Charged black holes in quadratic gravity

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Iterative solutions to fourth-order gravity describing static and electrically charged black holes are constructed. The obtained solutions are parametrized by two integration constants which are related to the electric charge and the exact location of the event horizon. Special emphasis is put on the extremal black holes. It is explicitly demonstrated that in the extremal limit the exact location of the (degenerate) event horizon is given by $r_+ = |e|$. Similarly to the classical Reissner-Nordström solution, the near-horizon geometry of the charged black holes in quadratic gravity, when expanded into the whole manifold, is simply that of Bertotti and Robinson. Similar considerations have been carried out for boundary conditions of the second type which employ the electric charge and the mass of the system as seen by a distant observer. The relations between results obtained within the framework of each method are briefly discussed.

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I. INTRODUCTION

According to our present understanding, the applicability of the conventional Einstein-Hilbert equations is limited to curvatures significantly less than the Planck scale and should be considered as the first approximation to a more fundamental theory. Although it is not clear how to construct this more fundamental theory, it seems reasonable to address the question of its possible low-energy impact. In the quest for imprints of quantum gravity effects in the classical regime, an especially helpful observation is that, regardless of the formulation of the fundamental theory, its low-energy effective action should consist of the classical gravity supplemented by covariant higher-curvature terms and higher-derivative terms involving other physical fields. The gravitational part of the total action can be written therefore as

$$I_G = I_0 + I_1 + I_2 + \cdots, \tag{1}$$

where I_0 is the Einstein-Hilbert action and I_i (for $i \geqslant 1$) denotes combination of operators of dimension $2i+2$ with numerical coefficients depending on the type of theory.

Among the various modifications of general relativity proposed so far, a prominent role is played by Lagrangians describing quadratic gravity. The interest in theories of this type is motivated by the fact that such theories appear, as expected, in the low-energy limit of string theory and in attempts to construct a renormalizable theory of gravity coupled to matter [1–3]. It should also be remembered that theories of this kind are almost as old as general relativity itself and like general relativity have a noble parentage [4–7].

Higher-derivative terms in the equations of the gravitational field have very important consequences. One of them consists in the obvious observation that such equations are presumably hard to solve even in cases when Einstein's equations admit solutions expressible in terms of known special functions. Further, it should be emphasized that although the family of solutions is rich one should be cautious in selecting physically meaningful solutions [8]. Naive acceptance of solutions may lead to interesting behavior of the system but the wrong physics. It is natural therefore that in order to gain insight into the nature of the nontrivial problem one has to refer to approximate methods. Such methods are expected to yield reasonable solutions to the problem and simultaneously select the physical ones [8–10].

For almost two decades perturbative methods have been extensively used in black hole physics in the context of the back reaction of the quantized fields (both massive and massless) on the metric [11–19]. The (one-loop) approximation to the stress-energy tensor of massive quantized fields in the large mass limit, for example, could be regarded as a higher-curvature term constructed from the curvature tensor, its contractions, and its covariant derivatives [20–24]. It should be noted, however, that back reaction analyses are, inevitably, limited to linearized equations. Similar considerations in the context of a string inspired action with quadratic and quartic terms constructed from the Riemann tensor have been carried out in Refs. [25,26].

A general perturbative method has been proposed in a series of papers [27–29], and, subsequently, successfully applied in a number of physically interesting cases, such as the de Sitter universe, cosmic strings, charged black holes, and gravitational waves [30]. The method is simple: assuming that the quadratic terms can be considered as small contribution to the effective stress-energy tensor, which is justified for small curvatures of spacetime, one can iteratively solve the resulting equations order by order starting with the classical Einstein field equations. Among various applications the most interesting from our point of view is the perturbative solution describing static, spherically symmetric, and electrically charged black hole constructed by Campanelli, Lousto, and Audretsch (CLA) in Ref. [29]. The calculations have been carried out up to third order with special emphasis put on the thermodynamical issues. However, their analysis is somewhat obscured by the choice of the boundary conditions, and, moreover, propagation of errors in the metric tensor caused errors in some of the results.

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In this paper we shall return to this problem and express the resultant solutions in terms of the electric charge and the *exact* location of the event horizon, r_+. It seems that such a choice is more natural and reveals the simplicity of the extremal configuration. It is also helpful in analyses of the near-horizon geometry of the extremal black hole. Our method is similar to that of York [11] and Lu and Wise [31] with a different choice of the boundary conditions.

The paper is organized as follows. In Sec. II we introduce the basic equations and briefly sketch the method employed. We choose the line element in the form propounded by Visser [32], which has proved to be a very useful representation. Since the functional dependence on the metric tensor of all terms appearing in the equations is known, we start with the exact location of the event horizon from the very beginning. This means that the higher-order terms do not contribute to r_+. To express our results in a more familiar form we also introduce the horizon defined mass. A discussion of the temperature and entropy of the nonextremal configuration is presented in Sec. III. A careful examination of the extremal configuration and the role played by the Bertotti-Robinson geometry is given in Sec. IV. Moreover, it is shown that the extremal black holes are characterized by $r_+=|e|$. In Sec. V we discuss the problem from the point of view of a distant observer and explicitly demonstrate the relations between the two choices of boundary conditions. Section VI contains final remarks.

