Quotients of $AdS_{p+1}\times S^q$: Causally well-behaved spaces and black holes

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Starting from the recent classification of quotients of Freund-Rubin backgrounds in string theory of the type $AdS_{p+1}\times S^q$ by one-parameter subgroups of isometries, we investigate the physical interpretation of the associated quotients by discrete cyclic subgroups. We establish which quotients have well-behaved causal structures, and of those containing closed timelike curves, which have interpretations as black holes. We explain the relation to previous investigations of quotients of asymptotically flat spacetimes and plane waves, of black holes in AdS spacetimes, and of Gödel-type universes.

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I. INTRODUCTION AND MOTIVATION

Taking quotients of smooth (super)gravity backgrounds has long been a fundamental tool in string theory, both in the context of Kaluza-Klein reduction, in which one quotients by the action of a continuous group, and in the orbifold context, in which the group is discrete. Riemannian singular quotients (orbifolds) provide exact string theory backgrounds which allow us to understand how string theory resolves certain types of timelike singularities. These techniques are also relevant in the Kaluza-Klein context: an early nontrivial example is the embedding of the Melvin universe [1] in string theory [2,3]. This work naturally suggests studying Lorentzian orbifolds, in the hope of reaching a similar understanding of certain types of spacelike singularities, in particular those related to the big bang. Although some progress has been achieved [4–19], the fate (and physics) of these singularities remains a very important area of research in string theory. Out of this effort we now have a complete list of smooth quotients of Minkowski spacetime. This classification was given in [20], recovering previous results on fluxbranes[1] [21–25] and uncovering the existence of an interesting nonstatic smooth quotient—the nullbrane—which can be understood as a desingularization of the parabolic orbifold [35], the supersymmetric toy model for a big-crunch–big-bang transition singularity, by the introduction of a new scale (modulus) that smooths the singularity.

In this paper, we will study discrete cyclic quotients of anti–de Sitter (AdS) backgrounds in gravity and in string theory. Because of its high degree of symmetry, the story for anti–de Sitter space is particularly interesting, and there is already a rich literature on physically interesting locally anti–de Sitter spacetimes, with much of the discussion having focused on the Bañados-Teitelboim-Zanelli (BTZ) black hole solutions [36,37] and their generalizations [38–40]. However, some examples of smooth quotients are also known [41–43].[2] Given the considerable interest of AdS backgrounds in string theory, the time seems ripe for a more systematic investigation of these questions.

In a recent pair of papers [54,55] we classified quotients of AdS by one-parameter subgroups of isometries. The emphasis in [54] being on AdS backgrounds in string theory, it was necessary to classify quotients of geometries of the form $AdS_{p+1}\times S^q$ by one-parameter subgroups of isometries. As such backgrounds are maximally supersymmetric, it was also natural to study the question of how much supersymmetry was preserved by the quotient and in [54] there is a detailed analysis of this question and the related issue of the existence of a spin structure on the quotient.

Our purpose in the present paper is to study the geometry of the discrete cyclic quotients associated with such one-parameter subgroups, paying close attention to their causal structure, and to develop a formalism to discuss the geometry and physical interpretation of all smooth quotients.

Many of the quotients classified in [54,55] contain closed timelike curves and, while there may be some interest in studying such quotients, we shall nevertheless concentrate our attention on those quotients for which there is a well-

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[1]Related work on the physics of fluxbranes can be found in [26–34].

[2]Some other work concerning orbifolds of AdS can be found in [44–53].

founded expectation that they will provide good backgrounds for string propagation. We will therefore focus on and discuss in detail two kinds of quotients that can be given a simple physical interpretation: smooth quotients with a well-behaved causal structure, and those which can be given a black hole interpretation following [36,37]. At the end of this work, we shall briefly comment on the relation between some of our spacetimes having closed causal curves and Gödel-type universes recently discussed in the literature [56–59]. The connection arises because certain quotients commute with the Penrose limit [60–63]. Thus, one can identify which discrete quotients of $\mathrm{AdS}_{p+1}\times S^q$ backgrounds give rise to compactified *pp* waves having closed timelike curves after taking the Penrose limit, the latter being *T* dual to Gödel-type universes.

We find that there are two types of quotients with well-behaved causal structures. First, there are quotients where an action on the AdS alone is well behaved. These are generalizations of the two cases studied previously.

(i) Self-dual orbifolds of AdS_3 [41,43] and their higher-dimensional generalizations, having no analogue in asymptotically flat configurations.

(ii) The AdS analogue of the flat nullbrane construction [42], consisting of a double null rotation action on $\mathrm{SO}(2,p)$, $p\geqslant 4$. This is the near horizon geometry of a stack of D3-branes in the nullbrane vacuum for $p=4$ and a stack of M5-branes in the same vacuum for $p=6$.

We give a comprehensive discussion of the structure of these quotients, extending previous results. For the double null rotation, we construct a new symmetry-adapted coordinate system, and find interesting relations to compactified plane waves. We comment on related issues in the nullbranes in Appendix B.

Secondly, there are quotients where the norm of the AdS isometry is non-negative, but not always positive, so the pure AdS action would have singularities or closed null curves. These can be removed by a suitable action on the transverse sphere if the latter is odd dimensional. This second type is qualitatively new. These nontrivial actions on AdS can be divided into three categories.

(i) Discrete quotients by rotations in AdS, the higher-dimensional analogues of the AdS_3 conical defects.

(ii) Discrete quotients by a null rotation, whose description in the Poincaré patch corresponds to a spacelike translation (in pure AdS_3, these would give rise to the massless BTZ black hole [37]) and whose sphere deformations are the near horizon limit of brane configurations in fluxbrane vacua classified in [64,65].

(iii) Discrete quotients defined by an everywhere null vector field in AdS_p ($p\geqslant 3$), whose description in the Poincaré patch corresponds to a "translation" along a lightlike direction. Once more, when deformed by a nontrivial action on a transverse sphere, this corresponds to the near horizon counterpart of the corresponding quotients classified in [64,65].

It is important to stress that any of the string theory backgrounds discussed in this paper are related to many others through *U* duality and by Kaluza-Klein reductions from or liftings to M theory. We shall not pursue this possibility in this paper, even though it is natural to wonder about the dual incarnations of our backgrounds.

In studying quotients with a black hole interpretation, we confirm and elucidate the conclusion of [38], that for $p>2$ the only locally AdS_{p+1} black hole solution is the higher-dimensional generalization of the nonrotating BTZ black hole, discussed previously in [39,40]. We explain the origin of this restriction in general. We discuss the relation to other recent work and comment on the proper interpretation of another solution presented in [40].

We begin in Sec. II by reviewing the classification of quotients, setting up the notation that will be used in the remainder of the paper, discussing Killing vectors on the sphere, and determining the conditions under which a discrete cyclic quotient of AdS×S will admit a spin structure. Section III explains the relation between the classification of Killing vectors in AdS and the existence of closed timelike curves in the resulting discrete quotients. In Sec. IV we discuss causally well-behaved quotients, and Sec. V demonstrates that the only black hole solution is the generalization of the nonrotating BTZ black hole. We finish with a small digression on Penrose limits of discrete quotients, and the relation between Gödel-type universes and some quotients of AdS having closed timelike curves. Some technical details are relegated to the Appendixes.

II. CONVENTIONS AND BACKGROUND MATERIAL

In this section we will briefly review the geometrical setup and the results of [54,55] in an attempt to make the present paper self-contained.

A. Anti–de Sitter isometries

The *dramatis personae* of this paper are quotients of AdS backgrounds, either of anti–de Sitter space AdS_{p+1} itself in the context of pure gravity, or of Freund-Rubin backgrounds of the form $\mathrm{AdS}_{p+1}\times \mathrm{S}^q$ in supergravity and string theory.

As usual in physics, throughout this paper AdS_{p+1} ($p\geqslant 2$) will denote the *simply connected* anti–de Sitter space. In other words, AdS_{p+1} (with radius of curvature R) is the universal cover of the quadric traced by the equation

$$-(x^1)^2-(x^2)^2+\sum_{i=3}^{p+2}(x^i)^2=-R^2 \tag{2.1}$$

in the pseudo-Euclidean space $\mathbb{R}^{2,p}$ with coordinates $(x^1,x^2,\ldots,x^{p+2})$. The isometry group of the quadric is $\mathrm{O}(2,p)$, which acts linearly on $\mathbb{R}^{2,p}$ and preserves the quadric. This is analogous to the case of the sphere S^q (of radius of curvature R), which can be identified with the corresponding quadric in the Euclidean space $\mathbb{R}^{q+1}$ and whose group of isometries is $\mathrm{O}(q+1)$ acting linearly in $\mathbb{R}^{q+1}$ and preserving the quadric. However, whereas the sphere (for $q>1$) is simply connected, the quadric (2.1) is not. Indeed, its fundamental group is $\mathbb{Z}$ if $p>2$ and $\mathbb{Z}\oplus\mathbb{Z}$ if $p=2$. This means that, although the isometry group of the quadric (2.1) is $\mathrm{O}(2,p)$, that of AdS_{p+1} is a nontrivial central extension by $\mathbb{Z}$ or $\mathbb{Z}\oplus\mathbb{Z}$.

In string theory, Freund-Rubin backgrounds of the form $AdS_{p+1}\times S^q$ are not fully specified by the geometry alone, but require in addition specifying fluxes, which in these backgrounds coincide with the volume forms of the relevant factors. In other words, both factors come with orientation. This means that the symmetries of a Freund-Rubin background are the orientation-preserving isometries of the underlying geometries. For S^q this is the Lie group $SO(q+1)$, whereas for AdS_{p+1} it is the infinite cover of $SO(2,p)$ obtained by centrally extending this group by the fundamental group of the quadric, as explained in [54, Sec. 5.1.2]. We will denote this group by $\widetilde{SO(2,p)}$. Annoyingly, it cannot be embedded in a matrix group; that is, it does not admit any finite-dimensional faithful linear representations. Crucially, however, $\widetilde{SO(2,p)}$ has two features in common with its quotient $SO(2,p)$. First of all, they share the same Lie algebra $\mathfrak{so}(2,p)$ and furthermore, since conjugation by central elements is trivial, the adjoint action of $\widetilde{SO(2,p)}$ on $\mathfrak{so}(2,p)$ factors through $SO(2,p)$. Similarly, the action of the spin cover $\widetilde{Spin(2,p)}$ of $\widetilde{SO(2,p)}$ on the spinor representations factors through $Spin(2,p)$. These happy facts allow a complete analysis of one-parameter subgroups and also the determination of the supersymmetry preserved by a quotient.

B. One-parameter subgroups of isometries of AdS_{p+1}

By definition, a one-parameter subgroup Γ of a Lie group G is the image under the exponential map of a one-dimensional subspace of its Lie algebra $\mathfrak{g}$. In other words, Γ consists of group elements of the form $\exp(tX)$, where $t\in\mathbb{R}$ and $X\in\mathfrak{g}$. The topology of Γ is either $\mathbb{R}$ or S^1, depending on whether or not $\exp(tX)$ is the identity element in G for some nonzero t. If $2\pi T>0$ is the smallest such t, then the exponential map defines a diffeomorphism of the circle $\mathbb{R}/2\pi T\mathbb{Z}$ with Γ, otherwise it defines a diffeomorphism of $\mathbb{R}$ with Γ.

Every one-parameter subgroup $\Gamma\subset G$ gives rise to an infinite family (indexed by the subgroup itself) of discrete cyclic subgroups Γ_γ generated by an element $\gamma\in\Gamma$. If γ has infinite order, then $\Gamma_\gamma\cong\mathbb{Z}$, whereas if the order is N, then $\Gamma_\gamma\cong\mathbb{Z}_N$. All infinite cyclic subgroups of G *in the image of the exponential map* are obtained in this way. In the cases when $\Gamma\cong S^1$, we will restrict our attention to elements γ of finite order. Quotienting a manifold M on which G acts by the action of Γ_γ consists in identifying points of M which are related by the action of γ. Since $\gamma=\exp(\ell X)$ for some $X\in\mathfrak{g}$ and some $\ell>0$, quotienting by Γ_γ consists in identifying points in M that are related by flowing along the integral curve of the Killing vector ξ_X corresponding to X for a time ℓ.

As explained, for example, in [20], if Γ and Γ' are conjugate subgroups of isometries of a space M, then their quotients M/Γ and M/Γ' are isometric, the isometry being induced from the isometry of M which conjugates Γ into Γ'. Therefore, to classify such quotients M/Γ, it is enough to classify subgroups up to conjugation. For one-parameter subgroups this corresponds to classifying adjoint orbits in the Lie algebra $\mathfrak{g}$. Furthermore, by reparametrizing the subgroup if needed, one can further projectivize the Lie algebra and declare collinear elements as equivalent.

TABLE I. The elementary blocks as two-forms.

Block	Two-form
$B^{(0,2)}(\varphi)$	$\varphi\boldsymbol{e}_{34}$
$B^{(1,1)}(\beta)$	$\beta\boldsymbol{e}_{13}$
$B^{(2,0)}(\varphi)$	$\varphi\boldsymbol{e}_{12}$
$B^{(1,2)}$	$\boldsymbol{e}_{13}-\boldsymbol{e}_{34}$
$B^{(2,1)}$	$\boldsymbol{e}_{12}-\boldsymbol{e}_{23}$
$B_\pm^{(2,2)}$	$\pm\boldsymbol{e}_{12}+\boldsymbol{e}_{13}\mp\boldsymbol{e}_{24}-\boldsymbol{e}_{34}$
$B_\pm^{(2,2)}(\beta)$	$\pm\boldsymbol{e}_{12}+\boldsymbol{e}_{13}\mp\boldsymbol{e}_{24}-\boldsymbol{e}_{34}+\beta(\boldsymbol{e}_{14}\mp\boldsymbol{e}_{23})$
$B_\pm^{(2,2)}(\varphi)$	$\pm\boldsymbol{e}_{12}+\boldsymbol{e}_{13}\mp\boldsymbol{e}_{24}-\boldsymbol{e}_{34}+\varphi(\pm\boldsymbol{e}_{12}+\boldsymbol{e}_{34})$
$B_\pm^{(2,2)}(\beta,\varphi)$	$\varphi(\pm\boldsymbol{e}_{12}-\boldsymbol{e}_{34})+\beta(\boldsymbol{e}_{14}\mp\boldsymbol{e}_{23})$
$B^{(2,3)}$	$\boldsymbol{e}_{12}-\boldsymbol{e}_{24}+\boldsymbol{e}_{13}-\boldsymbol{e}_{34}+\boldsymbol{e}_{15}-\boldsymbol{e}_{45}$
$B_\pm^{(2,4)}(\varphi)$	$\boldsymbol{e}_{15}-\boldsymbol{e}_{35}\pm\boldsymbol{e}_{26}-\boldsymbol{e}_{46}+\varphi(\mp\boldsymbol{e}_{12}+\boldsymbol{e}_{34}+\boldsymbol{e}_{56})$

Therefore to classify conjugacy classes of one-parameter subgroups of isometries of AdS_{p+1} for $p\geqslant 2$ it is equivalent to classify equivalence classes of elements $X\in\mathfrak{so}(2,p)$ under

$$X\sim tgXg^{-1}\quad\text{where}\quad t\in\mathbb{R}^\times\quad\text{and}\quad g\in SO(2,p). \tag{2.2}$$

Such a classification was established in [54,55] and we review it now.

Every $B\in\mathfrak{so}(2,p)$ defines a skew-symmetric endomorphism of $\mathbb{R}^{2,p}$, which we also denote by B. Associated with each such endomorphism there is an orthogonal decomposition

$$\mathbb{R}^{2,p}=\mathbb{V}_1\oplus\ldots\oplus\mathbb{V}_k$$

into nondecomposable nondegenerate subspaces stabilized by B; that is, for each i, $B(\mathbb{V}_i)\subset\mathbb{V}_i$, the inner product restricts nondegenerately to each $\mathbb{V}_i$, and the restriction B_i of B to $\mathbb{V}_i$ does not decompose further into nondegenerate blocks. Conversely, out of such *elementary blocks* B_i one can build the original endomorphism B. In this way, the original problem is essentially mapped into the classification of normal forms of skew-symmetric endomorphisms of $\mathbb{R}^{m,n}$ with $m\leqslant 2$ and $n\leqslant p$ up to conjugation by isometries. The latter are listed in Table I, where we found it convenient to identify the endomorphism with the corresponding bilinear form, and to write these in terms of the usual basis $\boldsymbol{e}_{ij}=\boldsymbol{e}_i\wedge\boldsymbol{e}_j$ for $\Lambda^2\mathbb{R}^{2,p}$ consisting of wedge products of the elements of the ordered frame $(\boldsymbol{e}_i)$, where $\boldsymbol{e}_1,\boldsymbol{e}_2$ denote the two timelike directions, the remaining ones being spacelike. The superscript (m,n) on the elementary blocks specifies the subspace $\mathbb{R}^{m,n}$ that they act on. The Killing vector in $\mathbb{R}^{2,p}$ associated with the two-form

$$X=\frac{1}{2}\sum_{i,j}B^{ij}\boldsymbol{e}_{ij}\in\Lambda^2\mathbb{R}^{2,p}\cong\mathfrak{so}(2,p)$$

is given by

$$\xi_X=\frac{1}{2}\sum_{i,j}B^{ij}(x_i\partial_j-x_j\partial_i)=\sum_{i,j}x^iB_i{}^j\partial_j\,.$$

It is clearly tangent to the quadric and it lifts to a Killing vector field on AdS_{p+1} which we also denote ξ_X.

Let us briefly discuss the interpretation of each of these elementary blocks to help the reader get used to our notation. We shall denote boost parameters by β and rotation parameters by φ. There are three inequivalent two-dimensional elementary blocks: a spacelike rotation $B^{(0,2)}(\varphi)$, a boost $B^{(1,1)}(\beta)$, and a timelike rotation $B^{(2,0)}(\varphi)$. In three dimensions, normal forms either reduce to the previous ones or preserve null directions. Since we work in a non-Lorentzian signature, we must distinguish among two different null rotations: a null rotation $B^{(1,2)}$ involving two spacelike directions and a null rotation $B^{(2,1)}$ involving two timelike directions. There are four types of nontrivial four-dimensional elementary blocks: a linear combination $B_{\pm}^{(2,2)}$ of timelike and spacelike null rotations, a deformation $B_{\pm}^{(2,2)}(\beta)$ of the latter by the addition of a linear combination of boosts, a different deformation $B_{\pm}^{(2,2)}(\varphi)$ involving the addition of a timelike rotation and a spacelike rotation, and finally a linear combination $B_{\pm}^{(2,2)}(\beta,\varphi)$ of two actions involving a timelike and spacelike rotation with parameter φ (up to signs) on one side and a linear combination of boosts on the other side. There is only one five-dimensional elementary block $B^{(2,3)}$, which can be interpreted as the linear combination of a timelike null rotation and two spacelike null rotations sharing the time direction and one of the spacelike directions. The last elementary block $B_{\pm}^{(2,4)}(\varphi)$ appears in six dimensions, and it consists of a double spacelike null rotation acting on orthogonal subspaces, deformed by a simultaneous rotation in the plane formed by the two timelike directions and two orthogonal spacelike planes.

Let us remark the appearance of pairs of elementary blocks $B_{\pm}^{(m,n)}$, with or without parameter, in the classification in Table I. It can be checked that one element of the pair is always mapped into the other by an orientation-reversing transformation. Therefore, no classification based on the isometry group $\mathrm{O}(2,p)$ can distinguish between these objects. Analogously, orientation-reversing transformations act nontrivially on the parameters (β,φ) in those elementary blocks which do not come in pairs, allowing us to restrict their range. In this section, we shall follow the $\mathrm{SO}(2,p)$ classification (unless otherwise stated), but in the rest of the paper, when discussing the geometrical interpretation of the different discrete quotients, we shall omit these distinctions. This is because the metric in AdS_{p+1} is invariant under orientation-reversing transformations; therefore the geometry itself will not change among the members of the pair. The distinction will arise in the signs of the fluxes that stabilize the classical configurations: the members of a pair will have opposite sign fluxes. This fact can certainly have consequences concerning the supersymmetry preserved by the members of the pair.

The small number of elementary blocks notwithstanding, the taxonomy of inequivalent discrete quotients increases quickly with dimension due to the possibility of combining the action of different blocks acting in orthogonal subspaces of $\mathbb{R}^{2,p}$. Lack of spacetime prevents us from discussing all possible quotients in detail. There are several criteria which we could employ to narrow our choice of quotients. For example, we could focus on supersymmetric quotients, everywhere spacelike and nonsingular quotients, etc. Our primary criterion will be that a quotient should have a well-behaved causal structure: our subsequent discussion will focus on those discrete quotients that either are free of closed timelike curves, or in which the closed timelike curves are "expungeable," in the sense that a spacetime free of closed timelike curves can be obtained by quotienting only part of AdS, and that the boundary so introduced lies behind a horizon. In the latter case, the resulting causally well-behaved singular spacetime is interpreted as an analogue of a black hole, following [36,37].

