular to them. Notably, there are models that are separable only in a noncomoving coordinate system [10]. Moreover, exact solutions to Einstein's equations in a noncomoving frame usually have a rich kinematics with shear, acceleration, and expansion. Such solutions are relatively rare in the comoving frame [3]; see also a recent discussion in [11]. Another point discussed in [10] is that comoving coordinates do not cover all the spacetime manifold for a specified energy-momentum tensor. Finally, it is worth mentioning that it is often difficult to do the mathematical transformation of a given solution from noncomoving coordinates to comoving ones, and even when the passage is made, there is no guarantee that the solution will continue to have a simple or explicit form.

In the present paper, we use the spherical null Bondi metric [25] to present models where a noncomoving frame is proven necessary. We explicitly demonstrate how a comoving frame leads to severe limitations.

Furthermore, we use the dust models in the noncomoving frame to outline a fitting procedure where observational data can be used to integrate explicitly for the metric functions. Using observational coordinates is particularly useful when one wants to compare directly an inhomogeneous model to observational data. Such an interesting program had been nicely developed in Refs. [12–17] where the authors used a general metric that can be written as a FLRW metric plus exact perturbations. The spherically symmetric dust solutions were considered in Ref. [14]. The authors also developed and used a fluid-ray tetrad formalism [13] in order to derive a fitting procedure where observations can be used to solve Einstein's field equations. After some necessary revisions [18,19], this program has been relaunched recently [19,20].

We consider here in our work the spherically symmetric case but using the Bondi metric [25] in a noncomoving frame. Also, we do not use the fluid-ray tetrad formalism [13] but the inverse approach to Einstein's equations developed in [4].

In the following section, we set the notation and recall some useful results. In Sec. III, we discuss observational coordinates and explain the cosmological construction around our world-line. We also discuss the physical meaning of the functions that appear in the metric used here. We provide in Sec. IV perfect fluid models in a noncomoving frame. In Sec. V, we show how dust models are not possible in a comoving frame. We describe dust models in a noncomoving frame and outline a fitting procedure in Sec. VI and summarize in Sec. VII.

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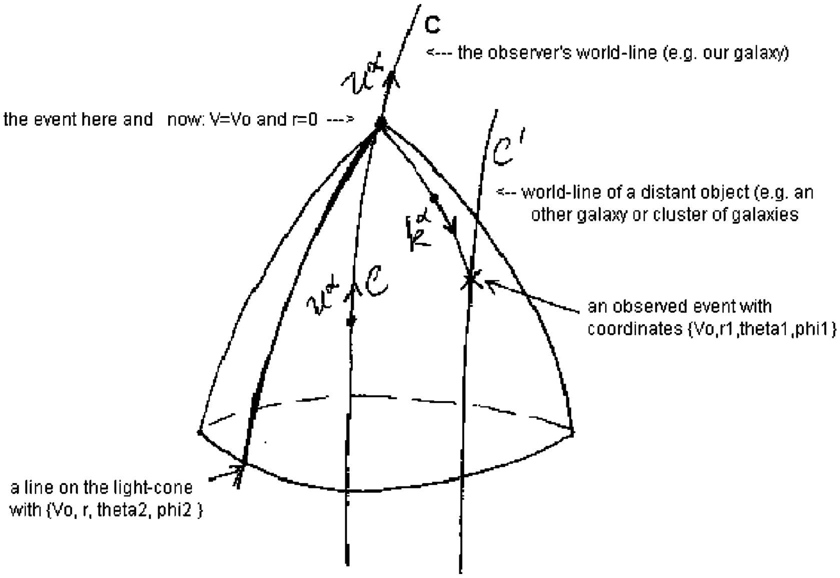

FIG. 1. Observational coordinates $\{v,r,\theta,\phi\}$. Our past null cone is defined by $v=v_o$. C is the observer world-line; C' is the world-line of another celestial object. The trajectory defined by v, θ, and ϕ constant is a radial null geodesic, u^α is the fluid velocity field vector, and k^α is the null tangent vector. The null rays are traveling in the opposite direction to k^α as drawn on the figure. The coordinate r increases along the trajectory CC' down the light cone.

