Off-equilibrium dynamics of the primordial perturbations in the inflationary universe: The $O(N)$ model

Wolung Lee* and Yeo-Yie Charng†
Institute of Physics, Academia Sinica, Taipei, Taiwan 115, Republic of China

Da-Shin Lee‡
Department of Physics, National Dong-Hwa University, Hua-lien, Taiwan 974, Republic of China

Li-Zhi Fang§
Department of Physics, University of Arizona, Tucson, Arizona 85721, USA

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Using the $O(N)$ model as an example, we investigate the self-interaction effects of inflaton on the dynamics of the primordial perturbations. When taking interactions into account, it is essential to employ a self-consistent off-equilibrium formalism to study the evolution of the inflationary background field and its fluctuations with back-reaction effects. Within the Hartree factorization scheme, we show that the $O(N)$ model has at least two observable remains left behind by the off-equilibrium processes: the running spectral index of primordial density perturbations and the correlations between perturbation modes in phase space. We find that the running of the spectral index is fully determined by the rate of the energy transfer from the inflationary background field to its fluctuations via particle creation processes as well as the dynamics of the background field itself. Furthermore, the amplitude of the field fluctuations turns out to be scale dependent due to the off-equilibrium evolution. As a consequence, the scale dependence of fluctuations yields a correlation between the phase-space modes of energy density perturbations, while the one-point function of the fluctuations in each Hartree mode is still Gaussian. More importantly, the mode-mode correlation of the primordial perturbations depends upon the dynamics of the self-interaction *as well as* the initial conditions of the inflation. Hence, we propose that the running spectral index and the correlation between phase-space modes would be two observable fossils to probe the epoch of inflation, even beyond.

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I. INTRODUCTION

In the inflationary scenario, the primordial perturbations of the Universe originate from vacuum fluctuations of the scalar field(s), the inflaton ϕ, driving the inflation. If the dynamics of the fluctuations is approximated by a single massless free field during the inflationary epoch, the power spectrum of curvature perturbations for a Fourier mode k can be obtained as

$$\mathcal{P}_{\mathcal{R}}(k)=\left[\left(\frac{H}{\dot{\phi}}\right)\left(\frac{H}{2\pi}\right)\right]^2_{k=aH}, \qquad (1)$$

where the inflationary scale factor $a=\exp(Ht)$ and the Hubble parameter $H=\sqrt{8\pi GV/3}$ are determined by the potential of the inflaton field $V(\phi)$. The first factor $H/\dot{\phi}$ on the right-hand side of Eq. (1) comes from the evolution of the background inflaton field ϕ—i.e., the expectation value of the quantum scalar field. Meanwhile, the second factor $H/2\pi$ is specified by the variance of the classical ϕ field fluctuations, $\delta\phi_k$, at a few Hubble times after horizon crossing. With slow-roll conditions, the derivative of the spectral index $dn(k)/d\ln k$ turns out to be very small or even negligible such that $n(k)=(d\ln\mathcal{P}_{\mathcal{R}}(k)/d\ln k)+1\simeq 1$ is roughly k independent [1].

However the k dependence, or the running of the spectral index, revealed by the recently released Wilkinson Microwave Anisotropy Probe (WMAP) data can be as large as $dn(k)/d\ln k=(d/d\ln k)^2\ln\mathcal{P}_{\mathcal{R}}(k)\simeq -0.055$ to -0.077 [2]. Hence, the origin of the primordial perturbations cannot be solely accommodated with the quantum fluctuations of a single free field. In a slow-roll inflation, one has $d/d\ln k =(1/H)d/dt=-(1/8\pi G)[V'(\phi)/V(\phi)](d/d\phi)$, and therefore the derivative of the power spectrum $(d/d\ln k)^n\ln\mathcal{P}_{\mathcal{R}}(k)$ will no longer be negligible for the second order $(n=2)$ as long as the third or higher derivatives of the inflaton potential become substantial. To fit in with the running of the spectral index, models beyond the single scalar field with quadratic potential $V(\phi)$ have been proposed accordingly [3]. In this context, the running index of the perturbation power spectrum is considered as an essential feature due to the interactions or the self-interactions of the inflaton(s).

It should be pointed out here that in deriving the derivative of the power spectrum (1), the tree level of the effective potential $V(\phi)$ is implicitly involved. However, when quantum fluctuations arising from the interactions of the inflaton become important, this approach may be problematic. In particular, the one-loop effective potential will turn complex

*Electronic address: leewl@phys.sinica.edu.tw
†Electronic address: charng@phys.sinica.edu.tw
‡Electronic address: dslee@mail.ndhu.edu.tw
§Electronic address: fanglz@physics.arizona.edu

within the region where the background field ϕ is constrained by $V''(\phi)<0$ [4]. The imaginary part of the effective potential would inevitably lead to a dynamically unstable state [5]. This so-called "spinodal instability" will allow long-wavelength fluctuations to grow nonperturbatively [6]. As such, the term $H/2\pi$ in Eq. (1) is no longer valid when the inflaton suffers from spinodal instabilities. Therefore, the primordial perturbations must be dealt with consistently with a different method to account for the amplification of vacuum fluctuations in the presence of spinodal instabilities.

This motivates us to study the self-interaction effects of the inflaton on the dynamics of primordial perturbations within a context of the self-consistent off-equilibrium formalism. We will use the $O(N)$ model of inflation as an example. When $N=1$, it reduces to the popular $\lambda\phi^4$ inflationary model which is disfavored by the WMAP. However, the $O(N)$ model with spontaneously broken symmetry is still a viable model in quantum field theory and off-equilibrium statistical mechanics [7]. It has been extensively used in modeling the quantum off-equilibrium processes in the early Universe, as well as the chiral phase transition in relativistic heavy ion collisions, etc. [8]. In particular, the dynamics of quantum fluctuations of the $O(N)$ model in the large-N limit has been developed with a Hartree-type factorization [9,10], which can be generalized for the finite-N situation. In this paper, we will focus on searching for the possible observable imprints caused by the off-equilibrium evolution of inflationary primordial perturbations.

The paper is organized as follows. In Sec. II, we briefly describe the $O(N)$ model of inflation and introduce the self-consistent, nonperturbative, and renormalized solutions to the nonlinear evolution of the background inflaton field as well as its quantum fluctuations within the Hartree approximation. Section III analyzes the dynamics of quantum fluctuations undergoing a quantum-to-classical transition and the statistical properties of the corresponding classical primordial perturbations. The spectral index along with the running of the perturbation power spectrum will be calculated in Sec. IV. Section V then discusses the correlation of primordial perturbations in phase space and its detection. Finally, we summarize our findings and give conclusions in Sec. VI.

