Bianchi type-I cosmology with scalar and spinor fields

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We consider a self-consistent system of interacting spinor and scalar fields within the framework of a Bianchi type-I (BI) cosmological model filled with perfect fluid. The interacting term in the Lagrangian is chosen in the form of derivative coupling, i.e., $\mathcal{L}_{\rm int}=(\lambda/2)\varphi_{,\alpha}\varphi^{,\alpha}F$. Here F is a power or trigonometric function of the invariants I and/or J constructed from bilinear spinor forms $S=\bar\psi\psi$ and $P=i\bar\psi\gamma^5\psi$. Self-consistent solutions to the spinor, scalar, and BI gravitational field equations are obtained. The problems of an initial singularity and the asymptotically isotropization process of the initially anisotropic space-time are studied. The role of the cosmological constant (Λ term) in the evolution of a BI Universe is studied. It is shown that a positive Λ generates an oscillatory mode of expansion of the BI model, whereas if F in $\mathcal{L}_{\rm int}$ is chosen to be a trigonometric function of its arguments, there exists a nonexponential mode of evolution even with a negative Λ. It is shown also that for a suitable choice of problem parameters the present model allows regular solutions without a broken dominant energy condition.

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I. INTRODUCTION

The nonlinear generalization of classical field theory remains one of the possible ways to overcome the difficulties of a theory which considers elementary particles as mathematical points. The gravitational field equation is nonlinear by nature and the field itself is universal and unscreenable. These properties lead to a definite physical interest in the gravitational field that goes with these matter fields.

Nonlinear self-couplings of the spinor fields may arise as a consequence of the geometrical structure of the space-time and, more precisely, because of the existence of torsion. As early as 1938, Ivanenko [1–3] showed that a relativistic theory imposes in some cases a fourth-order self-coupling. In 1950, Weyl [4] proved that, if the affine and the metric properties of the space-time are taken as independent, the spinor field obeys either a linear equation in a space with torsion or a nonlinear one in a Riemannian space. As the self-interaction is of spin-spin type, it allows the assignment of a dynamical role to the spin and offers a clue about the origin of the nonlinearities. A nonlinear spinor field, suggested by the symmetric coupling between nucleons, muons, and leptons, has been investigated by Finkelstein *et al.* [5] in the classical approximation. Although the existence of a spin-1/2 fermion is both theoretically and experimentally undisputed, these are described by quantum spinor fields. Possible justifications for the existence of classical spinors has been addressed in Ref. [6].

The present-day cosmology is based largely on the Friedmann solutions of Einstein equations, which describe the completely uniform and isotropic Universe (“closed” and “open” models). The main feature of these solutions is their nonstationarity. The idea of an expanding Universe, following from this property, is confirmed by the astronomical observations and it is now safe to assume that the isotropic model provides, in its general features, an adequate description of the present state of the Universe. Although the Universe seems homogeneous and isotropic at present, the large scale matter distribution in the observable Universe, largely manifested in the form of discrete structures, does not exhibit a high degree of homogeneity. Recent space investigations detect anisotropy in the cosmic microwave background. In fact, the theoretical arguments [7] and recent experimental data, which support the existence of an anisotropic phase that approaches an isotropic one, lead us to consider the models of Universe with anisotropic background. Zel’dovich was the first to assume that the early isotropization of the cosmological expanding process can take place as a result of quantum effect of particle creation near singularity [8]. This assumption was further justified by several authors [9–11].

The simplest of anistropic models, which, nevertheless, rather completely describe the anisotropic effects, are Bianchi type-I (BI) homogeneous models whose spatial sections are flat but the expansion or contraction rate is direction dependent. Moreover, a BI Universe falls within the general analysis of the singularity given by Belinskii *et al.* [12] and evolves into a Friedmann-Robertson-Walker (FRW) Universe [13] in the presence of a matter obeying the equation of state $p=\zeta\varepsilon,\ \zeta<1$. Since the modern-day Universe is almost isotropic at large, this feature of the BI Universe makes it a prime candidate for studying the possible effects of an anisotropy in the early Universe on present-day observations.

It should be noted that an important property of the isotropic model is the presence of a singular point in time in its space-time metric which means that the time is bounded from below. Is the presence of a singular point an inherent property of the relativistic cosmological models or is it only a consequence of specific simplifying assumptions underlying these models? To answer this question we studied a self-consistent system of the nonlinear spinor and BI gravitational fields in a series of papers [14–17]. It should be mentioned that a spinor field in a BI Universe was also studied by Belinskii and Khalatnikov [18]. Using Hamiltonian

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techniques, Henneaux studied class-A Bianchi Universes generated by a spinor source [19,20].

In this paper we consider a self-consistent system of the spinor, scalar, and Bianchi type-I gravitation fields in the presence of a perfect fluid and cosmological constant Λ. It should be noted that the inclusion of the Λ term adds a new dimension in the evolution of the Universe. Assuming that the Λ term may be both positive and negative, it opens a much wider range of possibilities in the search for a singularity-free solution of the field equations. Extending our previous studies [14–17], where the nonlinear term was taken to be a power law of $I=S^2=(\bar{\psi}\psi)^2$ and/or $J=P^2=(i\bar{\psi}\gamma^5\psi)^2$, in the present paper we consider the nonlinear term to be a trigonometric function of I (J), as well. In addition, a numerical analysis of the corresponding nonlinear differential equations has been performed.

II. DERIVATION OF BASIC EQUATIONS

Using the variational principle, in this section we derive the basic equations for the corresponding spinor, scalar, and gravitational fields from the action (2.1) and express corresponding spinor, scalar, and metric functions in terms of the volume scale τ (2.27) of the BI universe. From the gravitational field equations we also deduce the second-order multiparametric ordinary differential equation for τ. This last equation will be thoroughly studied both analytically and numerically in the following section.

We consider a system of the nonlinear spinor, scalar, and BI gravitational fields in the presence of a perfect fluid given by the action

$$\mathcal{S}(g;\psi,\bar{\psi},\varphi)=\int \mathcal{L}\sqrt{-g}\,d\Omega \tag{2.1}$$

with

$$\mathcal{L}=\mathcal{L}_{\rm g}+\mathcal{L}_{\rm sp}+\mathcal{L}_{\rm sc}+\mathcal{L}_{\rm int}+\mathcal{L}_{\rm pf}. \tag{2.2}$$

The gravitational part of the Lagrangian (2.2) $\mathcal{L}_{\rm g}$ is given by a Bianchi type-I (BI hereafter) space-time, whereas the terms $\mathcal{L}_{\rm sp}$, $\mathcal{L}_{\rm sc}$, and $\mathcal{L}_{\rm int}$ describe the spinor and scalar field Lagrangian and an interaction between them, respectively. The term $\mathcal{L}_{\rm pf}$ describes the Lagrangian density of the perfect fluid which minimally couples to the spinor and scalar fields through gravitational one.

A. Matter field Lagrangian

For a spinor field ψ, the symmetry between ψ and $\bar{\psi}$ appears to demand that one should choose the symmetrized Lagrangian [21]. Keeping this in mind we choose the spinor field Lagrangian as

$$\mathcal{L}_{sp}=\frac{i}{2}[\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi-\nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi]-m\bar{\psi}\psi, \tag{2.3}$$

with m being the spinor mass.

The massless scalar field Lagrangian is chosen to be

$$\mathcal{L}_{\rm sc}=\frac{1}{2}\varphi_{,\alpha}\varphi^{,\alpha}. \tag{2.4}$$

The interaction between the spinor and scalar fields is given by the Lagrangian [16]

$$\mathcal{L}_{\rm int}=\frac{\lambda}{2}\varphi_{,\alpha}\varphi^{,\alpha}F. \tag{2.5}$$

Here λ is the coupling constant and F is an arbitrary function of invariants generated from the real bilinear forms of the spinor field (a comprehensive description of the invariants can be found, e.g., in Ref. [15]). We choose $F=F(I,J)$ with $I=S^2=(\bar{\psi}\psi)^2$ and $J=P^2=(i\bar{\psi}\gamma^5\psi)^2$. By virtue of the Pauli-Fierz theorem [22] we claim that it describes the nonlinearity in the most general of its form [15]. Note that setting $\lambda=0$ in Eq. (2.5) we come to the case with minimal coupling between the spinor and scalar fields.