II. NOTATION AND PRELIMINARIES

We set here the notation and summarize results to be used in this paper. In Ref. [4], warped product spacetimes of class B_1 [21,22] were considered. These can be written in the form

$$ds^2_{\mathcal{M}}=ds^2_{\Sigma_1}(x^1,x^2)+C(x^\alpha)ds^2_{\Sigma_2}(x^3,x^4) \tag{1}$$

where $C(x^\alpha)=r(x^1,x^2)^2w(x^3,x^4)^2$, $\mathrm{sig}(\Sigma_1)=0$, and $\mathrm{sig}(\Sigma_2)=2\epsilon$ ($\epsilon=\pm1$). Although very special, these spaces include many of interest, for example, *all* spherical, plane, and hyperbolic spacetimes. For Σ_1, we write

$$ds^2_{\Sigma_1}=a(dx^1)^2+2b\,dx^1dx^2+c(dx^2)^2, \tag{2}$$

with a, b, and c functions of (x^1,x^2) only. Consider a congruence of unit timelike vectors (velocity field) $u^\alpha=(u^1,u^2,0,0)$ with an associated unit normal field n^α (in the tangent space of Σ_1) satisfying $n_\alpha u^\alpha=0, n_\alpha n^\alpha=1$ [23]. It was shown in [4] that u^α is uniquely determined from the zero flux condition

$$G^\beta_\alpha u^\alpha n_\beta=0, \tag{3}$$

where G^β_α is the Einstein tensor of the spacetime. The explicit forms for u^1 and u^2 were written out for canonical representations of Σ_1, including the null (Bondi) type of coordinates that we use in the present paper. With $G\equiv G^\alpha_\alpha$, $G1\equiv G^\beta_\alpha u^\alpha u_\beta$, and $G2\equiv G^\beta_\alpha n^\alpha n_\beta$, it was shown in [4] that the condition

$$G+G1=3G2 \tag{4}$$

is a necessary condition for a perfect fluid source, and that in some cases this condition is also sufficient. For example, in [5], Eq. (4) was used to derive an algorithm which generates all regular static spherically symmetric perfect fluid solutions of Einstein's equations. In this paper, we are interested in perfect fluid sources so it is important to recapitulate the following results from [4]. Consider a fluid with anisotropic pressure and shear viscosity but zero energy flux (nonconducting). The energy-momentum reads

$$T^\alpha_\beta=\rho u^\alpha u_\beta+p_1n^\alpha n_\beta+p_2\delta^\alpha_\beta+p_2(u^\alpha u_\beta-n^\alpha n_\beta)-2\eta\sigma^\alpha_\beta, \tag{5}$$

where ρ is the energy density and σ^α_β is the shear associated with u^α; η is the phenomenological shear viscosity; p_1 and p_2 are the pressures respectively parallel and perpendicular to n^α. When $p_1=p_2$ and the shear term vanishes the fluid is called perfect. It was shown in [4] that in the case where

$$\Delta\equiv\sigma^\beta_\alpha n^\alpha n^\beta\neq0, \tag{6}$$

we have

$$\rho=\frac{G1}{2\pi}, \tag{7}$$

$$p1=\frac{G2}{8\pi}+2\eta\Delta, \tag{8}$$

$$p2=\frac{G+G1-G2}{16\pi}-\eta\Delta \tag{9}$$

and η is a freely specified function. The procedure to impose a perfect fluid source in this degenerate case is to impose the condition (4) and also necessarily set $\eta\equiv0$. For other choices of η, the fluid is viscous.