II. $O(N)$ INFLATION

A. Model and the Hartree factorization

Consider the dynamics of an inflation driven by a field $\mathbf{\Phi}$ of the $O(N)$ vector model with spontaneous symmetry breaking. The action is defined by

$$\mathcal{S}=\int d^4x\mathcal{L}=\int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\mathbf{\Phi}\cdot\partial_\nu\mathbf{\Phi}-V(\mathbf{\Phi}\cdot\mathbf{\Phi})\right], \tag{2}$$

where $V(\mathbf{\Phi}\cdot\mathbf{\Phi})$ is a self-interaction potential given by

$$V(\mathbf{\Phi}\cdot\mathbf{\Phi})=\frac{\lambda}{8N}\left(\mathbf{\Phi}\cdot\mathbf{\Phi}-\frac{2Nm^2}{\lambda}\right)^2. \tag{3}$$

As an inflaton, the N components of the field generally are represented as $\mathbf{\Phi}=(\sigma,\vec{\pi})$, where $\vec{\pi}$ represent $N-1$ scalar fields. The cosmic inflation is characterized by the state in which the component σ has a spatially homogeneous expectation; i.e., σ can be decomposed into a background plus fluctuations around the background as

$$\sigma(\mathbf{x},t)=\sqrt{N}\phi(t)+\chi(\mathbf{x},t), \tag{4}$$

with the expectation value $\langle\sigma(\mathbf{x},t)\rangle=\sqrt{N}\phi(t)$ and, thus, $\langle\chi(\mathbf{x},t)\rangle=0$. During the inflationary epoch, the background space-time can be described by a spatially flat Friedmann-Robertson-Walker metric

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=dt^2-a^2(t)\delta_{ij}dx^idx^j, \tag{5}$$

where the scale factor $a=\exp(Ht)$ with the expansion rate $H=\sqrt{8\pi G\rho/3}$ determined by the mean of energy density of the inflaton field as

$$H^2(t)=\frac{8\pi N}{3M_{\rm Pl}^2}\left[\frac{1}{2}\dot{\phi}^2(t)+\frac{\lambda}{8}\left(\phi^2(t)-\frac{2m^2}{\lambda}\right)^2\right], \tag{6}$$

where $M_{\rm Pl}$ is the Planck mass. We have assumed that the effects of quantum fluctuations on the dynamics of the Hubble parameter can be ignored as we will justify this assumption later.

During inflation, the expectation value of the scalar field $\phi(t)$ undergoes an off-equilibrium evolution from the initial state $\phi(t)\sim 0$ where $V''(\phi)<0$. Accordingly, the mass square of the long-wavelength fluctuation modes will be negative, which leads to the nonperturbative growth of fluctuations. Therefore, a nonperturbative framework is necessary for taking account of the growth of fluctuations, especially in computing the power spectrum of perturbations later. We will employ the method of Hartree factorization, which approximates the potential $V(\phi)$ with an effective quadratic potential while keeping N finite [8,10]. The Hartree-factorized Lagrangian is

$$\mathcal{L}(t)=\int d^3xa^3(t)\left[\frac{1}{2}(\partial_\mu\chi)^2+\frac{1}{2}(\partial_\mu\vec{\pi})^2-\frac{1}{2}M_\chi^2(t)\chi^2-\frac{1}{2}M_\pi^2(t)\vec{\pi}^2-\chi V'(t)\right], \tag{7}$$

where

$$V'(t)=\sqrt{N}\{\ddot{\phi}(t)+3H\dot{\phi}(t)+[M_\chi^2-\lambda\phi^2(t)]\phi(t)\}. \tag{8}$$

The time-dependent effective masses $M_\chi(t)$ and $M_\pi(t)$ are obtained as

$$M_\chi^2(t)=-m^2+\frac{3\lambda}{2}\phi^2(t)+\frac{3\lambda}{2N}\langle\chi^2\rangle(t)+\frac{\lambda}{2}\left(1-\frac{1}{N}\right)\langle\psi^2\rangle(t),$$

$$M_\pi^2(t)=-m^2+\frac{\lambda}{2}\phi^2(t)+\frac{\lambda}{2N}\langle\chi^2\rangle(t)+\frac{\lambda}{2}\left(1+\frac{1}{N}\right)\langle\psi^2\rangle(t), \tag{9}$$

where $\langle\psi^2\rangle$ is defined by $\langle\vec{\pi}^2\rangle=(N-1)\langle\psi^2\rangle$.

The Hartree approximation we adopt here is equivalent to the Hartree-Fock approximation in the general Cornwall-Jackiw-Tomboulis (CJT) formalism [8]. By the requirement of a Hartree approximation, the one-loop corrections of the quantum field two-point Green's functions are canceled by the introduced ''mass counterterm.'' For the case of finite N, however, the $1/N$ corrections in the Hartree factorization do not include the contributions from the collision effects of the same order of $1/N$ [11]. Nevertheless, the factorization method provides a reliable resummation scheme that allows us to treat the growth of quantum fluctuations driven by spinodal instabilities self-consistently.

The equation of motion of the background field ϕ can be directly obtained from the term $\chi V'(t)$ in the Lagrangian [Eq. (7)]. By means of the tadpole condition $\langle\chi(\mathbf{x},t)\rangle=0$ and Eq. (8), we have

$$\ddot{\phi}(t)+3H\dot{\phi}(t)+[M_\chi^2(t)-\lambda\phi^2(t)]\phi(t)=0. \tag{10}$$

To see how the quantum effects influence the dynamics of the background field, it is critical to solve the background field $\phi(t)$ self-consistently by including the fluctuations $\langle\chi^2\rangle(t)$ and $\langle\psi^2\rangle(t)$ in the mass-squared term $M_\chi^2(t)$.

To find the dynamics of the fluctuation fields, we decompose $\chi(t)$ and $\vec{\pi}(t)$ in the Fourier basis. In the Heisenberg picture, one has

$$\chi(\mathbf{x},t)=\int\frac{d^3k}{8\pi^3}\chi(\mathbf{k},t)=\int\frac{d^3k}{8\pi^3}[b_{\mathbf{k}}f_{\chi,\mathbf{k}}(t)+b^\dagger_{-\mathbf{k}}f^\star_{\chi,-\mathbf{k}}(t)]e^{i\mathbf{k}\cdot\mathbf{x}},$$

$$\pi_i(\mathbf{x},t)=\int\frac{d^3k}{8\pi^3}\pi_i(\mathbf{k},t)=\int\frac{d^3k}{8\pi^3}[a_{i\mathbf{k}}f_{\pi,\mathbf{k}}(t)+a^\dagger_{i-\mathbf{k}}f^\star_{\pi,-\mathbf{k}}(t)]e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{11}$$

where $a_{i\mathbf{k}}$, $b_{\mathbf{k}}$ and $a^\dagger_{i\mathbf{k}}$, $b^\dagger_{\mathbf{k}}$ are the creation and annihilation operators which obey the commutation relations $[a_{i\mathbf{k}},a^\dagger_{j\mathbf{k}'}]=\delta_{i,j}\delta_{\mathbf{k},\mathbf{k}'}$ and $[b_{\mathbf{k}},b^\dagger_{\mathbf{k}'}]=\delta_{\mathbf{k},\mathbf{k}'}$. The equations of the mode functions $f_{\chi,\mathbf{k}}(t)$ and $f_{\pi,\mathbf{k}}(t)$ can be found from the Heisenberg field equations given by

$$\left[\frac{d^2}{dt^2}+3H\frac{d}{dt}+\frac{k^2}{a^2}+M_\chi^2(t)\right]f_{\chi,\mathbf{k}}(t)=0,$$

$$\left[\frac{d^2}{dt^2}+3H\frac{d}{dt}+\frac{k^2}{a^2}+M_\pi^2(t)\right]f_{\pi,\mathbf{k}}(t)=0. \tag{12}$$

Finally, to close these equations self-consistently, the terms $\langle\chi^2\rangle(t)$ and $\langle\psi^2\rangle(t)$ in the mass squared [Eq. (9)] can be determined by the mode functions as

$$\langle\chi^2\rangle(t)=\int\frac{d^3k}{8\pi^3}|f_{\chi,\mathbf{k}}(t)|^2,$$

$$\langle\psi^2\rangle(t)=\int\frac{d^3k}{8\pi^3}|f_{\pi,\mathbf{k}}(t)|^2. \tag{13}$$

Obviously, fluctuations $\langle\chi^2\rangle(t)$ and $\langle\psi^2\rangle(t)$ are created from the self-consistent background state $\phi(t)$ during inflation. The nonlinearity, or the self-interaction, is in fact encoded in the self-consistent solutions to Eqs. (10)–(13).