TABLE II. The elementary blocks and their norms.

Block	Norm
$B^{(0,2)}(\varphi)$	$\varphi^2(x_3^2+x_4^2)$
$B^{(1,1)}(\beta)$	$\beta^2(R^2+\Vert\boldsymbol{x}_\perp\Vert^2-x_2^2)$
$B^{(1,2)}$	$(x_1+x_4)^2$
$B^{(2,0)}(\varphi)$	$-\varphi^2(R^2+\Vert\boldsymbol{x}_\perp\Vert^2)$
$B^{(2,1)}$	$-(x_1+x_3)^2$
$B_{\pm}^{(2,2)}$	0
$B_{\pm}^{(2,2)}(\beta)$	$\beta^2(R^2+\Vert\boldsymbol{x}_\perp\Vert^2)+4\beta(x_1+x_4)(x_3\pm x_2)$
$B_{\pm}^{(2,2)}(\varphi)$	$-\varphi^2(R^2+\Vert\boldsymbol{x}_\perp\Vert^2)+2\varphi((x_1+x_4)^2+(x_3\pm x_2)^2)$
$B_{\pm}^{(2,2)}(\beta,\varphi)$	$(\beta^2-\varphi^2)(R^2+\Vert\boldsymbol{x}_\perp\Vert^2)-4\beta\varphi(x_1x_3\pm x_2x_4)$
$B^{(2,3)}$	$(x_4-x_1)^2-4(x_2+x_3)x_5$
$B_{\pm}^{(2,4)}(\varphi)$	$-\varphi^2(R^2+\Vert\boldsymbol{x}_\perp\Vert^2)+(x_1-x_3)^2+(x_4\mp x_2)^2$ $-4\varphi[(x_4\mp x_2)x_5+(x_1-x_3)x_6]$

The causal properties of the quotient are determined primarily by the norm of the Killing vector field generating it. It is therefore important to study the norm of the Killing vectors associated with the two-forms listed in Table I. These are given in Table II, where the following notation is used. We write explicitly the coordinates x_i of the subspace $\mathbb{W}\subset\mathbb{R}^{2,p}$ on which the elementary blocks act nontrivially and write $\boldsymbol{x}_\perp$ for the coordinates of the perpendicular subspace $\mathbb{W}^\perp$. The norm is defined on the quadric (2.1), but can be pulled back to functions on AdS which are invariant under the deck transformations generated by the fundamental group of the quadric.

We can see from Table II that some Killing vectors are timelike in some regions of AdS, leading to closed timelike curves in the associated discrete quotients. Indeed, we see that for $B^{(2,0)}(\varphi)$, $B^{(1,1)}(\beta)$, $B^{(2,1)}$, $B_{\pm}^{(2,2)}(\beta)$, $B_{\pm}^{(2,2)}(\beta,\varphi)$, $B^{(2,3)}$, $B_{\pm}^{(2,4)}(\varphi)$, and $B_{\pm}^{(2,2)}(\varphi<0)$, the norm is not bounded below. For $B_{\pm}^{(2,2)}(\varphi>0)$, the norm can be negative, but is bounded from below; whereas for $B^{(0,2)}(\varphi)$ and $B^{(1,2)}$ and $B_{\pm}^{(2,2)}$, the norm is always non-negative.

The Killing vector ξ which generates the quotient will be the sum of such elementary blocks and its norm on AdS will influence the causal structure of the quotient. We therefore consider the possible endomorphisms in the signature $(2,p)$ that can be constructed from elementary blocks acting in orthogonal subspaces. In Tables III, IV, and V we classify them in terms of the norms of the associated Killing vectors in AdS. It should be stressed that even though we used the

TABLE III. Killing vectors with everywhere non-negative norm.

Endomorphism
$\oplus_i B^{(0,2)}(\varphi_i)$
$B^{(1,1)}(\beta_1)\oplus B^{(1,1)}(\beta_2)\oplus_i B^{(0,2)}(\varphi_i)$ if $\lvert\beta_1\rvert=\lvert\beta_2\rvert>0$
$B^{(1,2)}\oplus_i B^{(0,2)}(\varphi_i)$
$B^{(1,2)}\oplus B^{(1,2)}\oplus_i B^{(0,2)}(\varphi_i)$
$B_{\pm}^{(2,2)}\oplus_i B^{(0,2)}(\varphi_i)$

notation adapted to an SO(2,p) classification, we have not constrained the range of the different parameters appearing in these endomorphisms. For a complete discussion concerning these constraints, we refer the reader to [54].

The quotients generated by the Killing vectors in Table V clearly contain closed timelike curves corresponding to the very orbits of the Killing vector in regions where it is timelike. Furthermore, even when we consider quotients of $AdS_{p+1}\times S^q$ by adding a nontrivial action on the sphere, the resulting Killing vector will still be timelike somewhere, so the quotients will still have closed timelike curves. Therefore the only way in which these quotients will enter into our discussion is in asking whether any of them lead to ''black hole'' spacetimes. We shall discuss this issue in Sec. V.

The quotients generated by the Killing vectors in Table IV also clearly contain closed timelike curves. This time, however, the Killing vector can be made everywhere spacelike by adding a suitable action on an odd-dimensional sphere. However, we will show in the next section that this is not sufficient to ensure the absence of closed timelike curves. Therefore the quotients of $AdS_{p+1}\times S^q$ associated with the Killing vectors in this table will not lead to causally regular quotients either. In summary, the only quotients we will consider in Sec. IV, where we discuss causally nonsingular quotients, are those in Table III.

C. Infinitesimal isometries of spheres

Here we set up the notation to describe the Killing vectors on spheres. For this purpose, we find it convenient to identify the q-sphere of radius R with the quadric traced by

$$\sum_{i=1}^{q+1} x_i^2=R^2 \tag{2.3}$$

in $\mathbb{R}^{q+1}$. This has the virtue that the isometry group of the quadric, O$(q+1)$, acts linearly in the ambient Euclidean space. As we did for AdS_{p+1}, we shall restrict this group to the subgroup SO$(q+1)$ which preserves the orientation.

TABLE IV. Killing vectors allowing negative norm but bounded below.

Endomorphism
$B^{(2,0)}(\varphi)\oplus_i B^{(0,2)}(\varphi_i)$ if p is even and $\lvert\varphi_i\rvert\geqslant\varphi>0$ for all i
$B_{\pm}^{(2,2)}(\varphi)\oplus_i B^{(0,2)}(\varphi_i)$ if $\lvert\varphi_i\rvert\geqslant\lvert\varphi\rvert\geqslant 0$ for all i

TABLE V. Killing vectors with norm unbounded below.

Endomorphism
$B^{(1,1)}(\beta_1)\oplus B^{(1,1)}(\beta_2)\oplus_i B^{(0,2)}(\varphi_i)$ unless $\lvert\beta_1\rvert=\lvert\beta_2\rvert>0$
$B^{(1,2)}\oplus B^{(1,1)}(\beta)\oplus_i B^{(0,2)}(\varphi_i)$
$B^{(2,0)}(\varphi)\oplus_i B^{(0,2)}(\varphi_i)$ unless p is even and $\lvert\varphi_i\rvert\geqslant\lvert\varphi\rvert$ for all i
$B^{(2,1)}\oplus_i B^{(0,2)}(\varphi_i)$
$B_{\pm}^{(2,2)}(\beta)\oplus_i B^{(0,2)}(\varphi_i)$
$B_{\pm}^{(2,2)}(\varphi)\oplus_i B^{(0,2)}(\varphi_i)$ unless $\lvert\varphi_i\rvert\geqslant\varphi>0$ for all i
$B_{\pm}^{(2,2)}(\beta,\varphi)\oplus_i B^{(0,2)}(\varphi_i)$
$B^{(2,3)}\oplus_i B^{(0,2)}(\varphi_i)$
$B_{\pm}^{(2,4)}(\varphi)\oplus_i B^{(0,2)}(\varphi_i)$

The conjugacy theorem for Cartan subalgebras of $\mathfrak{so}(q+1)$ allows us to bring any Killing vector ξ_S on S^q to the form

$$\xi_S=\sum_{i=1}^{r} \theta_i R_{2i-1,2i}, \tag{2.4}$$

where $r=\lfloor(q+1)/2\rfloor$, R_{ij} stands for a rotation in the ij plane, and the θ_i are real parameters specifying the rotation angles. This still leaves the freedom to conjugate by the Weyl group, which we can fix by arranging the parameters in such a way that

$$\theta_1\geqslant\theta_2\geqslant\cdots\geqslant\lvert\theta_r\rvert.$$

For odd-dimensional spheres, Killing vectors with all $\theta_i\neq 0$ are everywhere nonvanishing, whereas in even-dimensional spheres every vector field, Killing or not, has a zero.

It will be convenient in what follows to construct a coordinate system for S^q adapted to a given Killing vector ξ_S; that is, one in which $\xi_S=\partial_\psi$. Let us describe in detail the case of even-dimensional spheres. First, rewrite Eq. (2.3) as

$$\sum_{i=1}^{r} \lvert z_i\rvert^2+(x_{2r+1})^2=R^2, \tag{2.5}$$

in which we introduce r complex coordinates for the two-planes where the action of (2.4) may be nontrivial. A natural way to solve Eq. (2.5) is by

$$x_{2r+1}=R\cos\theta,$$

$$z_i=R\sin\theta\rho_i e^{i\varphi_i}\quad\text{where}\quad\sum_{i=1}^{r}\rho_i^2=1. \tag{2.6}$$

It is clear that in coordinates $\{\theta,\rho_i,\varphi_i\}$

$$\xi_S=\sum_{i=1}^{r}\theta_i\partial_{\varphi_i},$$

whence by a linear transformation in the space $\{\varphi_i\}$ we can rewrite ξ_S as ∂_ψ. Indeed, assume $\theta_1\neq 0$, and consider

$$\psi = \theta_1^{-1}\varphi_1,$$

$$\tilde{\varphi}_i = \varphi_i - \theta_i\theta_1^{-1}\varphi_1, \quad i = 2,\dots,r. \tag{2.7}$$

By construction, ξ_S becomes ∂_ψ.

The case of odd-dimensional spheres follows formally from the above by setting $\theta = \pi/2$ in the above expressions.

D. Spin structures and supersymmetry

A supergravity background must admit a spin structure, since the fermionic fields, although set to zero in a classical background, and the supersymmetry parameters are sections of (possibly twisted) spinor bundles. This is not necessarily the case in string or M theory as the phenomenon of "supersymmetry without supersymmetry" illustrates [66–68]. This has recently been discussed in [69] and in the present context of quotients in [54]. We will add nothing to this discussion here. Indeed, as in [54], we will adopt a conservative point of view and require the underlying spacetime of a supergravity background to be spin and will consider only supersymmetries that are realized geometrically as Killing spinors.

A natural question in this context is then the following. Let $(M,g,\dots)$ be a supergravity background with (M,g) a Lorentzian spin manifold and Γ a discrete (cyclic) group of orientation-preserving isometries acting freely and properly discontinuously on M (so that the quotient M/Γ is smooth). When will M/Γ be spin? Furthermore, if $(M,g,\dots)$ is a supersymmetric background, how much supersymmetry (if any at all) will the quotient preserve? These questions were answered in [54] for the case of Γ a one-parameter group: in principle for an arbitrary background, and explicitly for Freund-Rubin backgrounds of the form $\mathrm{AdS}_{p+1}\times \mathrm{S}^q$.

If Γ is a one-parameter group of isometries (hence automatically orientation preserving) acting freely on a spin manifold M with smooth quotient M/Γ, then M/Γ is spin if and only if the action of Γ on the bundle $P_{\mathrm{SO}}(M)$ of oriented orthonormal frames lifts to an action on the spin bundle $P_{\mathrm{Spin}}(M)$ in such a way that the natural surjection

$$\theta: P_{\mathrm{Spin}}(M) \to P_{\mathrm{SO}}(M)$$

is Γ equivariant. In this case, the spin bundle $P_{\mathrm{Spin}}(M/\Gamma)$ on the quotient is given by

$$P_{\mathrm{Spin}}(M/\Gamma) := P_{\mathrm{Spin}}(M)/\Gamma.$$

Indeed, equivariance guarantees that this bundle covers

$$P_{\mathrm{SO}}(M/\Gamma) := P_{\mathrm{SO}}(M)/\Gamma$$

twice and agrees fiberwise with the spin cover of the special orthogonal group.

The same is true for Γ a discrete group acting freely and properly discontinuously on a spin manifold M. For a general spin manifold M, it is not easy to determine when the action of Γ on $P_{\mathrm{SO}}(M)$ lifts equivariantly to the spin bundle; however, as explained in [54], for backgrounds of the form

$$M = \mathrm{AdS}_{p+1}\times \mathrm{S}^q$$

we fare much better. Indeed, for this geometry the criterion for the existence of a spin structure in M/Γ translates into a simple calculation in a Clifford algebra.

For simplicity we will consider discrete cyclic groups generated by an element γ in the image of the exponential map $\exp: \mathfrak{g} \to G$ between the Lie algebra and Lie group of (orientation-preserving) isometries of M; that is,

$$\gamma = \exp(\ell X)$$

for some $X \in \mathfrak{g}$ and some $\ell > 0$. Then Γ acts on the (unique) spin bundle on $\mathrm{AdS}_{p+1}\times \mathrm{S}^q$ if and only if Γ embeds isomorphically in $\mathrm{Spin}(2,p)\times_{\mathbb{Z}_2}\mathrm{Spin}(q+1) \subset C\ell(2,p+q+1)$. Since Γ is generated by γ, this is a simple criterion: does there exist

$$\tilde{\gamma} \in \mathrm{Spin}(2,p)\times_{\mathbb{Z}_2}\mathrm{Spin}(q+1) \subset C\ell(2,p+q+1)$$

which lifts γ and which has the same order?

The element γ has two possible lifts $\pm\tilde{\gamma}$. If γ has infinite order, so that $\Gamma \cong \mathbb{Z}$, then so does $\tilde{\gamma}$, and thus it also generates a group $\tilde{\Gamma} \cong \mathbb{Z}$ which therefore covers Γ isomorphically. Therefore, if $\Gamma \cong \mathbb{Z}$, the quotient

$$(\mathrm{AdS}_{p+1}\times \mathrm{S}^q)/\Gamma$$

is spin.

Now suppose that γ has finite order N. Then all we know is that $(\pm\tilde{\gamma})^N$ covers the identity, whence

$$(\pm\tilde{\gamma})^N = \pm\mathbb{1},$$

and the question is whether there exists a choice of lift such that $(\pm\tilde{\gamma})^N = \mathbb{1}$.

Clearly, if N is odd, then either $(\tilde{\gamma})^N = \mathbb{1}$ or $(-\tilde{\gamma})^N = \mathbb{1}$, whence if $\Gamma \cong \mathbb{Z}_N$, N odd, the quotient is spin.

The only possible obstruction arises when N is even. In this case the choice of lift is immaterial, and either $\tilde{\gamma}^N = \mathbb{1}$ or $\tilde{\gamma}^N = -\mathbb{1}$, and one needs to do a calculation to settle this issue.

This obstruction arises only if $\gamma = \exp(\ell X)$ for $\ell > 0$ and

$$X = \varphi_1 \boldsymbol{e}_{34} + \cdots + \varphi_r \boldsymbol{e}_{2r+1,2r+2} + \theta_1 R_{12} + \cdots + \theta_s R_{2s-1,2s},$$

where $r = \lfloor (p-1)/2 \rfloor$ and $s = \lfloor (q+1)/2 \rfloor$. Let $\gamma = \exp(\ell X)$. Then γ has order N if and only if

$$\ell\varphi_i = \frac{2\pi n_i}{N} \quad \text{and} \quad \ell\theta_j = \frac{2\pi m_j}{N},$$

where n_i, m_j are integers with

$$\gcd(n_1,\dots,n_r,m_1,\dots,m_s) = 1.$$

This last condition ensures that the order of γ is precisely N and not a smaller divisor. Let γ_i and Γ_i be the gamma matrices for $C\ell(2,p)$ and $C\ell(q+1)$, respectively, embedded in

$$C\ell(2,p+q+1) \cong C\ell(2,p)\hat{\otimes} C\ell(q+1),$$

where $\hat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product. Then the two lifts of γ in

$$\mathrm{Spin}(2,p)\times_{\mathbb{Z}_2}\mathrm{Spin}(q+1)\subset C\ell(2,p+q+1)$$

are given by $\pm\tilde{\gamma}$, where

$$\tilde{\gamma}=\left(\mathbb{1}\cos\frac{\ell\varphi_1}{2}+\gamma_{34}\sin\frac{\ell\varphi_1}{2}\right)\cdots\left(\mathbb{1}\cos\frac{\ell\varphi_r}{2}+\gamma_{2r+1,2r+2}\sin\frac{\ell\varphi_r}{2}\right)\left(\mathbb{1}\cos\frac{\ell\theta_1}{2}+\Gamma_{12}\sin\frac{\ell\theta_1}{2}\right)\cdots\left(\mathbb{1}\cos\frac{\ell\theta_s}{2}+\Gamma_{2s-1,2s}\sin\frac{\ell\theta_s}{2}\right),$$

whence

$$\tilde{\gamma}^N=\left(\mathbb{1}\cos\frac{N\ell\varphi_1}{2}+\gamma_{34}\sin\frac{N\ell\varphi_1}{2}\right)\cdots\left(\mathbb{1}\cos\frac{N\ell\varphi_r}{2}+\gamma_{2r+1,2r+2}\sin\frac{N\ell\varphi_r}{2}\right)\left(\mathbb{1}\cos\frac{N\ell\theta_1}{2}+\Gamma_{12}\sin\frac{N\ell\theta_1}{2}\right)\cdots\left(\mathbb{1}\cos\frac{N\ell\theta_s}{2}+\Gamma_{2s-1,2s}\sin\frac{N\ell\theta_s}{2}\right).$$

Using now that $N\ell\varphi_i=2\pi n_i$ and $N\ell\theta_j=2\pi m_j$, this evaluates to

$$\tilde{\gamma}^N=(-1)^{n_1+\cdots+n_r+m_1+\cdots+m_s}\mathbb{1}.$$

Therefore we conclude that when $\Gamma\cong\mathbb{Z}_N$, N even, the quotient is spin if and only if

$$\sum_{i=1}^{r} n_i+\sum_{j=1}^{s} m_j \quad \text{is even.}$$

III. CAUSAL PROPERTIES OF AdS_{p+1} QUOTIENTS AND THEIR DEFORMATIONS

In Sec. II B we reviewed the classification of one-parameter subgroups of isometries of AdS_{p+1}. We divided these into three different subsets according to whether the norm of the associated Killing vector field is non-negative (Table III); the norm can take negative values, but is bounded below (Table IV); or the norm can take arbitrarily negative values (Table V). As explained above, this distinction is important in the context of Freund-Rubin backgrounds of the form $AdS_{p+1}\times S^q$, since the spherical component of the Killing vector can in some cases render its norm positive everywhere. Indeed, odd-dimensional spheres admit Killing vectors whose norm is pinched away from zero, whence the total Killing vector

$$\xi=\xi_{AdS}+\xi_S \tag{3.1}$$

may be spacelike even if ξ_{AdS} is not. This can happen only if the norm of ξ_{AdS} is bounded below, since the norm of ξ_S is bounded above by compactness of S^q.

In this section, we will explain in detail the connection between this classification and the appearance of closed timelike curves in quotients involving these Killing vectors.

If we were just considering quotients of AdS, of course, the connection would be immediate. Indeed, the quotient consists in identifying points which are obtained by flowing along the integral curves of ξ_{AdS} for some time $\ell>0$. Let ξ_{AdS} be timelike in a nonempty region $D\subset AdS_{p+1}$ and let $x\in D$. Since the norm of ξ_{AdS} is constant along its integral curves, the integral curve passing through x is timelike and hence lies in D. Therefore the point $\gamma\cdot x$ is also in D and the segment of the integral curve from x to $\gamma\cdot x$ becomes, in the quotient, a closed timelike curve. A similar argument shows that the quotient has closed null curves in the region of AdS/Γ where ξ_{AdS} is null.

The situation for quotients of $AdS_{p+1}\times S^q$ is similar. Indeed, the same argument as for quotients of AdS shows that if $\xi=\xi_{AdS}+\xi_S$ is not everywhere spacelike, then any associated discrete cyclic quotient will have closed causal curves.