III. THE METRIC AND OBSERVATIONAL COORDINATES

We consider in the present paper the null coordinate system $\{x^a\}=\{v,r,\theta,\phi\}$. These are called observational (or cosmological) coordinates as we can construct them around our galaxy world-line C as indicated in Fig. 1. The trajectory defined by v, θ, and ϕ constant is a radial null geodesic and each hypersurface of constant v is a past light cone of events on C. We choose $v=v_o$ and $r=0$ to represent the vertex "here and now." The coordinate r is then set by construction to be the area distance as explained further, and is related to the luminosity distance d_L by $r=d_L/(1+z)^2$ (see, e.g., [24]). Finally, θ and ϕ are the spherical coordinates on the celestial sphere. The geometry of the models is represented by the general spherical Bondi metric in advanced coordinates [25]

$$ds^2=2c\,dr\,dv-c^2\left(1-\frac{2m}{r}\right)dv^2+r^2[d\theta^2+\sin(\theta)^2d\phi^2], \tag{10}$$

where $c\equiv c(r,v)>0$, $m\equiv m(r,v)$, and $r>2m$. The radial (θ and ϕ constant) ingoing null geodesic equation $v=$const is solved by construction (see Appendix A). The components of the mixed Einstein tensor G^α_β for Eq. (2) are given in Appendix B, and the structure of the Weyl tensor is discussed in Appendix C. Regularity of the metric and the Weyl invariants requires that $m(r,v)$ and $c(r,v)$ are C^3 at $r=0$ [e.g., see Eq. (C2)]. It follows that

$$\left.\left(1-\frac{2m(r,v)}{r}\right)\right|_{r=0}=1. \tag{11}$$

Also, we can use the freedom in the null coordinate v to normalize it by setting $c(0,v)=1$. As we will write further in this paper [see Eq. (52)], this means that we require that v measures the proper time τ along our galaxy world-line C.

Whereas the meaning of the metric function $m(r,v)$ is very well known, we are not aware of any previous literature where an interpretation for $c(r,v)$ was given. The function $m(r,v)$ represents the effective gravitational (geometrical) mass (e.g., [27–31]) and is given by

$$m\equiv\frac{g_{\theta\theta}^{3/2}}{2}R_{\theta\phi}{}^{\theta\phi}, \tag{12}$$

where $R_{\theta\phi}{}^{\theta\phi}$ is the mixed angular component of the Riemann curvature tensor. For the physical meaning of the function $c(r,v)$, it turns out to be useful to study the kinematics of null rays. These usually include the optical shear, vorticity, and rate of expansion, respectively, defined by [32]

$$\sigma^2_{optical}\equiv\frac{1}{2}k_{(\alpha;\beta)}k^{\alpha;\beta}-\frac{1}{4}(k^\alpha_{;\alpha})^2, \tag{13}$$

$$\omega^2_{optical}\equiv\frac{1}{2}k_{[\alpha;\beta]}k^{\alpha;\beta}, \tag{14}$$

$$\theta_{optical}\equiv\frac{1}{2}k^\alpha_{;\alpha} \tag{15}$$

where

$$k_\alpha=[1,0,0,0] \tag{16}$$

is the null four-vector tangent to the congruence of null geodesics. The physical meaning of the optical scalars can be understood in the following way [32,33]. If an opaque object is displaced an infinitesimal distance *dr* from a screen (perpendicularly to the beam of light), it will cast on the screen a shadow that is expanded by $\theta_{optical}dr$, rotated by $\omega_{optical}dr$, and sheared by $|\sigma_{optical}|dr$. As expected from the spherical symmetry of the geometry, the nonvanishing optical scalar for the null congruence k_α is the optical rate of expansion, from which we find

$$rc(r,v)=\frac{1}{\theta_{optical}}. \tag{17}$$

We identify from Eq. (17) that $rc(r,v)$ is a measure of the reciprocal of the expansion of null rays.

IV. MODELS IN A NONCOMOVING FRAME

A. The velocity field

We consider an observer moving with a fluid for which the streamlines are given by the general radial timelike vector $u^\alpha=[u^v(r,v),u^r(r,v),0,0]$. We assume that such a velocity field exists for which the energy-momentum tensor takes the perfect fluid form

$$T^\alpha_\beta=(\rho+p)u^\alpha u_\beta+p\,\delta^\alpha_\beta, \tag{18}$$

where ρ and p are respectively the energy-density and isotropic pressure associated with u^α. The velocity field is simply determined from the zero flux condition (3) and is given by

