Boundary value problem for five-dimensional stationary rotating black holes

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We study the boundary value problem for stationary rotating black hole solutions to the five-dimensional vacuum Einstein equation. Assuming the existence of two additional commuting rotational Killing vector fields and sphericity of the horizon topology, we show that a black hole with a regular event horizon is uniquely characterized by its mass and a pair of angular momenta.

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I. INTRODUCTION

In recent years there has been renewed interest in higher dimensional black holes in the context of both string theory and the brane world scenario. In particular, the possibility of black hole production in a linear collider has been suggested [1–4]. Such phenomena play a key role in getting insight into the structure of space-time; we might be able to prove the existence of extra dimensions and have some information about quantum gravity. Since the primary signature of black hole production in a collider will be Hawking emission from the stationary black hole, the classical equilibrium problem of black holes is an important subject. The black holes produced in colliders will be small enough compared with the size of the extra dimensions and generically have angular momenta; they will be well approximated by the higher dimensional rotating black hole solutions found by Myers and Perry [5]. The Myers-Perry black hole, which has an event horizon with spherical topology, can be regarded as a higher dimensional generalization of the Kerr black hole. One might expect that such a black hole solution describes the classical equilibrium state continued from the black hole production event, if it shows stability and uniqueness like the Kerr black hole in four dimensions. The purpose of this paper is to consider the uniqueness and nonuniqueness of rotating black holes in higher dimensions.

The uniqueness theorem states that a four-dimensional black hole with a regular event horizon is characterized only by mass, angular momentum, and electric charge [6,7]. Recently, the uniqueness and nonuniqueness properties of five- or higher dimensional black holes have also been studied. Emparan and Reall found a black ring solution of the five-dimensional vacuum Einstein equation, which describes a stationary rotating black hole with the event horizon homeomorphic to $S^2 \times S^1$ [8]. In a certain parameter region, a black ring and a (Myers-Perry) black hole can carry the same mass and angular momentum. This might suggest the nonuniqueness of higher dimensional stationary black hole solutions. For example, Reall [9] conjectured the existence of a stationary, asymptotically flat higher dimensional vacuum black hole admitting exactly two commuting Killing vector fields although all known higher dimensional black hole solutions have three or more Killing vector fields. In six or higher dimensions, the Myers-Perry black hole can have an arbitrarily large angular momentum for a fixed mass. The horizon of such a black hole greatly spreads out in the plane of rotation and looks like a black brane in the limit where the angular momentum goes to infinity. Hence, Emparan and Myers [10] argued that rapidly rotating black holes are unstable due to the Gregory-Laflamme instability [11] and decay to stationary black holes with rippled horizons, implying the existence of black holes with less geometric symmetry compared with Myers-Perry black holes. For supersymmetric black holes and black rings, a string theoretical interpretation was given by Elvang and Emparan [12]. They showed that the black hole and the black ring with the same asymptotic charges correspond to different configurations of branes, giving a partial resolution of the nonuniqueness problem of supersymmetric black holes in five dimensions. On the other hand, we have uniqueness theorems for black holes at least in the static case [13–18]. Furthermore, the uniqueness of stationary black holes is supported by an argument based on linear perturbation of higher dimensional static black holes [19,20]. There exist regular stationary perturbations that fall off in the asymptotic region only for vector perturbation, and thus the number of independent modes corresponds to the rank of the rotation group, namely, the number of angular momenta carried by the Myers-Perry black holes [21]. This suggests that higher dimensional stationary black holes have a uniqueness property in some sense, but some amendments will be required. Here we consider the possibility of a restricted black hole uniqueness which is consistent with any argument about uniqueness or nonuniqueness. Although the existence of the black ring solution explicitly violates black hole uniqueness, there is still a possibility of black hole uniqueness for fixed horizon topology [22]. Hence we restrict ourselves to stationary black holes with spherical topology.

In this paper, we consider the asymptotically flat, black hole solution to the five-dimensional vacuum Einstein equation with a regular event horizon homeomorphic to S^3, admitting two commuting spacelike Killing vector fields and a

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stationary (timelike) Killing vector field. The two spacelike Killing vector fields correspond to rotations in the (X^1-X^2) plane and (X^3-X^4) plane in the asymptotic region ($\{X^\mu\}$ are the asymptotic Cartesian coordinates), respectively, which commute with each other. According to the argument by Carter [23], it is possible to construct a timelike Killing vector field tangent to the fixed points (namely, the axis) of the axisymmetric Killing vector field from the given timelike Killing vector field. Repeating this procedure for each commuting spacelike Killing vector field, the timelike Killing vector field obtained also commutes with both spacelike Killing vector fields. Hence, it is natural to assume that all the three Killing vector fields commute with each other. The five-dimensional vacuum space-time admitting three commuting Killing vector fields is described by the nonlinear σ model [24]. Then the Mazur identity [25] for this system is derived. We show that the five-dimensional black hole solution with a regular event horizon of spherical topology is determined by three parameters under the appropriate boundary conditions.