The contribution of the perfect fluid to the system is performed by means of its energy-momentum tensor, which acts as one of the sources of the corresponding gravitational field equations. So here we do not need to write the Lagrangian density $\mathcal{L}_{\rm pf}$ explicitly. The reason for writing $\mathcal{L}_{\rm pf}$ in Eqs. (2.1) and (2.2) is to underline that we are dealing with a self-consistent system. An interesting discussion on the action and Lagrangian for a perfect fluid can be found in Refs. [23–25].

B. Gravitational field

As a gravitational field we consider the Bianchi type-I (BI) cosmological model. It is the simplest model of anisotropic Universe that describes a homogeneous and spatially flat space-time and if filled with perfect fluid with the equation of state $p=\zeta\varepsilon$, $\zeta<1$, it eventually evolves into a FRW Universe [13]. The isotropy of the present-day Universe makes the BI model a prime candidate for studying the possible effects of an anisotropy in the early Universe on modern-day data observations. In view of what has been mentioned above we choose the gravitational part of the Lagrangian (2.2) in the form

$$\mathcal{L}_{\rm g}=\frac{R}{2\kappa}, \tag{2.6}$$

where R is the scalar curvature, $\kappa=8\pi G$ being the Einstein's gravitational constant. The gravitational field in our case is given by a Bianchi type-I (BI) metric,

$$ds^2=dt^2-a^2dx^2-b^2dy^2-c^2dz^2, \tag{2.7}$$

with a, b, c being the functions of time t only. Here the speed of light is taken to be unity.

The metric (2.7) has the following nontrivial Christoffel symbols:

$$\Gamma_{10}^{1}=\frac{\dot a}{a},\quad \Gamma_{20}^{2}=\frac{\dot b}{b},\quad \Gamma_{30}^{3}=\frac{\dot c}{c},$$

$$\Gamma_{11}^{0}=a\dot a,\quad \Gamma_{22}^{0}=b\dot b,\quad \Gamma_{33}^{0}=c\dot c. \tag{2.8}$$

The nontrivial components of the Ricci tensors are

$$R_0^0=-\left(\frac{\ddot a}{a}+\frac{\ddot b}{b}+\frac{\ddot c}{c}\right), \tag{2.9a}$$

$$R_1^1=-\left[\frac{\ddot a}{a}+\frac{\dot a}{a}\left(\frac{\dot b}{b}+\frac{\dot c}{c}\right)\right], \tag{2.9b}$$

$$R_2^2=-\left[\frac{\ddot b}{b}+\frac{\dot b}{b}\left(\frac{\dot c}{c}+\frac{\dot a}{a}\right)\right], \tag{2.9c}$$

$$R_3^3=-\left[\frac{\ddot c}{c}+\frac{\dot c}{c}\left(\frac{\dot a}{a}+\frac{\dot b}{b}\right)\right]. \tag{2.9d}$$

From Eq. (2.9) one finds the following Ricci scalar for the BI Universe:

$$R=-2\left(\frac{\ddot a}{a}+\frac{\ddot b}{b}+\frac{\ddot c}{c}+\frac{\dot a}{a}\frac{\dot b}{b}+\frac{\dot b}{b}\frac{\dot c}{c}+\frac{\dot c}{c}\frac{\dot a}{a}\right). \tag{2.10}$$

The nontrivial components of Riemann tensors in this case read

$$R^{01}{}_{01}=\frac{\ddot a}{a},\quad R^{02}{}_{02}=\frac{\ddot b}{b},\quad R^{03}{}_{03}=\frac{\ddot c}{c},$$

$$R^{12}{}_{12}=-\frac{\dot a}{a}\frac{\dot b}{b},\quad R^{23}{}_{23}=-\frac{\dot b}{b}\frac{\dot c}{c},$$

$$R^{31}{}_{31}=-\frac{\dot c}{c}\frac{\dot a}{a}. \tag{2.11}$$

Now having all the nontrivial components of Ricci and Riemann tensors, one can easily write the invariants of gravitational field which we need to study the space-time singularity. We return to this study at the end of this section.

C. Field equations

Let us now write the field equations corresponding to the action (2.1).

Variation of Eq. (2.1) with respect to the spinor field ψ $(\bar\psi)$ gives the following spinor field equations:

$$i\gamma^\mu\nabla_\mu\psi-m\psi+\mathcal{D}\psi+\mathcal{G}i\gamma^5\psi=0, \tag{2.12a}$$

$$i\nabla_\mu\bar\psi\gamma^\mu+m\bar\psi-\mathcal{D}\bar\psi-\mathcal{G}i\bar\psi\gamma^5=0, \tag{2.12b}$$

where we use the notation

$$\mathcal{D}=\lambda S\varphi_{,\alpha}\varphi^{,\alpha}\frac{\partial F}{\partial I},\quad \mathcal{G}=\lambda P\varphi_{,\alpha}\varphi^{,\alpha}\frac{\partial F}{\partial J}.$$

Since the nonlinearity in the foregoing equations is generated by the interacting scalar field, Eqs. (2.12) can be viewed as the spinor field equations with induced nonlinearity.

Variation of Eq. (2.1) with respect to the scalar field yields the following scalar field equation:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}(\sqrt{-g}g^{\nu\mu}(1+\lambda F)\varphi_{,\mu})=0. \tag{2.13}$$

Finally, varying Eq. (2.1) with respect to metric tensor $g_{\mu\nu}$ one finds the Einstein's field equations. On account of the Λ term they have the form

$$R_\mu^\nu-\frac{1}{2}\delta_\mu^\nu R=\kappa T_\mu^\nu-\delta_\mu^\nu\Lambda. \tag{2.14}$$

In view of Eqs. (2.9) and (2.10) for the BI space-time (2.7) we rewrite Eq. (2.14) as

$$\frac{\ddot b}{b}+\frac{\ddot c}{c}+\frac{\dot b}{b}\frac{\dot c}{c}=\kappa T_1^1-\Lambda, \tag{2.15a}$$

$$\frac{\ddot c}{c}+\frac{\ddot a}{a}+\frac{\dot c}{c}\frac{\dot a}{a}=\kappa T_2^2-\Lambda, \tag{2.15b}$$

$$\frac{\ddot a}{a}+\frac{\ddot b}{b}+\frac{\dot a}{a}\frac{\dot b}{b}=\kappa T_3^3-\Lambda, \tag{2.15c}$$

$$\frac{\dot a}{a}\frac{\dot b}{b}+\frac{\dot b}{b}\frac{\dot c}{c}+\frac{\dot c}{c}\frac{\dot a}{a}=\kappa T_0^0-\Lambda. \tag{2.15d}$$

Here the overdot refers to a time derivative and T_ν^μ is the energy-momentum tensor of the matter field given by

$$T_\mu^\nu=T_{\mu(\mathrm{sp})}^{\ \nu}+T_{\mu(\mathrm{sc})}^{\ \nu}+T_{\mu(\mathrm{int})}^{\ \nu}+T_{\mu(\mathrm{pf})}^{\ \nu}. \tag{2.16}$$

Here $T_{\mu(\mathrm{sp})}^{\ \nu}$ is the energy-momentum tensor of the spinor field defined by

$$T_{\mu(\mathrm{sp})}^{\ \rho}=\frac{i}{4}g^{\rho\nu}(\bar\psi\gamma_\mu\nabla_\nu\psi+\bar\psi\gamma_\nu\nabla_\mu\psi-\nabla_\mu\bar\psi\gamma_\nu\psi-\nabla_\nu\bar\psi\gamma_\mu\psi)$$
$$-\delta_\mu^\rho\mathcal{L}_{\mathrm{sp}}. \tag{2.17}$$

The term $\mathcal{L}_{sp}$ in view of Eq. (2.12) takes the form

$$\mathcal{L}_{sp}=-(\mathcal{D}S+\mathcal{G}P). \tag{2.18}$$

The energy-momentum tensor of the scalar field is given by

$$T_{\mu(\mathrm{sc})}^{\ \nu}=\varphi_{,\mu}\varphi^{,\nu}-\delta_\mu^\nu\mathcal{L}_{\mathrm{sc}}. \tag{2.19}$$

For the interaction field we find

$$T_{\mu(\mathrm{int})}^{\ \nu}=\lambda F\varphi_{,\mu}\varphi^{,\nu}-\delta_\mu^\nu\mathcal{L}_{\mathrm{int}}. \tag{2.20}$$

$T_{\mu(\mathrm{pf})}^{\nu}$ is the energy-momentum tensor of a perfect fluid. For a Universe filled with a perfect fluid, in a comoving system of reference such that $u^\mu=(1,0,0,0)$ we have

$$T^{\nu}_{\mu(\mathrm{pf})}=(p+\varepsilon)u_\mu u^\nu-\delta^\nu_\mu p=(\varepsilon,\ -p,\ -p,\ -p). \tag{2.21}$$

The energy ε and the pressure p of the perfect fluid obey the following equation of state:

$$p=\zeta\varepsilon, \tag{2.22}$$

where ζ is a constant and lies in the interval $\zeta\in[0,1]$. Depending on its numerical value, ζ describes the following types of Universes [13]:

$$\zeta=0,\quad (\text{dust Universe}), \tag{2.23a}$$

$$\zeta=1/3,\quad (\text{radiation Universe}), \tag{2.23b}$$

$$\zeta\in(1/3,\ 1),\quad (\text{hard Universes}), \tag{2.23c}$$

$$\zeta=1,\quad (\text{Zel'dovich Universe or stiff matter}). \tag{2.23d}$$

Here once again we note that the perfect fluid is minimally coupled to the system. Being one of its sources the perfect fluid leaves its trace on the gravitational field which in turn influences the behavior of the spinor and scalar fields.