$$u^1\equiv u^v=\frac{1}{c(r,v)}\sqrt[4]{\frac{1}{[1-2m(r,v)/r]^2+4m^\bullet(r,v)/rc'(r,v)}} \tag{19}$$

$$u^2\equiv u^r=\frac{1}{2}\,\frac{[1-2m(r,v)/r]-\sqrt{\{[1-2m(r,v)/r]^2+4m^\bullet(r,v)/rc'(r,v)\}}}{\sqrt[4]{[1-2m(r,v)/r]^2+4m^\bullet(r,v)/rc'(r,v)}} \tag{20}$$

where the prime indicates $\partial/\partial r$ and the bold dot $\partial/\partial v$. The associated unit normal vector field n^{α} ($n_{\alpha}u^{\alpha}=0$ and $n_{\alpha}n^{\alpha}=1$) is given by

$$n_{\alpha}=c(r,v)[u^{r},-u^{v},0,0]. \tag{21}$$

Interestingly, the velocity field has shear $\sigma^{\alpha}_{\beta}\neq 0$, acceleration $\dot{u}^{\alpha}\neq 0$, and expansion rate scalar $\theta\neq 0$ in general.

B. The perfect fluid condition

It follows from the metric (10) that Δ, as defined in Eq. (6), is not zero and that for a perfect fluid source we must impose the condition (4) and set $\eta\equiv 0$. With $m\equiv m(r,v)$ and $c\equiv c(r,v)$, the condition (4) reads

$$\mathcal{L}-c^{2}\sqrt{c'\mathcal{N}}=0, \tag{22}$$

where

$$\mathcal{L}\equiv c^{3}(2m'-m''r)+c^{2}[3c'(m-rm')+c''r(r-2m)] + cr^{2}c'^{\bullet}-c'r^{2}c^{\bullet} \tag{23}$$

and

$$\mathcal{N}\equiv c'(r-2m)^{2}+4rm^{\bullet}. \tag{24}$$

The metric (10) along with the metric constraint (22) represent a perfect fluid model with

$$\rho=\frac{G1}{8\pi}=\frac{2(cm)'-c'r+\sqrt{c'\mathcal{N}}}{8\pi cr^{2}} \tag{25}$$

and

$$p(=p_{1}=p_{2})=\frac{-2(cm)'+c'r+\sqrt{c'\mathcal{N}}}{8\pi cr^{2}}. \tag{26}$$

V. MODELS IN A COMOVING FRAME

In this section, we specialize to models in a comoving frame of reference. We show how this frame fails in the realization of the cosmological construction proposed.

A. Perfect fluid models

With the metric function

$$g_{vv}=-c^{2}(r,v)\left(1-\frac{2m(r,v)}{r}\right)<0, \tag{27}$$

the requirement of comoving coordinates reads

$$u^{r}=0 \;\Leftrightarrow\; G^{r}_{v}=2\frac{\partial m(r,v)/\partial v}{r^{2}}=0. \tag{28}$$

Hence, the necessary and sufficient condition for a comoving frame is

$$m(r,v)=m(r). \tag{29}$$

It follows from Eqs. (19) and (29) that

$$u^{v}=\frac{1}{c(r,v)\sqrt{[1-2m(r)/r]}}, \tag{30}$$

$$n_{r}=\frac{-1}{\sqrt{[1-2m(r)/r]}}, \tag{31}$$

and $n_{v}=0$. With this velocity field the shear tensor vanishes; therefore, the necessary and sufficient condition for a perfect fluid model is Eq. (4), which can be written as

$$-c^{2}c'r+5c^{2}c'm+2c^{3}m'+c^{2}c''r^{2}-2c^{2}c''mr-3c^{2}c'm'r -c^{3}m''r-c'c^{\bullet}r^{2}+c'^{\bullet}cr^{2}=0. \tag{32}$$