II. EQUATIONS

The coupled system of electrodynamics and quadratic gravity with the cosmological term set to zero is described by the action

$$I=\frac{1}{16\pi}I_G+I_{em}, \tag{2}$$

where

$$I_G=\int g^{1/2}(R+\alpha R^2+\beta R_{ab}R^{ab})d^4x \tag{3}$$

and

$$I_{em}=-\frac{1}{16\pi}\int g^{1/2}F d^4x. \tag{4}$$

Here $F=F_{ab}F^{ab}$, $F_{ab}=A_{b,a}-A_{a,b}$, and all symbols have their usual meaning. The Kretschmann scalar has been removed from the action by means of the Gauss-Bonnet invariant.

For the numerical parameters α and β, we assume that they are small and of comparable order, otherwise they would lead to observational effects within our solar system. Their ultimate values should be determined from observations of light deflection, binary pulsars, and cosmological data [30,33,34]. Following [29], we shall restrict ourselves to spacetimes of small curvatures, for which the conditions

$$|\alpha R|\ll 1,\quad |\beta R_{ab}|\ll 1 \tag{5}$$

hold. Additional constraints could be obtained from nontachyon conditions, which are closely connected to the stability of the solutions [35]. Demanding the linearized equations to possess a real mass [36,37], one obtains

$$3\alpha-\beta\geqslant 0,\quad \beta\leqslant 0. \tag{6}$$

To simplify the calculations, especially to keep control of the order of terms in complicated series expansions, we shall introduce another (dimensionless) parameter ε, substituting $\alpha\rightarrow\varepsilon\alpha$ and $\beta\rightarrow\varepsilon\beta$. We shall put $\varepsilon=1$ in the final stage of the calculations. As the coefficient α does not appear in the final formulas, introduction of the additional parameter might appear as an unnecessary complication in the present context. However, it is really helpful when dealing with more general Lagrangians, such as, for example, those of nonlinear electrodynamics.

Functionally differentiating S with respect to the metric tensor, one obtains the system

$$G_a^b-\alpha\,{}^{(1)}H_a^b-\beta\,{}^{(2)}H_a^b=8\pi T_a^b, \tag{7}$$

where

$$\frac{1}{g^{1/2}}\frac{\delta}{\delta g_{ab}}\int g^{1/2}Rd^4x=-G^{ab}, \tag{8}$$

$$\begin{aligned}{}^{(1)}H^{ab}&=\frac{1}{g^{1/2}}\frac{\delta}{\delta g_{ab}}\int d^4x g^{1/2}R^2\\&=2R^{;\,ab}-2RR^{ab}+\frac{1}{2}g^{ab}(R^2-4\Box R)\end{aligned} \tag{9}$$

and

$$\begin{aligned}{}^{(2)}H^{ab}&=\frac{1}{g^{1/2}}\frac{\delta}{\delta g_{ab}}\int d^4x g^{1/2}R_{ab}R^{ab}\\&=R^{;\,ab}-\Box R^{ab}-2R_{cd}R^{cbda}+\frac{1}{2}g^{ab}(R_{cd}R^{cd}-\Box R).\end{aligned} \tag{10}$$

The Faraday tensor F_{ab} satisfies

$$F^{ab}{}_{;a}=0 \tag{11}$$

and

$$*F^{ab}{}_{;a}=0. \tag{12}$$

The stress-energy tensor T^{ab}, defined as

$$T^{ab}=\frac{2}{g^{1/2}}\frac{\delta}{\delta g_{ab}}S_{em}, \tag{13}$$

is therefore given by

$$T_a^b = \frac{1}{4\pi}\left(F_{ca}F^{cb} - \frac{1}{4}\delta_a^b F \right). \quad (14)$$

Let us consider a spherically symmetric and static geometry. As is well known, the spacetime metric can be cast in the form

$$ds^2 = -f(r)e^{2\psi(r)}dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (15)$$

with [32]

$$f(r) = 1 - \frac{2m(r)}{r}. \quad (16)$$

The metric has horizons at values of the radial coordinate satisfying

$$m(r_+) = \frac{r_+}{2}. \quad (17)$$

In what follows we shall restrict our analyses to the outermost horizon, i.e., the one for which

$$m(r) < \frac{r}{2} \quad (18)$$

for $r > r_+$.

Spherical symmetry places restrictions on the form of the tensor F_{ab}: the only nonvanishing components of the Faraday tensor are connected with the static radial electric and magnetic fields. Simple integration of the Maxwell equations gives

$$F_{01} = -\frac{a_1}{r^2}e^{\psi} \quad (19)$$

and

$$F_{23} = a_2 \sin\theta, \quad (20)$$

where a_1 and a_2 are integration constants interpreted as the electric and the magnetic charge, respectively, and the former will henceforth be denoted by e. In what follows we shall confine ourselves to solutions with an electric charge only.

The stress-energy tensor for the line element (15) and the Faraday tensor (19) is simply

$$T_t^t = T_r^r = -T_\theta^\theta = -T_\phi^\phi = -\frac{e^2}{8\pi r^4} \quad (21)$$

and in this form is independent of the metric potentials.