In Eqs. (2.12) and (2.17) ∇_μ is the covariant derivatives acting on a spinor field as [26,27]

$$\nabla_\mu\psi=\frac{\partial\psi}{\partial x^\mu}-\Gamma_\mu\psi,\quad \nabla_\mu\bar\psi=\frac{\partial\bar\psi}{\partial x^\mu}+\bar\psi\Gamma_\mu, \tag{2.24}$$

where Γ_μ are the Fock-Ivanenko spinor connection coefficients defined by

$$\Gamma_\mu=\frac{1}{4}\gamma^\sigma(\Gamma^\nu_{\mu\sigma}\gamma_\nu-\partial_\mu\gamma_\sigma). \tag{2.25}$$

For the metric (2.7) one has the following components of the spinor connection coefficients:

$$\Gamma_0=0,\quad \Gamma_1=\frac{1}{2}\dot a(t)\bar\gamma^1\bar\gamma^0,\quad \Gamma_2=\frac{1}{2}\dot b(t)\bar\gamma^2\bar\gamma^0,$$

$$\Gamma_3=\frac{1}{2}\dot c(t)\bar\gamma^3\bar\gamma^0. \tag{2.26}$$

The Dirac matrices $\gamma^\mu(x)$ of the curved space-time are connected with those of Minkowski as follows:

$$\gamma^0=\bar\gamma^0,\quad \gamma^1=\bar\gamma^1/a,\quad \gamma^2=\bar\gamma^2/b,\quad \gamma^3=\bar\gamma^3/c.$$

Here

$$\bar\gamma^0=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\quad \bar\gamma^i=\begin{pmatrix} 0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix},$$

$$\gamma^5=\bar\gamma^5=\begin{pmatrix} 0 & -I\\ -I & 0\end{pmatrix},$$

where σ_i are the Pauli matrices:

$$\sigma^1=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma^2=\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix},\quad \sigma^3=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

Note that the $\bar\gamma$ and the σ matrices obey the following properties:

$$\bar\gamma^i\bar\gamma^j+\bar\gamma^j\bar\gamma^i=2\eta^{ij},\quad i,j=0,1,2,3,$$

$$\bar\gamma^i\bar\gamma^5+\bar\gamma^5\bar\gamma^i=0,\quad (\bar\gamma^5)^2=I,\quad i=0,1,2,3,$$

$$\sigma^j\sigma^k=\delta_{jk}+i\varepsilon_{jkl}\sigma^l,\quad j,k,l=1,2,3,$$

where $\eta_{ij}=\{1,-1,-1,-1\}$ is the diagonal matrix, δ_{jk} is the Kronecker symbol, and ε_{jkl} is the totally antisymmetric tensor with $\varepsilon_{123}=+1$.

We study the space-independent solutions to the spinor and scalar field equations (2.12) and (2.13), so that $\psi=\psi(t)$ and $\varphi=\varphi(t)$. Defining

$$\tau=abc=\sqrt{-g} \tag{2.27}$$

from Eq. (2.13) for the scalar field we have

$$\varphi=C\int\frac{dt}{\tau(1+\lambda F)},\quad C=\text{const}. \tag{2.28}$$

The spinor field equation (2.12a) on account of Eqs. (2.24) and (2.26) takes the form

$$i\bar\gamma^0\left(\frac{\partial}{\partial t}+\frac{\dot\tau}{2\tau}\right)\psi-m\psi+\mathcal{D}\psi+\mathcal{G}i\gamma^5\psi=0. \tag{2.29}$$

Setting $V_j(t)=\sqrt{\tau}\psi_j(t)$, $j=1,2,3,4$, from Eq. (2.29) one deduces the following system of equations:

$$\dot V_1+i(m-\mathcal{D})V_1-\mathcal{G}V_3=0, \tag{2.30a}$$

$$\dot V_2+i(m-\mathcal{D})V_2-\mathcal{G}V_4=0, \tag{2.30b}$$

$$\dot V_3-i(m-\mathcal{D})V_3+\mathcal{G}V_1=0, \tag{2.30c}$$

$$\dot V_4-i(m-\mathcal{D})V_4+\mathcal{G}V_2=0. \tag{2.30d}$$

From Eq. (2.12) we write also the equations for the invariants S, P, and $A=\overline{\psi\gamma^5\bar\gamma^0}\psi$,

$$\dot S_0-2\mathcal{G}A_0=0, \tag{2.31a}$$

$$\dot P_0-2(m-\mathcal{D})A_0=0, \tag{2.31b}$$

$$\dot A_0+2(m-\mathcal{D})P_0+2\mathcal{G}S_0=0, \tag{2.31c}$$

where we use the notations $S_0=\tau S$, $P_0=\tau P$, and $A_0=\tau A$. From Eq. (2.31) we find the following relation between the invariants:

$$S^2+P^2+A^2=C^2/\tau^2,\quad C^2=\text{const}. \tag{2.32}$$

Given the concrete form of F the system (2.30) can be solved explicitly and using the solutions obtained one can write the components of spinor current:

$$j^{\mu}=\bar{\psi}\gamma^{\mu}\psi. \tag{2.33}$$

The component j^0

$$j^0=\frac{1}{\tau}[V_1^*V_1+V_2^*V_2+V_3^*V_3+V_4^*V_4], \tag{2.34}$$

defines the charge density of the spinor field that has the following chronometric-invariant form:

$$\rho=(j_0\cdot j^0)^{1/2}. \tag{2.35}$$

The total charge of the spinor field is defined as

$$Q=\int_{-\infty}^{\infty}\rho\sqrt{-{}^3g}\,dx\,dy\,dz=\rho\tau\mathcal{V}, \tag{2.36}$$

where $\mathcal{V}$ is the volume. From the spin tensor

$$S^{\mu\nu,\epsilon}=\frac{1}{4}\bar{\psi}\{\gamma^{\epsilon}\sigma^{\mu\nu}+\sigma^{\mu\nu}\gamma^{\epsilon}\}\psi \tag{2.37}$$

one finds the chronometric invariant spin tensor

$$S_{\rm ch}^{ij,0}=(S_{ij,0}S^{ij,0})^{1/2}, \tag{2.38}$$

and the projection of the spin vector on the k axis

$$S_k=\int_{-\infty}^{\infty}S_{\rm ch}^{ij,0}\sqrt{-{}^3g}\,dx\,dy\,dz=S_{\rm ch}^{ij,0}\tau V. \tag{2.39}$$

Let us now solve the Einstein equations. In doing so we first write the expression for the components of the energy-momentum tensor explicitly:

$$T_0^0=mS+\frac{C^2}{2\tau^2(1+\lambda F)}+\varepsilon,$$

$$T_1^1=T_2^2=T_3^3=\mathcal{D}S+\mathcal{G}P-\frac{C^2}{2\tau^2(1+\lambda F)}-p. \tag{2.40}$$

On account of Eq. (2.40), subtracting Eq. (2.15a) from Eq. (2.15b), one finds the following relation between a and b:

$$\frac{a}{b}=D_1\exp\left(X_1\int\frac{dt}{\tau}\right). \tag{2.41}$$

Analogously,

$$\frac{a}{c}=D_2\exp\left(X_2\int\frac{dt}{\tau}\right),\quad \frac{b}{c}=D_3\exp\left(X_3\int\frac{dt}{\tau}\right). \tag{2.42}$$