The remainder of the paper is organized as follows. In Sec. II A, we give the field equations for the five-dimensional vacuum space-time admitting three commuting Killing vector fields. In Sec. II B, we introduce the matrix form of the field equations to clarify the hidden symmetry of this system, following Maison [24]. Then the Mazur identity that is useful to show the coincidence of two solutions is derived in Sec. III. In Sec. IV, we determine the boundary conditions. We summarize this paper and give a discussion of related matters in Sec. V.

II. FIVE-DIMENSIONAL VACUUM SPACE-TIME ADMITTING THREE COMMUTING KILLING VECTOR FIELDS

Assuming symmetry of space-time, the Einstein equations reduce to equations for scalar fields defined on a three-dimensional space. Then we show that the system of scalar fields is described by a nonlinear σ model.

A. Weyl-Papapetrou metrics

We consider the five-dimensional space-time admitting two commuting Killing vector fields $\xi_I=\partial_I$ $(I=4,5)$. The metric can be written in the form

$$g=f^{-1}\gamma_{ij}dx^idx^j+f_{IJ}(dx^I+w^I_idx^i)(dx^J+w^J_jdx^j), \quad (1)$$

where $i,j=1,2,3$, $f=\det(f_{IJ})$. The three-dimensional metric γ_{ij}, the functions w^I_i and f_{IJ} are independent of the coordinates x^I ($x^4=\phi, x^5=\psi$, and we will later identify ξ_4 and ξ_5 as Killing vector fields corresponding to two independent rotations in the case of asymptotically flat space-time). We define the twist potential ω_I by

$$\omega_{I,\mu}=ff_{IJ}\sqrt{|\gamma|}\epsilon_{ij\mu}\gamma^{im}\gamma^{jn}\partial_m w^J_n, \quad (2)$$

where $\mu=1,\ldots,5$, $\gamma=\det(\gamma_{ij})$, γ^{ij} is the inverse metric of γ_{ij}, and $\epsilon_{\lambda\mu\nu}$ denotes a totally skew-symmetric symbol such that $\epsilon_{123}=1, \epsilon_{I\mu\nu}=0$. Then the vacuum Einstein equation reduces to the field equations for the five scalar fields f_{IJ} and ω_I defined on the three-dimensional space:

$$D^2f_{IJ}=f^{KL}Df_{IK}\cdot Df_{JL}-f^{-1}D\omega_I\cdot D\omega_J, \quad (3)$$

$$D^2\omega_I=f^{-1}Df\cdot D\omega_I+f^{JK}Df_{IJ}\cdot D\omega_K, \quad (4)$$

and the Einstein equations on the three-dimensional space:

$${}^{(\gamma)}R_{ij}=\frac{1}{4}f^{-2}f_{,i}f_{,j}+\frac{1}{4}f^{IJ}f^{KL}f_{IK,i}f_{JL,j}+\frac{1}{2}f^{-1}f^{IJ}\omega_{Ii}\omega_{Jj}, \quad (5)$$

where D is the covariant derivative with respect to the three-metric γ_{ij} and the dot denotes the inner product determined by γ_{ij}.

Here we assume the existence of another Killing vector field $\xi_3=\partial_3$ which commutes with the other Killing vectors as $[\xi_3,\xi_I]=0$ (we will later identify ξ_3 as the stationary Killing vector field in the case of asymptotically flat space-time). Then the metric can be written in Weyl-Papapetrou–type form [26] as

$$g=f^{-1}e^{2\sigma}(d\rho^2+dz^2)-f^{-1}\rho^2dt^2+f_{IJ}(dx^I+w^Idt)(dx^J+w^Jdt), \quad (6)$$

where we denote $x^3=t$, and all the metric functions depend only on ρ and z. Once the five scalar fields f_{IJ},ω_I are determined, the other metric functions σ and w^I are obtained by solving the following partial derivative equations:

$$\frac{2}{\rho}\sigma_{,\rho}=\frac{1}{4}f^{-2}[(f_{,\rho})^2-(f_{,z})^2]+\frac{1}{4}f^{IJ}f^{MN}(f_{IM,\rho}f_{JN,\rho}-f_{IM,z}f_{JN,z})+\frac{1}{2}f^{-1}f^{IJ}(\omega_{I,\rho}\omega_{J,\rho}-\omega_{I,z}\omega_{J,z}), \quad (7)$$

$$\frac{1}{\rho}\sigma_{,z}=\frac{1}{4}f^{-2}f_{,\rho}f_{,z}+\frac{1}{4}f^{IJ}f^{MN}f_{IM,\rho}f_{JN,z}+\frac{1}{2}f^{-1}f^{IJ}\omega_{I,\rho}\omega_{J,z}, \quad (8)$$