How about if ξ is everywhere spacelike? The property of being spacelike everywhere is a necessary condition for the absence of closed causal curves, but it is certainly not sufficient (see [70] for another example where it fails to be sufficient and a statement of a sufficient condition, and [71] for a discussion on this topic and its relation to U duality). Indeed, we will show presently that even when ξ is everywhere spacelike, if ξ_{AdS} is timelike in some region $D\subset AdS_{p+1}$, then any discrete cyclic quotient associated with $\xi=\xi_{AdS}+\xi_S$ will have closed timelike curves in the region $(D\times S^q)/\Gamma$ of the quotient. The key point in the argument is to exploit the fact that the sphere has a bounded diameter in order to construct a timelike curve between two points identified by the action of Γ which, as in [70], is different from the integral curve of ξ.

Let us first illustrate this construction with a simple example, which is depicted in Fig. 1. Let $C=(\mathbb{R}/2\pi\mathbb{Z})\times\mathbb{R}$ denote a Lorentzian cylinder coordinatized by (θ,τ) and flat metric $d\theta^2-d\tau^2$. Let $\xi=\partial_\theta+\alpha\partial_\tau$ be a spacelike Killing vector, so that $\alpha^2<1$. The integral curve of ξ through a point (θ_0,τ_0) is the curve

$$t\mapsto(\theta_0+t,\tau_0+\alpha t).$$

Let us define an action of $\mathbb{Z}$ on C, generated by the operation of flowing along the integral curves of ξ for a time $\ell>0$:

$$(\theta,\tau)\mapsto(\theta+\ell,\tau+\alpha\ell).$$

Consider the two points (θ,τ) and $(\theta+N\ell,\tau+\alpha N\ell)$, which are identified in the quotient $C/\mathbb{Z}$. The geodesic joining this point to (θ,τ) is the straight line

$$t\mapsto([\theta+tN\ell],\tau+\alpha N\ell),$$

where $[\cdots]$ denotes the residue modulo 2π. The norm of the velocity of this curve is therefore

$$[N\ell]^2-N^2\alpha^2\ell^2\leqslant4\pi^2-N^2\alpha^2\ell^2,$$

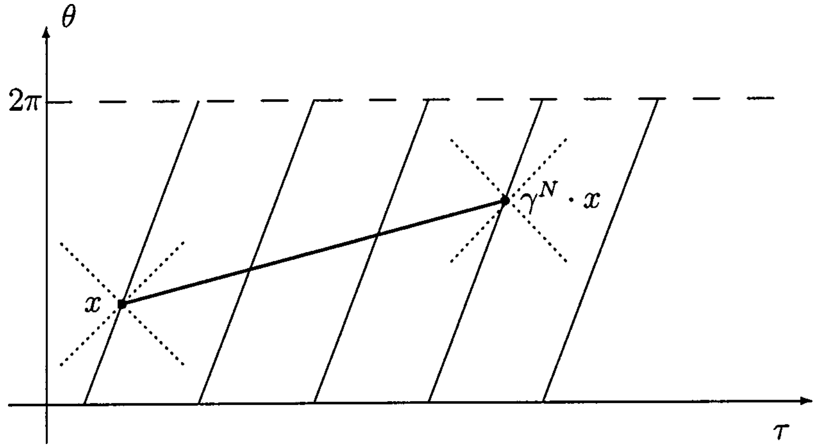

FIG. 1. Closed timelike curve in a discrete quotient of the Lorentzian cylinder. The dotted lines represent the "light cones" at x and at $\gamma^N \cdot x$. Notice that, although the orbit of ξ is spacelike, the straight line between x and $\gamma^N \cdot x$ is timelike.

which is clearly negative for N large enough. This curve is therefore a closed timelike curve in the quotient $C/\mathbb{Z}$.

Now let us go back to the general case. Let $\gamma = \exp(\ell X)$ for some $X \in \mathfrak{g}$ and $\ell > 0$, and let $\xi = \xi_{\mathrm{AdS}} + \xi_{\mathrm{S}}$ be the Killing vector corresponding to X, with ξ_{AdS} timelike in some nonempty region $D \subset \mathrm{AdS}_{p+1}$. Let $x \in D \times \mathrm{S}^q$. Since the norms of each component ξ_{AdS} and ξ_{S} are separately conserved along the integral curves of ξ, these belong to $D \times \mathrm{S}^q$, and hence so does $\gamma \cdot x$. For those Killing vectors with AdS component in Table IV, the associated discrete cyclic groups Γ have infinite order, so we can consider points x and $\gamma^N \cdot x$ for N arbitrarily large, which will give rise to the same point in the quotient. We will construct a curve

$$c:[0,N\ell] \to \mathrm{AdS}_{p+1} \times \mathrm{S}^q$$

between $c(0) = x = (x_{\mathrm{AdS}}, x_{\mathrm{S}})$ and $c(N\ell) = \gamma^N \cdot x = ((\gamma^N \cdot x)_{\mathrm{AdS}}, (\gamma^N \cdot x)_{\mathrm{S}})$ which will be timelike for N sufficiently large and hence becomes a closed timelike curve in the quotient.

The curve c is uniquely specified by its two components: c_{AdS} on AdS_{p+1} and c_{S} on S^q. We will take c_{AdS} to be the integral curve of ξ_{AdS}, and c_{S} to be a minimum-length geodesic between x_{S} and $(\gamma^N \cdot x)_{\mathrm{S}}$. Let L denote the diameter of the sphere; that is, the supremum of the geodesic distances between any two points. Then the arclength along c_{S} satisfies

$$\int_0^{N\ell} \|\dot{c}_{\mathrm{S}}\| dt = N\ell \|\dot{c}_{\mathrm{S}}\| \leq L,$$

where the equality is because $\|\dot{c}_{\mathrm{S}}\|$ is constant along c_{S} and the inequality is because c_{S} is length minimizing. Therefore,

$$\|\dot{c}\|^2 = \|\dot{c}_{\mathrm{AdS}}\|^2 + \|\dot{c}_{\mathrm{S}}\|^2 \leq \|\xi_{\mathrm{AdS}}\|^2 + \frac{L^2}{N^2\ell^2},$$

which is negative in $D \times \mathrm{S}^q$ for N large enough.

Let us remark that this argument applies to any Freund-Rubin background of the form $\mathrm{AdS} \times N$, or more generally $M \times N$, with M Lorentzian admitting such isometries, at least when N is complete. Indeed, the supergravity equations of motion force N to be Einstein with positive scalar curvature. By the Bonnet-Myers theorem (see, e.g., [72, Sec. 9.3], if N is complete, then it has bounded diameter.

This leaves the cases in Table III, where the AdS Killing vector is nowhere timelike. It is clear that the above argument for closed timelike curves fails in this case. One should note that this still does not directly imply the absence of closed timelike curves; however, we will see in the next section that there are in fact no closed timelike curves in any of these cases.

We should also note that in the cases where the Killing vector is null somewhere, namely, $\oplus_i B^{(0,2)}(\varphi_i)$, $B^{(1,2)} \oplus_i B^{(0,2)}(\varphi_i)$, and $B_{\pm}^{(2,2)} \oplus_i B^{(0,2)}(\varphi_i)$, we can use a similar argument to see that *some* quotients of $\mathrm{AdS}_{p+1} \times \mathrm{S}^q$ still produce closed causal curves. The point is that if we choose ℓ such that $\exp(\ell X_{\mathrm{S}}) \in \mathrm{SO}(q+1)$ has order N, then x and $x' = \gamma^N \cdot x$ can be null separated, as $x'_{\mathrm{S}} = x_{\mathrm{S}}$, and the separation in the AdS factor is null if $\|\xi_{\mathrm{AdS}}\| = 0$ at x. Physically, this corresponds to deforming by a rotation with rational angles on S^q.

Clearly, however, deformations for which γ_{S} does not have finite order do exist, and will not lead to closed causal curves by any of our arguments above. Hence, we should discuss all the cases listed in Table III in the next section, as they can all give rise to causally nonsingular quotients.

IV. CAUSALLY NONSINGULAR QUOTIENTS

In this section, we shall discuss in detail the geometry of the discrete quotients that are free of closed causal curves. These are based on the two-forms listed in Table III, conveniently deformed when necessary by some nontrivial action on an odd sphere leaving no invariant directions, so that the full Killing vector field (3.1) is spacelike everywhere.

Before initiating such a task, we would like to comment on the general philosophy that we shall apply in each of the particular geometries to be discussed. Just by inspection of Table III, we know that, given any two-form in that list, we can study the geometry of the corresponding discrete quotient in different dimensional AdS spacetimes, starting with the minimal (n,m) signature in the embedding space $\mathbb{R}^{(n,m)}$ that allows the action of the corresponding decomposable block. In addition to that, we can also study further deformations on the sphere sector of the discrete quotient. It is therefore natural to start our analysis in the lowest-dimensional $AdS_{p+1}\times S^q$ spacetime allowing our causally nonsingular quotients, and afterward, extend such an analysis to higher dimensions.

This latter extension is entirely straightforward. Indeed, given some adapted coordinate system describing the action of ξ_{AdS} in AdS_{n+1}, it is very simple to construct an adapted coordinate system describing the action of the same Killing vector field in AdS_{p+1} with $p>n$. This is just obtained by considering the standard AdS_{n+1} foliation of AdS_{p+1} given in terms of the embedding coordinates by[3]

$$x^i=\cosh\chi\hat{x}^i,\quad i=1,\dots,n+2,$$
$$x^m=\sinh\chi\hat{x}^m,\quad m=1,\dots,p-n, \tag{4.1}$$

where χ is noncompact and $\{\hat{x}^i\}$ satisfy the quadric defining relation giving rise to AdS_{p+1}, whereas $\{\hat{x}^m\}$ parametrize an S^{p-n-1} sphere of unit radius. For $p=n+1$, the range of χ is given by $-\infty<\chi<+\infty$, whereas for $p-n\geqslant 2$, it is simply given by $\chi\geqslant 0$. The metric description of AdS_{p+1} in the AdS_{n+1} foliation defined in Eq. (4.1) is

$$g_{AdS_{p+1}}=(\cosh\chi)^2 g_{AdS_{n+1}}+(d\chi)^2+(\sinh\chi)^2 g_{S^{p-n-1}}. \tag{4.2}$$

The foliation given by Eq. (4.2) also gives us an interesting description of the asymptotic boundary. If we assume $p-n\geqslant 2$, taking the limit $\chi\to\infty$ and conformally rescaling by a factor of $e^{-2\chi}$, we can describe the asymptotic boundary in terms of an $AdS_{n+1}\times S^{p-n-1}$ metric,[4]

$$g_\partial=g_{AdS_{n+1}}+g_{S^{p-n-1}}. \tag{4.3}$$

To see the relation of this coordinate system to the usual Einstein static universe description of the conformal boundary, let us write the AdS_{n+1} metric in global coordinates,

$$g_{AdS_{n+1}}=-\cosh^2\rho\, dt^2+d\rho^2+\sinh^2\rho\, g_{S^{n-1}}. \tag{4.4}$$

Then defining $\cos\theta=1/\cosh\rho$, we can rewrite Eq. (4.3) as

$$g_\partial=\frac{1}{\cos^2\theta}(-dt^2+d\theta^2+\sin^2\theta g_{S^{n-1}}+\cos^2\theta g_{S^{p-n-1}}). \tag{4.5}$$

This shows that the metric in Eq. (4.3) is indeed conformal to the Einstein static universe metric on $\mathbb{R}\times S^{p-1}$, where we are writing the S^{p-1} as an S^{p-n-1} fibered over an S^n. The coordinates of Eq. (4.3) cover all of the Einstein static universe apart from the $\mathbb{R}\times S^{n-1}$ submanifold where $\cos\theta=0$, which is conformally rescaled to become the boundary of the AdS_{n+1} factor in Eq. (4.3).

If there is a globally adapted coordinate system for the action of ξ_{AdS} on AdS_{n+1}, we can use the above foliation to construct an adapted coordinate system for the action on AdS_{p+1}. If we deform the action by $B^{(0,2)}(\varphi_i)$ blocks, these will act as rotations of the S^{p-n-1} factor in the above foliation.

When we consider the deformation of our AdS quotient by some nontrivial action on the transverse sphere, we have two approaches to the construction of an overall adapted coordinate such that the total Killing vector $\xi=\partial_\varphi$ for some coordinate φ. In most of the cases we consider,[5] there is a globally well-defined adapted coordinate on AdS_{p+1} such that $\xi_{AdS}=\partial_\phi$. As noted in Sec. II C, there is always a globally adapted coordinate system for the Killing vectors in the sphere, in which ξ_S acts by a simple "translation," i.e., $\xi_S=\partial_\psi$. Consequently, the full generator of the discrete quotient is

$$\xi=\partial_\phi+\gamma\partial_\psi. \tag{4.6}$$

By a linear transformation, $\varphi=\phi,\psi'=\psi-\gamma\phi$, we are able to write $\xi=\partial_\varphi$. This coordinate system is very convenient for studying the causal structure and asymptotic structure of the resulting quotient, so this is the technique we shall mostly employ.

Unfortunately, there are examples where there is no such globally adapted coordinate system on AdS. The example of this type we shall be concerned with is the quotient by a Killing vector with a single $B^{(1,2)}$ block. In this case, we need to use a different technique, exploiting the existence of adapted coordinates on the sphere. The full Killing vector field (3.1) can always be written as

$$\xi=\partial_\psi+\xi_{AdS}. \tag{4.7}$$

We can therefore write ξ as a dressed version of its "translation" component according to

$$\xi=U\partial_\psi U^{-1}\quad \text{where}\quad U=\exp(-\psi\xi_{AdS}). \tag{4.8}$$

Consequently, if the original coordinate system were given by $\{\psi,z^l\}$, where z^l stand for all the remaining coordinates describing the manifold $AdS_{p+1}\times S^q$, it is natural to change coordinates to an adapted coordinate system defined by

$$y=Uz, \tag{4.9}$$

which indeed satisfies the property $\xi y=0$, so that $\{y^l\}$ are good coordinates for the space of orbits. Equivalently, $\xi=\partial_\psi$ in the coordinates (4.9). Thus, we obtain an adapted coordinate system on the full quotient for any AdS Killing vector. For the case at hand, we split the coordinates $\{z^l\}$ appearing in the above discussion into $\{z^l\}=\{\tilde{\varphi}_i,\vec{x}\}$, where

[3]In the following, we shall set the radius of curvature R to 1.

[4]For $p-n=1$, we would have $-\infty<\chi<\infty$, and, conformally rescaling by a factor of $e^{-2|\chi|}$ as we take the limit $|\chi|\to\infty$, we would get a description of the boundary in terms of two AdS_p patches, each covering one of the hemispheres of the S^{p-1} in the usual Einstein static universe $\mathbb{R}\times S^{p-1}$ description of the boundary of AdS_{p+1}.

[5]The only exceptions are where the AdS Killing vector has fixed points.

$\{\vec{x}\}$ stand for the embedding coordinates of AdS_{p+1} in $\mathbb{R}^{2,p}$. Since ξ_{AdS} is a Lorentz transformation in $\mathbb{R}^{2,p}$, its action on $\vec{x}$ can be defined by

$$\xi_{\mathrm{AdS}}\vec{x}=B\vec{x}, \tag{4.10}$$

where B is a $(p+2)\times(p+2)$ constant matrix. Thus, $\vec{y}(\psi,\vec{x})=e^{-\psi B}\vec{x}$, so that

$$d\vec{x}=e^{\psi B}(d\vec{y}+B\vec{y}d\psi). \tag{4.11}$$

One can now compute the metric in adapted coordinates $\{\psi,\tilde{\varphi}_i,\vec{y}\}$. This can be written as

$$g=\|\xi_{\mathrm{S}}\|^2(d\psi+B_1)^2+\tilde{g}+g_{\mathrm{AdS}_{p+1}}+2d\psi\cdot\hat{\xi}_{\mathrm{AdS}}+\|\xi_{\mathrm{AdS}}\|^2d\psi^2, \tag{4.12}$$

where the first two terms are just describing the metric on S^q in the adapted coordinate system $\{\psi,\tilde{\varphi}_i\}$ introduced in Sec. II C, and $\hat{\xi}_{\mathrm{AdS}}$ stands for the one-form associated with the Killing vector ξ_{AdS}, that is,

$$\hat{\xi}_{\mathrm{AdS}}=\eta_{ij}\xi^j_{\mathrm{AdS}}dy^i=\eta_{ij}(B\cdot y)^jdy^i. \tag{4.13}$$

After these general considerations, we shall now proceed to discuss the different geometries that appear in these discrete quotients of $\mathrm{AdS}_{p+1}\times\mathrm{S}^q$.

A. Non-everywhere-spacelike ξ_{AdS}

Let us first discuss the three cases in which ξ_{AdS} is not always spacelike. The first of these is where the two-form is $\oplus_i B^{(0,2)}(\varphi_i)$, corresponding to the quotient of AdS_{p+1} by some combination of rotations in orthogonal two-planes $\mathbb{R}^2$ in the embedding space. These quotients produce special cases of the conical defects, which were discussed extensively in, for example [73]. An interesting discussion of the properties of the supersymmetric orbifolds in string theory is also given in [74,75]. We will not discuss this case further here, except to note that it is for these quotients where the existence of a spin structure is not guaranteed. The condition for the existence of a spin structure was stated at the end of Sec. II D.

To consider the other two cases in Table III which are not always spacelike, $B^{(1,2)}\oplus_i B^{(0,2)}(\varphi_i)$ and $B^{(2,2)}_{\pm}\oplus_i B^{(0,2)}(\varphi_i)$, we follow our general strategy, and start by describing the action of $B^{(1,2)}$ or $B^{(2,2)}_{\pm}$ in AdS_3. The action of a more general Killing vector of this form on AdS_{p+1} can then be built up by considering the AdS_3 action deformed by the rotations $B^{(0,2)}(\varphi_i)$ on the S^{p-3} in the $\mathrm{AdS}_3\times\mathrm{S}^{p-3}$ foliation of Eq. (4.2). We will then add in the deformation on a transverse sphere S^q to obtain an everywhere spacelike quotient.

For the quotient of AdS_3 by $B^{(1,2)}$, the relevant Killing vector is

$$\xi_{\mathrm{AdS}}=\boldsymbol{e}_{13}-\boldsymbol{e}_{34}. \tag{4.14}$$

This Killing vector is spacelike almost everywhere, $||\xi_{\mathrm{AdS}}||^2=(x_1+x_4)^2$. There is a single other Killing vector in $\mathfrak{so}(2,2)$ which commutes with this one, $\xi_1=\boldsymbol{e}_{12}-\boldsymbol{e}_{24}$. It has norm $||\xi_1||^2=-(x_1+x_4)^2$. The most convenient coordinate system for studying this quotient is Poincaré coordinates. The form of the Killing vectors in Poincaré coordinates is reviewed in Appendix A. It is easy to see from those expressions that in the case of $B^{(1,2)}$ we can orient the coordinates so that $\xi_{\mathrm{AdS}}=\partial_x$ and $\xi_1=\partial_t$, where the AdS_3 metric in Poincaré coordinates is

$$g_{\mathrm{AdS}_3}=\frac{1}{z^2}(-dt^2+dz^2+dx^2). \tag{4.15}$$

We see that the effect of the quotient is simply to make the coordinate x periodic. The Killing vector ξ_{AdS} becomes null on the Poincaré horizon $z=\infty$ where this coordinate system breaks down. In terms of the embedding coordinates, this is the surface $x_1+x_4=0$, where $\xi_{\mathrm{AdS}}=x_3(\partial_1-\partial_4)$. We note that this symmetry has a null line of fixed points at $x_1+x_4=x_3=0$ (parametrized by x_1-x_4). Away from the fixed points, the identification along ξ_{AdS} will generate closed null curves in the Poincaré horizon. These can be eliminated by deforming this quotient by a suitable action on an odd-dimensional sphere. Since we do not have a good global coordinate system on this quotient, the best way to describe the causally regular deformed quotient will be to use the coordinates adapted to the action on the transverse sphere, as described at the end of the last subsection. We will not give the details of the application of this general technique for this particular case; we just remark that for this case, the matrix B defined in Eq. (4.10) is

$$B=\begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}. \tag{4.16}$$

Following the supersymmetry analysis in [54], it is easy to conclude that for a suitable choice of sphere deformation, the above quotient preserves $\nu=\frac{1}{4}$ of the vacuum supersymmetry, that is, it has four supercharges.

For the case where we introduce a deformation on a transverse S^3, we can interpret the quotient as the near horizon geometry of a D1-D5 system that has been quotiented by the action generated by

$$\xi=\partial_x+\theta_1R_{12}+\theta_2R_{34},$$

in which x stands for the common direction shared by the D1-D5 system, and R_{ij} stand for rotations transverse to the D1-D5's. In the language developed in [64,65], this asymptotically flat spacetime would correspond to a D1-D5 system in a generic intersection of flux seven-brane vacuum. Whenever $\theta_1=\pm\theta_2$, it would be interpreted as a D1-D5 system in the flux five-brane vacuum, which also has four supercharges. Note that the standard supersymmetry enhancement due to the near horizon limit is lost in this quotient, as the generator ∂_x, which does not break any supersymmetry in the asymptotically flat spacetime construction, becomes a null rotation generator from the AdS perspective, which breaks one-half of the supersymmetry.