B. Renormalization and nonequilibrium equations of motion

Before proceeding further, one needs to understand the issue of renormalization regarding the divergences associated with the loop integrals. The divergences can be determined from the large loop momentum behavior of the mode functions, which can be found from the WKB-type solutions to Eq. (12) [10]. It turns out that the self-consistent loop integrals in Eq. (13) contain both quadratic and logarithmic divergences. To get the self-consistent renormalized equations, one has to subtract the divergences from both the bare mass and the coupling constant. However, for a weak coupling $\lambda<10^{-14}$ in a typical inflation model, the logarithmic subtractions can be neglected. Thus, the mass renormalization can be simplified as

$$m_R^2=m^2+\frac{\lambda}{2}\left(1+\frac{2}{N}\right)\left[-\frac{1}{8\pi^2}\frac{\Lambda^2}{a^2}\right], \tag{14}$$

with a negligible renormalization for the coupling constant. As a result, it leads to the renormalized masses $M_{\chi,R}$ and $M_{\pi,R}$ which are given by

$$M_{\chi,R}^2(t)=-m_R^2+\frac{3\lambda}{2}\phi^2(t)+\frac{3\lambda}{2N}\langle\chi^2\rangle_R(t)+\frac{\lambda}{2}\left(1-\frac{1}{N}\right)\langle\psi^2\rangle_R(t),$$

$$M_{\pi,R}^2(t)=-m_R^2+\frac{\lambda}{2}\phi^2(t)+\frac{\lambda}{2N}\langle\chi^2\rangle_R(t)+\frac{\lambda}{2}\left(1+\frac{1}{N}\right)\langle\psi^2\rangle_R(t), \tag{15}$$

where

$$\langle\chi^2\rangle_R(t)=\int^\Lambda\frac{d^3k}{8\pi^3}|f_{\chi,\mathbf{k}}(t)|^2-\frac{1}{8\pi^2}\frac{\Lambda^2}{a^2},$$

$$\langle\psi^2\rangle_R(t)=\int^\Lambda\frac{d^3k}{8\pi^3}|f_{\pi,\mathbf{k}}(t)|^2-\frac{1}{8\pi^2}\frac{\Lambda^2}{a^2}. \tag{16}$$

Hereafter, we will drop the subscript R, as the integrals of mode functions are always in terms of the renormalized quantities.

III. CLASSICAL FLUCTUATIONS OF THE INFLATON

A. Quantum decoherence

The quantum-to-classical transition of the quantum fluctuations during the inflation can be investigated by the Schrödinger wave function approach [12,13]. The Hamiltonian describing the evolution of quantum wave functions of the fluctuations can be obtained from Eq. (7). In general, the developments of different fluctuation modes are separable due to the quadratic feature of the Hartree-factorized Hamiltonian. Let us first consider a mode $\mathbf{k}$ of the χ field. In the Schrödinger picture, the initial (time t_0) vacuum state is specified by

$$b_{\mathbf{k}}|t_0\rangle_S=0, \tag{17}$$

for all $\mathbf{k}$. At time t, the evolved Schrödinger state $|t\rangle_S$ is given by

$$U(t,t_0)b_{\mathbf{k}}U^{-1}(t,t_0)|t\rangle_S=0, \tag{18}$$

where $U(t,t_0)=\exp-(i/\hbar)\int_{t_0}^{t}\mathcal{H}(t)dt$. In the coordinate representation, the conjugate momentum can be written as $\Pi(\mathbf{k})=-i\hbar\,\partial/\partial\chi(-\mathbf{k})$, and thus Eq. (18) reduces to

$$\left[\chi(\mathbf{k})-\hbar\,\gamma_k^{-1}(t)\frac{\partial}{\partial\chi(-\mathbf{k})}\right]\Psi[\chi(\mathbf{k}),t]=0, \tag{19}$$

where

$$\gamma_k(t)=\frac{1}{2|f_{\chi,k}(t)|^2}[1-2iF_{\chi,\mathbf{k}}(t)], \tag{20}$$

$$F_{\chi,\mathbf{k}}(t)=\frac{a^3(t)}{2}\frac{d}{dt}|f_{\chi,\mathbf{k}}(t)|^2, \tag{21}$$

and $\Psi[\chi(\mathbf{k}),t]=\langle\chi(\mathbf{k})|t\rangle_S$ is the wave function for the field $\chi(\mathbf{k})$. The solution to the equation can be obtained straightforwardly as

$$\begin{aligned}\Psi[\chi(\mathbf{k}),\mathbf{t}]&=\mathcal{N}_{\mathbf{k}}^{1/2}(t)\exp\left(-\frac{1}{\hbar}\gamma_k(t)|\chi(\mathbf{k})|^2\right)\\&=\mathcal{N}_{\mathbf{k}}^{1/2}(t)\exp\left(-\frac{1}{\hbar}\frac{|\chi(\mathbf{k})|^2}{2|f_{\chi,\mathbf{k}}(t)|^2}[1-i2F_{\chi,\mathbf{k}}(t)]\right),\end{aligned} \tag{22}$$

where $\mathcal{N}_{\mathbf{k}}^{1/2}(t)$ is the normalization coefficient. The wave function of the quantum fluctuations is then the direct product of the wave functions for all $\mathbf{k}$ modes of the χ field. Notice that for an initial vacuum state which is a pure state, it will remain a pure state under unitary evolution. Therefore, the density matrix in the coordinate representation becomes the wave function times its complex conjugate given by

$$\begin{aligned}\prod_{\mathbf{k}}\rho(\chi(\mathbf{k}),\overline{\chi}(\mathbf{k}),t)&=\prod_{\mathbf{k}}\Psi[\chi(\mathbf{k}),t]\Psi^{\star}[\overline{\chi}(\mathbf{k}),t]\\&=\prod_{\mathbf{k}}\mathcal{N}_{\mathbf{k}}(t)\exp\left\{-\frac{\chi(\mathbf{k})\chi(-\mathbf{k})+\overline{\chi}(\mathbf{k})\overline{\chi}(-\mathbf{k})}{2|f_{\chi,\mathbf{k}}(t)|^2}\right.\\&\quad\left.\times[1-i2F_{\chi,\mathbf{k}}(t)]\right\}.\end{aligned} \tag{23}$$

The density matrix for each mode $\mathbf{k}$ can be decomposed into diagonal and off-diagonal elements as follows:

$$\begin{aligned}&\rho[\chi(\mathbf{k})+\delta(\mathbf{k}),\overline{\chi}(\mathbf{k})-\delta(\mathbf{k}),t]\\&=\mathcal{N}_{\mathbf{k}}(t)\exp\left\{-\frac{1}{|f_{\chi,\mathbf{k}}(t)|^2}\{|\overline{\chi}(\mathbf{k})|^2\right.\\&\quad\left.-i2F_{\chi,\mathbf{k}}(t)[\overline{\chi}(\mathbf{k})\delta(-\mathbf{k})+\overline{\chi}(-\mathbf{k})\delta(\mathbf{k})]+|\delta(\mathbf{k})|^2\}\right\}.\end{aligned} \tag{24}$$

One can immediately recognize that $F_{\chi,\mathbf{k}}(t)$ is the phase of the off-diagonal elements. A quantum-to-classical transition is implied if the following condition is satisfied:

$$F_{\chi,\mathbf{k}}(t)\gg 1. \tag{25}$$

Similarly for mode π, the condition of the quantum-to-classical transition is given by

$$F_{\pi,\mathbf{k}}(t)=\frac{a^3(t)}{2}\frac{d}{dt}|f_{\pi,\mathbf{k}}(t)|^2\gg 1. \tag{26}$$

Hence, the quantum-to-classical transition is determined by the behavior of the time dependence of the mode functions $f_{\chi,\mathbf{k}}(t)$ and $f_{\pi,\mathbf{k}}(t)$.