Now we are going to construct an approximate solution describing a static and spherically symmetric charged black hole. The system of differential equations for $m(r)$ and $\psi(r)$ is to be supplemented with the appropriate, physically motivated boundary conditions. Our preferred choice, requiring knowledge of the exact location of the event horizon, r_+, is

$$m(r_+) = \frac{r_+}{2}, \quad (22)$$

which, as we shall see, is related to the horizon defined mass of the black hole. With such a choice we express the solution in terms of the exact location of the event horizon and the electric charge. On the other hand it seems natural to express the results in term of the mass of the system as seen by a distant observer:

$$\lim_{r\to\infty} m(r) = M_\infty. \quad (23)$$

For the function $\psi(r)$ we shall adopt the natural condition

$$\lim_{r\to\infty} \psi(r) = 0. \quad (24)$$

Let us select boundary conditions of the first type. For the functions $m(r)$ and $\psi(r)$, we assume that they can be expanded as

$$m(r) = M_0(r) + \sum_{k=1}^{n} \varepsilon^k M_k(r) + \mathcal{O}(\varepsilon^{n+1}) \quad (25)$$

and

$$\psi(r) = \sum_{k=1}^{n} \varepsilon^k \psi_k(r) + \mathcal{O}(\varepsilon^{n+1}). \quad (26)$$

It should be noted that $\psi_0(r) = 0$.

Since we have assumed the expansion of $m(r)$ in the form given by Eq. (25), the condition (22) can be rewritten in the form

$$M_i(r_+) = \begin{cases} \frac{r_+}{2} & \text{if } i = 0, \\ 0 & \text{if } i \geqslant 1, \end{cases} \quad (27)$$

and such a choice is a typical mathematical procedure [38]. Now, inserting the line element (15) into Eqs. (7), making use of the expansions (25) and (26), and finally collecting the terms with like powers of the parameter ε, one obtains a system of differential equations for M_0, M_k, and ψ_k ($k \geqslant 1$) of ascending complexity. The zeroth-order equations reduce to those of an Einstein-Maxwell system, and, after simple manipulations, lead to the following integral:

$$M_0 = \frac{1}{4}\int r^2 F dr + C_1, \quad (28)$$

which, when combined with the condition (22), yields

$$M_0(r) = \frac{r_+}{2} + \frac{e^2}{2r_+} - \frac{e^2}{2r}. \quad (29)$$

The first-order equation constructed from the radial component of Eq. (7) reads

$$M_1' = \beta\left(\frac{2M_0'}{r^2} - \frac{8M_0M_0'}{r^3} + \frac{2M_0'^2}{r^2} - \frac{2M_0''}{r} + \frac{5M_0M_0''}{r^2} - \frac{M_0'M_0''}{r} + \frac{M_0''^2}{2} + M_0^{(3)} - \frac{M_0M_0^{(3)}}{r} - M_0'M_0^{(3)} + rM_0^{(4)} - 2M_0M_0^{(4)}\right). \tag{30}$$

On the other hand, the first-order equation constructed from the time component of Eq. (7), when combined with Eq. (30), can be easily integrated to yield

$$\psi_1(r) = \beta\left(M_0^{(3)} - \frac{4}{r^2}M_0'\right) + C_2, \tag{31}$$

where C_2 is an integration constant.

Inserting Eq. (29) into Eq. (30) and integrating the equation thus obtained, one has

$$M_1(r) = \beta\left(\frac{2e^2}{r^3} - \frac{3e^2r_+}{2r^4} - \frac{3e^4}{2r_+r^4} + \frac{6e^4}{5r^5} - \frac{e^2}{2r_+^3} + \frac{3e^4}{10r_+^5}\right). \tag{32}$$

Further, substituting the zeroth-order solution to Eq. (31) and making use of the boundary condition (22) gives

$$\psi_1(r) = \beta\frac{e^2}{r^4}. \tag{33}$$

Starting from the second order, the differential equations become more and more complicated and we shall display their solutions only. After some algebra one has

$$M_2(r) = \beta^2\left(\frac{351}{4}\frac{e^4r_+}{r^8} - \frac{36e^2}{r^5} - \frac{1156e^4}{7r^7} + \frac{76e^2r_+}{r^6} + \frac{76e^4}{r^6r_+} - \frac{704}{15}\frac{e^6}{r^9} - 40\frac{e^2r_+^2}{r^7} - 40\frac{e^6}{r_+^2r^7} + \frac{351}{4}\frac{e^6}{r_+r^8} + \frac{3}{2}\frac{e^4}{r_+^3r^4} - \frac{9}{10}\frac{e^6}{r^4r_+^5} + \frac{1}{12}\frac{e^6}{r_+^9} - \frac{3}{28}\frac{e^4}{r_+^7}\right) \tag{34}$$

and

$$\psi_2 = \beta^2\left(32\frac{e^2r_+}{r^7} + 32\frac{e^4}{r^7r_+} - 24\frac{e^2}{r^6} - 41\frac{e^4}{r^8}\right) \tag{35}$$