We would also like to understand the boundary of this quotient. In the Poincaré coordinates (4.15), the global AdS boundary is written in terms of an infinite series of flat space patches,

$$g_\partial=-dt^2+dx^2. \tag{4.17}$$

The action of the Killing vector on the AdS boundary compactifies the spatial coordinate x; it might therefore seem that the quotient will have an infinite sequence of boundaries. However, the Killing vector only has isolated fixed points on the boundary, at the points where the line of fixed points $x_1+x_4=x_3=0$ meets the boundary. In Poincaré coordinates, these correspond to the points at past and future timelike infinity and at spacelike infinity. The different boundary patches are therefore connected. We can extend the Poincaré coordinates to cover more of the boundary by defining

$$v=t-x, \quad \tan T=t. \tag{4.18}$$

The boundary metric then becomes

$$g_\partial=\frac{1}{\cos^2 T}(-2dv\,dT+\cos^2 T dv^2), \tag{4.19}$$

and the Killing vector we quotient along is $\xi_{\mathrm{AdS}}=\partial_v$. Since we have only a conformal structure on the boundary, we can ignore the overall factor in this metric. In the resulting metric, we see that the direction we quotient along is spacelike except when $T=(n+1/2)\pi$, where it becomes null. These points correspond to one-half of future and past null infinity in the original Poincaré coordinates. This coordinate system covers the whole of the conformal boundary with the exception of a null line corresponding to one-half of past and future null infinity in each Poincaré patch. We could construct a similar coordinate system by defining $u=t+x$—it would then cover that half but not the one where $t-x$ remains finite. We can think of the field theory dual to the quotient along a null rotation as living on the cylindrical space described in Eq. (4.19), which has closed null curves at $T=(n+1/2)\pi$.[6] Since the deformation by an action on a transverse sphere does not alter the action on the boundary, it cannot remove these closed lightlike curves in the dual theory.

A more interesting example of a not everywhere spacelike quotient is $B_\pm^{(2,2)}$, where the Killing vector we quotient along is

$$\xi_{\mathrm{AdS}}^{\pm}=\pm(\boldsymbol{e}_{12}-\boldsymbol{e}_{24})+(\boldsymbol{e}_{13}-\boldsymbol{e}_{34}), \tag{4.20}$$

respectively. Both are null everywhere, $\|\xi_{\mathrm{AdS}}^{\pm}\|^2=0$. From now on, we shall focus on ξ_{AdS}^{+}; there is an analogous discussion and structure for ξ_{AdS}^{-}. There are three other Killing vectors in $\mathfrak{so}(2,2)$ commuting with ξ_{AdS}^{+},

$$\xi_1=\boldsymbol{e}_{24}+\boldsymbol{e}_{13}, \quad \xi_2=\boldsymbol{e}_{12}+\boldsymbol{e}_{34}, \quad \xi_3=\boldsymbol{e}_{14}-\boldsymbol{e}_{23}. \tag{4.21}$$

These satisfy

$$[\xi_i,\xi_j]=2\epsilon_{ijk}\xi_k, \tag{4.22}$$

so they define an $\mathfrak{sl}(2,\mathbb{R})$ symmetry which commutes with ξ_{AdS}^{+}. This $\mathfrak{sl}(2,\mathbb{R})$ structure appears because when we write $\mathfrak{so}(2,2)=\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$, the $B_\pm^{(2,2)}$ Killing vector lies entirely in one of the $\mathfrak{sl}(2,\mathbb{R})$ factors. A similar structure will reappear for the same reason in our discussion of the self-dual orbifold in Sec. IV B; it was first identified in that context in [41].

We would like to adopt a coordinate system adapted to this symmetry. Since the ξ_i do not commute, we can adapt our coordinates to only one of them. We note that $\|\xi_1\|^2=\|\xi_3\|^2=1$, $\|\xi_2\|^2=-1$. Since our interest is in causal structure, it seems natural to adapt the coordinates to the timelike vector ξ_2. We therefore want to construct a coordinate system (t,v,ρ) on AdS_3 such that $\xi_{\mathrm{AdS}}^{+}=\partial_v$ and $\xi_2=\partial_t$. This requires

$$\begin{aligned}
&\frac{\partial(x^4-x^1)}{\partial v}=0, \quad \frac{\partial(x^4+x^1)}{\partial v}=-2(x^3-x^2),\\
&\frac{\partial(x^3-x^2)}{\partial v}=0, \quad \frac{\partial(x^3+x^2)}{\partial v}=2(x^4-x^1),\\
&\frac{\partial(x^4-x^1)}{\partial t}=x^3-x^2, \quad \frac{\partial(x^4+x^1)}{\partial t}=x^3+x^2,\\
&\frac{\partial(x^3-x^2)}{\partial t}=-(x^4-x^1), \quad \frac{\partial(x^3+x^2)}{\partial t}=-(x^4+x^1).
\end{aligned} \tag{4.23}$$

A combination which is thus independent of t,v is $(x^4-x^1)^2+(x^3-x^2)^2$. We will choose the ρ coordinate so that this combination is $e^{2\rho}$. A suitable coordinate system satisfying these criteria and the condition $-x_1^2-x_2^2+x_3^2+x_4^2=-1$ defining the AdS_3 embedding is

$$\begin{aligned}
&x^4-x^1=e^{\rho}\sin t,\\
&x^4+x^1=-e^{-\rho}\sin t-2ve^{\rho}\cos t,\\
&x^3-x^2=e^{\rho}\cos t,\\
&x^3+x^2=-e^{-\rho}\cos t+2ve^{\rho}\sin t.
\end{aligned} \tag{4.24}$$

The inverse coordinate transformation is given by

[6] There are some obvious similarities between this construction and the Milne coordinate system on the orbifold of flat space by a boost.

$$e^{2\rho}=(x^4-x^1)^2+(x^3-x^2)^2,$$

$$\tan t=\frac{x^4-x^1}{x^3-x^2},$$

$$v=e^{-2\rho}\{[(x^3+x^2)+e^{-2\rho}(x^3-x^2)]^2 + [(x^4+x^1)+e^{-2\rho}(x^4-x^1)]^2\}. \tag{4.25}$$

Since these give finite values of t,v,ρ for all points in AdS_3, this coordinate system covers the whole spacetime. In terms of these coordinates, the metric is

$$g_{AdS_3}=-dt^2+d\rho^2-2e^{2\rho}dv\,dt. \tag{4.26}$$

In this coordinate system, the other two Killing vectors are

$$\xi_1=\sin 2t\,\partial_\rho+\cos 2t(\partial_t-e^{-2\rho}\partial_v),$$

$$\xi_3=-\cos 2t\,\partial_\rho+\sin 2t(\partial_t-e^{-2\rho}\partial_v). \tag{4.27}$$

We see that making identifications along the Killing vector ∂_v will produce closed null curves. To eliminate these closed null curves, we should introduce a deformation by a rotation on the transverse sphere. To simplify the discussion, we shall work it out explicitly for a transverse S^3, having in mind the standard way of embedding AdS_3 in type IIB string theory, as the near horizon geometry of the D1-D5 system, giving rise to $AdS_3\times S^3\times\mathbb{T}^4$. As discussed in Sec. II C, there are several inequivalent quotients that one can take of S^3. We will focus on a particular quotient which preserves supersymmetry, namely, the quotient where $\xi_S=\partial_\psi$ when we write the S^3 metric as

$$g_{S^3}=d\theta^2+d\psi^2+d\varphi^2+2\cos 2\theta d\psi\cdot d\varphi. \tag{4.28}$$

Thus, we consider the quotient along a total Killing vector $\xi=\xi_{AdS}+\gamma\xi_S=\partial_v+\gamma\partial_\psi$. Since we have a globally adapted coordinate system (4.26) on the AdS part of the quotient, it is convenient to construct the global coordinate system on the full $AdS_3\times S^3$ quotient by defining $\psi'=\psi-\gamma v$. The six-dimensional metric is then

$$g=-dt^2+d\rho^2-2e^{2\rho}dv\,dt+d\theta^2+(d\psi'+\gamma dv)^2+d\varphi^2 +2\cos 2\theta(d\psi'+\gamma dv)\cdot d\varphi. \tag{4.29}$$

The quotient is now along $\xi=\partial_v$. We can see that this is an everywhere spacelike direction; $\|\xi\|^2=\gamma^2$. This is a necessary but not a sufficient condition for the absence of closed causal curves, but it is easy to check explicitly that there are no closed causal curves in the bulk of the quotient manifold in this case. As shown in [54], the corresponding type IIB configuration preserves $\nu=\frac{1}{8}$ of the vacuum supersymmetry, that is, it has four supercharges. It is interesting to point out that if we had considered the action on the three-sphere (4.28) generated by $\xi_S=\partial_\varphi$, the corresponding quotient $\xi=\xi_{AdS}+\gamma\xi_S$ would have preserved $\nu=\frac{1}{4}$ of the full type IIB supersymmetry.

It is interesting to note that, like the null rotation, the $B_\pm^{(2,2)}$ Killing vector also has a simple action in Poincaré coordinates. We can orient the coordinates so that $\xi_{AdS}=\partial_t+\partial_x$ in the metric (4.15). The additional symmetry $\partial_t-\partial_x$ that is manifest in these coordinates can be written in terms of the $\mathfrak{sl}(2,\mathbb{R})$ Killing vectors (4.21) as the combination $\xi_2-\xi_1$. Although the Poincaré coordinates are not a global coordinate system for the quotient, they allow us to relate these quotients and quotients of branes in asymptotically flat spacetimes: the $B_\pm^{(2,2)}$ quotients can be understood as the near horizon geometries of a D1-D5 system quotiented by the discrete action generated by

$$\xi=\pm\partial_t+\partial_x+\theta_1R_{12}+\theta_2R_{34}. \tag{4.30}$$

The physical interpretation of these quotients is unclear. They can be supersymmetric, and they are free from closed causal curves. It might be possible to give them some interpretation using a limiting procedure in which one finally identifies bulk points along a "null translation," by infinitely boosting a spacelike translation. In this case, there is still a supersymmetry enhancement since the asymptotically flat quotient has four supercharges.

To discuss the conformal boundary of this quotient, we will use a technique that will be used again in Sec. IV C, and relate the spacetime to a plane wave. If we set $r=e^{-\rho}$, the metric (4.29) becomes

$$g=\frac{1}{r^2}\{-2dv\,dt-r^2dt^2+dr^2+r^2[d\theta^2+(d\psi'+\gamma dv)^2 +d\varphi^2+2\cos 2\theta(d\psi'+\gamma dv)\cdot d\varphi]\}. \tag{4.31}$$

The conformally related metric in curly brackets is a symmetric six-dimensional plane wave, written in a polar coordinate system deformed so that ∂_v is a mixture of the null translation symmetry of the plane wave and a rotation in the four transverse spacelike coordinates.

The conformal mapping between an $AdS_3\times S^3$ space and a plane wave is implicit in previous work [76], which showed that such plane waves can be conformally mapped onto the Einstein static universe. That is, since both spaces are conformally flat, we would expect them to be conformally related. It is interesting to note the relative simplicity of the relation: $AdS_3\times S^3$ corresponds to the plane wave with the axis $r=0$ excluded, rescaled by a factor of $1/r^2$.

More important for our present purpose is that the Killing vector we wish to quotient along, ∂_v, annihilates the conformal factor (as does $\xi_2=\partial_t$), so we can use this conformal map to study the boundary of the quotient spacetime, and not just to study global $AdS_3\times S^3$. Note that, unlike the double null rotation in Sec. IV C, the other Killing symmetries ξ_1 and ξ_2 of this quotient do not also commute with the conformal rescaling. They will hence appear as conformal isometries in the boundary theory.

The conformal boundary of the quotient (4.31) lies at $r=0$, and has the metric (up to conformal transformations)

$$g_\partial=-2dv\,dt. \tag{4.32}$$

Since v is periodically identified in the quotient, there is a compact null direction through every point in the boundary. As in the null rotation case, these closed null curves in the conformal boundary cannot be removed by a sphere deformation. This fact can explicitly be checked in Eq. (4.29). It is interesting to note that we get the same metric on the conformal boundary here as on either of the two boundaries in the self-dual orbifold discussed in the next subsection.

If we regard Eq. (4.29) simply as a coordinate system on $AdS_3\times S^3$, we can relate this description of the conformal boundary to the usual two-dimensional $\mathbb{R}\times S^1$ Einstein static universe boundary of global $AdS_3\times S^3$. In global coordinates, the Killing vector field is given by

$$\xi=[1+\cos(\tau-\varphi)](\partial_\tau-\partial_\varphi), \tag{4.33}$$

where we are using the global coordinates introduced in Appendix A, and further writing $\hat{x}_3=\cos\varphi$, $\hat{x}_4=\sin\varphi$, so that the metric on the boundary reads

$$g_\partial=-d\tau^2+d\varphi^2. \tag{4.34}$$

We see that the quotient is along a null direction, and has a single null line of fixed points at $\tau-\varphi=\pi \pmod{2\pi}$. While the coordinate system (4.29) covers all of global $AdS_3\times S^3$, it does not cover all of its conformal boundary, as these symmetry-adapted coordinates break down on the fixed points of ξ_{AdS}. The coordinates of Eq. (4.29) cover all of the boundary apart from this null line. They are related to the global description above in the same way that a symmetric plane wave is related to the Einstein static universe in higher-dimensional cases [76] (in two dimensions, there is no nontrivial plane wave). Thus we see that Eq. (4.32) provides a natural description of the asymptotic boundary of the quotient, corresponding to excluding these fixed points in discussing the quotient.

While it is clear that the deformed quotient (4.29) is free of closed causal curves, we can show that this quotient does not preserve the stable causality of the original $AdS_3\times S^3$ space. If we write Eq. (4.29) in the form appropriate for Kaluza-Klein reduction along v,

$$\begin{aligned}g=&-(1+\gamma^{-2}e^{4\rho})dt^2+d\rho^2+d\theta^2+\sin^2 2\theta d\varphi^2\\&+2\gamma^{-1}e^{2\rho}dt(d\psi'+\cos 2\theta d\varphi)\\&+(\gamma dv+d\psi'+\cos 2\theta d\varphi-\gamma^{-1}e^{2\rho}dt)^2,\end{aligned} \tag{4.35}$$

we see that the lower-dimensional metric obtained by Kaluza-Klein reduction along v will have closed null curves, since the compact circle parametrized by ψ' is null. This implies that there can be no time function τ on $AdS_3\times S^3$ such that $\mathcal{L}_\xi\tau=0$, for if there was, the Kaluza-Klein reduced metric would be stably causal, which is inconsistent with the appearance of closed null curves in the latter. Thus, the discrete quotient cannot satisfy the condition of [70], and does not preserve stable causality.

Following the discussion around Eq. (4.2), it is straightforward to describe the quotient generated by ξ^+_{AdS} in higher-dimensional AdS_{p+1} spaces. By construction, the global symmetries of such a higher-dimensional quotient will be the ones discussed before times $SO(p-2)$, corresponding to the rotational symmetry transverse to the subspace where ξ^+_{AdS} acts. Notice that in this case, the metric on the boundary is conformally equivalent to a plane wave metric,

$$g_\partial=-2dv\,dt-r^2dt^2+dr^2+r^2g_{S^{p-3}}. \tag{4.36}$$

In higher dimensions, there exists the possibility to deform the quotient by rotations, i.e., $\oplus_i B^{(0,2)}(\varphi_i)$. Let us focus on AdS_5, for algebraic simplicity. The metric for AdS_5 in the AdS_3 foliation adapted to the action of ξ^+_{AdS} is given by

$$g_{AdS_5}=\cosh^2\chi(-dt^2+d\rho^2-2e^{2\rho}dv\,dt)+d\chi^2+\sinh^2\chi d\theta^2. \tag{4.37}$$

The deformation consists in acting on the angular direction θ through the generator $\xi=\varphi\partial_\theta$. Thus, it is convenient to introduce the coordinate $\theta'=\theta-\varphi v$, so that $\xi^+_{AdS}+\xi=\partial_v$. The metric on the deformed quotient is

$$\begin{aligned}g_{AdS_5/\Gamma}=&\cosh^2\chi(-dt^2+d\rho^2-2e^{2\rho}dv\,dt)+d\chi^2\\&+\sinh^2\chi(d\theta'+\varphi dv)^2,\end{aligned} \tag{4.38}$$

where, once again, $v\sim v+2\pi$. As expected, the periodic coordinate v becomes everywhere spacelike except at the fixed point of the deformed action. This is just a consequence of the fact that the norm of the deformed Killing vector is $\|\xi^+_{AdS}+\xi\|^2=\varphi^2[(x^5)^2+(x^6)^2]=\varphi^2\sinh^2\chi$, which certainly vanishes at the origin of the 56-plane, where the fixed point of ξ lies.

This particular deformation ($\varphi\neq 0$) breaks all the supersymmetry and it can be interpreted as the near horizon geometry of a bunch of parallel and coincident D3-branes quotiented by the action of a null translation plus a rotation. It is certainly possible to turn on supersymmetric deformations in higher-dimensional AdS spacetimes. In particular, it is possible to consider families of two-parameter deformations corresponding to $B^{(0,2)}(\varphi_1)\oplus B^{(0,2)}(\varphi_2)$ in AdS_7. Whenever $\varphi_1=\pm\varphi_2$, the quotient will preserve supersymmetry. The corresponding asymptotically flat interpretation would be in terms of parallel and coincident M5-branes quotiented by the action of a null translation plus a certain rotation in $\mathbb{R}^4$. The supersymmetric deformation would correspond to the action having an $\mathfrak{su}(2)$ holonomy.

B. Self-dual orbifolds and their deformations

The fifth two-form appearing in Table III, $B^{(1,1)}(\beta_1)\oplus B^{(1,1)}(\beta_2)\oplus_i B^{(0,2)}(\varphi_i)$ with $|\beta_1|=|\beta_2|$, can be interpreted as the deformation of the self-dual orbifolds of AdS_3, first introduced in [41], and recently discussed in [43]. The norm of ξ_{AdS} is spacelike everywhere. Therefore, one can study these geometries with or without any further nontrivial action on transverse spheres.

As already indicated above, the minimal dimension where this discrete quotient exists is for $p=2$, i.e., AdS_3. The addition of any rotation parameter φ_i would increase this dimension by 2. Since the elementary nondecomposable block acting on AdS_3 is a linear combination of boosts in $\mathbb{R}^{2,2}$, this discrete quotient does not have an analogue in an asymptotically flat spacetime, in the sense that there is no quotient whose near horizon limit gives rise to these self-dual orbifolds.

The anti–de Sitter action, including the deformation parameters $\{\varphi_1\}$, integrates to the following $\mathbb{R}$ action on $\mathbb{R}^{2,p}$:

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^{2i+5} \\ x^{2i+6} \end{pmatrix} \mapsto \begin{pmatrix} x^1 \cosh\beta t \pm x^3 \sinh\beta t \\ x^2 \cosh\beta t + \pm x^4 \sinh\beta t \\ x^3 \cosh\beta t \pm x^1 \sinh\beta t \\ x^4 \cosh\beta t \pm x^2 \sinh\beta t \\ x^{2i+5} \cos\varphi_i t - x^{2i+6} \sin\varphi_i t \\ x^{2i+6} \cos\varphi_i t + x^{2i+5} \sin\varphi_i t \end{pmatrix}, \quad \forall\ i, \tag{4.39}$$

where we set $\beta_1 = \beta$ and $\beta_2 = \pm\beta$. Notice that the above action is manifestly free of fixed points for any value of the boost and rotation parameters $\{\beta, \varphi_i\}$.

In the following, we shall review the main features of the self-dual orbifolds of AdS_3, extending the discussion to uncover their embeddings in higher-dimensional anti–de Sitter spacetimes and their deformations both by rotations in anti–de Sitter and nontrivial actions on transverse spheres, afterward.