The integration constants D_1, D_2, D_3, X_1, X_2, X_3 obey the following relations:

$$D_1D_2D_3=1,\quad X_1+X_2+X_3=0. \tag{2.43}$$

In view of Eq. (2.43), from Eqs. (2.41) and (2.42) we write the metric functions explicitly [15],

$$a(t)=(D_1^2D_3)^{1/3}\tau^{1/3}\exp\left[\frac{2X_1+X_3}{3}\int\frac{dt}{\tau(t)}\right], \tag{2.44a}$$

$$b(t)=(D_1^{-1}D_3)^{1/3}\tau^{1/3}\exp\left[-\frac{X_1-X_3}{3}\int\frac{dt}{\tau(t)}\right], \tag{2.44b}$$

$$c(t)=(D_1D_3^2)^{-1/3}\tau^{1/3}\exp\left[-\frac{X_1+2X_3}{3}\int\frac{dt}{\tau(t)}\right]. \tag{2.44c}$$

As one sees from Eqs. (2.44a)–(2.44c), for $\tau=t^n$ with $n>1$, the exponent tends to unity at large t, and the anisotropic model evolves into an isotropic one.

Further, we will investigate the existence of singularity (singular point) of the gravitational case, which can be done by investigating the invariant characteristics of the space-time. In general relativity these invariants are composed of the curvature tensor and the metric one. In a 4D Riemann space-time there are 14 independent invariants [15,28]. Instead of analyzing all 14 invariants, one can confine this study only in 3, namely the scalar curvature $I_1=R$, $I_2=R_{\mu\nu}R^{\mu\nu}$, and the Kretschmann scalar $I_3=R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ [29,30]. At any regular space-time point, these three invariants I_1, I_2, I_3 should be finite. Let us rewrite these invariants in detail.

For the BI metric one finds the scalar curvature

$$I_1=R=-2\left(\frac{\ddot a}{a}+\frac{\ddot b}{b}+\frac{\ddot c}{c}+\frac{\dot a}{a}\frac{\dot b}{b}+\frac{\dot b}{b}\frac{\dot c}{c}+\frac{\dot c}{c}\frac{\dot a}{a}\right). \tag{2.45}$$

Since the Ricci tensor for the BI metric is diagonal, the invariant $I_2=R_{\mu\nu}R^{\mu\nu}\equiv R_{\mu}^{\nu}R_{\nu}^{\mu}$ is a sum of squares of diagonal components of Ricci tensor, i.e.,

$$I_2=[(R_0^0)^2+(R_1^1)^2+(R_2^2)^2+(R_3^3)^2], \tag{2.46}$$

with the components of the Ricci tensor being given by Eq. (2.9).

Analogously, for the Kretschmann scalar in this case we have $I_3=R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\mu\nu}$, a sum of squared components of all nontrivial $R^{\mu\nu}{}_{\mu\nu}$, which in view of Eq. (2.11) can be written as

$$\begin{aligned}I_3&=4[(R^{01}{}_{01})^2+(R^{02}{}_{02})^2+(R^{03}{}_{03})^2+(R^{12}{}_{12})^2+(R^{23}{}_{23})^2+(R^{31}{}_{31})^2]\\&=4\left[\left(\frac{\ddot a}{a}\right)^2+\left(\frac{\ddot b}{b}\right)^2+\left(\frac{\ddot c}{c}\right)^2+\left(\frac{\dot a}{a}\frac{\dot b}{b}\right)^2+\left(\frac{\dot b}{b}\frac{\dot c}{c}\right)^2+\left(\frac{\dot c}{c}\frac{\dot a}{a}\right)^2\right].\end{aligned} \tag{2.47}$$

Let us now express the foregoing invariants in terms of τ. From Eqs. (2.44a)–(2.44c) we have

$$a_i = A_i \tau^{1/3} \exp\left((Y_i/3) \int \tau^{-1} dt \right), \qquad (2.48a)$$

$$\frac{\dot{a}_i}{a_i} = \frac{Y_i + \dot{\tau}}{3} \frac{1}{\tau}, \qquad (2.48b)$$

$$\frac{\ddot{a}_i}{a_i} = \frac{3\tau\ddot{\tau} - 2\dot{\tau}^2 - Y_i\dot{\tau} + Y_i^2}{9} \frac{1}{\tau^2}, \qquad (2.48c)$$

where $i = 1,2,3$ and a_1, a_2, and a_3 stand for a, b, and c, respectively. From Eqs. (2.48a)–(2.48c) one can easily verify that

$$I_1 \propto \frac{1}{\tau^2}, \quad I_2 \propto \frac{1}{\tau^4}, \quad I_3 \propto \frac{1}{\tau^4}.$$

Thus we see that at any space-time point, where $\tau = 0$, the invariants I_1, I_2, I_3 as well as the scalar and spinor fields become infinity, hence the space-time becomes singular at this point.

In what follows, we write the equation for τ and study it in detail.

Summation of the Einstein equations (2.15a), (2.15b), (2.15c), and (2.15d) multiplied by 3 gives

$$\frac{\ddot{\tau}}{\tau} = \frac{3}{2}\kappa(mS + \mathcal{D}S + \mathcal{G}P + \varepsilon - p) - 3\Lambda. \qquad (2.49)$$

For the right-hand-side of Eq. (2.49) to be a function of τ only, the solution of this equation is well known [31].

Let us demand the energy momentum to be conserved, i.e.,

$$T^{\nu}_{\mu;\nu} = T^{\nu}_{\mu,\nu} + \Gamma^{\nu}_{\rho\nu} T^{\rho}_{\mu} - \Gamma^{\rho}_{\mu\nu} T^{\nu}_{\rho} = 0, \qquad (2.50)$$

which in our case has the form

$$\frac{1}{\tau}(\tau T^0_0)^{\cdot} - \frac{\dot{a}}{a} T^1_1 - \frac{\dot{b}}{b} T^2_2 - \frac{\dot{c}}{c} T^3_3 = 0. \qquad (2.51)$$

On account of the equation of state $p = \zeta\varepsilon$ and

$$(m - \mathcal{D})\dot{S}_0 - \mathcal{G}\dot{P}_0 = 0,$$

which follows from Eq. (2.31), after a little manipulation from Eq. (2.51) we obtain

$$\varepsilon = \varepsilon_0/\tau^{1+\zeta}, \quad p = \zeta\varepsilon_0/\tau^{1+\zeta}, \qquad (2.52)$$

with ε_0 being the constant of integration. In view of Eq. (2.52), Eq. (2.49) can be written as

$$\frac{\ddot{\tau}}{\tau} = \frac{3}{2}\kappa[mS + \mathcal{D}S + \mathcal{G}P + (1-\zeta)\varepsilon_0/\tau^{1+\zeta}] - 3\Lambda. \qquad (2.53)$$

As it was mentioned earlier, we consider F to be a function of I, J, or $I \pm J$. In the section to follow we study Eq. (2.53) in detail.

III. EXACT SOLUTIONS AND NUMERICAL ANALYSIS

In the preceding section we solved the spinor, scalar, and gravitational field equations and wrote the solutions in terms of volume scale τ. It was also mentioned that if the right-hand side of Eq. (2.53) is a function of τ, then its solution can be written in quadrature. In what follows, we show that Eq. (2.53) is indeed an autonomous equation and explicitly write the corresponding solutions for a concrete choice of F.

A. Exact solutions

Here we consider the cases with minimal coupling and with F being the function of either I or J (with zero mass). In this subsection we simply write the solutions to the spinor field equations explicitly and present the solution for τ in quadrature.

1. Minimally coupled scalar and spinor fields

Let us first consider the case with minimal coupling when the scalar and spinor fields interact through gravitational one. In this case from Eq. (2.31) one finds $S = C_0/\tau$. The scalar field and the components of the spinor field in this case have the following explicit form:

$$\varphi = C \int \frac{dt}{\tau}, \qquad (3.1)$$

$$\psi_1(t) = \frac{C_1}{\sqrt{\tau}} e^{-imt}, \quad \psi_2(t) = \frac{C_2}{\sqrt{\tau}} e^{-imt},$$

$$\psi_3(t) = \frac{C_3}{\sqrt{\tau}} e^{imt}, \quad \psi_4(t) = \frac{C_4}{\sqrt{\tau}} e^{imt}, \qquad (3.2)$$

with the integration constants C_j satisfying $C_0 = C_1^2 + C_2^2 - C_3^2 - C_4^2$.