for the second order, whereas the third-order results read

$$M_3(r) = \beta^3\left(1440\frac{e^2}{r^7} - 5508\frac{e^2r_+}{r^8} + 21168\frac{e^4}{r^9} + \frac{12392664}{385}\frac{e^6}{r^{11}} + \frac{331624}{65}\frac{e^8}{r^{13}} + \frac{2229}{572}\frac{e^8}{r_+^{13}} - 4\frac{e^2}{r_+^7} + \frac{652}{55}\frac{e^4}{r_+^9} - \frac{2587}{220}\frac{e^6}{r_+^{11}} + 6816\frac{e^6}{r^9r_+^2} - 76\frac{e^4}{r^6r_+^3} - 2744\frac{e^2r_+^3}{r^{10}} + \frac{228}{5}\frac{e^6}{r^6r_+^5} - 5508\frac{e^4}{r_+r^8} - \frac{130732}{5}\frac{e^4r_+}{r^{10}} - \frac{130732}{5}\frac{e^6}{r^{10}r_+} + \frac{115176}{11}\frac{e^4r_+^2}{r^{11}} - \frac{51347}{4}\frac{e^6r_+}{r^{12}} + \frac{115176}{11}\frac{e^8}{r_+^2r^{11}} - \frac{51347}{4}\frac{e^8}{r_+r^{12}} - 2744\frac{e^8}{r_+^3r^{10}} + 80\frac{e^4}{r_+^2r^7} + 32\frac{e^6}{r_+^4r^7} + 6816\frac{e^2r_+^2}{r^9} - 48\frac{e^8}{r_+^6r^7} - \frac{351}{4}\frac{e^6}{r_+^3r^8} + \frac{1053}{20}\frac{e^8}{r_+^5r^8} - \frac{e^8}{4r^4r_+^9} + \frac{9}{28}\frac{e^6}{r_+^7r^4}\right) \tag{36}$$

and

$$\psi_3(r) = \beta^3\left(\frac{96}{5}\frac{e^6}{r^7r_+^5} - 3264\frac{e^2r_+}{r^9} - 3264\frac{e^4}{r^9r_+} - 32\frac{e^4}{r^7r_+^3} - \frac{74560}{11}\frac{e^4r_+}{r^{11}} - \frac{74560}{11}\frac{e^6}{r^{11}r_+} + 2352\frac{e^2r_+^2}{r^{10}} + \frac{48384}{5}\frac{e^4}{r^{10}} + 2352\frac{e^6}{r^{10}r_+^2} + \frac{69572}{15}\frac{e^6}{r^{12}} + 1080\frac{e^2}{r^8}\right). \tag{37}$$

It is possible to express this result in a more familiar form by introducing the horizon defined mass M, i.e., to represent the solution in terms of (e,M) rather than (e,r_+). This can be easily done by employing the equality

$$M = \frac{r_+}{2} + \frac{e^2}{2r_+} \tag{38}$$

and repeating the calculations with

$$M_0(r) = M - \frac{e^2}{2r}. \tag{39}$$

Now the *exact* location of the event horizon is related to the horizon defined mass by the classical formula

$$r_+ = M + (M^2 - e^2)^{1/2}. \tag{40}$$

It should be noted, however, that although the zeroth-order equation gives the exact location of the event horizon the same is not true for its second root,

$$r_c = \frac{e^2}{r_+}. \tag{41}$$

Indeed, it can be shown that the inner horizon r_- is given by

$$\begin{aligned} r_- = r_c &+ \frac{\beta}{5}\left(\frac{3e^4}{r_+^5} - \frac{2e^2}{r_+^3} - \frac{2}{r_+} - \frac{2r_+}{e^2} + \frac{3r_+^3}{e^4}\right) \\ &+ \beta^2\left(\frac{e^6}{6r_+^9} - \frac{214}{525}\frac{e^4}{r_+^7} + \frac{38}{525}\frac{e^2}{r_+^5} + \frac{206}{525r_+^3} - \frac{131}{105e^2 r_+} \right.\\ &\left. + \frac{58}{105}\frac{r_+}{e^4} + \frac{374}{525}\frac{r_+^3}{e^6} + \frac{542}{525}\frac{r_+^5}{e^8} - \frac{191}{150}\frac{r_+^7}{e^{10}}\right) + \mathcal{O}(\beta^3), \end{aligned} \tag{42}$$

and, moreover, a simple calculation shows that for $r_+ = |e|$ both horizons coincide to the required order. Such behavior strongly suggests that this very relation describes degenerate horizons of the extreme black hole. Since extreme black holes deserve more accurate treatment we shall postpone further analysis of this and related problems to Sec. IV.

III. TEMPERATURE AND ENTROPY

One of the most important characteristics of the black hole is its Hawking temperature T_H. To investigate how T_H is modified by the quadratic terms, we employ the standard procedure of the Wick rotation. The complexified line element thus obtained has no conical singularity as $r \to r_+$ provided the time coordinate is periodic with a period P, which is to be identified with the reciprocal of T_H. Elementary considerations carried out for the general static and spherically symmetric spacetime describing a black hole lead to the formula

$$P = \frac{1}{T_H} = 4\pi \lim_{r \to r_+} (g_{tt} g_{rr})^{1/2}\left(\frac{d}{dr} g_{tt}\right)^{-1}. \tag{43}$$

For the line element (15) and the boundary condition (22) the above result assumes the simple form

$$T_H = \frac{e^{\psi(r_+)}}{2\pi r_+^2}\left(\frac{r_+}{2} - r_+ \frac{dm(r)}{dr}\bigg|_{r=r_+}\right). \tag{44}$$