1. Pure AdS

Let us start our discussion by focusing on AdS_3, so that there are no $B^{(0,2)}(\varphi_i)$ blocks. In this case, as first described in [41], the quotient preserves an $\mathbb{R}\times\mathfrak{sl}(2,\mathbb{R})$ subalgebra of the original $\mathfrak{so}(2,2) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ isometry algebra. A suitable system of global coordinates adapted to the quotient and the timelike vector in $\mathfrak{sl}(2,\mathbb{R})$ is [41]

$$\begin{aligned} x^1 &= \cosh z \cosh\beta\phi \cos t - \sinh z \sinh\beta\phi \sin t, \\ x^2 &= \cosh z \cosh\beta\phi \sin t + \sinh z \sinh\beta\phi \cos t, \\ x^3 &= -\cosh z \sinh\beta\phi \cos t + \sinh z \cosh\beta\phi \sin t, \\ x^4 &= \pm(\cosh z \sinh\beta\phi \sin t - \sinh z \cosh\beta\phi \cos t). \end{aligned} \tag{4.40}$$

The sign ambiguity in the last line of Eq. (4.40) corresponds to the two distinct cases $\beta_2 = \pm\beta_1$ in the $SO(2,n)$ classification reviewed in Sec. II B. This illustrates explicitly that these two cases are related by an orientation-reversing symmetry of AdS_3, namely, the reflection $x_4 \to -x_4$. It is important to stress that, at this point, the coordinates $\{t,\phi,z\}$ are just some particular global description for AdS_3. All of them are defined in the range $-\infty < t,\phi,z < +\infty$. It is only when we identify points in AdS_3 along some discrete step generated by $\xi_{AdS} = \partial_\phi$ that our discrete quotients will differ from AdS_3 globally, by making the adapted coordinate ϕ a compact variable with period 2π in some normalization, i.e., $\phi \sim \phi + 2\pi$.

As first proved in [41] for AdS_3, corroborated in [43], and extended to any higher-dimensional AdS spacetime in [54], the supersymmetry preserved by these self-dual orbifolds is one-half of the original one.

The metric in adapted coordinates (4.40) looks like

$$g_{sd} = -dt^2 + \beta^2 d\phi^2 + dz^2 - 2\beta \sinh 2z\, dt\, d\phi. \tag{4.41}$$

Thus, it describes a nonstatic but stationary spacetime. One interesting feature that has not previously been noted is that t is a global time function, since $\nabla_\mu t \nabla^\mu t = -1/\cosh^2 2z$, so the self-dual orbifolds are stably causal and hence do not contain closed timelike curves. This metric can be interpreted as an S^1 fibration over AdS_2, as the following rewriting indicates:

$$g_{sd} = -\cosh^2 2z\, dt^2 + dz^2 + (\beta d\phi - \sinh 2z\, dt)^2. \tag{4.42}$$

This quotient was recently analyzed in detail in [43], where its isometries, geodesics, asymptotic structure, and holography in this background were extensively studied.

An important point to note from that analysis is the structure of the conformal boundaries. It was shown in [43] that the quotient has two disconnected conformal boundaries. If we consider the coordinate transformation

$$\sinh z = \tan\theta, \qquad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

the metric (4.41) becomes

$$g_{sd} = \frac{1}{\cos^2\theta}[\cos^2\theta(-dt^2 + \beta^2 d\phi^2) + d\theta^2 - 4\beta \sin\theta\, dt\, d\phi], \tag{4.43}$$

from which we learn that the metric on both conformal boundaries, located at $\theta \to \pm\pi/2$, is given by

$$g_\partial = \pm dt\, d\phi. \tag{4.44}$$

Thus, there are closed lightlike curves on the conformal boundary. The appearance of two disconnected boundaries can be further understood by noting that in the adapted coordinates (4.40) the original AdS_3 conformal boundary is covered by four connected patches located at $z \to \pm\infty$ and $\phi \to \pm\infty$. After the discrete identification, two of these patches no longer belong to our space, leaving as a consequence the existence of two boundaries at $z \to \pm\infty$ that are disconnected. These boundaries are causally connected through the bulk, as was shown in [43] by analyzing the geodesics in this space.

Unlike the previous cases, this quotient has no natural interpretation as arising from a quotient of an asymptotically flat spacetime. This is related to the fact that the quotient does not take a simple form in Poincaré coordinates. However, Strominger [77] showed that these self-dual orbifolds emerge as the local description of a very near horizon geometry when focusing on the vicinity of the horizon of an extremal BTZ black hole.

Thus, even though this quotient does not emerge directly from the D1-D5 perspective, it is nevertheless possible to set up an asymptotically flat spacetime which reproduces the self-dual orbifolds in two steps [43]. This is achieved by adding some momentum along the common direction shared by the D1's and D5's, and taking the standard near horizon limit, keeping the momentum density fixed. One then focuses on the vicinity of the horizon resulting from the previous limit. This procedure generalizes the construction in [78] to the D1-D5 system, and it provides an independent way of understanding the discrete light-cone quantization (DLCQ) holography proposed in [43].

Following our general discussion presented at the beginning of Sec. IV, it is straightforward to extend the analysis to higher-dimensional AdS_{p+1} spaces, for $p\geqslant 3$. Indeed, we can use the foliation in Eq. (4.1) and replace the $\{\hat{x}_i\}$ appearing there with the $R=1$ version of Eq. (4.40). The resulting metric is

$$g_{sd_{p+1}}=(\cosh\chi)^2 g_{sd}+(d\chi)^2+(\sinh\chi)^2 g_{\mathrm{S}^{p-3}}, \tag{4.45}$$

where g_{sd} is the metric given in (4.41).

This allows us to see that in these higher-dimensional cases the boundary of the quotient will be connected. The point is that the boundary of the quotient in higher dimensions is given in these coordinates by $\chi\to\infty$, as discussed earlier. Thus, the boundary of the higher-dimensional quotients naturally contains a copy of the bulk of the AdS_3 quotient. Since the AdS_3 quotient is connected, this implies that the boundary of the quotient is connected in higher dimensions. It also shows us that, unlike the AdS_3 case, in higher dimensions there is a natural nondegenerate metric on the boundary of the quotient.

2. Deformation by $B^{(0,2)}$

Even though we could discuss the turning on of the deformation parameters φ_i in the general case, we shall just briefly mention their main features in the string theory embeddings described above. This means that we shall concentrate on AdS_5 and AdS_7, since these deformations are not available for AdS_4.

This program is particularly simple to carry on already in the foliation defined by Eq. (4.1). As previously mentioned, $B^{(0,2)}(\varphi_i)$ blocks correspond to rotations in $\mathbb{R}^2$ planes in the embedding space, and in the coordinates of Eq. (4.2), these motions can be globally described as a single "translation" along one of the angular variables of the S^{n-1} factor. The definition of the adapted coordinate system in which $\oplus_i B^{(0,2)}(\varphi_i)$ takes the form of a single "translation" is precisely parallel to the discussion for the transverse S^q given in Sec. II C.

As an example, consider AdS_5. In this case, we can turn on only one parameter $\varphi_1=\varphi$. It is clear that rotations in $\mathbb{R}^2$ correspond to motions along the S^1 transverse to the AdS_3 foliation of AdS_5 in Eq. (4.2), for $p-n=2$. If we parametrize this circle by θ, the Killing vector field ξ_{AdS} generating the full action of the deformed discrete quotient is given by

$$\xi_{\mathrm{AdS}}=\partial_\phi+\varphi\partial_\theta, \tag{4.46}$$

in the adapted coordinates defined by Eqs. (4.1) and (4.40).

It is now just a matter of applying a linear transformation in the $\{\phi,\theta\}$ plane, which will generate an extra fibration, to rewrite the metric in a globally defined coordinate system adapted to the deformed Killing vector field ξ_{AdS}. This metric is given by

$$g=\cosh^2\chi g_{sd}+d\chi^2+\sinh^2\chi(d\theta+\varphi d\phi)^2. \tag{4.47}$$

By construction, this deformation will break all the spacetime supersymmetry.

The techniques for AdS_7 are exactly the same, but there is a richer structure of possibilities since we have an S^3 transverse to the AdS_3 action, which allows us to turn on two inequivalent parameters $\{\varphi_1,\varphi_2\}$

$$\varphi_1 R_{12}+\varphi_2 R_{34},$$

where R_{ij} stands for a rotation generator in the ij plane belonging to $\mathbb{R}^4$, where the three-sphere is embedded as a quadric. Let us describe this three-sphere in terms of standard complex coordinates

$$z_1=x^1+ix^2=\cos\theta e^{i(\psi+\varphi)},$$
$$z_2=x^3+ix^3=\sin\theta e^{i(\psi-\varphi)}. \tag{4.48}$$

A supersymmetric quotient [54] is given by the choice $\varphi_1=-\varphi_2=\theta_1$. The metric describing the global quotient is given by

$$g_{\mathrm{AdS}_7/\Gamma}=\cosh^2\chi\, g_{sd}+d\chi^2+\sinh^2\chi[d\theta^2+(d\varphi+\theta_1\phi)^2 + d\psi^2+2\cos 2\theta(d\varphi+\theta_1 d)\cdot d\psi]. \tag{4.49}$$

Adding a transverse four-sphere and a constant flux on it, the above configuration is supersymmetric. It actually preserves $\nu=\frac{1}{2}$ of the supersymmetries preserved by the original vacuum. Thus, it has 16 supercharges. It is worth mentioning that the deformation described by $\varphi_1=-\varphi_2$ does not break any further supersymmetry. It is a further action that we can consider in our spacetime for free, supersymmetrywise. Contrary to what intuition may suggest, as explained in more detail in [54], the deformation $\varphi_1=\varphi_2$ breaks all the supersymmetry.

3. Sphere deformations

Let us start our discussion on sphere deformations of self-dual orbifolds on the embedding of $\mathrm{AdS}_3\times\mathrm{S}^3$ in type IIB supergravity. The most general action that we can write down on S^3 is given in terms of two real parameters

$$\xi_{\mathrm{S}}=\theta_1 R_{12}+\theta_2 R_{34}. \tag{4.50}$$

Because of the freedom that we have to quotient by the action of the Weyl group, we can always choose to work on the fundamental region defined by $\theta_1\geqslant|\theta_2|$.

Among all these quotients, only a subset preserve supersymmetry. In particular, if we consider the action generated by $e_{13}\pm e_{24}$ on AdS_3, the only supersymmetric deformations are given by $\theta_1=\pm\theta_2$, the signs being correlated. Interestingly, such deformations still preserve the same amount of supersymmetry as the self-dual orbifolds themselves. Thus, these supersymmetric deformations are for free, as pointed out in [54], where the reader can also find the explanation for this phenomenon.

The discussion proceeds in an analogous way for higher-dimensional AdS spacetimes. If we consider the 11-dimensional configuration $AdS_4\times S^7$, their deformations are characterized by four real numbers

$$\xi_S=\theta_1 R_{12}+\theta_2 R_{34}+\theta_3 R_{56}+\theta_4 R_{78}. \tag{4.51}$$

Due to the Weyl group action, we can restrict ourselves to the region defined by $\theta_1\geqslant\theta_2\geqslant\theta_3\geqslant|\theta_4|$. As discussed in [54], there are several loci in this parameter space where supersymmetry is allowed. If $\theta_1=\theta_2$ and $\theta_3=-\theta_4$ the quotient preserves $\nu=\frac{1}{4}$. Whenever one of the relations

$$\theta_1-\theta_2+\theta_3+\theta_4=0,$$

$$\theta_1+\theta_2-\theta_3+\theta_4=0,$$

$$\theta_1-\theta_2-\theta_3-\theta_4=0$$

is satisfied, the supersymmetry will be $\nu=\frac{1}{8}$. Finally, there is enhancement whenever $\theta_1=\theta_2=\theta_3=-\theta_4$, giving rise to $\nu=\frac{3}{8}$.

The discussion for $AdS_5\times S^5$ is fairly simple. The action on the five-sphere is given in terms of three real parameters

$$\xi_S=\theta_1 R_{12}+\theta_2 R_{34}+\theta_3 R_{56}. \tag{4.52}$$

The deformation preserves $\nu=\frac{1}{4}$ for $\theta_1=\theta_2$ and $\theta_3=0$. It preserves $\nu=\frac{1}{8}$ if $\theta_1\pm\theta_2\pm\theta_3=0$, with uncorrelated signs. See [54] for more details.

The only supersymmetric deformation for $AdS_7\times S^4$ out of the two-parameter family

$$\xi_S=\theta_1 R_{12}+\theta_2 R_{34} \tag{4.53}$$

is given by $\theta_1=\theta_2$, also preserving $\nu=\frac{1}{4}$.

As an explicit example of a supersymmetric deformation of the self-dual orbifold, we shall present one particular example of the above discussion, one embedded in $AdS_5\times S^5$. More precisely, we shall focus on $\theta_1=2,\ \theta_2=\theta_3=1$. A simple description of this quotient can be obtained by parametrizing the five-sphere in terms of the coordinates

$$z_1=x^1+ix^2=\cos\theta_1 e^{i(\varphi_1+2\psi)},$$

$$z_2=x^3+ix^4=\sin\theta_1\cos\theta_2 e^{i(\psi+\varphi)},$$

$$z_3=x^5+ix^6=\sin\theta_1\sin\theta_2 e^{i(\psi-\varphi)}. \tag{4.54}$$

One can check that $\xi_S=\partial_\psi$. This is an example in which both ξ_{AdS} and ξ_S are described in terms of adapted coordinates. Thus, by a simple linear transformation, we can easily write the fully adapted ten-dimensional metric as

$$\begin{aligned} g=&\cosh^2\chi g_{sd}+d\chi^2+\sinh^2\chi d\theta^2+d\theta_1^2+\sin^2\theta_1 d\theta_2^2\\ &+\cos^2\theta_1[d\varphi_1+2(d\psi+d\phi)]^2+\sin^2\theta_1[(d\psi+d\phi)^2\\ &+d\varphi^2+2\cos 2\theta_2(d\psi+d\phi)\cdot\varphi]. \end{aligned} \tag{4.55}$$

As can be checked from the review of the results in [54] presented at the beginning of this subsection, this particular example preserves $\nu=\frac{1}{8}$ of the vacuum supersymmetry. Thus, it has four supercharges.

Of course, there is no conceptual difficulty in dealing with deformations that contain both two forms $\oplus_i B^{(0,2)}(\varphi_i)$ on AdS and nontrivial sphere actions. The supersymmetric quotients can also be found in [54].

C. Double null rotation and its deformations

The third two-form appearing in Table III, $B^{(1,2)}\oplus B^{(1,2)}\oplus_i B^{(0,2)}(\varphi_i)$, can be interpreted as a deformation, with deformation parameters $\{\varphi_i\}$, of the double null rotation discrete quotient considered in [42]. Indeed, it consists of the simultaneous action of two spacelike null rotations in transverse $\mathbb{R}^{1,2}$ subspaces, and a set of rotations with parameters φ_i in different transverse $\mathbb{R}^2$ planes. Since the norm of ξ_{AdS} is positive everywhere, even for $\varphi_i=0\ \forall\ i$, there is no need to deform the previous action by a nontrivial one on a transverse sphere to get an everywhere spacelike Killing vector field ξ in Eq. (3.1).

The minimal dimension where such an object exists is for $p=4$, i.e., AdS_5, in which case there are no $B^{(0,2)}(\varphi_i)$ blocks. The pure double null rotation discrete quotient has a very natural interpretation in the Poincaré patch: it consists of the combined action of a null rotation plus a spacelike translation. Consequently, it has a very straightforward origin in terms of the geometry of a bunch of parallel D3-branes: the pure double null rotation discrete quotient in AdS_5 is the near horizon geometry corresponding to a bunch of parallel D3-branes whose worldvolume is the nullbrane, i.e., $\mathbb{R}^{1,3}/\mathbb{Z}$, four-dimensional Minkowski spacetime modded out by the simultaneous discrete action of a null rotation in $\mathbb{R}^{1,2}$ and a spacelike translation along $\mathbb{R}$, which was first introduced in [20].

The full anti–de Sitter action, including the deformation parameters, integrates to the following $\mathbb{R}$ action on $\mathbb{R}^{2,p}$:

$$\begin{pmatrix} x^1\\ x^2\\ x^3\\ x^4\\ x^5\\ x^6\\ x^{2i+5}\\ x^{2i+6}\end{pmatrix}\mapsto\begin{pmatrix} x^1-tx^3+\frac{1}{2}t^2(x^1-x^4)\\ x^2-tx^5+\frac{1}{2}t^2(x^2-x^6)\\ x^3+t(x^4-x^1)\\ x^4-tx^3+\frac{1}{2}t^2(x^1-x^4)\\ x^5+t(x^6-x^2)\\ x^6-tx^5+\frac{1}{2}t^2(x^2-x^6)\\ x^{2i+5}\cos\varphi_i t-x^{2i+6}\sin\varphi_i t\\ x^{2i+6}\cos\varphi_i t+x^{2i+5}\sin\varphi_i t\end{pmatrix},\quad\forall\ i \tag{4.56}$$

which is manifestly free of fixed points for any value of the rotation parameters.

1. Pure AdS

Let us first consider the pure double null rotation in AdS_5. This was analyzed in [42]. We will extend this analysis by discussing the isometries preserved by the quotient, constructing suitable adapted coordinate systems, and examining the action on the boundary of AdS. In the process, we will uncover interesting relations to compactified plane waves.

The Killing vector that we quotient along is

$$\xi_{\mathrm{AdS}}=\boldsymbol{e}_{13}-\boldsymbol{e}_{34}+\boldsymbol{e}_{25}-\boldsymbol{e}_{56}. \tag{4.57}$$

Its norm is $||\xi_{\mathrm{AdS}}||^2=(x_1+x_4)^2+(x_2+x_6)^2$. This is clearly positive semidefinite, and the quadric $-(x_1+x_4)(x_1-x_4)-(x_2+x_6)(x_2-x_6)+x_3^2+x_5^2=-1$ defining the AdS embedding constrains the coordinates so that it is positive definite. There are four linearly independent commuting isometries in $\mathfrak{so}(2,4)$:

$$\begin{aligned}
\xi_1&=\boldsymbol{e}_{13}-\boldsymbol{e}_{34}-\boldsymbol{e}_{25}+\boldsymbol{e}_{56},\\
\xi_2&=\boldsymbol{e}_{15}+\boldsymbol{e}_{23}-\boldsymbol{e}_{36}+\boldsymbol{e}_{45},\\
\xi_3&=\boldsymbol{e}_{12}-\boldsymbol{e}_{24}+\boldsymbol{e}_{16}+\boldsymbol{e}_{46},\\
\xi_4&=\boldsymbol{e}_{35}-\boldsymbol{e}_{12}+\boldsymbol{e}_{46}.
\end{aligned} \tag{4.58}$$

These Killing vectors have the nontrivial commutation relations

$$[\xi_1,\xi_2]=-2\xi_3,\quad [\xi_1,\xi_4]=2\xi_2,\quad [\xi_2,\xi_4]=-2\xi_1. \tag{4.59}$$

They therefore form a Heisenberg algebra on which ξ_4 acts as an outer automorphism. The symmetry algebra of the quotient is hence $(\mathfrak{h}(1)\rtimes\mathbb{R})\oplus\mathbb{R}$. The norms of the Killing vectors are $||\xi_1||^2=||\xi_2||^2=||\xi_{\mathrm{Ads}}||^2$, $||\xi_3||^2=0$, $||\xi_4||^2=-1$.