To solve Eq. (12), we consider the early stage of inflation when fluctuations have not grown significantly yet in the presence of spinodal instabilities while the Hubble parameter remains constant. In this stage, the masses $M_\chi(t)$ and $M_\pi(t)$ can be approximated by $-m$ in Eq. (15). Thus, both mode equations (12) can be written as

$$\left[\frac{d^2}{dt^2}+3H\frac{d}{dt}+\frac{k^2}{a^2}-m^2\right]f_{\mathbf{k}}(t)=0. \tag{27}$$

The general solution can be expressed as a combination of the Hankel function $H_\nu(k/aH)$ and the Neumann function $N_\nu(k/aH)$ with $\nu=\sqrt{(9/4)+(m/H)^2}$. In particular, for fluctuations in the superhorizon regime where $k\ll aH$, the growing mode solution leads to

$$f_{\mathbf{k}} \simeq f_0(k) e^{(\nu-3/2)Ht}, \tag{28}$$

where $f_0(k)$ depends on the initial conditions of the mode functions as we set $t_0=0$. Apparently, $F_{\mathbf{k}}(t) \gg 1$ is valid after several e-foldings of inflation when the modes become superhorizon sized.

B. Equations of classical fluctuations

Using the density matrix, one can find the Wigner function from

$$\begin{aligned} W(\overline{\chi}(\mathbf{k}), \overline{\pi}_\chi(\mathbf{k}), t) &= \int d\left(\frac{\delta}{2\pi}\right) e^{-(i/\hbar)\overline{\pi}_\chi(\mathbf{k})\delta(\mathbf{k})} \\ &\quad \times \rho\left(\overline{\chi}(\mathbf{k}) - \frac{\delta(\mathbf{k})}{2}, \overline{\chi}(\mathbf{k}) + \frac{\delta(\mathbf{k})}{2}, t\right) \\ &= P[|\overline{\chi}(\mathbf{k})|]\left(N_{\mathbf{k}} \exp\left[-\frac{|f_{\chi,\mathbf{k}}(t)|^2}{\hbar} \right| \right. \\ &\quad \left. \times \overline{\pi}_\chi(\mathbf{k}) - \frac{F_{\chi,\mathbf{k}}(t)}{|f_{\chi,\mathbf{k}}(t)|^2} \overline{\chi}(\mathbf{k}) \right|^2 \Bigg] \Bigg), \end{aligned} \tag{29}$$

where

$$P[|\overline{\chi}(\mathbf{k})|] = \left(\frac{1}{\hbar \pi |f_{\chi,\mathbf{k}}(t)|^2} \right)^{1/2} \exp\left[-\frac{|\overline{\chi}(\mathbf{k})|^2}{\hbar |f_{\chi,\mathbf{k}}(t)|^2} \right]. \tag{30}$$

Since the Wigner function in Eq. (29) is definite positive, it can be interpreted as a distribution function in the phase space of stochastic field perturbations.

For the perturbation modes just crossing out the horizon, the dynamics of the classical fluctuations can be described by the Wigner function in the limit of $\hbar \to 0$ while keeping $\hbar |f_{\chi,\mathbf{k}}(\tau)|^2$ fixed. In fact, the term $\hbar |f_{\chi,\mathbf{k}}(\tau)|^2$ basically measures the variance of the field fluctuations $\overline{\chi}(\mathbf{k})$ and will grow nonperturbatively due to the spinodal instabilities. As a result, the Wigner function becomes

$$W(\overline{\chi}(\mathbf{k}), \overline{\pi}_\chi(\mathbf{k}), t) = P[|\overline{\chi}(\mathbf{k})|] \, \delta\left(\left| \overline{\pi}_\chi(\mathbf{k}) - \frac{F_{\chi,\mathbf{k}}(t)}{|f_{\chi,\mathbf{k}}(t)|^2} \overline{\chi}(\mathbf{k}) \right| \right). \tag{31}$$

The delta function in the above expression yields the equations of motion for the classical fluctuations at superhorizon scales:

$$\overline{\pi}_\chi(\mathbf{k},t) - \frac{F_{\chi,\mathbf{k}}(t)}{|f_{\chi,\mathbf{k}}(t)|^2} \overline{\chi}(\mathbf{k},t) = 0. \tag{32}$$

In terms of the classical fluctuation $\overline{\chi}(\mathbf{k},t)$, Eq. (32) can be rewritten as

$$\frac{d\overline{\chi}(\mathbf{k},t)}{dt} - \frac{1}{2}\left(\frac{d}{dt} \ln |f_{\chi,\mathbf{k}}(t)|^2 \right) \overline{\chi}(\mathbf{k},t) = 0. \tag{33}$$

With the mode functions $f_{\chi,\mathbf{k}}(t)$ derived self-consistently from Eqs. (11), (14), and (15), Eq. (33) describes the dynamical evolution of the mean value of the stochastic fields $\overline{\chi}(\mathbf{k},t)$ in the superhorizon regime. Obviously, the stationary solution to the probability distribution function (PDF) of variables $\overline{\chi}(\mathbf{k},t)$ can be obtained from Eq. (30).

In contrast to the usual second-order evolution equation for the field $\chi(\mathbf{k},t)$, Eq. (33) is a diffusion-type equation which typically occurs during the overdamped process of harmonic oscillators. To accommodate the overdamped behavior of the mean field $\overline{\chi}(\mathbf{k},t)$ into the mode equations, the equations for the mode function, Eqs. (12), at superhorizon scales can be further approximated by

$$\left[3H\frac{d}{dt} + \frac{k^2}{a^2(t)} + M_\chi^2(t) \right] f_{\chi,\mathbf{k}}(t) = 0. \tag{34}$$

As a consequence, Eq. (33) reduces to

$$3H \frac{d\overline{\chi}(\mathbf{k},t)}{dt} + \left[\frac{k^2}{a^2(t)} + M_\chi^2(t) \right] \overline{\chi}(\mathbf{k},t) = 0. \tag{35}$$

As expected, these equations are the classical Euler-Lagrange field equations for the superhorizon modes of fluctuations $\chi(\mathbf{k},t)$ in the overdamped regime.

Following the same procedure, we can obtain a similar equation for the fluctuating field $\pi(\mathbf{k},t)$ as

$$3H \frac{d\overline{\pi_i(\mathbf{k},t)}}{dt} + \left[\frac{k^2}{a^2(t)} + M_\pi^2(t) \right] \overline{\pi_i(\mathbf{k},t)} = 0. \tag{36}$$

C. PDF of classical fluctuations

The statistical properties of the classical fluctuation on mode $\mathbf{k}$ can be characterized by the PDF given by Eq. (30):

$$\begin{aligned} P[|\pi_i(\mathbf{k})|] &= \left(\frac{1}{\pi |f_{\pi,\mathbf{k}}(t)|^2} \right)^{1/2} \exp\left[-\frac{\pi_i^2(\mathbf{k})}{|f_{\pi,\mathbf{k}}(t)|^2} \right], \\ P[|\chi(\mathbf{k})|] &= \left(\frac{1}{\pi |f_{\chi,\mathbf{k}}(t)|^2} \right)^{1/2} \exp\left[-\frac{\chi^2(\mathbf{k})}{|f_{\chi,\mathbf{k}}(t)|^2} \right]. \end{aligned} \tag{37}$$

Apparently, the PDFs of the classical perturbation modes $\pi_i(\mathbf{k})$ and $\chi(\mathbf{k},t)$ are Gaussian with variances $|f_{\pi,\mathbf{k}}(t)|^2$ and $|f_{\chi,\mathbf{k}}(t)|^2$, respectively, under the Hartree approximation. It implies that

$$\begin{aligned} &\langle \chi^{2n+1}(\mathbf{k},t) \rangle = 0, \ \ \langle \pi_i^{2n+1}(\mathbf{k},t) \rangle = 0, \\ &\langle \chi^{2n}(\mathbf{k},t) \rangle = (2n-1)!! \langle \chi^2(\mathbf{k},t) \rangle^n, \\ &\langle \pi_i^{2n}(\mathbf{k},t) \rangle = (2n-1)!! \langle \pi_i^2(\mathbf{k},t) \rangle^n, \end{aligned} \tag{38}$$

where $n=1,2,\ldots$. However, the variances generally are **k** dependent. Such **k** dependence of variances eventually leads to nontrivial correlations between fluctuation modes which will be scrutinized later.