$$w^I_{,\rho}=\rho f^{-1}f^{IJ}\omega_{J,z}, \quad (9)$$

$$w^I_{,z}=-\rho f^{-1}f^{IJ}\omega_{J,\rho}. \quad (10)$$

The f_{IJ} and ω_I are given by the axisymmetric solution of the field equations (3) and (4) on the abstract flat three-space with the metric

$$\gamma=d\rho^2+dz^2+\rho^2d\varphi^2. \quad (11)$$

Thus the system is described by the action

$$S=\int d\rho dz\rho\left[\frac{1}{4}f^{-2}(\partial f)^2+\frac{1}{4}f^{IJ}f^{KL}\partial f_{IK}\cdot\partial f_{JL}+\frac{1}{2}f^{-1}f^{IJ}\partial\omega_I\cdot\partial\omega_J\right]. \tag{12}$$

B. Matrix representation

The action (12) is invariant under the global $SL(3,\mathbf{R})$ transformations as shown by Maison [24]. Instead of the nonlinear representation by the scalar fields f_{IJ} and ω_I, we introduce the $SL(3,\mathbf{R})$ matrix field Φ as

$$\Phi=\begin{pmatrix} f^{-1} & -f^{-1}\omega_\phi & -f^{-1}\omega_\psi \\ -f^{-1}\omega_\phi & f_{\phi\phi}+f^{-1}\omega_\phi\omega_\phi & f_{\phi\psi}+f^{-1}\omega_\phi\omega_\psi \\ -f^{-1}\omega_\psi & f_{\phi\psi}+f^{-1}\omega_\phi\omega_\psi & f_{\psi\psi}+f^{-1}\omega_\psi\omega_\psi \end{pmatrix}, \tag{13}$$

which is symmetric (${}^t\Phi=\Phi$) and unimodular ($\det\Phi=1$). Φ transforms as a covariant, symmetric, second-rank tensor field under global $SL(3,\mathbf{R})$ transformations. When the Killing vector fields ξ_ϕ and ξ_ψ are spacelike, all the eigenvalues of Φ are real and positive. Therefore, there is an $SL(3,\mathbf{R})$ matrix field g which is a square root of the matrix field Φ, namely,

$$\Phi=g\,{}^tg. \tag{14}$$

This square root matrix g is determined up to global $SO(3)$ rotation because the rotation $g\mapsto g\Lambda$ for any $\Lambda\in SO(3)$ is canceled by $\Lambda^{-1}={}^t\Lambda$. Since any $SL(3,\mathbf{R})$ matrix field g conversely defines a symmetric and unimodular matrix field by $\Phi=g\,{}^tg$, the matrix Φ defines a map from the two-dimensional ρ-z half plane (base space) to the coset space $SL(3,\mathbf{R})/SO(3)$.

The inverse matrix of Φ is explicitly given by

$$\Phi^{-1}=\begin{pmatrix} f+f^{IJ}\omega_I\omega_J & f^{\phi J}\omega_J & f^{\psi J}\omega_J \\ f^{\phi J}\omega_J & f^{\phi\phi} & f^{\phi\psi} \\ f^{\psi J}\omega_J & f^{\phi\psi} & f^{\psi\psi} \end{pmatrix}, \tag{15}$$

and transforms as a second-rank contravariant tensor field on the base space.

The current matrix defined by

$$J_i=\Phi^{-1}\partial_i\Phi \tag{16}$$

transforms linearly according to the adjoint representation of $SL(3,\mathbf{R})$. This current is conserved, namely, every element of D_iJ^i independently vanishes due to the field equations (3) and (4).

The action (12) can be expressed in terms of J_i or Φ as

$$S=\frac{1}{4}\int d\rho dz\rho\,\mathrm{tr}(J_iJ^i), \tag{17}$$

$$=\frac{1}{4}\int d\rho dz\rho\,\mathrm{tr}(\Phi^{-1}\partial_i\Phi\Phi^{-1}\partial^i\Phi). \tag{18}$$

This action takes a nonlinear σ-model form.

III. MAZUR IDENTITY

Let us consider two different sets of the field configurations $\Phi_{[0]}$ and $\Phi_{[1]}$ satisfying the field equations (3) and (4). To show the coincidence of the two solutions, we will derive the Mazur identity for the nonlinear σ model on the symmetric space $SL(3,\mathbf{R})/SO(3)$.