For a perfect fluid source in this frame the pressure p is a function of both r and v while the energy density is a function only of r,

$$\rho=\frac{G1}{8\pi}=\frac{m'(r)}{4\pi r^{2}} \tag{33}$$

and

$$p=\frac{c'(r,v)[r+2m(r)]+c(r,v)m'(r)}{4\pi r^{2}c(r,v)}. \tag{34}$$

The four-acceleration $\dot{u}^{\alpha}$ has the nonvanishing components

$$\dot{u}^{v}=\frac{rc'(r,v)[r-2m(r)]+c[m(r)-m'(r)r]}{rc^{2}(r,v)[r-2m(r)]} \tag{35}$$

and

$$\dot{u}^{r}=\frac{rc'(r,v)[r-2m(r)]+c[m(r)-m'(r)r]}{r^{2}c(r,v)}. \tag{36}$$

A caveat in this frame (comoving) is that the expansion scalar vanishes and the model is not suitable for describing an expanding Universe.

B. The zero-pressure case

The present matter dominated (as opposed to radiation dominated) Universe is well approximated by a zero-pressure model, commonly referred to as ‘‘dust.’’ In this case (comoving) $\Delta=0$ and the zero-pressure conditions follow from Eqs. (8) and (9) as

$$G2=0 \tag{37}$$

and

$$G+G1=0. \tag{38}$$

With Eq. (29), Eqs. (37) and (38) read

$$\frac{c'}{c}-\frac{m'}{r-2m}=0 \tag{39}$$

and

$$c^2c'r+c^2c'm+c^2c''r^2-2c^2c''rm-3c^2c'rm'-c^3m''r \\ -c'c^{\bullet}r^2+c'^{\bullet}cr^2=0. \tag{40}$$

Integrating Eq. (39) gives

$$c(r,v)=f(v)\exp\left(\int\frac{m'(r)}{r-2m(r)}dr\right), \tag{41}$$

which when put into Eq. (40) gives

$$\frac{f(v)^3\exp\left\{3\int[m'(r)/r-2m(r)]dr\right\}m'(r)m(r)}{r^2[r-2m(r)]}=0. \tag{42}$$

With $f(v)>0$ [from $c(r,v)>0$] the zero-pressure model reduces to the following two cases:

(i) $m(r)=0$ and the spacetime reduces to the Minkowski flat spacetime ($R_{\alpha\beta\gamma\delta}=0$), or

(ii) $m'(r)=0$ (m is constant) and the spacetime reduces to the Schwarzschild vacuum in Eddington-Finkelstein coordinates ($R_{\alpha\beta}=0$, $R_{\alpha\beta\gamma\delta}\neq 0$).

Therefore, a dust model is not possible in a comoving frame using the observational coordinates and the Bondi metric (10). We turn in the following section to a noncomoving frame for dust models.

VI. DUST MODELS IN A NONCOMOVING FRAME

A. The velocity field

We are interested in building dust models using spherical observational coordinates and a noncomoving frame. In a $1+3$ decomposition of the spacetime, these models are given by the general Lemaître-Tolman-Bondi solution [34,1]. In this noncomoving case $\Delta\neq 0$ in general, so we must set $\eta\equiv 0$ and impose the zero-pressure conditions (37) and (38) which can be written as

$$m^{\bullet}=cm'\left[\frac{cm'}{c'r}-\left(1-\frac{2m}{r}\right)\right] \tag{43}$$

and Eq. (38) can be written as

$$c'^{\bullet}=c\left[\frac{1}{r}(3c'm'+cm'')-\frac{c'}{r}\left(1+\frac{m}{r}\right)-c''\left(1-\frac{2m}{r}\right)\right] \\ +\frac{c'c^{\bullet}}{c}. \tag{44}$$

The metric (10) with constraints (43) and (44) represents a class of inhomogeneous dust models in spherical observational coordinates. Using Eq. (7), the energy density is given by

$$4\pi\rho(r,v)=\frac{2m'(r,v)}{r^2}-\frac{c'(r,v)}{rc(r,v)}\left(1-\frac{2m(r,v)}{r}\right). \tag{45}$$

This result can also be obtained from the effective gravitational mass equation (12). The velocity field follows from Eqs. (19) and (20):

$$u^v=\frac{1}{c(r,v)}\frac{1}{\sqrt{[1-2m(r,v)/r]+2m^{\bullet}(r,v)/m'c}} \tag{46}$$

or equivalently by using Eq. (43)

$$u^v=\frac{1}{c(r,v)}\frac{1}{\sqrt{2m'(r,v)c(r,v)/rc'(r,v)-[1-2m(r,v)/r]}} \tag{47}$$

and

$$u^r=\frac{m^{\bullet}(r,v)}{m'(r,v)}u^v. \tag{48}$$

We verified that the acceleration four-vector field $\dot{u}^{\alpha}$ vanishes as the dust fluid is moving geodesically. Interestingly, the velocity field remains with nonvanishing shear and expansion rate.