We want to construct adapted coordinates to describe this quotient; it is convenient for studying causality to adapt them to ξ_{AdS}, ξ_3, and ξ_4. Let us therefore seek to choose coordinates (t,u,ϕ,ρ,γ) so that $\xi_3=\partial_v$, $\xi_4=-\partial_t$, and $\xi_{\mathrm{AdS}}=\partial_\phi$. This requires

$$\begin{aligned}
&\frac{\partial(x^4-x^1)}{\partial\phi}=0,\quad \frac{\partial(x^4+x^1)}{\partial\phi}=-2x^3,\\
&\frac{\partial(x^6-x^2)}{\partial\phi}=0,\quad \frac{\partial(x^6+x^2)}{\partial\phi}=-2x^5,\\
&\frac{\partial x^3}{\partial\phi}=x^4-x^1,\quad \frac{\partial x^5}{\partial\phi}=x^6-+x^2,\\
&\frac{\partial(x^4-x^1)}{\partial v}=0,\quad \frac{\partial(x^4+x^1)}{\partial v}=-2(x^6-x^2),\\
&\frac{\partial(x^6-x^2)}{\partial v}=0,\quad \frac{\partial(x^6+x^2)}{\partial v}=-2(x^4-x^1),\\
&\frac{\partial x^3}{\partial v}=0,\quad \frac{\partial x^5}{\partial v}=0,\\
&\frac{\partial(x^4-x^1)}{\partial t}=(x^6-x^2),\quad \frac{\partial(x^4+x^1)}{\partial t}=(x^6+x^2),\\
&\frac{\partial(x^6-x^2)}{\partial t}=-(x^4-x^1),\quad \frac{\partial(x^6-x^2)}{\partial t}=-(x^4+x^1),\\
&\frac{\partial x^3}{\partial t}=x^5,\quad \frac{\partial x^5}{\partial t}=-x^3.
\end{aligned} \tag{4.60}$$

There are two quantities independent of $\{t,v,\phi\}$: $(x^4-x^1)^2+(x^6-x^2)^2$ and $x^3\cdot(x^6-x^2)-x^5\cdot(x^4-x^1)$. We will choose coordinates $\{\rho,\psi\}$ so that

$$\begin{aligned}
&(x^4-x^1)^2+(x^6-x^2)^2=e^{2\rho},\\
&x^3\cdot(x^6-x^2)-x^5\cdot(x^4-x^1)=e^{\rho}\psi;
\end{aligned} \tag{4.61}$$

we must take $-\infty<\rho<\infty$ and $-\infty<\psi<\infty$ to obtain coordinates that cover the whole spacetime. A coordinate system satisfying all these conditions is

$$\begin{aligned}
&x^4-x^1=e^{\rho}\sin t,\\
&x^4+x^1=-e^{\rho}(2\phi\psi+2v)\cos t-(e^{-\rho}+(\psi^2+\phi^2)e^{\rho})\sin t,\\
&x^6-x^2=e^{\rho}\cos t,\\
&x^6+x^2=e^{\rho}(2\phi\psi+2v)\sin t-(e^{-\rho}+(\psi^2+\phi^2)e^{\rho})\cos t,\\
&x^3=e^{\rho}(\psi\cos t+\phi\sin t),\\
&x^5=e^{\rho}(-\psi\sin t+\phi\cos t).
\end{aligned} \tag{4.62}$$

The AdS_5 metric in these coordinates is

$$g_{dnr}=-dt^2+d\rho^2+e^{2\rho}(d\psi^2+d\phi^2-2dtdv-4\psi dtd\phi), \tag{4.63}$$

and the other two Killing vectors are

$$\begin{aligned}
\xi_1&=-\cos 2t(\partial_\phi-2\psi\partial_v)+\sin 2t\,\partial_\psi,\\
\xi_2&=\sin 2t(\partial_\phi-2\psi\partial_v)+\cos 2t\,\partial_\psi.
\end{aligned} \tag{4.64}$$

Even though we will not give the explicit details, it is easy to check by working out the inverse coordinate transformation that this coordinate system covers the whole of AdS. Before any identification, the range of all adapted coordinates is noncompact. The double null rotation quotient is simply described by making the coordinate ϕ compact.

We would also like to understand the conformal boundary of this quotient. First, we should note that, even though the quotient is free of fixed points in the bulk, its boundary has a continuous line of them. The action generated by $B^{(1,2)}\oplus B^{(1,2)}$ integrates to the real line, so the only possible fixed points are the ones for which ξ_{AdS} vanishes. These points are given by

$$x^4 - x^1 = x^6 - x^1 = x^3 = x^5 = 0.$$

The above does not belong to AdS_5, since the points do not satisfy the quadric equation (2.1). This is indeed true for the bulk of AdS (finite noncompact spacelike direction in global AdS), but there is a continuous curve of fixed points on an infinite cylinder of axis, global time τ, and a maximal circle base. To see this, consider the standard global description of AdS_5,

$$x^1 = \cosh \chi \cos \tau,$$

$$x^2 = \cosh \chi \sin \tau,$$

$$x^i = \sinh \chi \hat{x}^i, \quad i = 3, \ldots, 6,$$

where $\{\hat{x}^i\}$ parametrize a three-sphere of unit radius. It is easy to see that any solution to the fixed point conditions requires $\chi \to \infty$, from which we already learn that such points belong to the boundary of AdS_5. It is also clear that $\hat{x}^3 = \hat{x}^5 = 0$. Thus, such fixed points belong to a maximal circle in the x^4-x^6 plane. If the angular variable describing such a maximal circle is φ $(0 \leqslant \varphi < 2\pi)$, the continuous line of fixed points is determined by

$$\tau = \varphi \pmod{2\pi}.$$

Thus, the action of the quotient is well defined on the global boundary of AdS (i.e., the Einstein static universe) with a single null line deleted. However, we know that the Einstein static universe with a null line deleted is conformal to a symmetric plane wave [76]. This suggests that the boundary of (4.63) should be described in terms of a plane wave.

Inspired by this and the analysis of the $B_{\pm}^{(2,2)}$ case in Sec. IV A, let us now make a coordinate transformation $Z = e^{-\rho}$ in Eq. (4.63). The metric then becomes

$$g_{dnr} = \frac{1}{Z^2}(-2dtdv - Z^2 dt^2 + dZ^2 + d\psi^2 + d\phi^2 - 4\psi dt d\phi), \tag{4.65}$$

where $0 < Z < \infty$ covers the whole of AdS_5. By rescaling the metric by a factor of Z^2, we can conformally map global AdS_5 into the space with metric

$$\bar{g} = -2dtdv - Z^2 dt^2 + dZ^2 + d\psi^2 + d\phi^2 - 4\psi dt d\phi, \tag{4.66}$$

with the conformal boundary lying at $Z = 0$. Since $\xi_{\mathrm{AdS}} = \partial_\phi$ annihilates the conformal factor, this embedding commutes with the quotient; we can regard the double null rotation as conformally embedded in (4.66) with ϕ compactified.[7]

Now, the space (4.66) is simply a symmetric plane wave. This can be made obvious by making the further coordinate transformation[8]

$$V = v + \psi\phi,$$

$$U = t,$$

$$X = \psi \cos t + \phi \sin t,$$

$$Y = -\psi \sin t + \phi \cos t, \tag{4.67}$$

under which the metric becomes

$$\bar{g} = -2dUdV - (X^2 + Y^2 + Z^2)dU^2 + dX^2 + dY^2 + dZ^2. \tag{4.68}$$

This provides an interesting alternative description of the double null rotation, of interest independent of the question of the conformal boundary. As in Sec. IV A, this relation between the symmetric plane wave and AdS is anticipated by previous work, since they are both conformally flat spaces and hence conformally embedded in the Einstein static universe. We see also that AdS covers half of the plane wave at $Z > 0$, as we would expect, since it covers half the Einstein static universe. What is remarkable is that the isometry we want to quotient along commutes with the conformal rescaling, as noted above. In fact, not only does it do so; all the unbroken symmetries of the double null rotation also do so, since they do not involve ∂_ρ. Thus, they are all symmetries of the conformally related plane wave metric (4.68). If we introduce the usual basis for the Killing vectors of the plane wave,

$$\xi_{e_i} = -\cos U \partial_{X^i} + X^i \sin U \partial_V,$$

$$\xi_{e_i^*} = -\sin U \partial_{X^i} - X^i \cos U \partial_V,$$

$$\xi_{e_V} = \partial_V,$$

$$\xi_{e_U} = -\partial_U, \tag{4.69}$$

we can identify the isometries of the double null rotation quotient as

[7]Note that this conformal embedding does not provide a true compactification of the spacetime, since Eq. (4.66) is itself not compact. As noted above, this represents the necessary exclusion of the fixed points of the quotient in the Einstein static universe.

[8]It is worth noting that there is a simple relation between these and the embedding coordinates for AdS_5: $x^4 - x^1 = (\sin U)/Z$, $x^4 + x^1 = -[V \cos U + (X^2+Y^2+Z^2)\sin U]/Z$, $x^6 - x^2 = (\cos U)/Z$, $x^6 + x^2 = [V \sin U - (X^2+Y^2+Z^2)\cos U]/Z$, $x^3 = X/Z$, $x^5 = Y/Z$.

$$\begin{aligned}\xi_{AdS}&=-\xi_{e_1^*}-\xi_{e_2},\\ \xi_1&=-\xi_{e_1^*}+\xi_{e_2},\\ \xi_2&=-\xi_{e_1}-\xi_{e_2^*},\\ \xi_3&=\xi_{e_V},\\ \xi_4&=\xi_{e_U}-\xi_{M_{12}}.\end{aligned} \tag{4.70}$$

Thus, the double null rotation is conformally related to a compactification of the plane wave of the type considered in [79].

To return to the question of the conformal boundary of the double null rotation, we see that it is given by the surface at $Z=0$ in Eq. (4.66), with metric

$$g_\partial=-2dtdv+d\psi^2+d\phi^2-4\psi dtd\phi. \tag{4.71}$$

This is itself a compactified plane wave, as can be seen by the application of the coordinate transformation (4.67). One might be puzzled by this result, as one would have expected to find the nullbrane as the conformal boundary of the double null rotation. We demonstrate in Appendix B that the nullbrane is in fact related to Eq. (4.71) by a further conformal transformation. Thus, Eq. (4.71) and the nullbrane describe the same conformal structure on the boundary. The description in terms of the compactified plane wave (4.71) is preferable to the nullbrane for two reasons: First, the nullbrane covers only a part of the boundary [it corresponds to the region $-\pi/2<t<\pi/2$ in Eq. (4.71)], so the former description is more global. Second, the further conformal transformation to the nullbrane does not commute with the symmetry ξ_4 of the double null rotation. If we work with Eq. (4.71), all the unbroken symmetries of the bulk spacetime after we perform the quotient are realized as symmetries of the boundary (rather than conformal isometries). This should be a helpful simplification in studying the holographic relation for this spacetime.

The connection to plane waves also makes it easy to identify a time function for the double null rotation. Writing the double null rotation metric (4.65) in the form suitable for Kaluza-Klein reduction along ϕ,

$$g=\frac{1}{Z^2}[-2dvdt-(Z^2+4\psi^2)dt^2+d\psi^2+(d\phi-2\psi dt)^2], \tag{4.72}$$

we see that the lower-dimensional spacetime would again be a plane wave (up to conformal factor). Hence, applying the results of [70], where time functions were found for general plane waves, we can deduce that a suitable time function for the nullbrane is

$$\tau=t+\frac{1}{2}\tan^{-1}\left(\frac{4v}{1+Z^2+4\psi^2}\right). \tag{4.73}$$

It is easy to check that

$$\nabla_\mu\tau\nabla^\mu\tau=-\frac{4Z^2}{[(1+Z^2+4\psi^2)^2+16v^2]}. \tag{4.74}$$

Thus, τ is a good time function on AdS. Since $\mathcal{L}_{\xi_{AdS}}\tau=0$, its existence shows that the double null rotation quotient of AdS preserves the property of stable causality by the general argument of [70].

As recently discussed in [54],[9] the supersymmetry preserved by this double null rotation quotient in AdS_5, and actually in any higher-dimensional AdS spacetime embedded in a supergravity theory, is $\nu=\frac{1}{2}$. That is, this configuration has 16 supercharges. It is interesting to comment on the relation with the single null rotation quotient. In that case, we argued that the standard enhancement of supersymmetry when taking the near horizon geometry was lost after the identification. This may suggest that the same phenomenon is taking place in the double null rotation, since the action generated by the latter is the combination of two commuting null rotations. However, the general solution to the eigenvalue problem

$$N\varepsilon=N_1\cdot N_2\varepsilon=0,$$

where N stands for the full double null rotation generator in the spinorial representation, and N_i, $i=1,2$, stand for nilpotent operators, is not given in terms of the intersection of kernels of the nilpotent operators associated with each of the null rotations, which would give rise to $\nu=\frac{1}{4}$, but there exist nontrivial solutions [54] that enhance supersymmetry to one-half. Thus, in this case, the double null rotation quotient preserves the same amount of supersymmetry as the corresponding asymptotically flat analogue in terms of parallel and coincident D3-branes in the nullbrane vacuum.

Deformation by $B^{(0,2)}$. In order to turn on any deformation parameter, we must consider higher-dimensional AdS spacetimes. In particular, it is natural to consider AdS_7, since this is very naturally obtained in M theory from the near horizon limit of M5-branes. If we denote by α the deformation parameter, the deformed seven-dimensional quotient can be written as

$$g_{AdS_7/\Gamma}=\cosh^2\chi g_{dnr}+d\chi^2+\sinh^2\chi(d\varphi_1+\alpha d\phi)^2, \tag{4.75}$$

where g_{dnr} stands for Eq. (4.63).

Since we turned on only a single deformation parameter α, the corresponding seven-dimensional quotient, when embedded in string theory, will break supersymmetry. It is certainly possible to construct supersymmetric versions of the latter by deforming the orbifold action with a nontrivial action on S^4.

[9]In [42], it was claimed that the amount of supersymmetry preserved by the double null rotation quotient was $\nu=1/4$, but as shown in [54], the latter is actually enhanced to $\nu=1/2$.

2. Sphere deformations

Let us start our discussion on sphere deformations of the double null rotation quotient by focusing on $AdS_5\times S^5$. The family of deformations is described by Eq. (4.52), that is, by three real parameters. As discussed in [54], the only supersymmetric loci in the fundamental region defined by the action of the Weyl group is, in addition to the origin, given either by $\theta_1=\theta_2$ and $\theta_3=0$, preserving $\nu=\frac{1}{4}$, or by $\theta_1-\theta_2\pm\theta_3=0$, preserving $\nu=\frac{1}{8}$.

The discussion for $AdS_7\times S^4$ is analogous. In this case, there exists a two-parameter family of deformations, given by Eq. (4.53). The only supersymmetric loci in the fundamental region defined by the action of the Weyl group is either the origin, corresponding to the double null rotation quotient itself, or the line $\theta_1=\theta_2$, which preserves $\nu=\frac{1}{4}$.

As an explicit example of a sphere deformation of the double null rotation quotient, we shall focus on a supersymmetric deformation on $AdS_5\times S^5$. We will focus on the same sphere action considered in Sec. IV B 3. As before, we apply the general formalism developed in Eq. (4.8) for the full Killing vector $\xi=\xi_{AdS}+\xi_S$. If we introduce adapted coordinates so that $\xi=\partial_\phi$ by defining $\psi'=\psi-\gamma\phi$, the full ten-dimensional metric on the quotient space will be

$$g=g_{dnr}+d\theta_1^2+\sin^2\theta_1 d\theta_2^2+\cos^2\theta_1[d\varphi_1+2(d\psi'+\gamma d\phi)]^2 +\sin^2\theta_1[(d\psi'+\gamma d\phi)^2 +d\varphi^2+2\cos 2\theta_2(d\psi'+\gamma d\phi)\cdot\varphi], \tag{4.76}$$

where g_{dnr} denotes the metric on the quotient of AdS_5 given in Eq. (4.63).

We could consider quotients involving both two-forms $\oplus_i B^{(0,2)}(\varphi_i)$ acting on AdS and sphere deformations. The techniques required to deal with them are exactly the same as those used above. The reader can find an analysis of their supersymmetry in [54].

V. BLACK HOLES AS QUOTIENTS

In the previous section, we discussed causally regular quotients, which arise in some cases where the Killing vector defining the AdS orbifold is nowhere timelike. One might think that these are the cases of primary interest, since any other quotient will have at least a region of closed timelike curves. However, as is well known, certain causally ill-behaved quotients can be given an interpretation as an analogue of black holes [36,37].

The idea is that one can excise regions where closed timelike curves will arise from the original spacetime, and consider the quotient just of the remaining portion of AdS_{p+1}. The resulting geometry will be causally regular by construction, but will clearly not be geodesically complete, having a "singularity" corresponding to the boundary of the excised region. This singularity is not a curvature singularity in the classical geometry, but extending the spacetime beyond it would introduce causal pathologies; it is therefore expected on the basis of the chronology protection conjecture that quantum corrections will lead to a true singularity at this location. The interesting question is whether this singularity is naked—that is, visible from infinity—or concealed by an event horizon. If it is behind an event horizon, we view the quotient geometry as a black hole, generalizing the BTZ solution [36,37].

In this section, we will study which quotients can lead to black holes of this type. Unlike in the previous section, where deformation on the sphere introduced qualitatively new possibilities, we find that the quotients with a black hole interpretation are the BTZ quotients in AdS_3, and the higher-dimensional generalization of the nonrotating BTZ quotients, coupled with some action on the sphere.

First, we need to establish what region of the spacetime we remove. In [38], where quotients acting just on the AdS factor were considered, it was argued that we should remove the region where the Killing vector ξ_{AdS} fails to be spacelike. Clearly, the quotient will contain closed timelike curves in this region. However, it is not in general true that all closed timelike curves will pass inside this region. In particular, for cases with $B^{(0,2)}(\varphi_i)$ components, this does not remove all the closed timelike curves.

Closed timelike curves in the region where ξ_{AdS} is spacelike can be constructed by an argument very similar to that used in Sec. III. As discussed at the beginning of Sec. IV, for any of our quotients, we can construct a natural coordinate system (4.1) on the AdS part, in which we decompose AdS_{p+1} in terms of an AdS_{n+1} and a S^{p-n-1} factors, where the Killing vector generating the quotient is $\xi_{AdS}=\xi_{AdS_{n+1}}+\xi_r$, with $\xi_{AdS_{n+1}}$ acting only on the AdS_{n+1} part of the metric (4.2) and containing the nontrivial block or blocks, while the ξ_r is a combination of rotations [the $B^{(0,2)}(\varphi_i)$ blocks] acting on the unit sphere S^{p-n-1}. Now consider an orbit where ξ_{AdS} is spacelike, but $\xi_{AdS_{n+1}}$ is timelike. As in Sec. III, we can construct a closed curve which follows the orbit of $\xi_{AdS_{n+1}}$ on the AdS_{n+1} factor and a length-minimizing geodesic on the S^{p-n-1} factor. There are identified points that are separated by an arbitrarily large timelike distance in the AdS_{n+1} factor; since the separation on S^{p-n-1} is bounded, this closed curve will be timelike for sufficiently large separation on the AdS_{n+1} factor. Obviously, a similar argument applies when we consider the deformation on the transverse sphere; there will be closed timelike curves wherever the norm of the nontrivial blocks taken on their own is timelike.

Thus, it would seem that a natural region to excise is the region where $\xi_{AdS_{n+1}}$ is timelike. That is, the region to excise is determined by the norm of the nontrivial blocks, omitting all the rotations [both $B^{(0,2)}(\varphi_i)$ and the rotations on transverse spheres]. Note, however, that this is still not sufficient to eliminate the closed timelike curves in all cases. That is, the resulting quotient is not guaranteed to be causally regular. However, this is the only possibility we will consider here. It represents the natural generalization of the construction of black hole solutions of [36,37] to higher dimensions. We will focus on seeing what black analogues can be constructed by removing this portion of the quotient. We will see that the resulting spacetime in the black hole examples are in fact free of closed causal curves.

The singularity surface we consider is then where

$\xi_{AdS_{n+1}}\cdot\xi_{AdS_{n+1}}=0$ in $AdS_{p+1}\times S^q$. Our main concern for the rest of this section is to establish in which cases this singularity surface is naked, and in which cases it is concealed by an event horizon. Since $\xi_{AdS_{n+1}}$ is a Killing field,

$$\nabla_{\xi_{AdS_{n+1}}}(\|\xi_{AdS_{n+1}}\|^2)=2i_{\xi_{AdS_{n+1}}}(\nabla_{\xi_{AdS_{n+1}}}\xi_{AdS_{n+1}})=0, \tag{5.1}$$

so $\xi_{AdS_{n+1}}$ is always tangent to surfaces defined by $\|\xi_{AdS_{n+1}}\|^2=$const. Hence, the "singularity" defined by $\|\xi_{AdS_{n+1}}\|^2=0$ has a null tangent, and must be a timelike or null surface. We think of such a quotient as an analogue of a black hole if there is a nontrivial event horizon $\dot{J}^-(\mathcal{J}^+)$ in the quotient. Since the singularity surface is timelike or null, this can only happen if the singularity surface divides the future null infinity $\mathcal{J}^+$ of the AdS_{p+1} spacetime into disconnected regions. The behavior of the Killing vector on the asymptotic boundary of the AdS spacetime is therefore essential in determining if a given case is a black hole or not.

A. AdS_3 black holes

For the AdS_3 case, the addition of a deformation on the sphere does not significantly modify the analysis of [37]: the only quotients which lead to black holes are the ones whose AdS Killing vector field is associated with the two-forms $B^{(1,1)}(\beta_1)\oplus B^{(1,1)}(\beta_2)$, for $|\beta_1|\neq|\beta_2|$, and $B^{(2,2)}(\beta)$ for $\beta\neq 0$, corresponding to nonextremal and extremal black holes, respectively. These AdS Killing vectors correspond to type I_b and type II_a in the notation of [37].[10] When embedding these black holes in string theory, it is certainly natural to embed them in type IIB, in terms of $AdS_3\times S^3\times\mathbb{T}^4$, coming from the near horizon of the D1-D5 system. Thus, the most general Killing vector field giving rise to black holes is given by

$$\xi=\xi_{BTZ}+\theta_1R_{12}+\theta_2R_{34}, \tag{5.2}$$

where we are using the notation introduced in Sec. II C.