Equation (2.53) in this case takes the form

$$\ddot{\tau} = \frac{3}{2}\kappa(mC_0 + \varepsilon_0(1-\zeta)/\tau^{\zeta}) - 3\Lambda\tau, \qquad (3.3)$$

with the solution

$$\int \frac{d\tau}{\sqrt{\kappa(mC_0\tau + \varepsilon_0\tau^{1-\zeta}) - \Lambda\tau^2 + E}} = \sqrt{3}t. \qquad (3.4)$$

Here E is the constant of integration. Let us note that being the volume scale τ cannot be negative. On the other hand, the radical in Eq. (3.4) should be positive. This leads to the fact that for a positive Λ the value of τ should be bound from above giving rise to an oscillatory mode of expansion of the BI space-time.

2. Case with $F = F(I)$

Here we consider the interacting system of the scalar and spinor fields with the interaction given by $\mathcal{L}_{int} = (\lambda/2)\varphi_{\mu}\varphi^{\mu}F(I)$. As in the case with minimal coupling from Eq. (2.31a) one finds

$$S=\frac{C_0}{\tau},\quad C_0=\text{const.} \tag{3.5}$$

For components of the spinor field we find [15]

$$\psi_1(t)=\frac{C_1}{\sqrt{\tau}}e^{-i\beta},\quad \psi_2(t)=\frac{C_2}{\sqrt{\tau}}e^{-i\beta},$$

$$\psi_3(t)=\frac{C_3}{\sqrt{\tau}}e^{i\beta},\quad \psi_4(t)=\frac{C_4}{\sqrt{\tau}}e^{i\beta}, \tag{3.6}$$

with C_i being the integration constants and related to C_0 as $C_0=C_1^2+C_2^2-C_3^2-C_4^2$. Here we use the notation $\beta=\int(m-\mathcal{D})dt$.

For the components of the spin current from Eq. (2.33) we find

$$j^0=\frac{1}{\tau}[C_1^2+C_2^2+C_3^2+C_4^2],$$

$$j^1=\frac{2}{a\tau}[C_1C_4+C_2C_3]\cos(2\beta),$$

$$j^2=\frac{2}{b\tau}[C_1C_4-C_2C_3]\sin(2\beta),$$

$$j^3=\frac{2}{c\tau}[C_1C_3-C_2C_4]\cos(2\beta),$$

whereas for the projection of spin vectors on the X, Y, and Z axis we find

$$S^{23,0}=\frac{C_1C_2+C_3C_4}{bc\tau},\quad S^{31,0}=0,$$

$$S^{12,0}=\frac{C_1^2-C_2^2+C_3^2-C_4^2}{2ab\tau}.$$

The total charge of the system in a volume $\mathcal{V}$ in this case is

$$Q=[C_1^2+C_2^2+C_3^2+C_4^2]\mathcal{V}. \tag{3.7}$$

Thus for $\tau\neq 0$ the components of spin current and the projection of spin vectors are singularity free and the total charge of the system in a finite volume is always finite.

The equation for determining τ in this case has the form

$$\ddot{\tau}=\frac{3}{2}\kappa[mC_0+\mathcal{D}C_0+\varepsilon_0(1-\zeta)/\tau^{\zeta}]-3\Lambda\tau. \tag{3.8}$$

Recalling that $\mathcal{D}=\lambda C_0C^2F_I/\tau^3[1+\lambda F(I)]^2$ the solution to Eq. (3.8) can be written in quadrature,

$$\int\frac{d\tau}{\sqrt{\kappa[mC_0\tau+C^2/2(1+\lambda F)+\varepsilon_0\tau^{1-\zeta}]-\Lambda\tau^2+E}}=\sqrt{3}t, \tag{3.9}$$

with E being the integration constant. Given the explicit form of $F(I)$ we find various modes of expansion depending on the sign of Λ. Later we numerically study this case in detail.

3. Case with $F=F(J)$

Here we consider the interacting system of the scalar and spinor fields with the interaction given by $\mathcal{L}_{int}=(\lambda/2)\varphi_\mu\varphi^\mu F(J)$. In the case considered we assume the spinor field to be massless. Note that, in the unified nonlinear spinor theory of Heisenberg, the massive term remains absent, and according to Heisenberg, the particle mass should be obtained as a result of quantization of spinor prematter [32]. In the nonlinear generalization of classical field equations, the massive term does not possess the significance that it possesses in the linear one, as it by no means defines total energy (or mass) of the nonlinear field system. Thus without losing the generality we can consider the massless spinor field putting $m=0$. Then from Eq. (2.31b) one gets

$$P=D_0/\tau,\quad D_0=\text{const.} \tag{3.10}$$

In this case the spinor field components take the form

$$\psi_1=\frac{1}{\sqrt{\tau}}(D_1e^{i\sigma}+iD_3e^{-i\sigma}),\quad \psi_2=\frac{1}{\sqrt{\tau}}(D_2e^{i\sigma}+iD_4e^{-i\sigma}),$$

$$\psi_3=\frac{1}{\sqrt{\tau}}(iD_1e^{i\sigma}+D_3e^{-i\sigma}),\quad \psi_4=\frac{1}{\sqrt{\tau}}(iD_2e^{i\sigma}+D_4e^{-i\sigma}). \tag{3.11}$$

The integration constants D_i are connected to D_0 by $D_0=2(D_1^2+D_2^2-D_3^2-D_4^2)$. Here we set $\sigma=\int\mathcal{G}dt$.

For the components of the spin current from Eq. (2.33) we find

$$j^0=\frac{2}{\tau}[D_1^2+D_2^2+D_3^2+D_4^2],$$

$$j^1=\frac{4}{a\tau}[D_2D_3+D_1D_4]\cos(2\sigma),$$

$$j^2=\frac{4}{b\tau}[D_2D_3-D_1D_4]\sin(2\sigma),$$

$$j^3=\frac{4}{c\tau}[D_1D_3-D_2D_4]\cos(2\sigma),$$

whereas, for the projection of spin vectors on the X, Y, and Z axis we find

$$S^{23,0}=\frac{2(D_1D_2+D_3D_4)}{bc\tau},\quad S^{31,0}=0,$$

$$S^{12,0}=\frac{D_1^2-D_2^2+D_3^2-D_4^2}{2ab\tau}.$$

For τ in this case we have

$$\ddot{\tau}=\frac{3}{2}\kappa[\mathcal{G}C_0+\varepsilon_0(1-\zeta)/\tau^{\zeta}]-3\Lambda\tau. \tag{3.12}$$

In view of Eq. (3.10), $\mathcal{G}$ in this case takes the form analogous to that taken by $\mathcal{D}$ in the previous case with F_I replaced by F_J. Then the solution of Eq. (3.12) we write in quadrature as

$$\int \frac{d\tau}{\sqrt{\kappa[C^2/2(1+\lambda F)+\varepsilon_0\tau^{1-\zeta}]-\Lambda\tau^2+E}}=\sqrt{3}t. \tag{3.13}$$

Depending on the form of F and Λ we have a different mode of expansion of the BI Universe as in the previous case. In what follows we numerically study the aforementioned cases.

B. Numerical experiments

In this subsection we study Eq. (3.8) for different choices of F. As it was mentioned earlier, setting $\lambda=0$ in Eq. (3.8) we come to the case with minimal coupling given by Eq. (3.3), whereas, assuming $m=0$ we get Eq. (3.12). Let us first rewrite Eq. (3.8):

$$\ddot{\tau}=\mathcal{F}(\tau,p), \tag{3.14}$$

where we use the notation

$$\mathcal{F}\equiv\frac{3}{2}\kappa[mC_0+\mathcal{D}C_0+\varepsilon_0(1-\zeta)/\tau^{\zeta}]-3\Lambda\tau, \tag{3.15}$$

and $p\equiv\{\kappa,\lambda,m,C_0,C,\varepsilon_0,\zeta,\Lambda\}$ is the set of the parameters. Since in the examples we consider $F=F(S)$, let us rewrite $\mathcal{D}$ in terms of S. On account of $S=C_0/\tau$ for $\mathcal{D}$ we have

$$\mathcal{D}=\lambda C^2F_S/2\tau^2[1+\lambda F(S)]^2.$$

From a mechanical point of view Eq. (3.14) can be interpreted as an equation of motion of a single particle with unit mass under the force $\mathcal{F}(\tau,p)$. Then the following first integral exists [35]:

$$\dot{\tau}=\sqrt{2[E-\mathcal{U}(\tau,p)]}. \tag{3.16}$$

Here E is the integration constant and

$$\mathcal{U}\equiv-\frac{3}{2}\{\kappa[mC_0\tau+C^2/2(1+\lambda F)+\varepsilon_0\tau^{-\zeta}]-\Lambda\tau^2\}$$

is the potential of the force $\mathcal{F}$. We note that the radical expression must be non-negative. The zeros of this expression, which depend on all the problem parameters p, define the boundaries of the possible rates of changes of $\tau(t)$. In what follows we analyze Eqs. (3.14) and (3.15) for different a choice of $F(I)$ as well as for different problem parameters p.