B. The conformally flat case

It is a well known result that a cosmological model that satisfies the Einstein equations with a perfect fluid source, a barotropic equation of state, i.e., $p=p(\rho)$ (including $p=0$), which is conformally flat ($C_{\alpha\beta\gamma\delta}=0$) and has nonzero expansion is a Lemaître-Friedmann-Robertson-Walker model (LFRW) [1,3,35]. For dust models in the noncomoving frame, the condition $C_{\alpha\beta\gamma\delta}=0$ (see Appendix C) reduces to

$$\frac{c'}{c}\left(1-\frac{2m}{r}\right)-\frac{2m'}{r}+\frac{3m}{r^2}=0. \tag{49}$$

Therefore, the metric (10) along with the constraints (43), (44), and (49) represents the homogeneous and isotropic (LFRW) limit of the models. We are interested here in more general inhomogeneous models.

C. Basic observable quantities

1. The redshift

The light emitted with a wavelength λ_e from a point on the light cone is observed at the vertex "here and now" (see Fig. 1) with a wavelength λ_o. The redshift is then given by (see, e.g., [36,24])

$$1+z=\frac{(k_{\alpha}u^{\alpha})_{emitter}}{(k_{\beta}u^{\beta})_{observer}}=\frac{d\tau_{observer}}{d\tau_{emitter}}=\frac{\lambda_o}{\lambda_e}, \tag{50}$$

where u^α is the normalized timelike velocity vector field and k_α is the null vector as given previously by Eq. (16). It follows that

$$k_\alpha u^\alpha = u^v(r,v_o), \tag{51}$$

where $u^v(r,v_o)$ is given by Eq. (19). It follows from the regularity condition (11) and the timelike normalization condition $u^\alpha u_\alpha = -1$ evaluated at $r=0$ (observer) that

$$(k_\alpha u^\alpha)|_{observer} = u^v(0,v_o) = \frac{1}{c(0,v_o)} = 1, \tag{52}$$

where in the last step we used the freedom in the null coordinate v to set $c(0,v)=0$. Finally,

$$1+z = \frac{u^v(r,v_o)_{emitter}}{u^v(r,v_o)_{observer}} = u^v(r,v_o)_{emitter}. \tag{53}$$

For the dust case, Eq. (53) gives

$$1+z = \frac{1}{c(r,v_o)} \frac{1}{\sqrt{2m'(r,v_o)c(r,v_o)/rc'(r,v_o) - (1-2m(r,v_o)/r)}} \tag{54}$$

or equivalently, by using the constraint (43),

$$1+z = \frac{1}{c(r,v_o)} \frac{1}{\sqrt{m'(r,v_o)c(r,v_o)/rc'(r,v_o) + m^{\bullet}(r,v_o)/m'(r,v_o)c(r,v_o)}}, \tag{55}$$

where we have set $r \equiv r_{emitter}$ in Eqs. (54) and (55).

2. The observer area distance

The coordinate r in the model is set by construction to be the observer area distance [24,36] which is defined by $dA = r^2 d\Omega$ where dA is the cross-sectional area of an emitting object, and $d\Omega$ is the solid angle subtended by that object at the observer. The area distance r is related to the luminosity distance d_L by $r = d_L/(1+z)^2$ [24]. The luminosity distance can be determined by comparing the observed luminosity of an object to its known intrinsic luminosity: $4\pi d_L^2 = L/F$ where F is the observed (measured) flux of light received and L is the object's intrinsic luminosity.