In summary, after a quantum-to-classical transition, the dynamics of the classical stochastic fields $\overline{\chi(\mathbf{k},t)}$, $\overline{\pi_i(\mathbf{k},t)}$ must be based on Eqs. (35) and (36) with the evolution of the self-consistent mode functions. The corresponding PDF of the perturbations is governed by Eq. (37). In fact, by adding a noise term to the right-hand side of both Eqs. (35) and (36), they become Langevin-like equations which are capable of describing the stochastic properties of perturbations. Thus, the noise correlations can be determined by the variances of the PDFs. For the case of a single free inflaton, the dynamics of the superhorizon-sized fluctuations is governed by a Langevin equation [13], which is obtained by coarse graining the degrees of freedom of all subhorizon modes [14]. According to the results we obtained here, the coarse-grained Langevin equation approach is effective not only for the situation of a free scalar inflaton field, but also for models with a self-interaction such as the $O(N)$ inflation.

Moreover, it should be pointed out that Eqs. (34) and (35) and their counterparts for the fluctuating $\pi(\mathbf{k},t)$ field are not closed if considering only the classical fluctuations or the superhorizon modes. By virtue of $M_\chi^2(t)$ and $M_\pi^2(t)$, the subhorizon modes also contribute towards these diffusion-type equations. This feature is anticipated because of the inevitable coupling between all wavelength modes under a substantial self-interaction. Thus, the renormalized mass-squared terms play critical roles as phenomenological potentials. Consequently, Eqs. (34) and (35) and their counterparts for the $\pi(\mathbf{k},t)$ field provide a possible scheme to study the interactions of the inflaton by comparing the phenomenological potentials given by data fitting with the predicted $M_\chi^2(t)$ and $M_\pi^2(t)$. This characteristic is the main result of this paper.

We now turn our attention to the cosmological implications from the off-equilibrium dynamics of the $O(N)$ inflationary model in the next section.

IV. POWER SPECTRUM OF PRIMORDIAL PERTURBATIONS

A. Mass density perturbations

The power spectrum of primordial perturbations is described by $\mathcal{P}(k)=\langle|\delta_\mathbf{k}|^2\rangle$ where the mass density perturbations are determined by the gauge invariant quantity [17]

$$\delta_\mathbf{k}=\frac{\delta\rho}{\rho+p}\bigg|_{k=aH}. \tag{39}$$

The mass density fluctuation $\delta\rho$ originates from the field fluctuations π_i and χ given by

$$\begin{aligned}\frac{\delta\rho}{N}=&\left(1-\frac{1}{N}\right)\left[\frac{1}{2}\langle\dot\psi^2\rangle+\frac{1}{2a^2}\langle(\nabla\psi)^2\rangle-\frac{1}{2}m^2\langle\psi^2\rangle\right.\\&\left.+\frac{\lambda}{4}\phi^2\langle\psi^2\rangle+\frac{\lambda}{4N}\langle\chi^2\rangle\langle\psi^2\rangle+\frac{\lambda}{8}\left(1+\frac{1}{N}\right)\langle\psi^2\rangle^2\right]\\&+\frac{1}{N}\left[\frac{1}{2}\langle\dot\chi^2\rangle+\frac{1}{2a^2}\langle(\nabla\chi)^2\rangle-\frac{1}{2}m^2\langle\chi^2\rangle+\frac{3\lambda}{4}\phi^2\langle\chi^2\rangle\right.\\&\left.+\frac{\lambda}{4}\left(1-\frac{1}{N}\right)\langle\psi^2\rangle\langle\chi^2\rangle+\frac{3\lambda}{8N}\langle\chi^2\rangle^2\right],\end{aligned} \tag{40}$$

with [10]

$$\begin{aligned}\langle\dot\psi^2\rangle(t)=&\int\frac{d^3\mathbf{k}}{(2\pi)^3}|\dot f_{\pi,\mathbf{k}}(t)|^2-\left[\frac{1}{8\pi^2}\frac{\Lambda^4}{a^4}\right.\\&\left.+\frac{1}{8\pi^2}\frac{\Lambda^2}{a^2}\left(M_\chi^2(t)-\frac{R}{6}+2\frac{\dot a^2}{a^2}\right)\right],\\\langle\dot\chi^2\rangle(t)=&\int\frac{d^3\mathbf{k}}{(2\pi)^3}|\dot f_{\chi,\mathbf{k}}(t)|^2\\&-\left[\frac{1}{8\pi^2}\frac{\Lambda^4}{a^4}+\frac{1}{8\pi^2}\frac{\Lambda^2}{a^2}\left(M_\pi^2(t)-\frac{R}{6}+2\frac{\dot a^2}{a^2}\right)\right],\end{aligned} \tag{41}$$

$$\begin{aligned}\langle(\nabla\psi)^2\rangle(t)=&\int\frac{d^3\mathbf{k}}{(2\pi)^3}k^2|f_{\pi,\mathbf{k}}(t)|^2\\&-\left[\frac{1}{8\pi^2}\frac{\Lambda^4}{a^2}-\frac{\Lambda^2}{8\pi^2}\left(M_\chi^2(t)-\frac{R}{6}\right)\right],\\\langle(\nabla\chi)^2\rangle(t)=&\int\frac{d^3\mathbf{k}}{(2\pi)^3}k^2|f_{\chi,\mathbf{k}}(t)|^2\\&-\left[\frac{1}{8\pi^2}\frac{\Lambda^4}{a^2}-\frac{1\,\Lambda^2}{8\pi^2}\left(M_\pi^2(t)-\frac{R}{6}\right)\right],\end{aligned} \tag{42}$$

where the Ricci scalar $R=6(\dot a^2/a+\ddot a/a)=12H^2$. Similar to the renormalization of Eq. (16), we have ignored logarithmic subtractions in the renormalization of the above equations. The term summing up the energy density and the pressure, $\rho+p$, in Eq. (39) can be obtained by

$$\begin{aligned}\frac{\rho+p}{N}=&\dot\phi^2+\left(1-\frac{1}{N}\right)\left[\langle\dot\psi^2\rangle+\frac{1}{a^2}\langle(\nabla\psi)^2\rangle\right]\\&+\frac{1}{N}\left[\langle\dot\chi^2\rangle+\frac{1}{a^2}\langle(\nabla\chi)^2\rangle\right].\end{aligned} \tag{43}$$

Apparently, the first term $\dot\phi^2$ in Eq. (43) comes from the background inflaton field. The other terms, however, are contributions from the fluctuations.

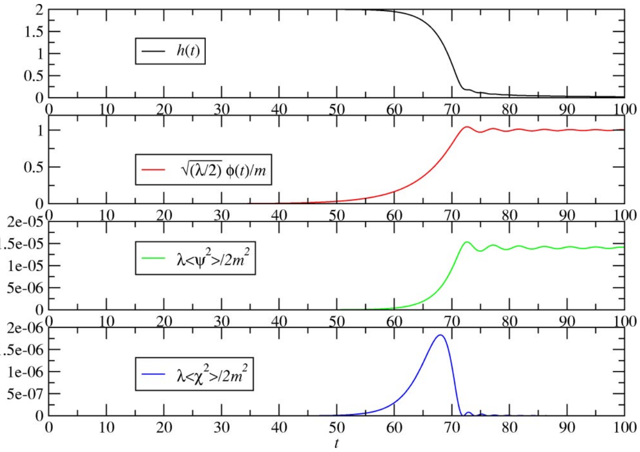

FIG. 1. The evolution of $h=H/m$, $(\sqrt{\lambda/2})\phi/m$, $(\lambda/2m^2)\langle\psi^2\rangle$, and $(\lambda/2m^2)\langle\chi^2\rangle$, respectively, vs t (in units of m^{-1}) for initial conditions $H(0)=2m$, $\phi(0)\approx 0$, $\dot{\phi}(0)=0$, $M_\chi^2(0)=M_\pi^2(0)=m^2$ with $\lambda=10^{-14}$ in the case of $N=4$.