Making use of Eqs. (29) and (32)–(37), and collecting the terms with like powers of β, after massive simplification one obtains

$$\begin{aligned} T_H = &\frac{1}{4\pi r_+}\left(1 - \frac{e^2}{r_+^2}\right) + \frac{\beta e^2}{4\pi r_+^5}\left(1 - \frac{e^2}{r_+^2}\right) - \frac{\beta^2 e^4}{8\pi r_+^9}\left(1 - \frac{e^2}{r_+^2}\right) \\ &+ \frac{\beta^3 e^2}{440\pi r_+^{13}}\left(1 - \frac{e^2}{r_+^2}\right)(880 r_+^4 - 3232 e^2 r_+^2 + 2455 e^4) \\ &+ \mathcal{O}(\beta^4). \end{aligned} \tag{45}$$

We suspect that the common factor in the above expression appears in all higher-order terms and for $r_+ = |e|$ the black hole temperature approaches zero. We shall return to this problem later in the text.

We have not, as yet, imposed any constraints on the parameter β, but now we are going to examine the consequences of the nontachyon condition, which, in the case at hand, is simply $\beta < 0$. For the nonextremal black hole the terms proportional to β and β^2 in the right hand side of Eq. (45) are strictly negative, and, therefore, one concludes that the temperature (to this order) is lower as compared with the Reissner-Nordström black hole described by the same values of r_+ and e. On the other hand the higher-order terms proportional to β^k (for $k \geq 3$) change sign; however, for reasonable values of the coupling constant their contribution to the total temperature is negligible.

The entropy of the black hole in quadratic gravity may be calculated using various methods. Here we shall employ Wald's Noether charge technique [39], which, as has been shown in Refs. [40,41], may be safely applied for Lagrangians of the type (2) and leads to the remarkably simple and elegant general result

$$S = \frac{1}{4}\int_\Sigma d^2x \sqrt{h}[1 + 2\alpha R + \beta(R - h^{ab} R_{ab})], \tag{46}$$

where h_{ab} is the induced metric on Σ and the surface integral is taken across an arbitrary section of the event horizon. In our representation, the entropy after some algebra may be compactly written as

$$\begin{aligned} S = &\frac{A}{4} - \frac{8\pi^2\beta}{A} e^2 - \frac{1024\pi^4 e^2 \beta^3}{A^5}(A - 4\pi e^2)^2 \\ &+ \frac{16384\pi^5 e^2 \beta^4}{A^7}(A - 4\pi e^2)^2 (26\pi e^2 - 5A) + \mathcal{O}(\beta^5), \end{aligned} \tag{47}$$

where $A = 4\pi r_+^2$ is the surface of the event horizon. We attribute discrepancies between our result and the entropy computed by CLA to errors in their metric tensor.

IV. THE EXTREMAL CONFIGURATION AND ITS NEAR-HORIZON GEOMETRY

In this section we shall investigate the important issue of extremal black holes. On general grounds one expects that the extremal configuration is the one in which (at least) two horizons merge. This means that in the simplest case the first two terms of the expansion

$$f(r)=f(r_+)+\left.\frac{df}{dr}\right|_{r=r_+}(r-r_+)+\frac{1}{2}\left.\frac{d^2f}{dr^2}\right|_{r=r_+}(r-r_+)^2+\cdots \tag{48}$$

vanish. It is evident that if the first n terms in the above expansion are absent one has an n-fold merging of n horizons. In view of the further applications, we also expect regularity of the function $\psi(r)$ as the degenerate event horizon is approached:

$$|\psi(r_+)|<\infty, \quad \left|\left.\frac{d\psi(r)}{dr}\right|_{r=r_+}\right|<\infty. \tag{49}$$

In order to determine the location of the event horizon, i.e., to relate the integration constants r_+ and e, one has to consider the consequences of the vanishing of the black hole temperature (surface gravity). Equation (45) strongly indicates that $r_+=|e|$. To demonstrate the validity of this relation let us consider the (rr) component of Eq. (7). It can be shown that if $f'(r_+)=0$, the conditions (49) are satisfied, and, additionally,

$$|f^{(3)}(r_+)|<\infty, \quad |\psi^{(3)}(r_+)|<\infty; \tag{50}$$

then the second derivative of f computed at $r=r_+$ satisfies the constraint equation

$$\frac{1}{r_+^2}+\frac{\beta}{r_+^4}-\frac{\beta}{4}\left(\left.\frac{d^2f}{dr^2}\right|_{r_+}\right)^2=\frac{e^2}{r_+^4}. \tag{51}$$

The metric of the electrically charged black hole must be independent of the coupling constant α [27,42]. Repeating the calculations for the (rr) component of the tensor ${}^{(1)}H_a^b$ at $r=r_+$ one has

$${}^{(1)}H_r^r(r_+)=\frac{2}{r_+^4}-\frac{1}{2}\left(\left.\frac{d^2f}{dr^2}\right|_{r_+}\right)^2. \tag{52}$$