1. $F=S^n$

Let us first choose F to be a power law of S(or I), setting $F=S^n$. In this case setting $C_0=1$ and $C=1$ we rewrite $\mathcal{F}$ as

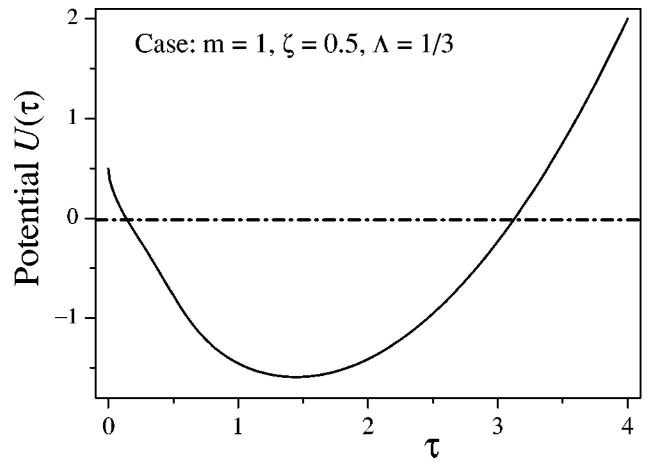

FIG. 1. View of the potential $\mathcal{U}(\tau)$ [Eq. (3.18)] with BI space-time being filled with perfect fluid describing a hard Universe.

$$\mathcal{F}=\frac{3\kappa}{2}\left(m+\frac{\lambda n\tau^{n-1}}{2(\lambda+\tau^n)^2}+\varepsilon_0\frac{(1-\zeta)}{\tau^{\zeta}}\right)-3\Lambda\tau, \tag{3.17}$$

with the potential

$$\mathcal{U}=-\frac{3}{2}\left\{\kappa\left[m\tau-\frac{\lambda}{2(\lambda+\tau^n)}+\varepsilon_0\tau^{1-\zeta}\right]-\Lambda\tau^2\right\}. \tag{3.18}$$

Note that the nonnegativity of the radical in Eq. (3.16) in view of Eq. (3.18) imposes a restriction on τ from above in the case of $\Lambda>0$. It means that in the case of $\Lambda>0$ the value of τ runs between 0 and some $\tau_{\max}$, where $\tau_{\max}$ is the maximum value of τ for the given p. This equation has been studied for different values of parameters p. Here we demonstrate the evolution of τ for different choices of τ_0 for fixed "energy" E and vice versa.

As the first example, we consider a massive spinor field with $m=1$. Other parameters are chosen in the following

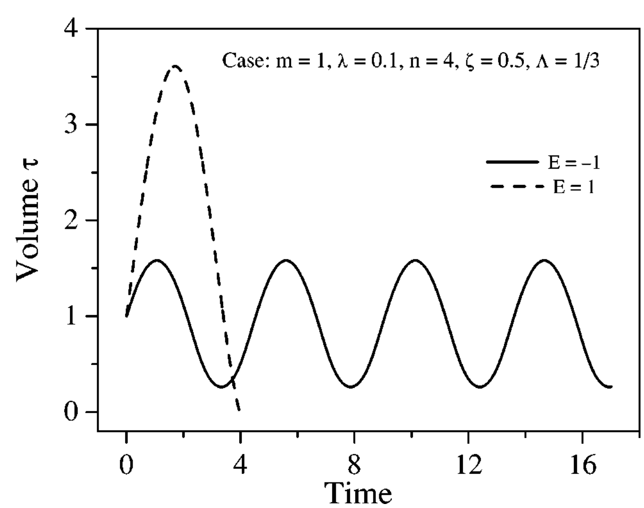

FIG. 2. Evolution of the BI space-time corresponding to the potential given in Fig. 1 for a different choice of E.

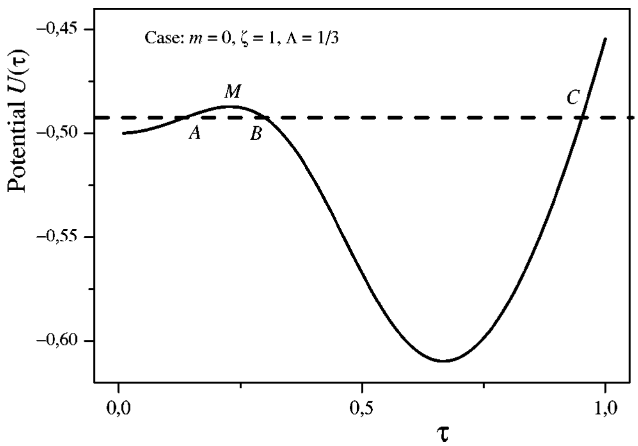

FIG. 3. View of the potential $\mathcal{U}(\tau)$ [Eq. (3.18)] with BI space-time being filled with a stiff matter.

way: coupling constant $\lambda=0.1$, power of nonlinearity $n=4$, and cosmological constant $\Lambda=1/3$. We also choose $\zeta=0.5$ describing a hard Universe.

In Fig. 1 we plot corresponding potential $\mathcal{U}(\tau)$ multiplied by the factor 2/3. As is seen from Figs. 1 and 2, choosing the integration constant E we may obtain two different types of solutions. For $E>0.5$ solutions are nonperiodic, whereas for $E_{\rm min}<E\leqslant 0.5$ the evolution of the Universe is oscillatory.

As a second example we consider the massless spinor field. Other parameters of the problem are left unaltered with the exception of ζ. Here we choose $\zeta=1$ describing stiff matter. It should be noted that this particular choice of ζ gives rise to a local maximum. This results in two types of solutions for a single choice of E.

As one sees from Fig. 3, if E is taken to be above the level M there exists only nonperiodic solutions, whereas for $E_{\rm min}<E<\mathcal{U}(\tau=0)=-0.5$ the solutions are always oscillatory. For $E\in(-0.5,M)$ there exits two types of solutions depending on the choice of τ_0. In Fig. 4 we plot the evolution of τ for $E\in(-0.5,M)$. As is seen, for $\tau_0\in(0,A)$ (here $\tau_0=0.1$) we have mathematical solutions that are oscillatory and τ in this case becomes negative in some interval of time. Since by

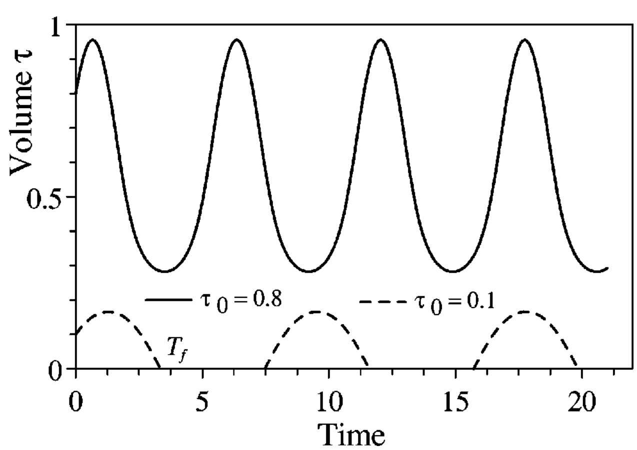

FIG. 4. Evolution of the BI space-time corresponding to the potential given in Fig. 3 in the case of a massless spinor field for different choices of τ_0 with $E\in(-0.5,M)$.