The metric on these solutions is easily constructed. For simplicity, we shall focus again on the deformation for which $\theta_1=\theta_2=\gamma$. Let us adopt BTZ coordinates on the AdS space, so that $\xi_{AdS_3}=\partial_\phi$, and adapted coordinates on the sphere, so that $\xi_S=\partial_\psi$. Then the metric is

$$g=-\frac{(r^2-r_+^2)(r^2-r_-^2)}{r^2}dt^2+\frac{r^2dr^2}{(r^2-r_+^2)(r^2-r_-^2)}+r^2\left(d\phi-\frac{r_-r_+}{r^2}dt\right)^2+d\theta^2+d\chi^2+d\psi^2+2\cos 2\theta d\chi d\psi, \tag{5.3}$$

and the quotient introduces the periodic identifications $\phi\sim\phi+2\pi m, \psi\sim\psi+2\pi\gamma m$, $m\in\mathbb{Z}$. If we introduce a coordinate $\tilde{\psi}=\psi-\gamma\phi$, then $\xi=\partial_\phi$ and the metric in fully adapted coordinates is

$$g=-\frac{(r^2-r_+^2)(r^2-r_-^2)}{r^2}dt^2+\frac{r^2dr^2}{(r^2-r_+^2)(r^2-r_-^2)}+r^2\left(d\phi-\frac{r_-r_+}{r^2}dt\right)^2+\gamma^2d\phi^2+2\gamma d\phi(d\tilde{\psi}+\cos 2\theta d\chi)+d\theta^2+d\chi^2+d\tilde{\psi}^2+2\cos 2\theta d\chi d\tilde{\psi}. \tag{5.4}$$

Note that the deformation on the sphere does not affect the leading r^2 part of the metric at large distances, so the structure of the asymptotic boundary of the black hole is not changed. From the point of view of Kaluza-Klein reduction over the sphere, this geometry is described as the rotating BTZ black hole with a flat $SU(2)_L\subset SO(4)$ gauge connection $A^3_\phi=\gamma$ turned on, in analogy with previous discussions of conical defects [73]. Since the gauge field has zero stress-energy, it does not modify the three-dimensional metric. Its presence does however modify the supersymmetry conditions [73]. Unlike in the conical defect case, we cannot make nonsupersymmetric black hole solutions supersymmetric by adding a deformation on the sphere, as we cannot balance the hyperbolic black hole holonomy by a holonomy in $SU(2)$.

B. Higher-dimensional black holes

Let us now investigate what happens in higher dimensions. For the excision we are studying, the singularity is determined by the nontrivial part of the AdS action, $\xi_{AdS_{n+1}}$, and the presence of horizons is determined by considering the intersection of this singularity surface with the AdS boundary. We therefore focus on the AdS part of the story, and only add in the sphere at the end.

We want to know if there is an event horizon in the quotient. Since the location of the singularity is determined by $\xi_{AdS_{n+1}}$, it is natural to study this using the decomposition (4.1). This considerably simplifies the task of studying the higher-dimensional cases, by relating it to the lower-dimensional classification. It would require considerable work to determine directly from the form of the Killing vectors whether or not event horizons exist. By relating this question to the existence of horizons in lower dimensions, we can avoid most of this work and also gain some valuable insight into the differences between the AdS_3 case and higher dimensions.

For a Killing vector that does not contain a $B^{(2,3)}$ block, a $B^{(2,4)}_\pm(\varphi)$ block, two $B^{(1,2)}$ blocks, or a $B^{(1,2)}$ and a $B^{(1,1)}$ block, we can adapt the coordinate system of Eq. (4.2) with $n=2$; that is, we can decompose AdS_{p+1} in terms of AdS_3 and S^{p-3} factors. The Killing vector then decomposes as $\xi_{AdS}=\xi_{AdS_3}+\xi_r$, where ξ_{AdS_3} acts only on the AdS_3 part of the metric (4.2) and contains the nontrivial block or blocks,

[10] Note that the $M=J=0$ black hole solutions of [37], obtained by quotienting by $B^{(1,2)}$, do not have a generalization to include rotation on the sphere, as the associated AdS Killing vectors are nowhere timelike, so these give causally regular quotients once a nontrivial ξ_{S^3} is included, as described in the previous section.

while the ξ_r is a combination of rotations [the $B^{(0,2)}(\varphi_i)$ blocks] acting on the unit sphere S^{p-3}. Furthermore, ξ_{AdS_3} is precisely the Killing vector associated with the same type of quotient in the analysis of [37].

We would like to exploit this decomposition to simplify the problem of finding horizons. We will show that there is a simple condition on the action in AdS_3 which will imply that the singularity is naked in AdS_{p+1}. The existence of a nontrivial event horizon in the quotient spacetime implies that there are points in the singularity surface $||\xi_{AdS_3}||^2=0$ which cannot be connected to the same asymptotic region in both the past and the future. Conversely, if a point in AdS with $||\xi_{AdS_3}||^2=0$ lies on some timelike curve which lies entirely in the region where $||\xi_{AdS_3}||^2\geqslant 0$ in the bulk and starts and ends in some connected component of the region of the boundary where $||\xi_{AdS_3}||^2>0$, this point on the singularity will be naked in the quotient. Thus, the existence of such a curve implies the nakedness of the singularity.

Now, in the coordinates (4.2), we can consider the restriction to the AdS_3 factor at some fixed point on the sphere factor that ξ_r acts on, and ask if there is such a curve which in addition stays in this submanifold. This will supply a sufficient condition for nakedness of the singularity which can be expressed in AdS_3 terms. We therefore want to look for a timelike curve in AdS_3 which connects points in the same connected component of the region of the boundary where $||\xi_{AdS_3}||^2>0$ through the region where $||\xi_{AdS_3}||^2\geqslant 0$ in the bulk, and passing through a point at $||\xi_{AdS_3}||^2=0$. But this is the same thing as the condition for a naked singularity in AdS_3: cases that do not lead to black holes in AdS_3 do not lead to black holes in higher dimensions either. Horizons can arise only in the cases where there is a horizon in the AdS_3 quotient.

Consider now the cases which give black holes in AdS_3; that is, the $B^{(1,1)}(\beta_1)\oplus B^{(1,1)}(\beta_2)$ for $|\beta_1|\neq|\beta_2|$, and $B^{(2,2)}(\beta)$ for $\beta\neq 0$. Consider first the rotating black holes. We will see that there will be no horizons in the higher-dimensional cases. In the quotient of AdS_3, we obtained a solution with an inner horizon and a timelike singularity, so any point on the singularity surface was connected to the boundary to both the past and future, but it was connected to different components of the boundary, so this did not imply the absence of a horizon. In higher dimensions, however, we can describe the asymptotic boundary in terms of an $AdS_3\times S^{p-3}$ metric,

$$g_\partial = g_{AdS_3} + g_{S^{p-3}}. \tag{5.5}$$

Since the portion of the bulk of AdS_3 where ξ_{AdS_3} is spacelike is connected, the portion of the boundary of AdS_{p+1} where ξ_{AdS_3} is spacelike will be connected, and hence the curves that link a point on the singularity to the boundary have their end points in a single connected component of the region of the boundary where $||\xi_{AdS_3}||^2>0$. Thus, they imply that the singularity is naked in the higher-dimensional quotients, as noted for the case $p=3$ in [38].

This leaves only the cases where we quotient by a Killing vector with a single $B^{(1,1)}$ factor, which would correspond to a nonrotating black hole in AdS_3. We will see shortly that this case does have a nontrivial event horizon for AdS_{p+1}, $p\geqslant 2$. This is thus the only case involving AdS_3 blocks with an event horizon in higher dimensions.[11]

It remains to consider the Killing vectors containing blocks $B^{(2,3)}$ and $B_\pm^{(2,4)}(\varphi)$, and the cases containing two $B^{(1,2)}$ blocks or a $B^{(1,2)}$ block and a $B^{(1,1)}$ block. However, these do not lead to any more examples with horizons. For two $B^{(1,2)}$ blocks, this is obvious, as the Killing vector is nowhere timelike. For the $B^{(2,3)}$ block, we can observe that it was shown in [38] (where this case is called type V) that there is no horizon in this case in AdS_4; this can easily be extended to show that there is no horizon in higher dimensions by the arguments used above. For a $B^{(1,2)}$ block and a $B^{(1,1)}$ block, we can similarly appeal to the analysis of [38].

For the $B_\pm^{(2,4)}(\varphi)$ blocks, we analyze the situation in AdS_5, and appeal to the argument set forth above to extend the conclusion to general dimensions. In AdS_5, the Killing vector is

$$\xi_{AdS} = \boldsymbol{e}_{15} - \boldsymbol{e}_{35} \pm \boldsymbol{e}_{26} - \boldsymbol{e}_{46} + \varphi(\mp \boldsymbol{e}_{12} + \boldsymbol{e}_{34} + \boldsymbol{e}_{56}). \tag{5.6}$$

The norm of this Killing vector is

$$\begin{aligned}\|\xi_{AdS}\|^2 = &-\varphi^2 + 4\varphi[x_6(x_3-x_1) - x_5(x_4\mp x_2)]\\ &+(x_3-x_1)^2 + (x_4\mp x_2)^2,\end{aligned} \tag{5.7}$$

where $\{x_1,\dots,x_6\}$ are the $\mathbb{R}^{2,4}$ embedding coordinates. Adapting a global coordinate system on AdS_5,

$$\begin{aligned}&x_1=\cosh\rho\cos t,\quad x_2=\cosh\rho\sin t,\\ &x_3=\sinh\rho\cos\theta\cos\phi,\quad x_4=\sinh\rho\cos\theta\sin\phi,\\ &x_5=\sinh\rho\sin\theta\cos\psi,\quad x_6=\sinh\rho\sin\theta\sin\psi,\end{aligned} \tag{5.8}$$

the norm becomes

$$\begin{aligned}\|\xi_{AdS}\|^2 = &-\varphi^2 + 4\varphi\sinh\rho\sin\theta[-\cosh\rho\sin(\psi\pm t)\\ &+\sinh\rho\cos\theta\sin(\psi-\phi)] + \cosh^2\rho + \sinh^2\rho\cos^2\theta\\ &-2\cosh\rho\sinh\rho\cos\theta\cos(\phi\pm t).\end{aligned}$$

Thus, we see that the global time dependence of the norm is simply a simultaneous rotation in the two angles ϕ,ψ on the S^3 in AdS_5. Thus, the region of the boundary where the norm of the Killing vector is spacelike is clearly connected, and this case does not give rise to a black hole in any dimension.

Thus, the only quotient with a black hole interpretation for $p>2$ is the quotient by an AdS Killing vector $B^{(1,1)}(\beta)\oplus_i B^{(0,2)}(\varphi_i)$. The resulting quotient is the higher-

[11]We are again excluding the case of $B^{(1,2)}$, corresponding to an $M=0$ black hole, on the grounds that once we include rotation on the sphere, this will become a causally regular quotient.

dimensional generalization of the nonrotating BTZ black hole. Special cases of this solution for $p=3,4$ have been discussed before in [38–40].[12] As above, the natural coordinate system on these quotients in general is the one given by the decomposition (4.1). If we adopt adapted coordinates for the $B^{(1,1)}(\beta)$ action on the AdS_3 factor, this is

$$g=\cosh^2\chi\left(-(r^2-1)dt^2+\frac{dr^2}{r^2-1}+r^2d\phi^2\right)+d\chi^2+\sinh^2\chi d\Omega_{p-3}, \tag{5.9}$$

where we have reabsorbed the length scale r_+ associated with the black hole by rescaling coordinates, so the period of the angular coordinate ϕ depends on r_+. The quotient makes identifications in ϕ with some twist on the S^{p-3} determined by the φ_i. We note that although these are deformations of the higher-dimensional BTZ quotient by rotations, they do not look like rotating black holes in the usual sense: ∂_t is still hypersurface orthogonal, and there is a single horizon.

The special case where we consider a simple boost, so $\varphi_i=0$, was considered in detail in [38–40]. In this case the quotient preserves, in addition to the symmetry associated with ξ, an $SO(1,p-1)$ symmetry in the orthogonal subspace. Various coordinate systems were defined on the quotient which are adapted to make some or all of this symmetry manifest in [39,40]. We would like to briefly connect to that work by showing how our preferred coordinate system above which makes the AdS_3 structure manifest is connected to one of those coordinate systems.

In [40], "spherical" coordinates were defined, in which the metric takes the form

$$g=(\rho^2-1)[-\sin^2\theta dt^2+d\theta^2+\cos^2\theta d\Omega_{p-3}]+\frac{d\rho^2}{(\rho^2-1)}+\rho^2d\phi^2. \tag{5.10}$$

These coordinates are one example of coordinates adapted to the $SO(1,p-1)\times SO(1,1)$ symmetry of this spacetime. They are related to Eq. (5.9) by the coordinate transformation

$$\cos\theta=\frac{\sinh\chi}{\sqrt{\rho^2-1}},\quad \rho=r\cosh\chi. \tag{5.11}$$

It is interesting to note that this shows that the $SO(1,1)$ manifest in Eq. (5.10) is precisely the time translation of the BTZ black hole. Note that the spherical coordinates of Eq. (5.10) cover more of the spacetime than the BTZ coordinates of Eq. (5.9). This illustrates that, while the coordinates we have constructed adapted to the decomposition of the Killing vector in terms of lower-dimensional quotients are useful, they are not the best coordinate system for every purpose.

Another interesting coordinate system on this quotient is the "de Sitter" coordinates of [40], which were used in [80,81], where this locally AdS_{p+1} black hole arises as the asymptotic behavior of the bubble of nothing solution. In that context, it is convenient to adopt a coordinate system in which the metric is

$$g=(1+R^2)d\phi^2+\frac{dR^2}{1+R^2}+R^2[-d\tau^2+\cosh^2\tau(d\tilde\theta^2+\sin^2\tilde\theta d\Omega_{p-3})]. \tag{5.12}$$

These coordinates are adapted to the same $SO(1,p-1)\times SO(1,1)$ symmetry as in Eq. (5.10). The coordinate transformation relating Eq. (5.12) to Eq. (5.10) is

$$\rho^2=1+R^2,\quad \cos\theta=\cosh\tau\sin\tilde\theta,\quad \tanh t=\frac{\tanh\tau}{\cos\tilde\theta}. \tag{5.13}$$

These "de Sitter" coordinates have the advantage that they cover the whole exterior region of the black hole. They demonstrate that the black hole is not a static solution in higher dimensions; there is no Killing vector which is timelike everywhere outside the black hole event horizon.

As in the three-dimensional case, when we consider the quotient of $AdS_{p+1}\times S^q$, we can write the AdS and sphere factors in adapted coordinates separately, so that $\xi_{\mathrm{AdS}}=\partial_\phi$, and $\xi_{\mathrm{S}}=\partial_\psi$. Fully adapted coordinates are then obtained by setting $\tilde\psi=\psi-\gamma\phi$, which introduces $O(1)$ cross terms between AdS and sphere coordinates. Again, from the Kaluza-Klein reduced point of view, what we are doing is introducing a flat $SO(q+1)$ gauge connection $A^a_\phi=\gamma$ on the black hole solution above, without modifying the metric.

One other issue deserves a comment on the subject of black holes: in [40], it was claimed that a rotating black hole solution could be constructed by taking a quotient of AdS_5. We want to point out that this is not the same as the deformation by $B^{(0,2)}(\varphi_i)$ discussed above; in fact, this quotient is not a black hole. The solution of [40] was given by considering AdS_5 in the coordinates

$$g=\sinh^2\rho[-\cos^2\theta d\tilde t^2+d\theta^2+\sin^2\theta d\psi^2]+d\rho^2+\cosh^2\rho d\tilde\phi^2, \tag{5.14}$$

and making identifications along $\phi=\tilde\phi$ at fixed $t=(r_+\tilde t-r_-\tilde\phi)/(r_+^2-r_-^2)$. This gives a "black hole" metric of the form

[12]Note that in [38] it was claimed that this does not lead to a black hole for $\varphi_i\neq 0$. This is because [38] took the singularity surface to be $||\xi_{\mathrm{AdS}}||^2=0$, which does not eliminate all closed timelike curves in this case. We take the singularity surface to be $||\xi_{\mathrm{AdS}_3}||^2=0$, cutting out more of the global AdS spacetime; this gives a causally regular spacetime which can be interpreted as a black hole.

$$g=\cos^2\theta\left[-\frac{(r^2-r_+^2)\cdot(r^2-r_-^2)}{r^2}dt^2 + r^2\left(d\phi-\frac{r_-}{r_+\cdot r^2}(r^2-r_+^2)dt\right)^2\right] + \frac{r^2dr^2}{(r^2-r_+^2)\cdot(r^2-r_-^2)}+\frac{(r^2-r_+^2)}{(r_+^2-r_-^2)}(d\theta^2+\sin^2\theta d\psi^2) + \frac{r_+^2(r^2-r_-^2)}{(r_+^2-r_-^2)}\sin^2\theta d\phi^2, \tag{5.15}$$

where $r^2=r_+^2\cosh^2\rho-r_-^2\sinh^2\rho$. Since the coordinates $\tilde{t}$ and $\tilde{\phi}$ in Eq. (5.14) both parametrize SO(1,1) symmetries [while χ parametrizes an SO(2) symmetry], we can easily see that this quotient corresponds to the rotating BTZ black hole type of quotient: that is, to a quotient by a Killing vector formed from $B^{(1,1)}(\beta_1)\oplus B^{(1,1)}(\beta_2)$, with $\beta_1\beta_2\neq 0$. This can be seen explicitly by noting that, defining the new coordinates $\chi,\overline{r}$ by

$$\sinh^2\chi=\frac{(r^2-r_+^2)}{(r_+^2-r_-^2)}\sin^2\theta,$$

$$\overline{r^2}-r_-^2=\frac{r^2-r_-^2}{\cosh^2\chi}, \tag{5.16}$$

we can rewrite Eq. (5.15) as

$$g=\cosh^2\chi\left[-\frac{(\overline{r^2}-r_+^2)\cdot(\overline{r^2}-r_-^2)}{\overline{r^2}}dt^2 + \overline{r^2}\left(d\phi+\frac{r_-}{r_+\overline{r^2}}(\overline{r^2}-r_+^2)dt\right)^2 + \frac{\overline{r^2}d\overline{r^2}}{(\overline{r^2}-r_+^2)\cdot(\overline{r^2}-r_-^2)}\right]+d\chi^2+\sinh^2\chi d\psi^2, \tag{5.17}$$

showing that the quotient space has a rotating BTZ black hole factor and a circle factor, as expected for this type of quotient. Now, we have argued above that the presence of a rotating BTZ black hole factor implies that the region of the boundary of AdS_5 where the Killing vector we are quotienting along is spacelike is connected. Thus, this quotient cannot lead to an event horizon. The apparent presence of an event horizon in the coordinates (5.15) is attributable to those coordinates not covering the whole of infinity.

VI. ON PENROSE LIMITS OF DISCRETE QUOTIENTS

In Sec. III, we determined the subset of quotients of $AdS_{p+1}\times S^q$ spacetimes having closed timelike curves. In the main body of this work, we focused on the quotients which are free of closed causal curves, or on those having them, but allowing a black hole interpretation. We would like to finish our work with some short discussion regarding the relation of a subset of quotients of $AdS_{p+1}\times S^q$ having closed timelike curves and not falling in the black hole category and *compactified* plane waves and Gödel-type universes, both having closed timelike curves. The relation between closed timelike curves in quotients of $AdS_{p+1}\times S^q$ and compactified plane waves was already briefly commented on in [70].

That such a relation should exist is very intuitive, given the existing relation between Penrose limits of $AdS_{p+1}\times S^q$ and plane waves [61–63], and the *T*-duality relation between the latter and Gödel-type universes [58,59,71].[13] One possible motivation to make this connection more precise could be the fact that AdS/conformal field theory [82] could shed some light on the issue of physics in the presence of closed timelike curves.

In general, the operation consisting on taking the Penrose limit of a given configuration $\mathcal{M}$ does not commute with the operation of considering a discrete quotient in $\mathcal{M}$. Even though we do not have a general statement, it turns out that for Abelian discrete quotients whose generator belongs to the maximal compact subgroup of AdS, that is, for two-forms $B^{(2,0)}(\varphi)\oplus_i B^{(0,2)}(\varphi_i)$, the following diagram commutes:

$$\begin{array}{ccc} (\mathrm{AdS}_p\times \mathrm{S}^q)/\Gamma & \xrightarrow{T\text{-duality}} & \mathcal{N} \\ \Big\downarrow \text{Penrose limit} & & \Big\downarrow \text{Penrose limit} \\ (\text{plane wave})/\Gamma & \xrightarrow{T\ \text{duality}} & \text{Gödel type.} \end{array} \tag{6.1}$$

Let us make the connection more explicit. Even though we could develop the discussion in general, we shall focus on $AdS_3\times S^3$ for algebraic simplicity. Consider the quotient generated by

$$\xi_c=\beta A_-\partial_\tau+\beta A_+\partial_\psi+\beta(\partial_\varphi+\partial_\chi), \tag{6.2}$$

where $\{\tau,\rho,\varphi\}$ are global coordinates in AdS_3 and $\{\theta,\psi,\chi\}$ are global coordinates in S^3, whereas β is any nonvanishing real number and $A_\pm$ are defined as

$$A_\pm=\left(1\pm\frac{1}{4\beta^2R^2}\right). \tag{6.3}$$

The norm of such Killing vector field is given by

$$\|\xi_c\|^2=\frac{1}{16\beta^2R^2}\{\cosh^2\rho(8\beta^2R^2-1)+\cos^2\theta(8\beta^2R^2+1)\}. \tag{6.4}$$

[13]The interplay between Penrose limits and quotients of AdS was also considered in [50], although their physical motivation was not related to closed timelike curves.