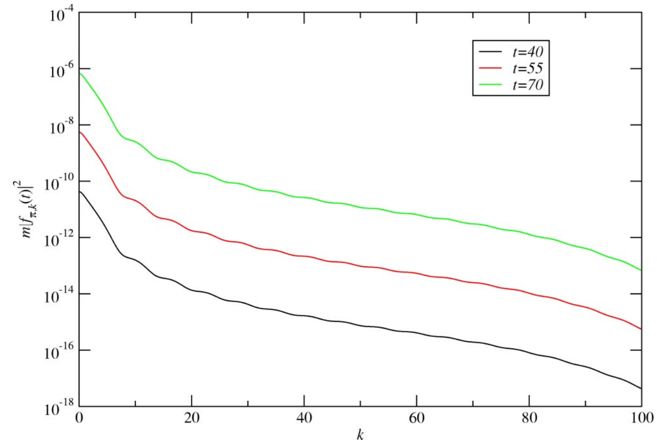

FIG. 2. Mode function $|f_{\pi,\mathbf{k}}(t)|^2$ vs k (in units of m) at $t=40$, 55, 70 (in units of m^{-1}) under the same initial conditions and parameters as in Fig. 1 for the case of $N=4$.

B. Numerical examples

We present numerical examples in this section to demonstrate the self-interaction effect of the $\phi(t)$ field upon the power spectrum. To begin with, we need to find self-consistent solutions to the background field ϕ and the mode functions $f_{\chi,\mathbf{k}}(t)$ and $f_{\pi,\mathbf{k}}(t)$ from Eqs. (10) and (12). We assume $\phi(0)\approx 0$ and $\dot{\phi}(0)=0$ for the background field at the onset of inflation. The initial $(t=0)$ conditions of the mode functions can be specified as effective free massive scalar fields in an expanding universe:

$$f_{\chi,\mathbf{k}}(0)=\frac{1}{\sqrt{2[k^2+M_\chi^2(0)]}},\qquad \dot{f}_{\chi,\mathbf{k}}(0)=-i\sqrt{\frac{k^2+M_\chi^2(0)}{2}},$$

$$f_{\pi,\mathbf{k}}(0)=\frac{1}{\sqrt{2[k^2+M_\pi^2(0)]}},\qquad \dot{f}_{\pi,\mathbf{k}}(0)=-i\sqrt{\frac{k^2+M_\pi^2(0)}{2}},\tag{44}$$

where we have set $a(0)=1$ and $M_\chi^2(0)$, $M_\pi^2(0)$ are the effective mass squared of the field components χ and π, respectively. The values of $M_\chi^2(0)$ and $M_\pi^2(0)$ can be zero or a positive number, which depends on the details of the onset of inflation. Although this uncertainty really is not decisive to the evolution of the mode functions $f_{\chi,\mathbf{k}}(t)$ and $f_{\pi,\mathbf{k}}(t)$, it turns out to become crucial for correlating the fluctuations as we will see later.

Figure 1 plots the expansion rate $h(t)$ of the Universe, the self-consistent solution to the background field $\phi(t)$, and the fluctuations $\langle\psi^2\rangle(t)$, $\langle\chi^2\rangle(t)$ where the parameters are taken to be $\lambda=10^{-14}$, $N=4$, and $H(0)=2m$, $M_\chi^2(0)=M_\pi^2(0)\simeq m^2>0$. The value for $H(0)$ is determined from the energy density of the inflaton field given by Eq. (6). During inflation, the energy density of the inflaton is dominated by that of the potential energy which leads to $h^2(0)=(H/m)^2\approx Nm^2/M_{\rm Pl}^2\lambda\approx\mathcal{O}(1)$ for $N\approx\mathcal{O}(1)$, $\lambda\approx 10^{-14}$, and the inflaton mass $m\approx 10^{12}$ GeV. As a consequence, the Hubble parameter remains approximately a constant $h\simeq 2$ at $t\leqslant 60m^{-1}$, but drops dramatically after $t\geqslant 70m^{-1}$. Apparently, $t\leqslant 70m^{-1}$ specifies the inflationary epoch.

It has been pointed out earlier that we treat the dynamics of the Hubble parameter $H(t)$ classically without any quantum correction involved in Eq. (6). It turns out that the background field does follow its classical equation of motion as the quantum corrections in Eq. (10) give rise to a small $\delta\rho/\rho\approx 10^{-4}$ in our case. Thus, when the semiclassical Einstein equations are employed for a truly self-consistent approach, it yields a similar solution of h as shown in [10].

Figure 1 also shows that $\phi(t)\simeq\langle\psi^2\rangle(t)\simeq\langle\chi^2\rangle(t)\simeq 0$ during the first stage of inflation, $t<40m^{-1}$, in the spinodal regime where the initially positive M_χ^2 and M_π^2, turn negative quickly. The spinodal instabilities become important eventually and lead to a significant growth of both $\langle\psi^2\rangle(t)$ and $\langle\chi^2\rangle(t)$ starting at $t=55m^{-1}$. As expected, these properties are not sensitive to the choice of the initial effective masses $M_\chi^2(0)$ and $M_\pi^2(0)$. It can also be seen from the mode function $|f_{\pi,\mathbf{k}}(t)|^2$ shown in Fig. 2.

Evidently, the mode function undergoes a substantial increase in the period of $55m^{-1}<t<70m^{-1}$, but varies much slower after $t=70m^{-1}$. This is owing to the predomination of the criterion $M_\pi^2<0$ in the spinodal regions spawned during the slow rolling of the inflaton. On the other hand, the spinodal condition is only weakly satisfied over a span of $70m^{-1}\leqslant t\leqslant 80m^{-1}$ where the background ϕ starts a rapid falling into the valley of the inflaton potential. Therefore, the process of spinodal instability terminates at time t_e given by

$$M_\pi^2(t_e)=-m^2+\frac{\lambda}{2}\phi^2(t_e)+\frac{\lambda}{2N}\langle\chi^2\rangle(t_e)+\frac{\lambda}{2}\left(1+\frac{1}{N}\right)\langle\psi^2\rangle(t_e)\approx 0.\tag{45}$$

It renders $t_e\simeq(70\text{–}80)m^{-1}$.

The sharp decline of h signifies the end of inflation. Therefore, the number of total inflationary e-foldings is $N_e\approx Ht=2mt\approx 140$. The perturbation just about to enter the

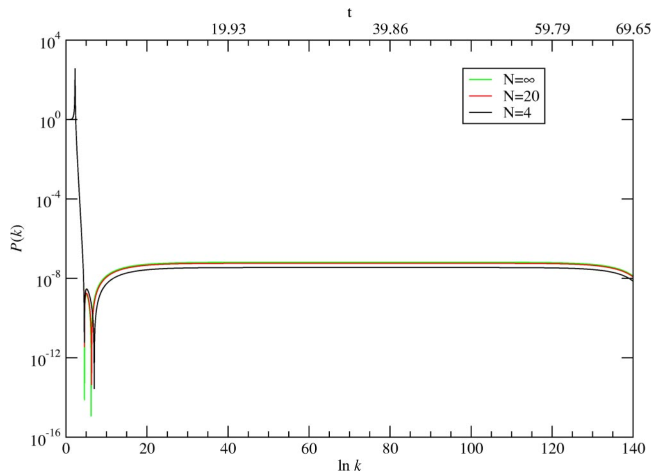

FIG. 3. The power spectrum of primordial density perturbations vs $\ln k$ (k in units of m) with the same initial conditions and parameters as in Fig. 1 for the case of $N=4$, 20, and ∞ respectively.