3. Galaxy number counts

Another observable of interest is the source number counts as a function of the redshift. An observer at the vertex "here and now" will count on the light cone a number dN of sources between redshifts z and $z+dz$ in a solid angle $d\Omega$. It follows that

$$\frac{dN}{dz} = f_c n(v_o,r) r^2 d\Omega \frac{dr}{dz}, \tag{56}$$

where $n(v_o,r)$ is the number density of sources and f_c is a fractional number indicating the efficiency of the counts (completeness) [14]. This number corrects for errors in source selection and detection; see, e.g., [14,37]. For simplicity, we can assume that the necessary adjustments for the dark matter can be incorporated via f_c. The energy-density follows

$$\rho(v_o,r) = n(v_o,r)M, \tag{57}$$

where M is the average rest mass for the counted sources.

D. A fitting procedure algorithm

As discussed earlier, the approach used allows us to integrate the models explicitly, given observational data. As a first step, we rearrange the model equations. We combine Eqs. (45) evaluated at $v=v_o$ with Eq. (54) and use $c(0,v_o)=1$ to obtain

$$\frac{c'(r,v_o)}{c^3(r,v_o)} = 4\pi(1+z)^2(r,v_o)\rho(r,v_o), \tag{58}$$

which integrates to

$$c(r,v_o) = \frac{1}{\sqrt{1-8\pi\int (1+z)^2(r,v_o) r\rho(r,v_o) dr}}. \tag{59}$$

Integrating Eq. (45) for $m(r,v_o)$ gives

$$m(r,v_o) = \frac{1}{2c(r,v_o)}\left(\int [c'(r,v_o) + 4\pi r\rho(r,v_o)c(r,v_o)] r dr\right), \tag{60}$$

where we also used $m(0,v_o)=0$. Now, the observations provided as polynomial functions $\rho(z)$ and $r(z)$ fitted to the data can be used to integrate explicitly for $m(r,v)$ and $c(r,v)$. The steps for the fitting algorithm are as follows.

Express cosmological data as polynomial functions for two quantities. (i) The energy-density $\rho(r,v_o)$ from galaxy number counts. Many projects are accumulating very large amounts of data. See, for example, [38] for the Sloan Digital Sky Survey. (ii) The observer area distance $r(z,v_o)$ from "standard candles" projects in which it is possible to measure the redshift and the distance independently. The accumulating data from the supernovae cosmology projects are very promising. See, for example, [39] for the High-Z SN Search project, [40] for the Supernova Cosmology Project, and [41] for the Supernova Acceleration Probe project.

Invert the function $r(z,v_o)$ to obtain $z(r,v_o)$.

This can in turn be used to write the energy-density polynomial function as $\rho(z(r,v_o),v_o)$.

Now, with $z(r,v_o)$ and $\rho(r,v_o)$ expressed as functions of r (and not z), integrate Eq. (59) over r to obtain $c(r,v_o)$ on the light cone.

With $c(r,v_o)$ determined, integrate Eq. (60) over r to obtain $m(r,v_o)$ on the light cone.

Finally, with $c(r,v_o)$ and $m(r,v_o)$ determined, use Eqs. (43) and (44) to integrate over v.

The level of difficulty of this last step can be monitored using the analytical forms used for $\rho(z)$ and $r(z)$ and it remains a tractable problem, while integrating the null geodesic equation in the standard 1+3 form of the LTB models is not tractable (see, e.g., [18]), and one has to have recourse to numerical integrations [42]. Moreover, the fitting procedure has the interesting feature of incorporating the observations in the process of integrating explicitly for the metric functions.

It is worth mentioning that in principle the information on our light cone cannot determine its future evolution uniquely. We need to make the reasonable assumption that there will be no future events in the cosmic evolution that will invalidate the entire data obtained from our light cone (see, e.g., [43]). Furthermore, one must keep in mind the usual limitation of the underlying models used here as they are spherically symmetric around our world-line and more general inhomogeneous models should be considered in future studies of fitting procedures.