A bull's eye $\odot$ denotes the difference between the value of the functional obtained from the field configuration $\Phi_{[1]}$ and the value obtained from $\Phi_{[0]}$, e.g.,

$$\overset{\odot}{J}{}^i=J^i_{[1]}-J^i_{[0]}=\Phi^{-1}_{[1]}\partial^i\Phi_{[1]}-\Phi^{-1}_{[0]}\partial^i\Phi_{[0]}. \tag{19}$$

The deviation matrix Ψ is defined by

$$\Psi=\overset{\odot}{\Phi}\Phi^{-1}_{[0]}=\Phi_{[1]}\Phi^{-1}_{[0]}-\mathbf{1}, \tag{20}$$

where $\mathbf{1}$ is the unit matrix. The deviation Ψ vanishes if and only if the two sets of field configurations ([1] and [0]) coincide with each other. Differentiating Ψ,

$$D^i\Psi=\Phi_{[1]}\overset{\odot}{J}{}^i\Phi^{-1}_{[0]} \tag{21}$$

and taking the divergence, we obtain

$$D_i(D^i\Psi)=\Phi_{[1]}D_i\overset{\odot}{J}{}^i\Phi^{-1}_{[0]}+\Phi_{[1]}\{J_{[1]i}J^i_{[1]}-2J_{[1]i}J^i_{[0]}+J_{[0]i}J^i_{[0]}\}\Phi^{-1}_{[0]}. \tag{22}$$

Due to the current conservation $D_iJ^i=0$, the first term on the right hand side of Eq. (22) vanishes. Since ${}^tJ^i=\Phi J^i\Phi^{-1}$, the second term on the right hand side can be rewritten as

$$\Phi_{[1]}\{J_{[1]i}J^i_{[1]}-2J_{[1]i}J^i_{[0]}+J_{[0]i}J^i_{[0]}\}\Phi^{-1}_{[0]}$$

$$=\Phi_{[1]}(J^i_{[1]}\overset{\odot}{J}_i-\overset{\odot}{J}_iJ^i_{[0]})\Phi^{-1}_{[0]} \tag{23}$$

$$={}^tJ^i_{[1]}\Phi_{[1]}\overset{\odot}{J}_i\Phi^{-1}_{[0]}-\Phi_{[1]}\overset{\odot}{J}_i\Phi^{-1}_{[0]}\,{}^tJ^i_{[0]}. \tag{24}$$

Then, taking the trace, we obtain the identity

$$(D_iD^i\mathrm{tr}\,\Psi)=\mathrm{tr}\{{}^t\overset{\odot}{J}{}^i\Phi_{[1]}\overset{\odot}{J}_i\Phi^{-1}_{[0]}\}. \tag{25}$$

Since D is covariant derivative with respect to the abstract flat three-metric (11) and all quantities are independent of φ, the above identity (25) is

$$\partial_a(\rho\partial^a \mathrm{tr}\,\Psi) = \rho h_{ab}\mathrm{tr}\{{}^t\overset{\odot}{J}{}^a\Phi_{[1]}\overset{\odot}{J}{}^b\Phi_{[0]}^{-1}\}, \tag{26}$$

where h_{ab} is the flat two-dimensional metric

$$h = d\rho^2 + dz^2. \tag{27}$$

Integrating Eq. (26) over the relevant region $\Sigma = \{(\rho,z)|\rho \geqslant 0\}$ in the ρ-z plane, and using Green's theorem, we find

$$\oint_{\partial\Sigma}\rho\partial^a \mathrm{tr}\,\Psi\, dS_a = \int_\Sigma \rho h_{ab}\mathrm{tr}\{{}^t\overset{\odot}{J}{}^a\Phi_{[1]}\overset{\odot}{J}{}^b\Phi_{[0]}^{-1}\}d\rho dz, \tag{28}$$

where the boundary $\partial\Sigma$ corresponds to the horizon, the two planes of rotation, and infinity. Since the matrix Φ has a square root matrix g like Eq. (14), the integrand of the right hand side of Eq. (28) is written as

$$\rho h_{ab}\mathrm{tr}\{{}^t\overset{\odot}{J}{}^a\Phi_{[1]}\overset{\odot}{J}{}^b\Phi_{[0]}^{-1}\} = \rho h_{ab}\mathrm{tr}\{g_{[0]}^{-1}{}^t\overset{\odot}{J}{}^a g_{[1]}{}^t g_{[1]}\overset{\odot}{J}{}^b{}^t g_{[0]}^{-1}\}. \tag{29}$$

Thus, we obtain the Mazur identity

$$\oint_{\partial\Sigma}\rho\partial^a \mathrm{tr}\,\Psi\, dS_a = \int_\Sigma \rho h_{ab}\mathrm{tr}\{\mathcal{M}^a{}^t\mathcal{M}^b\}d\rho dz, \tag{30}$$

where the matrix $\mathcal{M}$ is defined by

$$\mathcal{M}^a = g_{[0]}^{-1}{}^t\overset{\odot}{J}{}^a g_{[1]}. \tag{31}$$

When the current difference $J\odot^a$ is not zero, the right hand side of the identity (30) is positive. Hence we must have $J\odot^a = 0$ if the boundary conditions under which the left hand side of Eq. (30) vanishes are imposed at $\partial\Sigma$. Then the difference Ψ is a constant matrix over the region Σ. The limiting value of Ψ is zero on at least one part of the boundary $\partial\Sigma$ is sufficient to obtain the coincidence of two solutions $\Phi_{[0]}$ and $\Phi_{[1]}$.