The absence of terms proportional to α^k $(k=0,1,2,\ldots)$ requires ${}^{(1)}H_a^b$ to vanish identically, and, since both conditions are to be satisfied simultaneously, one obtains the expected exact result:

$$r_+=|e|. \tag{53}$$

Equivalently, expressing the location of r_+ in terms of the horizon defined mass yields

$$r_+=M. \tag{54}$$

The relations obtained are identical with those that characterize the geometry of the extremal Reissner-Nordström black hole. As the particular form of the traceless stress-energy tensor played a decisive role in our derivation, one should not expect simple generalizations. In a more general case, such as for example that of nonlinear electrodynamics, there is an explicit dependence on the coupling constant α, and the stress-energy tensor lacks the simplicity of its Maxwellian analogue. Indeed, repeating the calculations for a line element of the type (15) and an arbitrary stress-energy tensor, one has

$$(\beta+2\alpha)\left[\left(\frac{1}{2}f''(r_+)\right)^2-\frac{1}{r_+^4}\right]-\frac{1}{r_+^2}=8\pi T_r^r(r_+) \tag{55}$$

and

$$-(\beta+2\alpha)\left[\left(\frac{1}{2}f''(r_+)\right)^2-\frac{1}{r_+^4}\right]+\frac{1}{2}f''(r_+)=8\pi T_\theta^\theta(r_+). \tag{56}$$

It should be noted that in order to obtain Eq. (56) one has to assume $|f^{(4)}(r_+)|<\infty$ and $|\psi^{(4)}(r_+)|<\infty$. The spherical symmetry and the field equations at the event horizon, respectively, give

$$T_\theta^\theta=T_\phi^\phi, \quad T_t^t=T_r^r. \tag{57}$$

Combining the above equations, one obtains

$$f''(r_+)-\frac{2}{r_+}=8\pi T_a^a(r_+), \tag{58}$$

where T_a^a is the trace of the stress-energy tensor.

As is well known the closest vicinity of the event horizon of the extremal Reissner-Nordström black hole, after the coordinate transformation

$$r=r_+\left(1+\frac{r_+}{y}\right), \tag{59}$$

can be approximated by the Bertotti-Robinson line element [43,44]

$$ds^2=\frac{r_+^2}{y^2}(-dt^2+dy^2+y^2d\Omega^2). \tag{60}$$

Employing new static coordinates, the Bertotti-Robinson line element can be rewritten in the form

$$ds^2=r_+^2(-\sinh^2\chi d\tau^2+d\chi^2+d\theta^2+\sin^2\theta d\phi^2) \tag{61}$$

or

$$ds^2=-\left(\frac{x^2}{r_+^2}-1\right)dT^2+\left(\frac{x^2}{r_+^2}-1\right)^{-1}dx^2 +r_+^2(d\theta^2+\sin^2\theta d\phi^2). \tag{62}$$

The latter form is particularly useful in demonstrating that the topology of the Bertotti-Robinson solution is $AdS_2\times S^2$, i.e., it is a simple topological product of a (1+1)-dimensional anti–de Sitter spacetime and a two-sphere of radius r_+.

One expects that this spacetime also plays some role for extremal black holes in quadratic gravity. Indeed, first observe that, because of simplicity of the Bertotti-Robinson line element, the tensors ${}^{(1)}H_{ab}$ and ${}^{(2)}H_{ab}$ as well as the curvature scalar vanish. The only contribution to the left hand side of Eq. (7) comes, therefore, from the Ricci tensor, which for the metric (60) is given by

$$R_t^t = R_r^r = -R_\theta^\theta = -R_\phi^\phi = -\frac{1}{r_+^2}. \tag{63}$$

As the stress-energy tensor of the electromagnetic field is simply

$$T_t^t = T_r^r = -T_\theta^\theta = -T_\phi^\phi = -\frac{1}{8\pi r_+^2}, \tag{64}$$

one concludes that the line element (60) is an exact solution if $r_+ = |e|$. Moreover, inspection of Eq. (7) suggests that, after redefinition of the time coordinate, the dominant contribution to the metric potentials g_{tt} and g_{rr} near the event horizon comes from the function $M_0(r)$. It follows then that locally it resembles the geometry of the extremal Reissner-Nordström black hole.

To investigate the role played by the Bertotti-Robinson solution in more detail let us return to the expansion (48) and ask when the near-horizon geometry of the extremal black hole is that described by the line element (60). In the vicinity of the event horizon the line element may be written as

$$ds^2 = -e^{2\psi(r_+)}F(r-r_+)^2 dt^2 + \frac{1}{F(r-r_+)^2}dr^2 + r_+^2 d\Omega^2, \tag{65}$$

where

$$F = \frac{1}{2}\left.\frac{d^2 f}{dr^2}\right|_{r=r_+}. \tag{66}$$

It can be easily demonstrated that, by expressing the line element in terms of a new coordinate y defined by means of the relation

$$r = r_+\left(1 + \frac{r_+}{e^{\psi(r_+)}y}\right), \tag{67}$$

one obtains the line element (60) provided

$$\left.\frac{d^2 f}{dr^2}\right|_{r=r_+} = \frac{2}{r_+^2}. \tag{68}$$

Inspection of Eq. (52) shows that it is precisely the relation that is satisfied by an extremal black hole. It should be noted that from Eq. (58) our demonstration requires the trace of the stress-energy tensor to vanish at $r = r_+$ only.