Thus, $\|\xi_c\|^2>0$ whenever $8\beta^2R^2>0\ \forall\ \rho,\theta$. Even if this property is satisfied, we know the corresponding discrete quotient will have closed timelike curves, as proved in Sec. III.

It is convenient for our purposes to make the change of variables

$$\tau=\beta A_- u+x^-,\quad \varphi=\hat{\varphi}+\beta u,$$
$$\psi=\beta A_+ u-x^-,\quad \chi=\hat{\chi}+\beta u, \tag{6.5}$$

in which $\xi_c=\partial_u$. The global metric describing the above quotient of $\mathrm{AdS}_3\times \mathrm{S}^3$ consists in rewriting the metric in the new adapted coordinate system and making u compact. The result is

$$g=-R^2(\cosh^2\rho-\cos^2\theta)(dx^-)^2+R^2(d\rho^2+\sinh^2\rho d\hat{\varphi}^2 + d\theta^2+\sin^2\theta d\hat{\chi}^2)+2\beta R^2 du[\sin^2\theta d\hat{\chi}+\sinh^2\rho d\hat{\varphi} -(A_-\cosh^2\rho+A_+\cos^2\theta)dx^-]+\|\xi_c\|^2du^2. \tag{6.6}$$

The full type IIB configuration certainly includes a transverse $\mathbb{T}^4$ and some fluxes. It will not be necessary for our purposes to write these explicitly, but we shall keep in mind that we are working with a vacuum in which no Neveu-Schwarz–Neveu-Schwarz (NS-NS) three-form field strength is turned on.

We shall first show that the Penrose limit of Eq. (6.6) is indeed a quotient of a plane wave. The procedure is by now standard. Thus, we shall just state that one needs to rescale $x^-=R^{-2}v$, take the limit $R\to\infty$ while focusing on the lightlike geodesic sitting at $\rho=\theta=0$. Thus, we also need the rescalings $\rho=r/R$ and $\theta=y/R$. Following this prescription, and having in mind that u is compact, we can afterward apply a T-dual transformation giving rise to a Gödel-type spacetime, in particular, to one dual version of $\mathcal{G}_5$, following the conventions introduced in [59]. Of course, the dual configuration will have a nonvanishing NS-NS two-form potential, by construction, due to the crossed terms in the metric (6.6).

We would be interested in determining the spacetime that we get after applying the upper horizontal transformation in the diagram above. This corresponds to applying a T-duality transformation along the orbits of ∂_u. The T-dual metric that we get in this way is given by

$$g'=-R^2(\cosh^2\rho-\cos^2\theta)(dx^-)^2+R^2(d\rho^2+\sinh^2\rho d\hat{\varphi}^2 + d\theta^2+\sin^2\theta d\hat{\chi}^2)-\frac{\beta^2R^4}{\|\xi_c\|^2}[\sin^2\theta d\hat{\chi}+\sinh^2\rho d\hat{\varphi} -(A_+\cos^2\theta+A_-\cosh^2\rho)dx^-]^2+\|\xi_c\|^{-2}du^2. \tag{6.7}$$

It is a straightforward exercise to check that the Penrose limit of the above metric gives rise to $\mathcal{G}_5$. The corresponding fluxes can also be matched. Note that $\|\xi_c\|^2\to 1$ in the Penrose limit, which matches the construction given, for instance, in [58].

Thus, indeed, it is possible to understand the physics of Gödel-type universes as describing the physics of certain sectors of the dual field theory associated with the discrete quotient of the original $\mathrm{AdS}_3\times \mathrm{S}^3$, following [83]. However, we also see that the dual field theory is living in a space with closed timelike curves. One easy way to realize this fact is to note that the action of the Killing vector field ξ_c acts in the same way at any value of the noncompact spacelike coordinate ρ in AdS_3, in particular at its conformal boundary. Actually, the argument applies to any AdS_{p+1} spacetime. We thus learn that if $\mathrm{AdS}_{p+1}/\Gamma$ is the geometry of the bulk, where Γ stands for the discrete group associated with the discrete quotient generated by $B^{(2,0)}(\varphi)\oplus_i B^{(0,2)}(\varphi_i)$, its conformal boundary is given by $(\mathbb{R}\times \mathrm{S}^{p-1})/\hat{\Gamma}$, where $\hat{\Gamma}$ stands for the restriction of Γ on the boundary. The conformal boundary quotient would possibly include a nontrivial action on the fields coming from the R symmetry group. Thus, whenever $\hat{\Gamma}$ acts nontrivially on the real timelike $\mathbb{R}$ axis, the boundary theory will be defined in a base space having closed timelike curves, and as such, it will be nonglobally hyperbolic. Therefore, any holographic description for these scenarios involves an understanding of field theory in nonglobally hyperbolic spaces, which we are generically missing.

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APPENDIX A: GLOBAL VS POINCARÉ PATCH IN AdS

In this appendix, we will review the global coordinate and Poincaré patch descriptions of AdS_{p+1}. We wish to remind the reader of the expressions for the Killing vector fields generating the isometries in these two coordinate systems. For the Poincaré patch, this will be useful for understanding the relation between certain global AdS quotients and the near horizon limit of the corresponding discrete quotients of brane geometries in supergravity. For global coordinates, this will be useful for understanding the action of the Killing vectors on the Einstein static universe boundary of AdS.

Considering first the Poincaré coordinates, let us define $\{y^\mu,z\}$ $\mu=2,\ldots,p+1$ in terms of the flat embedding coordinates in $\mathbb{R}^{2,p}$ introduced in Eq. (2.1) by

$$x^\mu=\frac{1}{z}y^\mu,\quad \mu=2,\ldots,p+1,$$

$$x^1=\frac{1}{2z}[z^2+(1+\eta_{\mu\nu}y^\mu y^\nu)],$$

$$x^{p+2}=\frac{1}{2z}[z^2-(1-\eta_{\mu\nu}y^\mu y^\nu)]. \tag{A1}$$

In these coordinates, the AdS_{p+1} metric is

$$g=\frac{1}{z^2}(\eta_{\mu\nu}dy^\mu dy^\nu+dz^2). \tag{A2}$$

The explicit symmetries in this form of the metric are the Poincaré symmetries acting on the slices of constant z. Using the identities

$$\frac{\partial x^\mu}{\partial y^\nu}=\frac{1}{z}\delta^\mu_\nu,\quad \frac{\partial x^{p+2}}{\partial y^\nu}=\frac{\partial x^1}{\partial y^\nu}=\eta_{\nu\mu}x^\mu,\quad x^1-x^{p+2}=\frac{1}{z},$$

we see that these are related to the usual $\mathfrak{so}(2,p)$ basis by

$$P_\mu=\partial_{y^\mu}\to-(\boldsymbol{e}_{1\mu}-\boldsymbol{e}_{\mu p+2}),$$

$$L_{\mu\nu}=y_\mu\partial_{y^\nu}-y_\nu\partial_{y^\mu}\to\boldsymbol{e}_{\mu\nu}. \tag{A3}$$

Therefore, timelike translations in the Poincaré patch correspond to a null rotation with two timelike directions in global AdS, which is mapped to the two-form $B^{(2,1)}$. On the other hand, spacelike translations in the Poincaré patch correspond to a standard null rotation with two spacelike directions, or equivalently to $B^{(1,2)}$. Finally, Lorentz transformations in the Poincaré patch are mapped to Lorentz transformations in $\mathbb{R}^{2,p}$.

The other symmetries in $\mathfrak{so}(2,p)$ are realized as conformal symmetries acting on the slices of constant z together with a suitable ∂_z component:

$$\boldsymbol{e}_{1\mu}+\boldsymbol{e}_{\mu p+2}=-\eta_{\sigma\nu}y^\sigma y^\nu\partial_{y^\mu}+2y_\mu y^\nu\partial_{y^\nu}+2zy_\mu\partial_z, \tag{A4}$$

$$\boldsymbol{e}_{1p+2}=-y^\mu\partial_{y^\mu}-z\partial_z. \tag{A5}$$

A convenient global coordinate system on AdS_{p+1} is defined in terms of the embedding coordinates by

$$x_1=\cosh\chi\sin\tau,$$

$$x_2=\cosh\chi\cos\tau,$$

$$x_m=\sinh\chi\hat{x}_m,\quad m=3,\ldots,p+2, \tag{A6}$$

where the $\hat{x}_m$ are embedding coordinates for an S^{p-1}, $\Sigma_m\hat{x}_m^2=1$. The metric in this coordinate system is

$$g=-\cosh^2\chi d\tau^2+d\chi^2+\sinh^2\chi d\Omega_{p-1}. \tag{A7}$$

The explicit symmetries of this form of the metric are the time translation

$$\boldsymbol{e}_{12}=\partial_\tau, \tag{A8}$$

and the $\mathfrak{so}(p)$ symmetries of the sphere,

$$\boldsymbol{e}_{mn}=\hat{x}_m\partial_{\hat{x}_n}-\hat{x}_n\partial_{\hat{x}_m},\quad m,n=3,\ldots,p+2. \tag{A9}$$

The other Killing vectors are

$$\boldsymbol{e}_{1m}=\cos\tau\tanh\chi\hat{x}_m\partial_\tau+\sin\tau\hat{x}_m\partial_\chi+\sin\tau\coth\chi(\delta_{mn}-\hat{x}_m\hat{x}_n)\partial_{\hat{x}_n},$$

$$\boldsymbol{e}_{2m}=-\sin\tau\tanh\chi\hat{x}_m\partial_\tau+\cos\tau\hat{x}_m\partial_\chi+\cos\tau\coth\chi(\delta_{mn}-\hat{x}_m\hat{x}_n)\partial_{\hat{x}_n}, \tag{A10}$$

where $m,n=3,\ldots,p+2$.

APPENDIX B: SYMMETRY-ADAPTED COORDINATES FOR NULLBRANES

As a by-product of our investigations of the quotients of anti–de Sitter space in this paper—most particularly, the studies of the double null rotations in Sec. IV C—we were led to realize that there is a rich structure of symmetries in the nullbrane quotients of flat space which has not been fully exploited in previous work on these solutions.

The nullbrane is a quotient of flat $\mathbb{R}^{1,3}$ by a combination of a null rotation and a translation [20],

$$\xi=\partial_4-\boldsymbol{e}_{12}+\boldsymbol{e}_{23}=\partial_4+(x^1-x^3)\partial_2+x^2(\partial_1+\partial_3), \tag{B1}$$

where x^1 is the timelike coordinate and $\{x_2,x_3,x_4\}$ are spacelike ones. The norm of this Killing vector is $\|\xi\|^2=(x_1-x_3)^2+1$, so it is spacelike everywhere. This quotient was shown to be free of closed causal curves in [20]. There are three Killing vectors in the $\mathfrak{so}(1,3)\ltimes\mathbb{R}^4$ Poincaré algebra on $\mathbb{R}^{1,3}$ which commute with this ξ,

$$
\begin{aligned}
\xi_1 &= -\partial_4 - \boldsymbol{e}_{12} + \boldsymbol{e}_{23},\\
\xi_2 &= \partial_2 - (\boldsymbol{e}_{14} + \boldsymbol{e}_{34}),\\
\xi_3 &= \partial_1 + \partial_3 .
\end{aligned} \tag{B2}
$$

These have norms $\|\xi_1\|^2 = \|\xi_2\|^2 = \|\xi\|^2$ and $\|\xi_3\|^2 = 0$. The only nontrivial commutation relation is $[\xi_1, \xi_2] = -2\xi_3$. The coordinates defined on the nullbrane in [20] do not make any of these additional symmetries manifest. We will now construct an adapted coordinate system which makes the ξ_2 and ξ_3 symmetries manifest: that is, we want $\xi = \partial_{\bar{\phi}}$, $\xi_2 = \partial_{\bar{\psi}}$, and $\xi_3 = \partial_{\bar{v}}$. This requires

$$
\begin{aligned}
&\frac{\partial x^1}{\partial \bar{\phi}} = \frac{\partial x^3}{\partial \bar{\phi}} = x^2, \quad \frac{\partial x_2}{\partial \bar{\phi}} = x^1 - x^3, \quad \frac{\partial x^4}{\partial \bar{\phi}} = 1,\\
&\frac{\partial x^1}{\partial \bar{\psi}} = \frac{\partial x^3}{\partial \bar{\psi}} = x^2, \quad \frac{\partial x_2}{\partial \bar{\psi}} = 1, \quad \frac{\partial x^4}{\partial \bar{\psi}} = x^1 - x^3,\\
&\frac{\partial x^1}{\partial \bar{v}} = \frac{\partial x^3}{\partial \bar{v}} = 1.
\end{aligned} \tag{B3}
$$

Since $x^1 - x^3$ is independent of $\bar{\phi}, \bar{\psi}, \bar{v}$, we will choose to define coordinates so that $x^1 - x^3 = \bar{u}$. A suitable coordinate system is

$$
\begin{aligned}
x^1 + x^3 &= 2\overline{\phi\psi} + \bar{u}(\bar{\phi}^2 + \bar{\psi}^2) + 2\bar{v},\\
x^1 - x^3 &= \bar{u},\\
x^2 &= \bar{\psi} + \overline{u\phi},\\
x^4 &= \bar{\phi} + \overline{u\psi}.
\end{aligned} \tag{B4}
$$

In these coordinates, the flat metric is

$$
g = -2d\bar{u}d\bar{v} + (1+\bar{u}^2)(d\bar{\psi}^2 + d\bar{\phi}^2) + 4\bar{u}d\bar{\phi}d\bar{\psi}. \tag{B5}
$$

The nullbrane is constructed by compactifying the $\bar{\phi}$ coordinate. The determinant of the metric is $-\det g = (1-\bar{u}^2)^2$, so this coordinate system breaks down at $\bar{u} = \pm 1$, where the expressions for x^2 and x^4 lose their linear independence. Thus, although these are symmetry-adapted coordinates, they do not provide global coordinates for the spacetime.

It is interesting to note that in these coordinates the solution resembles a plane wave written in Rosen coordinates. For the uncompactified solution, this is not unexpected; flat space is a trivial plane wave. The interesting observation is that the compactification of ϕ preserves this structure. By a slight change in the coordinate system, we can make a more direct relation to a nontrivial plane wave, and at the same time obtain global coordinates. Instead of Eq. (B4), we set

$$
\begin{aligned}
x^1 + x^3 &= 2\phi\psi + u(\phi^2 + \psi^2) + 2v,\\
x^1 - x^3 &= u,\\
x^2 &= \psi + u\phi,\\
x^4 &= \phi - u\psi.
\end{aligned} \tag{B6}
$$

The flat metric is now

$$
g = -2du dv + (1+u^2)(d\psi^2 + d\phi^2) - 4\psi d\phi du. \tag{B7}
$$

The determinant of the metric is $-\det g = (1+u^2)^2$, so this is now a global coordinate system.

The price we pay is that the symmetry ξ_2 is no longer manifest; on the other hand, this form treats the two Killing vectors ξ_1 and ξ_2 more symmetrically. In these coordinates, $\xi = \partial_\phi$, $\xi_3 = \partial_v$, while the other two Killing vectors are

$$
\begin{aligned}
\xi_1 &= -\frac{1-u^2}{1+u^2}\partial_\phi + \frac{2u}{1+u^2}\partial_\psi + 2\psi\frac{1-u^2}{1+u^2}\partial_v,\\
\xi_2 &= \frac{2u}{1+u^2}\partial_\phi + \frac{1-u^2}{1+u^2}\partial_\psi - 2\psi\frac{2u}{1+u^2}\partial_v .
\end{aligned} \tag{B8}
$$

The inverse coordinate transformation is

$$
\begin{aligned}
u &= x^1 - x^3,\\
\phi &= \frac{x^4 + (x^1-x^3)x^2}{[1+(x^1-x^3)^2]},\\
\psi &= \frac{x^2 - (x^1-x^3)x^4}{[1+(x^1-x^3)^2]},\\
2v &= (x^1+x^3) - \frac{(x^1-x^3)}{1+(x^1-x^3)^2}[(x^2)^2 + (x^4)^2]\\
&\quad - \frac{2}{[1+(x^1-x^3)^2]^2}[x^4 + (x^1-x^3)x^2]\\
&\quad \times [x^2 - (x^1-x^3)x^4].
\end{aligned} \tag{B9}
$$

The advertised relation to the plane wave can be seen if we now set $u = \tan U$. Then

$$
g = \frac{1}{\cos^2 U}[-2dUdv + d\psi^2 + d\phi^2 - 4\psi d\phi dU]. \tag{B10}
$$

The metric in square brackets is a conformally flat plane wave. Furthermore, the symmetry $\xi = \partial_\phi$ that we quotient along annihilates the conformal factor, so we can think of the nullbrane as conformally related to a compactified plane wave. The plane wave nature of this solution can be instantly recognized after the further coordinate transformation

$$V = v + \psi\phi,$$

$$X = \psi\cos U + \phi\sin U,$$

$$Y = -\psi\sin U + \phi\cos U, \tag{B11}$$

which brings the metric to the form

$$g = \frac{1}{\cos^2 U}[-2dUdV - (X^2+Y^2)dU^2 + dX^2 + dY^2]. \tag{B12}$$

This form makes little of the symmetry explicit. The Killing vector we are quotienting along is

$$\xi = \sin U\partial_X + \cos U\partial_Y + (X\cos U - Y\sin U)\partial_V, \tag{B13}$$

and the other symmetries of the quotient are

$$\xi_1 = \sin U\partial_X - \cos U\partial_Y + (X\cos U + Y\sin U)\partial_V,$$

$$\xi_2 = \cos U\partial_X + \sin U\partial_Y + (-X\sin U + Y\cos U)\partial_V,$$

$$\xi_3 = \partial_V. \tag{B14}$$

Note that not only does ξ annihilate the conformal factor; so do the other isometries. Thus, all the isometries of the nullbrane are related to isometries of the conformally related compactified plane wave. We can recognize them as

$$\xi = -\xi_{e_1^*} - \xi_{e_2},$$

$$\xi_1 = -\xi_{e_1^*} + \xi_{e_2},$$

$$\xi_2 = -\xi_{e_1} - \xi_{e_2^*},$$

$$\xi_3 = \xi_{e_V}, \tag{B15}$$

where we write the isometries of the plane wave in the usual basis:

$$\xi_{e_i} = -\cos U\partial_{X^i} + X^i\sin U\partial_V,$$

$$\xi_{e_i^*} = -\sin U\partial_{X^i} - X^i\cos U\partial_V,$$

$$\xi_{e_V} = \partial_V,$$

$$\xi_{e_U} = -\partial_U. \tag{B16}$$

Thus, the quotient of the plane wave that is conformally related to the nullbrane is of the type considered in [79]. The additional symmetry ξ_{e_U} that would be present in the plane wave is broken by the conformal factor. As we saw in Sec. IV C, this is precisely the additional symmetry that appears in the double null rotation.

As in Sec. IV C, in addition to exposing this relation to the plane waves, the global coordinates (B10) allow us to easily find a global time function for the nullbrane, hence demonstrating that it is a stably causal solution. We first rewrite the nullbrane metric in a form suitable for Kaluza-Klein reduction along ϕ,

$$g = \frac{1}{\cos^2 U}[-2dUdv - 4\psi^2 dU^2 + d\psi^2 + (d\phi - 2\psi dU)^2]. \tag{B17}$$

We see that Kaluza-Klein reduction will give a plane wave metric in one dimension lower (up to conformal factor). Hence, applying the results of [70], a suitable time function for the nullbrane is

$$\tau = U + \frac{1}{2}\tan^{-1}\left(\frac{4v}{1+4\psi^2}\right). \tag{B18}$$

It is easy to check that

$$\nabla_\mu\tau\nabla^\mu\tau = -\frac{4\cos^2 U}{[(1+4\psi^2)^2+16v^2]} = -\frac{4(1+u^2)}{[(1+4\psi^2)^2+16v^2]}. \tag{B19}$$

Thus, τ is a good time function on flat space, and since $\mathcal{L}_\xi\tau = 0$, the nullbrane is stably causal by the general argument of [70].

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