IV. BOUNDARY CONDITIONS AND COINCIDENCE OF SOLUTIONS

When one use the Mazur identity, the boundary conditions for the fields Φ (i.e., f_{IJ} and ω_I) are needed at infinity, the two planes of rotation, and the horizon. We will require asymptotic flatness, regularity at the two planes of rotation, and regularity at the spherical horizon. Under these conditions, the Mazur identity shows coincidence of the solutions.

An asymptotically flat space-time with mass $M = 3\pi m/8G$ and angular momenta $J_\phi = \pi m a/4G$ and $J_\psi = \pi m b/4G$ (where we restrict ourselves to the case in which $m > a^2 + b^2 + 2|ab|$) has a metric in the following form:

$$\begin{aligned} g = &-\left[1-\frac{m}{r^2}+O(r^{-3})\right]dt^2 - \left[\frac{2ma}{r^4}+O(r^{-5})\right]dt(ydx \\ &-xdy) - \left[\frac{2mb}{r^4}+O(r^{-5})\right]dt(wdz - zdw) \\ &+\left[1+\frac{m}{2r^2}+O(r^{-3})\right][dx^2+dy^2+dz^2+dw^2]. \end{aligned} \tag{32}$$

Here we introduce the coordinates

$$x = \sqrt{r^2+a^2}\sin\theta\cos[\bar{\phi} - \tan^{-1}(a/r)], \tag{33}$$

$$y = \sqrt{r^2+a^2}\sin\theta\sin[\bar{\phi} - \tan^{-1}(a/r)], \tag{34}$$

$$z = \sqrt{r^2+b^2}\cos\theta\cos[\bar{\psi} - \tan^{-1}(b/r)], \tag{35}$$

$$w = \sqrt{r^2+b^2}\cos\theta\sin[\bar{\psi} - \tan^{-1}(b/r)], \tag{36}$$

and proceed with the further coordinate transformations

$$d\bar{\phi} = d\phi - \frac{a}{r^2+a^2}dr, \tag{37}$$

$$d\bar{\psi} = d\psi - \frac{b}{r^2+b^2}dr, \tag{38}$$

then one obtains

$$\begin{aligned} g = &-\left[1-\frac{m}{r^2}+O(r^{-3})\right]dt^2 + \left[\frac{2ma(r^2+a^2)}{r^4}\sin^2\theta \right. \\ &\left. +O(r^{-3})\right]dtd\phi + \left[\frac{2mb(r^2+b^2)}{r^4}\cos^2\theta + O(r^{-3})\right]dtd\psi \\ &+\left[1+\frac{m}{2r^2}+O(r^{-3})\right]\times\left[\frac{r^2+a^2\cos^2\theta+b^2\sin^2\theta}{(r^2+a^2)(r^2+b^2)}r^2dr^2\right. \\ &+(r^2+a^2\cos^2\theta+b^2\sin^2\theta)d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 \\ &\left.+(r^2+b^2)\cos^2\theta d\psi^2\right]. \end{aligned} \tag{39}$$

Here the metric (39) admits two orthogonal planes of rotation $\theta = \pi/2$ and $\theta = 0$, which are specified by the azimuthal angles ϕ and ψ, respectively. The planes $\theta = 0$ and $\theta = \pi/2$ are invariant under rotation with respect to the Killing vector fields ∂_ϕ and ∂_ψ, respectively. Both angles ϕ and ψ have period 2π. Comparing the asymptotic form (39) with the Weyl-Papapetrou form (6), we derive boundary conditions.

The regularity on the invariant planes requires

$$g_{\phi\phi} = f_{\phi\phi} = \sin^2\theta\tilde{f}_{\phi\phi}, \tag{40}$$

$$g_{\psi\psi} = f_{\psi\psi} = \cos^2\theta\tilde{f}_{\psi\psi}, \tag{41}$$

$$g_{\phi\psi} = f_{\phi\psi} = \sin^2\theta\cos^2\theta\tilde{f}_{\phi\psi}, \tag{42}$$

TABLE I. Boundary conditions for f_{IJ}.

	ϕ-invariant plane $\mu \to +1$	ψ-invariant plane $\mu \to -1$	Horizon $\lambda \to c$	Infinity $\lambda \to +\infty$
$\tilde{f}_{\phi\phi}$	$O(1)$	$O(1)$	$O(1)$	$2\lambda+(a^2-b^2+m)/2+O(\lambda^{-1/2})$
$\tilde{f}_{\phi\psi}$	$O(1)$	$O(1)$	$O(1)$	$O(\lambda^{-1/2})$
$\tilde{f}_{\psi\psi}$	$O(1)$	$O(1)$	$O(1)$	$2\lambda+(b^2-a^2+m)/2+O(\lambda^{-1/2})$

where the quantities with a tilde are regular at both the invariant planes and the black hole horizon.