On the other hand, if $f''(r_+)$ does not obey Eq. (68), the topology of the solution (65) is still a simple product of $AdS_2 \times S^2$, but with a different modulus of curvature. Indeed, it can be easily shown that

$$R = K_{AdS_2} + K_{S^2}, \tag{69}$$

where

$$K_{AdS_2} = -2F \quad \text{and} \quad K_{S^2} = \frac{2}{r_+^2}. \tag{70}$$

Upon the substitution

$$r = r_+ + \frac{1}{e^{\psi(r_+)}Fy}, \tag{71}$$

one obtains

$$ds^2 = \frac{1}{Fy^2}(-dt^2 + dy^2) + r_+^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{72}$$

Finally, observe that vanishing of the curvature scalar as $r \to r_+$ yields Eq. (68), and the line element (72) reduces to (60).

V. A DISTANT OBSERVER POINT OF VIEW

In this section we shall briefly examine the consequences of the second choice of the boundary conditions as given by Eqs. (23) and (24). This problem was considered earlier by CLA. Unfortunately, their metric tensor and hence other characteristics of the charged black holes in quadratic gravity contain errors in the terms proportional to β^3.

Before proceeding further let us return to the function $m(r)$ constructed with the aid of the condition (22). Its limit as $r \to \infty$ is simply

$$M_\infty = \frac{r_+}{2} + \frac{e^2}{2r_+} + \beta\left(\frac{3e^4}{10r_+^5} - \frac{e^2}{2r_+^3}\right) + \beta^2\left(\frac{e^6}{12r_+^9} - \frac{3e^4}{28r_+^7}\right) - \beta^3\left(\frac{4e^2}{r_+^7} - \frac{652e^4}{55r_+^9} + \frac{2587e^6}{220r_+^{11}} - \frac{2229e^8}{572r_+^{13}}\right) + \mathcal{O}(\beta^4) \tag{73}$$

and is interpreted as the total mass of the system as seen by a distant observer and expressed in terms of the exact location of the event horizon and electric charge. On the other hand, one can represent the solution to the system (7) in terms of the total mass as seen from large distances and the electric charge from the very beginning. In this case the boundary conditions for the expansion (25) are to be rewritten in the form

$$\lim_{r\to\infty} M_i(r) = \begin{cases} M_\infty & \text{if } i=0, \\ 0 & \text{if } i \geqslant 1, \end{cases} \tag{74}$$

while the condition (24) remains, of course, intact. Repeating the calculations order by order with the new boundary conditions, one obtains

$$m(r)=M_\infty-\frac{e^2}{2r}+\beta\left(2\frac{e^2}{r^3}-3\frac{e^2M_\infty}{r^4}+\frac{6e^4}{5r^5}\right)+\beta^2\left(152\frac{e^2M_\infty}{r^6}-\frac{596}{7}\frac{e^4}{r^7}-\frac{704}{15}\frac{e^6}{r^9}-160\frac{e^2M_\infty^2}{r^7}+\frac{351}{2}\frac{e^4M_\infty}{r^8}-36\frac{e^2}{r^5}\right)$$
$$+\beta^3\left(27264\frac{e^2M_\infty^2}{r^9}-11016\frac{e^2M_\infty}{r^8}+1440\frac{e^2}{r^7}+\frac{4330344}{385}\frac{e^6}{r^{11}}-\frac{179144}{5}\frac{e^4M_\infty}{r^{10}}+\frac{331624}{65}\frac{e^8}{r^{13}}+\frac{460704}{11}\frac{e^4M_\infty^2}{r^{11}}\right.$$
$$\left.-\frac{51347}{2}\frac{e^6M_\infty}{r^{12}}-21952\frac{e^2M_\infty^3}{r^{10}}+7536\frac{e^4}{r^9}\right)+\mathcal{O}(\beta^4) \tag{75}$$

and

$$\psi(r)=\beta\frac{e^2}{r^4}-\beta^2\left(\frac{24e^2}{r^6}-\frac{64e^2M_\infty}{r^7}+\frac{41e^4}{r^8}\right)$$
$$+\beta^3\left(\frac{24864e^4}{5r^{10}}+\frac{9408e^2M_\infty^2}{r^{10}}+\frac{69572e^6}{15r^{12}}+\frac{1080e^2}{r^8}\right.$$
$$\left.-\frac{6528e^2M_\infty}{r^9}-\frac{149120e^4M_\infty}{11r^{11}}\right)+\mathcal{O}(\beta^4). \tag{76}$$

Equations (75) and (76) are sufficient to determine g_{tt} and g_{rr} to $\mathcal{O}(\beta^4)$. The metric tensor thus obtained differs from that obtained by CLA. To demonstrate that our calculations lead to correct results, let us make use of a consistency check. Inserting Eq. (73) into Eqs. (75) and (76), one obtains, as expected, Eqs. (29) and (32)–(37). Since the calculations have been carried out independently we conclude that our results are correct.