VII. SUMMARY

We expressed inhomogeneous cosmological models in null spherical noncomoving coordinates using the Bondi spherical metric. A known difficulty in using inhomogeneous models is that the null geodesic equation is not integrable in general. Our choice of null coordinates solves the radial null geodesic by construction. We identified the general meaning of the metric function $c(r,v)$ to be the reciprocal of the optical expansion. We used an approach where the velocity field is uniquely calculated from the metric rather than put in by hand. Conveniently, this allowed us to explore models in a noncomoving frame of reference. In this frame, we find that the velocity field has shear, acceleration, and expansion rate in general. In this set of coordinates, we showed that a comoving frame is not compatible with expanding perfect fluid models and dust models are simply not possible in this frame. We then described perfect fluid and dust models in a noncomoving frame. The framework developed allows one to outline a fitting procedure where observational data can be used directly to integrate explicitly for the models.

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APPENDIX A: NULL GEODESIC EQUATION

The paths of light rays are described by null geodesic trajectories under the eikonal assumption [26]. The geodesic trajectories are determined by solving the null geodesic equation

$$k_{\alpha;\beta}k^{\beta}=0, \tag{A1}$$

where k^{α} is a null vector ($k^{\alpha}k_{\alpha}=0$) tangent to the null geodesics, $k^{\alpha}=dx^{\alpha}/d\lambda$ where λ is an affine parameter. For the Bondi metric (10), the four equations (A1) are all satisfied for $v=$const.

APPENDIX B: EXPRESSIONS FOR G^{α}_{β} COMPONENTS

The expressions for the components of the mixed Einstein tensor are as follows:

$$G^r_r=\frac{2c'}{cr}\left(1-\frac{2m}{r}\right)-\frac{2m'}{r^2}, \tag{B1}$$

$$G^v_r=\frac{2c'(r,v)}{c^2 r}, \tag{B2}$$

$$G^r_v=\frac{2m^{\bullet}}{r^2}, \tag{B3}$$

$$G^v_v=-\frac{2m'}{r^2}, \tag{B4}$$

$$G^{\theta}_{\theta}=G^{\phi}_{\phi}=\frac{c'}{cr}\left(1-3m'+\frac{m}{r}-\frac{c^{\bullet}r}{c^2}\right)+\frac{c''}{c}\left(1-\frac{2m}{r}\right)-\frac{m''}{r}+\frac{c^{\bullet\prime}}{c^2}, \tag{B5}$$

where the bold dot indicates $\partial/\partial v$ and the prime $\partial/\partial r$, $c\equiv c(r,v)$, and $m\equiv m(r,v)$. We note that these components are related by

$$G^r_r-G^v_v=G^v_r c(r,v)\left(1-\frac{2m(r,v)}{r}\right). \tag{B6}$$

APPENDIX C: THE WEYL TENSOR AND THE CONDITION FOR CONFORMAL FLATNESS

The structure of the Weyl tensor $C_{\alpha\beta\gamma\delta}=0$ is usually explored to derive the conformally flat case of a cosmological solution (i.e., $C_{\alpha\beta\gamma\delta}=0$). This can reveal the limits of the model's parameters for which it reduces to a Lemaître-Friedmann-Robertson-Walker model. The nonvanishing components of the mixed Weyl tensor for the metric (10) are related and given by

$$C_{rv}{}^{rv}=C_{\theta\phi}{}^{\theta\phi}=2C_{r\theta}{}^{r\theta}=2C_{r\phi}{}^{r\phi}$$
$$=2C_{v\theta}{}^{v\theta}=2C_{v\phi}{}^{v\phi}=\mathcal{W}(r,v), \tag{C1}$$

where

$$\mathcal{W}(r,v)\equiv\frac{1}{c^3r^3}\Bigg[c^3r\left(m''r-4m'+\frac{6m}{r}\right)-c^2c''r^3\left(1-\frac{2m}{r}\right)$$
$$+c^2c'r^2\left(1+3m'-\frac{5m}{r}\right)+c^{\bullet}c'r^3-c^{\bullet\prime}cr^3\Bigg]. \tag{C2}$$

The condition for conformal flatness of the models is therefore $\mathcal{W}(r,v)=0$.

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