The asymptotic behavior of $\tilde{f}_{\phi\phi}$ and $\tilde{f}_{\psi\psi}$ is derived from Eq. (39), and $\tilde{f}_{\phi\psi}$ is at most $O(r^{-1})$ since the Killing vectors ∂_ϕ and ∂_ψ are asymptotically orthogonal:

$$\tilde{f}_{\phi\phi}=r^2+a^2+\frac{m}{2}+O(r^{-1}), \tag{43}$$

$$\tilde{f}_{\psi\psi}=r^2+b^2+\frac{m}{2}+O(r^{-1}), \tag{44}$$

$$\tilde{f}_{\phi\psi}=O(r^{-1}). \tag{45}$$

Since $f_{\phi\psi}$ is negligible as compared with $f_{\phi\phi}$ and $f_{\psi\psi}$ in the asymptotic region, the leading terms of $g_{t\phi}$ and $g_{t\psi}$ are $f_{\phi\phi}w^\phi$ and $f_{\psi\psi}w^\psi$, respectively. Then we have

$$f_{\phi\phi}w^\phi=\frac{ma\sin^2\theta}{r^2}+O(r^{-3}), \tag{46}$$

$$f_{\psi\psi}w^\psi=\frac{mb\cos^2\theta}{r^2}+O(r^{-3}). \tag{47}$$

Thus we obtain

$$w^\phi=\frac{ma}{r^4}+O(r^{-5}), \tag{48}$$

$$w^\psi=\frac{mb}{r^4}+O(r^{-5}). \tag{49}$$

Similarly, we have

$$g_{tt}=-f^{-1}\rho^2+f_{\phi\phi}w^\phi w^\phi+2f_{\phi\psi}w^\phi w^\psi+f_{\psi\psi}w^\psi w^\psi \tag{50}$$

$$=-1+\frac{m}{r^2}+O(r^{-3}). \tag{51}$$

Here the $O(r^{-2})$ term must come from the $-f^{-1}\rho^2$ term since the w^I are $O(r^{-4})$. Therefore ρ behaves as

$$\rho^2=[r^4+(a^2+b^2)r^2+O(r)]\sin^2\theta\cos^2\theta. \tag{52}$$

ρ^2 vanishes not only at the ϕ-invariant plane ($\sin\theta=0$) and ψ-invariant plane ($\cos\theta=0$), but also at the horizon due to the form of the metric (6). Since the horizon has the topology of S^3, let us introduce the spheroidal coordinates on Σ as

$$z=\lambda\mu, \tag{53}$$

$$\rho^2=(\lambda^2-c^2)(1-\mu^2), \tag{54}$$

where $\mu=\cos 2\theta$. Then the relevant region is $\Sigma=\{(\lambda,\mu)|\lambda \geqslant c, -1\leqslant\mu\leqslant 1\}$. The boundaries $\lambda=c$, $\lambda=+\infty$, $\mu=1$, and $\mu=-1$ correspond to the horizon, infinity, the ϕ-invariant plane, and the ψ-invariant plane, respectively. In these coordinates, the two-dimensional metric on Σ is given by

$$h=d\rho^2+dz^2=(\lambda^2-c^2\mu^2)\left(\frac{d\lambda^2}{\lambda^2-c^2}+\frac{d\mu^2}{1-\mu^2}\right). \tag{55}$$

The boundary integral in the left hand side of the Mazur identity (30) is explicitly written as

$$\begin{aligned}\oint_{\partial\Sigma}\rho\partial^a \mathrm{tr}\,\Psi\, dS_a=&\int_c^\infty d\lambda\left(\sqrt{\frac{h_{\lambda\lambda}}{h_{\mu\mu}}}\rho\frac{\partial\,\mathrm{tr}\,\Psi}{\partial\mu}\right)\Bigg|_{\mu=-1}\\&+\int_{-1}^{+1}d\mu\left(\sqrt{\frac{h_{\mu\mu}}{h_{\lambda\lambda}}}\rho\frac{\partial\,\mathrm{tr}\,\Psi}{\partial\lambda}\right)_{\lambda=\infty}\\&+\int_\infty^c d\lambda\left(\sqrt{\frac{h_{\lambda\lambda}}{h_{\mu\mu}}}\rho\frac{\partial\,\mathrm{tr}\,\Psi}{\partial\mu}\right)\Bigg|_{\mu=+1}\\&+\int_{+1}^{-1}d\mu\left(\sqrt{\frac{h_{\mu\mu}}{h_{\lambda\lambda}}}\rho\frac{\partial\,\mathrm{tr}\,\Psi}{\partial\lambda}\right)_{\lambda=c},\end{aligned} \tag{56}$$

where

$$\frac{\partial\,\mathrm{tr}\,\Psi}{\partial x^a}=\frac{\partial}{\partial x^a}[f_{[1]}^{-1}(-\overset{\odot}{f}+f_{[0]}^{IJ}\overset{\odot}{\omega}_I\overset{\odot}{\omega}_J)+f_{[0]}^{IJ}\overset{\odot}{f}_{IJ}]$$
$$\text{for } x^a=\lambda,\mu. \tag{57}$$