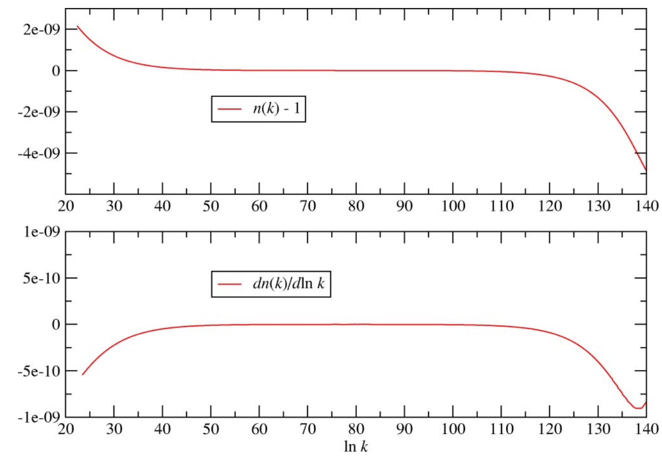

FIG. 4. Spectral index $n(k)$ and its running $dn(k)/d\ln k$ vs $\ln k$ (k in units of m) with the same initial conditions and parameters as in Fig. 1 for the case of $N=4$.

present Hubble horizon should be transgressing the horizon in the inflationary era around $t=t_{60}\simeq 40m^{-1}$, some 60 e-folds before the end of inflation. We find that most of energy density of the background ϕ state is transferred to the fluctuations $\langle\psi^2\rangle$ via the production of π modes during the inflationary epoch. Since $M_\pi^2(t_e)\simeq 0$, these π modes are in fact the massless Goldstone modes of the broken symmetry phase. The Goldstone theorem is satisfied dynamically.

Using the numerical solutions, we calculate the power spectrum of primordial perturbations. The result is shown in Fig. 3, in which N is taken to be 4, 20, and ∞. The N dependence of the power spectrum is trivial. We find that the power is generally k independent for modes k which cross the horizon when $t<55m^{-1}$. While the energy transfer from the background field to the fluctuations driven by the spinodal instability leads to a swift increase in $\langle\psi^2\rangle$, the inflaton field at later times rolls down the potential hill which also results in an increase of $\dot\phi$. This in turn renders the gradual decrease in power for shorter-wavelength modes crossing out of the Hubble horizon at $t>55m^{-1}$ as the increase of $\dot\phi$ dominates.

The spectral index $n(k)-1=dP(k)/d\ln k$ and its k dependence $dn(k)/d\ln k$ are given in Fig. 4. The value of $n(k)$ varies from unity at the larger scales as $\ln k<110$ to about $n<1$ at the smaller scale as $\ln k>110$. This leads to an index running $dn(k)/d\ln k<0$ in the wavelength range corresponding to the horizon-crossing times during $55<mt<70$. Once again, the physical reason for the running of the spectral index is due to the energy transfer from the inflationary background field to the fluctuations as well as the evolution of the inflaton field. Although one may refer this energy transfer to the third- or even higher-order derivative of the effective potential $V'''(\phi)$ of the inflaton, the index running shown in Fig. 4 can only be obtained through a proper off-equilibrium dynamics, but not by the classical effective potential approach.

V. THE CORRELATION OF PRIMORDIAL PERTURBATIONS

A. Correlation function and initial condition of classical field fluctuations

The equal-time two-point correlation functions of fluctuations have been extensively used to study applications of the $O(N)$ model to the off-equilibrium chiral phase transition in relativistic heavy ion collisions where the coupling constant is of order 1 [15]. The correlation caused by the off-equilibrium process is generic, because the evolution of fluctuations is typically with respect to a time-dependent background. In our case, the mode functions of fluctuations $f_{\chi,\mathbf{k}}(t)$ and $f_{\pi,\mathbf{k}}(t)$ are not trivial plane waves; they depend rather on the dynamics of interaction or of self-interaction. Hence, it would be interesting to search for the correlations of the primordial perturbations.

By means of Eq. (11), the two-point correlation functions of the fluctuations are given by

$$\begin{aligned}\langle\chi(\mathbf{x},t)\chi(\mathbf{x}',t)\rangle &= \int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}|f_{\chi,\mathbf{k}}(t)|^2\\ &= \frac{1}{2\pi^2|\mathbf{x}-\mathbf{x}'|}\int^\Lambda kdk\,\sin k|\mathbf{x}-\mathbf{x}'||f_{\chi\mathbf{k}}(t)|^2,\\ \langle\pi_i(\mathbf{x},t)\pi_i(\mathbf{x}',t)\rangle &= \int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}|f_{\pi,\mathbf{k}}(t)|^2\\ &= \frac{1}{2\pi^2|\mathbf{x}-\mathbf{x}'|}\int^\Lambda kdk\,\sin k|\mathbf{x}-\mathbf{x}'||f_{\pi\mathbf{k}}(t)|^2,\end{aligned} \tag{46}$$

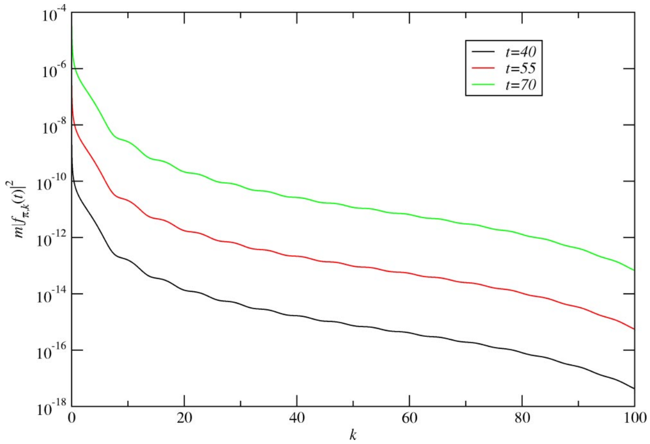

FIG. 5. Mode function $|f_{\pi,\mathbf{k}}(t)|^2$ vs $k(m)$ at $t=40$, 55, 70 in units of m^{-1}. All parameters are specified the same as in Fig. 2 except the initial conditions, Eqs. (44), where $M_\chi^2(0)=M_\pi^2(0)=0$ are assumed.

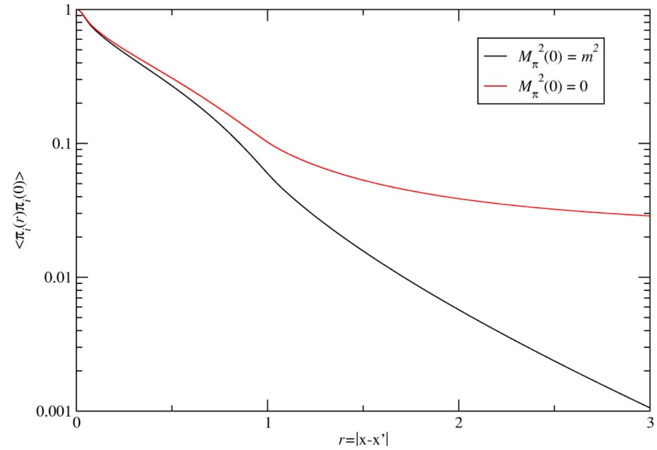

FIG. 6. $\langle\pi(\mathbf{x},t)\pi(\mathbf{x}',t)\rangle$ vs $r=|\mathbf{x}-\mathbf{x}'|$ with initial conditions $M_\pi^2(0)=m^2$ (lower line) and zero (upper line) for modes crossing the horizon at $t=40m^{-1}$. The separation r is in units of m^{-1}.

where we have considered that the dynamics of $|f_{\pi,\mathbf{k}}(t)|^2$ and $|f_{\chi,\mathbf{k}}(t)|^2$ is isotropic in $\mathbf{k}$ space.