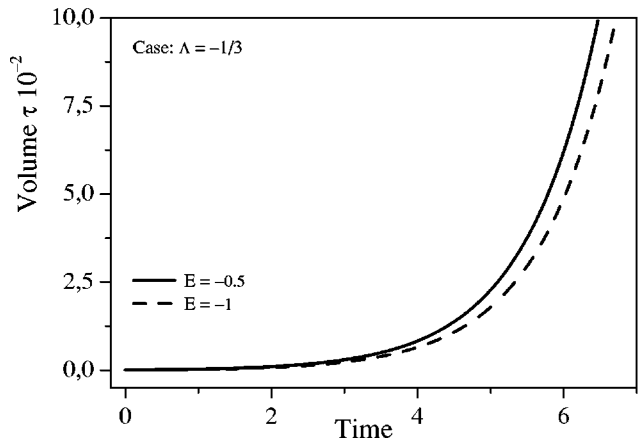

FIG. 5. Evolution of the BI Universe for a negative Λ. As one sees, the evolution of the Universe in this case takes exponential character and the initial anisotropy of the BI space-time quickly dies away.

definition τ is non-negative, we plot only the part of the solution where $\tau\geqslant 0$ (cf. Fig. 4, dashed curve). Note that only that part of τ defined in the interval of time $t\in(0,T_f)$ is physically relevant. For $\tau_0\in(B,C)$ we again have the oscillatory mode of the evolution of τ. These two regions are separated by the no-solution zone (A,B).

Let us also consider the case with $\Lambda<0$. For a negative Λ, as well as in the absence of the Λ term, the evolution of τ is always exponential as it is seen in Fig. 5. In this case the initial anisotropy of the BI space-time quickly dies away and the Universe becomes an isotropic one.

Let us analyze the dominant energy condition in the Hawking-Penrose theorem [33,34]. For a BI Universe the dominant energy condition can be written in the form [15]

$$T_0^0\geqslant T_1^1a^2+T_2^2b^2+T_3^3c^2, \tag{3.19a}$$

$$T_0^0\geqslant T_1^1a^2, \tag{3.19b}$$

$$T_0^0\geqslant T_2^2b^2, \tag{3.19c}$$

$$T_0^0\geqslant T_3^3c^2. \tag{3.19d}$$

Let us note that in Ref. [15] we considered a self-consistent system of nonlinear spinor and BI gravitational fields in the presence of a perfect fluid and a Λ term. It was shown that in this case the regular solutions can be obtained by virtue of the spinor field nonlinearity and/or a positive Λ term. It was shown also that the absence of initial singularity in the considered cosmological solution is consistent with the violation of the dominant energy condition in the Hawking-Penrose theorem. Note that regular solutions obtained for a linear spinor field by means of a positive Λ term do not violate this condition. Let us now analyze the dominant energy condition for the system in hand. To analyze this condition for the system of the interacting spinor and scalar fields we rewrite the components of the energy-momentum tensor. For energy density in this case we have

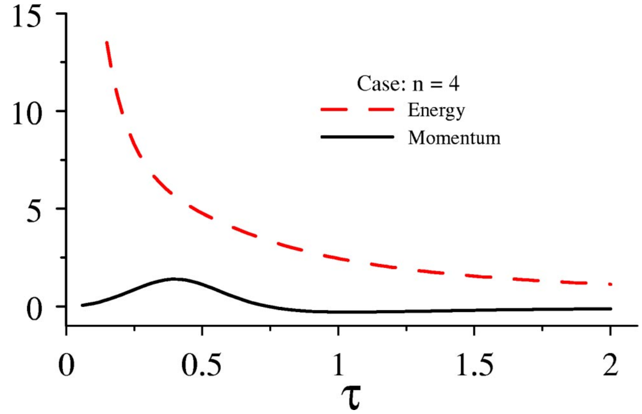

FIG. 6. Comparing T_0^0 and T_1^1 for a positive n, one sees that for a small value of n it is possible to construct a regular solution without violating the dominant energy condition.

$$T_0^0=\frac{mC_0}{\tau}+\frac{C^2\tau^{n-2}}{2(\tau^n+\lambda C_0^n)}+\frac{\varepsilon_0}{\tau^{1+\zeta}}. \tag{3.20}$$

As one sees from Eq. (3.20) for any positive value of τ energy density is always positive definite. As $\tau\to 0$, $T_0^0\to\infty$, whereas T_0^0 decreases as τ increases. For the pressure components in this case we have

$$T_1^1=T_2^2=T_3^3=\frac{C^2\tau^{n-2}}{2(\tau^n+\lambda C_0^n)^2}[\lambda C_0^n(n-1)-\tau^n]-\frac{\zeta\varepsilon_0}{\tau^{1+\zeta}}. \tag{3.21}$$

The second term in Eq. (3.21) is always positive, it means that T_1^1 has a greater value when the BI Universe is filled with dust, i.e., when $\zeta=0$. To investigate the dominant energy condition we study the pressure term (since $T_1^1=T_2^2=T_3^3$, hereafter we mention it as T_1^1) at length. For simplicity we set $C=1$ and $C_0=1$. It is clear from Eq. (3.21) that if

$$\tau^n>\lambda(n-1), \tag{3.22}$$

we have $T_1^1<0$. In this case the dominant energy condition remains unbroken. From Eq. (3.22) we see for $\lambda=0$ that the foregoing inequality holds for any $\tau>0$. It means that like the linear spinor field [15], the system with minimally coupled scalar and spinor fields possesses regular solutions without broken dominant energy condition. For an interacting system this condition holds for any negative n with a positive λ and vice versa. Let us now see what happens when both n and λ are positive (negative). Note that the coupling constant λ may take any value. The magnitude of λ defines the strength of interaction.

Let us go back to Eq. (3.22). As one sees, for any reasonable value of λ the inequality (3.22) holds at large τ. On the other hand, as $\tau\to 0$, the corresponding energy density T_0^0 tends to infinity. So the conditions (3.19) hold for small τ as well. Finally, let us analyze the situation in the neighborhood of $\tau=1$. The energy density T_0^0 at this point is reasonably small, whereas, as it is shown in Fig. 7, violation of the dominant energy condition, i.e., the situation when T_1^1 dominates T_0^0, may occur only for a relatively large value of n. Thus we conclude that in case of interacting spinor and scalar fields it is possible to construct regular solutions without violating dominant energy condition of Hawking-Penrose theorem (see Fig. 6).

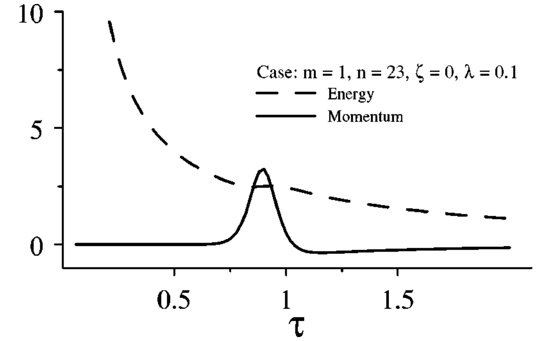

FIG. 7. For a large n there exists some value of τ where the pressure component prevails energy. In this case the dominant energy condition breaks down.

2. $F=\sin S$

Let us now consider the case with F being a trigonometric function of S, namely, $F=\sin S$. In this case for $\mathcal{F}$ we have

$$\mathcal{F}=\frac{3\kappa}{2}\left(m+\frac{\lambda\cos S}{2\tau^2(\lambda+\sin S)^2}+\varepsilon_0\frac{(1-\zeta)}{\tau^{\zeta}}\right)-3\Lambda\tau,\qquad S=\frac{1}{\tau}, \tag{3.23}$$

with the potential

$$\mathcal{U}=-\frac{3}{2}\left\{\kappa\left[m\tau+\frac{1}{2(1+\lambda\sin S)}+\varepsilon_0\tau^{1-\zeta}\right]-\Lambda\tau^2\right\}. \tag{3.24}$$

It should be noted that unlike the case with F being a power law of $S=1/\tau$, where the nonlinearity appears in the region with a large value of τ, in the case under consideration, a number of interesting properties emerge in the region where $0<\tau<1$, namely, in the vicinity of the singular point $\tau=0$. A

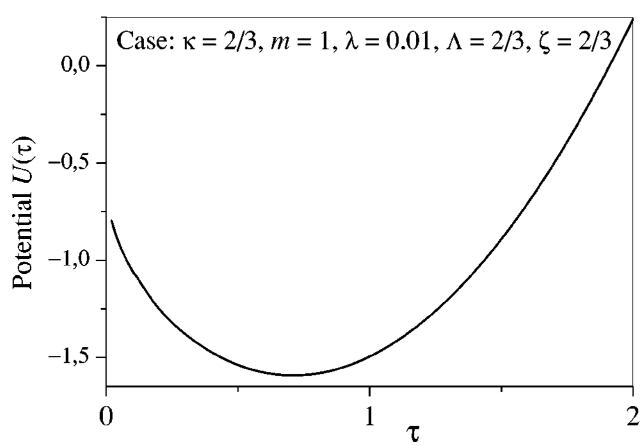

FIG. 8. The potential $\mathcal{U}(\tau)$ [Eq. (3.24)] with BI space-time being filled perfect fluid describing a hard Universe.