Here the relation between λ and r is given by

$$\lambda=\frac{r^2}{2}+\frac{a^2+b^2}{4}+O(r^{-1}) \tag{58}$$

or

TABLE II. Boundary conditions for ω_I.

	ϕ-invariant plane $\mu\to+1$	ψ-invariant plane $\mu\to-1$	Horizon $\lambda\to c$	Infinity $\lambda\to+\infty$
$\tilde{\omega}_\phi$	$O((1-\mu)^2)$	$O(1+\mu)$	$O(1)$	$O(\lambda^{-1/2})$
$\tilde{\omega}_\psi$	$O(1-\mu)$	$O((1+\mu)^2)$	$O(1)$	$O(\lambda^{-1/2})$

$$r=\sqrt{2}\lambda^{1/2}\left[1-\frac{a^2+b^2}{8\lambda}+O(\lambda^{-3/2})\right]. \tag{59}$$

The boundary conditions for f_{IJ} are summarized in Table I, where

$$f_{\phi\phi}=\frac{(1-\mu)}{2}\tilde{f}_{\phi\phi}, \tag{60}$$

$$f_{\phi\psi}=\frac{(1-\mu)(1+\mu)}{4}\tilde{f}_{\phi\psi}, \tag{61}$$

$$f_{\psi\psi}=\frac{(1+\mu)}{2}\tilde{f}_{\psi\psi}. \tag{62}$$

Next, let us derive the boundary conditions for the twist potentials. By the definition of twist potentials, Eq. (2),

$$\frac{\partial\omega_\phi}{\partial\lambda}=-\frac{ff_{\phi J}}{\lambda^2-c^2}\frac{\partial w^J}{\partial\mu},\quad \frac{\partial\omega_\phi}{\partial\mu}=\frac{ff_{\phi J}}{1-\mu^2}\frac{\partial w^J}{\partial\lambda}, \tag{63}$$

$$\frac{\partial\omega_\psi}{\partial\lambda}=-\frac{ff_{\psi J}}{\lambda^2-c^2}\frac{\partial w^J}{\partial\mu},\quad \frac{\partial\omega_\psi}{\partial\mu}=\frac{ff_{\psi J}}{1-\mu^2}\frac{\partial w^J}{\partial\lambda}. \tag{64}$$

From the μ dependence of f_{IJ} and w^I, we can obtain the μ dependence of the derivatives of the twist potentials: at $\mu=+1$,

$$\frac{\partial\omega_\phi}{\partial\lambda}=\frac{\partial\omega_\phi}{\partial\mu}=\frac{\partial\omega_\psi}{\partial\lambda}=0 \tag{65}$$

and

$$\frac{\partial\omega_\psi}{\partial\mu}\neq 0 \tag{66}$$

if $ma\neq 0$, and at $\mu=-1$,

$$\frac{\partial\omega_\psi}{\partial\lambda}=\frac{\partial\omega_\psi}{\partial\mu}=\frac{\partial\omega_\phi}{\partial\lambda}=0 \tag{67}$$

and

$$\frac{\partial\omega_\phi}{\partial\mu}\neq 0 \tag{68}$$

if $mb\neq 0$.

In the asymptotic region $(\lambda\to+\infty)$, the derivatives of the twist potentials behave as

TABLE III. Quantities appearing in boundary integral (56).

	ϕ-invariant plane $\mu\to+1$	ψ-invariant plane $\mu\to-1$	Horizon $\lambda\to c$	Infinity $\lambda\to+\infty$
$\partial\,\mathrm{tr}\,\Psi/\partial\lambda$	—	—	$O(1)$	$O(\lambda^{-5/2})$
$\partial\,\mathrm{tr}\,\Psi/\partial\mu$	$O(1)$	$O(1)$	—	—
ρ	$O(\sqrt{1-\mu})$	$O(\sqrt{1+\mu})$	$O(\sqrt{\lambda-c})$	$O(\lambda)$
$\sqrt{h_{\mu\mu}/h_{\lambda\lambda}}$	—	—	$O(\sqrt{\lambda-c})$	$O(\lambda)$
$\sqrt{h_{\lambda\lambda}/h_{\mu\mu}}$	$O(\sqrt{1-\mu})$	$O(\sqrt{1+\mu})$	—	—

$$\frac{\partial\omega_\phi}{\partial\lambda}=O(\lambda^{-3/2}), \tag{69}$$

$$\frac{\partial\omega_\phi}{\partial\mu}=-\frac{ma}{2}(1-\mu)+O(\lambda^{-1/2}). \tag{70}$$

Thus we obtain

$$\omega_\phi=-\frac{ma}{4}\mu(2-\mu)+(1-\mu)^2(1+\mu)O(\lambda^{-1/2}), \tag{71}$$

and similarly

$$\omega_\psi=-\frac{mb}{4}\mu(2+\mu)+(1-\mu)(1+\mu)^2O(\lambda^{-1/2}). \tag{72}$$

Then, of course, the condition that ω_I are regular on the horizon is required.