To determine the location of the event horizon one can either iteratively solve the equation

$$g_{tt}(r_+)=0 \tag{77}$$

or invert the relation (73). Both methods give the same result, which reads

$$r_+=r_0+\beta\frac{e^2(5r_0^2-3e^2)}{5r_0^3(r_0^2-e^2)}$$
$$-\beta^2\frac{e^4(2925r_0^6-6515r_0^4e^2+5095r_0^2e^4-1337e^6)}{1050r_0^7(r_0^2-e^2)^3}$$
$$-\beta^3\frac{e^2}{(r_0^2-e^2)^5}\left(\frac{17268913}{75075}\frac{e^{10}}{r_0^7}-\frac{61234643}{750750}\frac{e^{12}}{r_0^9}\right.$$
$$-\frac{137993}{770}\frac{e^4}{r_0}+\frac{3265043}{10010}\frac{e^6}{r_0^3}-\frac{26686967}{75075}\frac{e^8}{r_0^5}$$
$$\left.+\frac{436497}{35750}\frac{e^{14}}{r_0^{11}}-8r_0^3+\frac{3064}{55}e^2r_0\right)+\mathcal{O}(\beta^4), \tag{78}$$

where

$$r_0=M_\infty+(M_\infty^2-e^2)^{1/2}. \tag{79}$$

Repeating the steps of Sec. III necessary to compute the Hawking temperature and expressing the final result in terms of e and r_0, one has

$$T_H=\frac{1}{4\pi r_0^3}(r_0^2-e^2)+\beta\frac{e^4(2r_0^2-e^2)}{5r_0^7(r_0^2-e^2)}$$
$$+\beta^2\frac{e^4}{\pi r_0^{11}(r_0^2-e^2)^3}\left(\frac{8}{25}e^8-\frac{4}{21}e^2r_0^6+\frac{113}{75}e^4r_0^4\right.$$
$$\left.-\frac{676}{525}e^6r_0^2-\frac{3}{7}r_0^8\right)+\mathcal{O}(\beta^3). \tag{80}$$

Although the extremal configuration can be studied in (e,M_∞) representation it is not the best choice. Indeed, simple calculations give for the location of the event horizon

$$r_+=M_\infty+\frac{1}{5M_\infty}\beta-\frac{17}{1050M_\infty^3}\beta^2+\frac{317}{68250M_\infty^5}\beta^3+\mathcal{O}(\beta^4), \tag{81}$$

whereas the relation between the total mass as seen by a distant observer and the electric charge has the form

$$M_\infty^2=e^2-\frac{2}{5}\beta-\frac{4}{525e^2}\beta^2-\frac{8}{1365e^4}\beta^3+\mathcal{O}(\beta^4). \tag{82}$$

This can be contrasted with the simple relation between r_+, and $|e|$ given by Eqs. (53) and (54). Finally, observe that, since the entropy as given by Eq. (47) is expressed in terms of the surface of the event horizon and the electric charge, we expect that it should be described by the same formula for both choices of the conditions (22) and (23).

VI. CONCLUSION AND SUMMARY

In this paper we have constructed iterative solutions describing spherically symmetric and static black holes to the equations of fourth-order gravity with the source term given by the stress-energy tensor of the electromagnetic field. The line elements obtained are parametrized by two integration constants which are related to the electric charge and the exact location of the event horizon. The metric potentials thus computed enabled construction of the basic characteristics of the black hole: its Hawking temperature and entropy.

Special emphasis has been put on extremal black holes. Specifically, it has been explicitly demonstrated that in the extremal limit the exact location of the (degenerate) event horizon is given by $r_+ = |e|$. It was shown that, similarly to the classical Reissner-Nordström solution, the near-horizon geometry of the charged black holes in quadratic gravity, when expanded into the whole manifold, is simply that of Bertotti and Robinson. As a by-product of our investigations we obtained a simple equation that relates the horizon value f'' with the trace of the stress-energy tensor, which, for solutions of the equations of quadratic gravity describing black holes, may serve as a useful criterion for possessing near-horizon geometry of the Bertotti-Robinson type.

Similar considerations have been carried out for boundary conditions of a second type, which employ the electric charge and the mass of the system as seen by a distant observer. Moreover, the method to relate the appropriate results obtained within the framework of each method has been explicitly demonstrated.

Returning to the extremal black holes, we observe that there are good reasons to believe that they are qualitatively different from the nonextremal ones. A proper distance, for example, between two points, one of which resides on the event horizon, is infinite. This can easily be seen from the integral

$$\int \sqrt{g_{rr}}dr \sim \frac{1}{r_+}\ln(r - r_+), \tag{83}$$

as it diverges in the limit $r \to r_+$. Moreover, the entropy remains nonzero as $A \to 4\pi r_+^2$ and depends on the electric charge. This behavior clearly violates the Nernst formulation of the third law of thermodynamics, which states that the entropy of a system must go to zero or a universal constant as $T \to 0$. On the other hand, even if the entropy of the extremal black hole vanishes [45], there are still problems simply because of its noncontinous nature. Indeed, zero is not the limit to which the entropy of the nearly extremal black holes tends. One can argue that this behavior should not be treated as worrisome simply because the Nernst formulation probably should not be considered as a fundamental law of thermodynamics. For a recent discussion of this issue, see [46]. On the other hand according to more radical opinions, this failure indicates that extremal black holes must not be treated as thermodynamic systems to which one can assign the notions of temperature and entropy [47]. Of course, we are unable here to judge which option should be treated seriously. All we can say now is that the entropy as given by Eq. (47) (probably) can be safely used for nonextremal black holes. However, the particular case of extreme black holes certainly deserves further study.

Finally, let us observe that the methods of this paper could easily be generalized to other sources, such as, for example, nonlinear electrodynamics. This group of problems is under active investigation and the results will be published elsewhere.

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