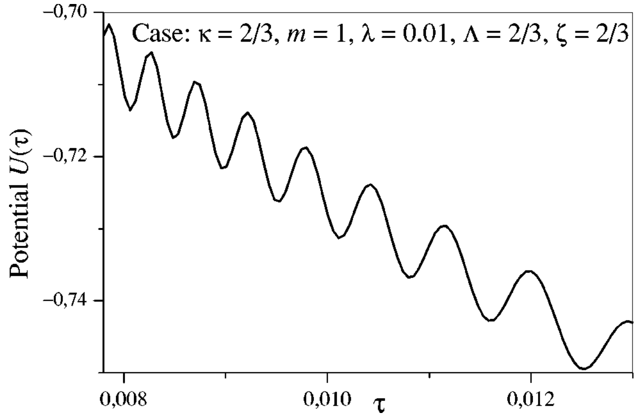

FIG. 9. Fragment of the potential (3.24) in the vicinity of the point $\tau=0$ that occurs due to the nonlinear term F.

graphical view of the potential $\mathcal{U}(\tau)$ [Eq. (3.24)] is given in Figs. 8 and 9. Here we choose the problem parameters as follows: $\kappa=2/3$, spinor mass $m=1$, coupling constant $\lambda=0.01$, cosmological constant $\Lambda=2/3$, $\varepsilon_0=1$ and $\zeta=2/3$. Since $S=1/\tau$ and $\mathcal{U}(\tau)\propto 1/\sin(S)$, a large number of small oscillations occurs as $\tau\to 0$ [cf. Fig. 9].

It is clear from Figs. 8 and 9 that depending on the choice of integration constant E we have two types of solutions demonstrated in Fig. 2. Moreover, for some values of E there exists more than one periodic solution.

Let us now study the system for a negative Λ. Contrary to the case with $F=S^n$, where all the solutions for a negative Λ grow exponentially, in this case an interesting situation occurs for some special choice of parameters.

As one sees from Fig. 10, depending on the integration constant and the initial value of τ, the mode of evolution can be both finite and exponential. For the integration constant being at the level AB in Fig. 10 (here it is -3), with $\tau_0\in(0,\tau_A)$ the evolution of τ is finite and similar to the one illustrated in Fig. 2 corresponding to $E=1$, whereas, for $\tau_0>\tau_B$ we have an exponentially expanding τ. Thus we conclude that for the interacting term being a trigonometric function of its arguments, the system even with a negative Λ admits a nonexponential mode of evolution.

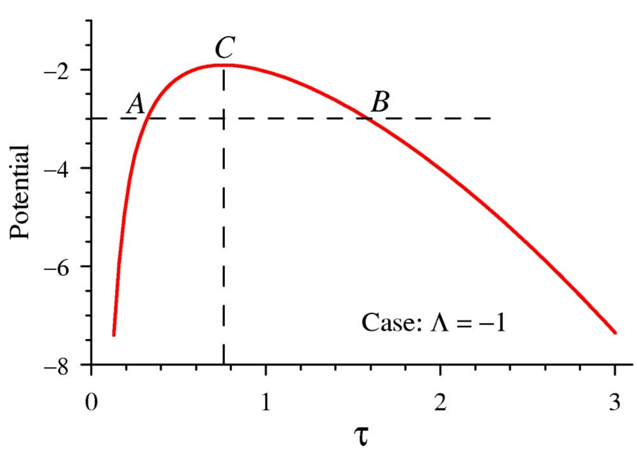

FIG. 10. View of the potential $\mathcal{U}(\tau)$ [Eq. (3.24)] with a negative Λ.

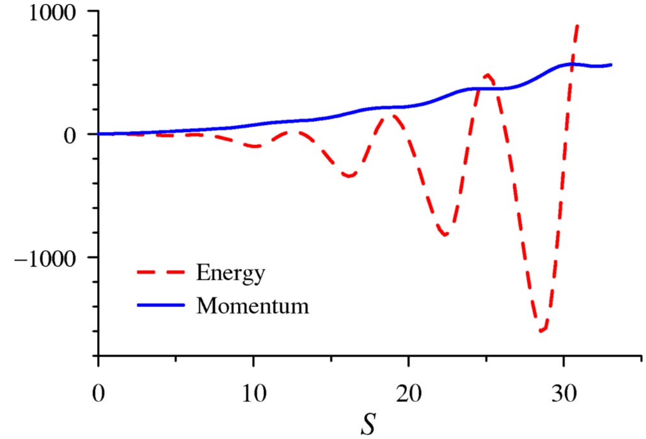

FIG. 11. Relative behavior of T_0^0 and T_1^1 with $F=\sin(S)$. This picture clearly shows the violation of the dominant energy condition that takes place in the case considered.

To investigate the dominant energy condition let us write the components of the energy-momentum tensor. For simplicity we set $C_0=1$ and in terms of S for the energy density we write

$$T_0^0=mS+\frac{S^2}{2(1+\lambda\sin S)}+\varepsilon_0 S^{1+\zeta}. \tag{3.25}$$

Since τ is a positive quantity, S is positive as well. As one sees from Eq. (3.25) for any positive value of S and $\lambda<1$ energy density is always positive definite and proportional to S^2. Since $S=1/\tau$, it means that T_0^0 has its maximum as $\tau\to 0$ and tends to zero as $\tau\to\infty$.

For the pressure components we have

$$T_1^1=T_2^2=T_3^3=\frac{\lambda S^3\cos S}{2(1+\lambda\sin S)^2}-\frac{S^2}{2(1+\lambda\sin S)}-\varepsilon_0\zeta S^{1+\zeta}. \tag{3.26}$$

As one sees, for a $\lambda<1$, the pressure T_1^1 may be both positive and negative depending on the sign of $\cos S$. Moreover, its maximum value is proportional to S^3. Thus, in the case of $F=\sin S$, for any ζ defined as in Eq. (2.23) and any nontrivial λ, there exist intervals (S_i,S_{i+1}) such that for $S\in(S_i,S_{i+1})$ the inequality $T_0^0<T_1^1$ takes place as it is shown in Fig. 11. Therefore we conclude that the regular solutions obtained in this case result in the broken dominant energy condition.

IV. CONCLUSION

Within the framework of the simplest model of interacting spinor and scalar fields it is shown that the Λ term plays a very important role in the evolution of the BI cosmology. In particular, it invokes oscillations in the model. For a nonpositive Λ with F being the power law of its arguments we find a Universe expanding exponentially, hence the initial anisotropy of the model quickly dies away. In the case of F being the trigonometric function of its arguments, a negative Λ beside the exponential ones allows a nonexponential mode of evolution. For a positive Λ with a suitable choice of integration constant $\mathcal{E}$ one finds the oscillatory mode of expansion of the Universe. In this case it is possible to construct solutions that are regular at all space-time points. It should be emphasized that if the spinor field nonlinearity is generated by self-interaction as in Ref. [15], the regularity of the solutions obtained results in the violation of the dominant energy condition of the Penrose-Hawking theorem [15], whereas in the case considered here, when the spinor field nonlinearity is induced by the scalar one, regular solutions can be obtained even without breaking the aforementioned condition. It should be noted that the dominant energy condition holds for F being the power law of I or J, whereas it is not the case when F is chosen to be a trigonometric function of its arguments. Note that in the presence of the Λ term the role of other parameters such as order of nonlinearity n, perfect fluid parameter ζ, and spinor mass in the evolution process are rather local. The global process is totally determined by the Λ term. For example, if $\Lambda > 0$, we have physically allowable solutions that are either oscillatory or defined on some finite interval of time. In the case of $\Lambda < 0$ solutions are generally inflationlike though for some special choices of problem parameters the oscillatory mode of evolution can be attained.

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