The boundary conditions for ω_I are summarized in Table II, where

$$\omega_\phi=-\frac{ma}{4}\mu(2-\mu)+\tilde{\omega}_\phi, \tag{73}$$

$$\omega_\psi=-\frac{mb}{4}\mu(2+\mu)+\tilde{\omega}_\psi. \tag{74}$$

The behavior of the quantities which appear in the boundary integral (56) are easily calculated as shown in Table III.

Then the boundary integral (56) vanishes. The difference matrix Ψ is constant and has asymptotic behavior as

$$\Psi\to\begin{pmatrix} O(\lambda^{-3/2}) & O(\lambda^{-7/2}) & O(\lambda^{-7/2})\\ O(\lambda^{-1/2}) & O(\lambda^{-3/2}) & O(\lambda^{-3/2})\\ O(\lambda^{-1/2}) & O(\lambda^{-3/2}) & O(\lambda^{-3/2})\end{pmatrix},\quad (\lambda\to+\infty). \tag{75}$$

Ψ vanishes at infinity, and thus Ψ is zero over Σ. Thus the two configurations $\Phi_{[0]}$ and $\Phi_{[1]}$ coincide with each other.

V. SUMMARY AND DISCUSSION

We show the uniqueness of the asymptotically flat, black hole solution to the five-dimensional vacuum Einstein equation with a regular event horizon homeomorphic to S^3, admitting two commuting spacelike Killing vector fields and a stationary Killing vector field. The solution of this system is determined by only three asymptotic charges, the mass $M = 3\pi m/8G$ and the two angular momenta $J_\phi = \pi m a/4G$ and $J_\psi = \pi m b/4G$. The five-dimensional Myers-Perry black hole solution is unique in this class.

The vacuum black ring solution satisfies the above conditions other than that on the topology of the horizon. There exist two black ring solutions which have the same mass and angular momentum, which means the uniqueness property fails for the $S^2 \times S^1$ event horizon. It is intriguing to investigate how this nonuniqueness occurs.

It will be impossible to extend our argument using the Mazur identity to six- or higher dimensional Myers-Perry black hole solutions. An n-dimensional space-time admitting $(n-3)$ commuting Killing vector fields is always described by the nonlinear σ model as shown by Maison [24]. To derive the Mazur identity for this nonlinear σ model, all the $(n-3)$ Killing vector fields have to be spacelike. However, the n-dimensional Myers-Perry black hole space-time has only $[(n-1)/2]$ commuting spacelike Killing vector fields. Thus our method cannot be used except for a five-dimensional Myers-Perry black hole.

The rigidity theorem in four dimensions claims that asymptotically flat, stationary analytic space-time is also axisymmetric [27]. However, the existence of additional space-time Killing vector fields is no longer justified in the case of five-dimensional black holes. Therefore the uniqueness shown in the present work does not exclude the possibility of the existence of black hole solutions with less symmetry, as suggested by Reall [9].

We would like to speculate on an extension of the present work to the case of other theories. The configuration of the five-dimensional Einstein-Maxwell fields admitting three commuting Killing vector fields is encoded in eight scalar fields [26]. This system is described in terms of a harmonic (scalar) map onto eight-dimensional target space, but it is not the nonlinear σ model in the sense that the target space is not homogeneous. Therefore the Mazurs method utilized here, which is applicable only when the target space is homogeneous, does not work for five-dimensional Einstein-Maxwell theory. We also note that another method given by Bunting [28], which can be applied to harmonic maps onto the target space with negative curvature, fails, since it turns out that the target space corresponding to the five-dimensional Einstein-Maxwell theory [26] does not always have negative curvature. However, such technical difficulties might be improved when the dilatonic scalar fields are taken into account. Recently, Emparan [29] has shown the existence of an infinite number of black ring solutions with the same mass and angular momentum as the Myers-Perry black hole [5] and the neutral black ring solution [8] in five-dimensional Einstein-Maxwell-dilaton theories. This suggests that the uniqueness or nonuniqueness property of the stationary black holes might be somewhat complicated in dilatonic theories. We hope that the boundary value formulation of the dilatonic black hole and the black ring might be helpful in understanding this infinite breakdown of the black hole uniqueness.

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