It can be seen from Eqs. (46) that both correlation functions $\langle\chi(\mathbf{x},t)\chi(\mathbf{x}',t)\rangle$ and $\langle\pi_i(\mathbf{x},t)\pi_i(\mathbf{x}',t)\rangle$ over large spatial distances $|\mathbf{x}-\mathbf{x}'|$ are sensitive to the long-wavelength behavior of $|f_{\chi,\mathbf{k}}(t)|^2$ and $|f_{\pi,\mathbf{k}}(t)|^2$. However, the infrared behavior of the mode functions during the early stage of the inflation are actually governed by the initial conditions of the fluctuations as shown in Eq. (28). For instance, if we take the initial values $M_\chi^2(0)=M_\pi^2(0)\simeq m^2$ in Eq. (44), then one can show that with Eq. (28) the correlations decay exponentially as $\exp\{-(m|\mathbf{x}-\mathbf{x}'|)\}=\exp\{-(H|\mathbf{x}-\mathbf{x}'|/2)\}$ when we set $H=2m$ to mimic the constant expansion rate during the inflation. Hence, the correlation between the inflationary perturbation modes is negligible if the spatial distance between the modes is larger than a horizon size, $a|\mathbf{x}-\mathbf{x}'|>H^{-1}=0.5m^{-1}$. On the other hand, if the initial conditions for the mode functions are chosen as $M_\chi^2(0)=M_\pi^2(0)=0$, then the solution to the mode function can be approximated by

$$|f_{\mathbf{k}}(t)|^2\simeq\frac{\left(\nu+\frac{3}{2}\right)^2}{k}e^{2[\nu-(3/2)]Ht}. \qquad (47)$$

This function is singular as $1/k$ when $k\to0$. For such infrared behavior of the mode function, we have

$$\frac{\langle\chi(\mathbf{x},t)\chi(\mathbf{x}',t)\rangle}{\langle\chi^2\rangle}\simeq\frac{\langle\pi(\mathbf{x},t)\pi(\mathbf{x}',t)\rangle}{\langle\pi^2\rangle}\propto\frac{1}{(Ha|\mathbf{x}-\mathbf{x}'|)^2}. \qquad (48)$$

This correlation clearly covers a spatial range much larger than the horizon size H^{-1}.

The numerical solutions of $|f_{\pi,\mathbf{k}}(t)|^2$ with the vanishing initial mass terms are plotted in Fig. 5. Obviously the mode functions given in Fig. 5 have shown very different infrared behavior from those in Fig. 2, where we have taken $M_\chi^2(0)=M_\pi^2(0)\simeq m^2$. However, we expect that the tail of the correlation function depends sensitively on the initial conditions of the inflaton fluctuations. To illustrate this feature, we plot in Fig. 6 the correlation functions $\langle\pi(\mathbf{x},t)\pi(\mathbf{x}',t)\rangle$ obtained by two different initial conditions $M_\pi^2(0)>0$ and $M_\pi^2(0)=0$. We see that the respective behaviors of $\langle\pi(\mathbf{x},t)\pi(\mathbf{x}',t)\rangle$ at scales $r=|\mathbf{x}-\mathbf{x}'|$ larger than $0.5m^{-1}$ (which equals the horizon distance $1/H$) are quite distinguishable. With $M_\pi^2=0$ initially, the correlation function scales as a power law:

$$\langle\pi_i(\mathbf{x},t)\pi_i(\mathbf{x}',t)\rangle\propto|\mathbf{x}-\mathbf{x}'|^{-1.8}. \qquad (49)$$

On the other hand, the correlation function clearly possesses an exponential falloff with distance when the initial condition is taken to be $M_\pi^2(0)>0$. This example shows that the initial conditions can be recalled from the tail of the correlation function of classical perturbations despite the fact that the original quantum fluctuations for those superhorizon modes have gone through a quantum-to-classical transition during off-equilibrium evolution.

B. Detection of the two-point space-scale correlation

The primordial density perturbations are governed by fluctuations of the fields χ and π_i. Thus, the field correlation functions $\langle\chi(\mathbf{x},t)\chi(\mathbf{x}',t)\rangle$ and $\langle\pi_i(\mathbf{x},t)\pi_i(\mathbf{x}',t)\rangle$ will inevitably lead to a correlation of the primordial density perturbation $\langle\delta(\mathbf{x},t)\delta(\mathbf{x}',t)\rangle$. When perturbations of scale k cross the horizon at a time t, the equal-time two-point correlation function between $(\mathbf{x},t)$ and $(\mathbf{x}',t)$ will yield a correlation between two space-scale modes $(\mathbf{x},k)$ and $(\mathbf{x}',k)$ via the mapping formula $k=a(t)H$ [Eq. (39)]. The density contrast $\delta_{\mathbf{k}}(\mathbf{x})$ represents the fluctuations in total energy density at the spatial point $\mathbf{x}$ with a scale $\mathbf{k}$.

Obviously, one cannot measure the density (energy) perturbations precisely on a scale $\mathbf{k}$ and at a spatial point $\mathbf{x}$ simultaneously. The essence of $\delta_{\mathbf{k}}(\mathbf{x})$ is as follows. When a perturbation crosses the horizon at the scale $k=aH$, the position of the perturbation has an uncertainty typically given by the size of the horizon—i.e., $\Delta x=(aH)^{-1}$. By virtue of the uncertainty relation $\Delta x\Delta k\simeq2\pi$, the scale of the perturbation thus lies within the band from $k-(1/2)\Delta k$ to $k+(1/2)\Delta k$ where $\Delta k \simeq aH$. Therefore, $\delta_{\mathbf{k}}(\mathbf{x})$ describes the perturbation in a phase-space cell $(\mathbf{x},\mathbf{k})$ with a size confined by the spatial range from $\mathbf{x}$ to $\mathbf{x}+\Delta\mathbf{x}$ and by the scale range from $\mathbf{k}$ to $\mathbf{k}+\Delta\mathbf{k}$. The volume of this phase-space cell is characterized by $\Delta\mathbf{x}\cdot\Delta\mathbf{k}\simeq 2\pi$. Consequently, $\langle \delta_{\mathbf{k}}(\mathbf{x})\delta_{\mathbf{k}}(\mathbf{x}')\rangle$ governs the correlation between two perturbation modes localized within the cells centering at $(\mathbf{x},\mathbf{k})$ and $(\mathbf{x}',\mathbf{k})$ in the phase ($\mathbf{x}$-$\mathbf{k}$) space. In order to unveil the effects of $\langle \chi(\mathbf{x},t)\chi(\mathbf{x}',t)\rangle$ and $\langle \pi(\mathbf{x},t)\pi(\mathbf{x}',t)\rangle$, it is indispensable to decompose the mass density perturbations into the $\mathbf{x}$-$\mathbf{k}$ modes in phase space. The discrete wavelet transform (DWT) is designed to do such a space-scale [$\mathbf{x}$-$\mathbf{k}$] decomposition [16].

In the formulation of DWT, there are two sets of spatially localized bases given by the scaling functions $|\mathbf{j},\mathbf{l}\rangle_s$ and the wavelet functions $|\mathbf{j},\mathbf{l}\rangle_w$; both are characterized by the indices $\mathbf{j}$ and $\mathbf{l}$. For a 1D sample with spatial size L, the index $j=0,1,2,\ldots$ stands for a scale from k_j to $k_j+\Delta k_j$ in which $k_j=2\pi 2^j/L$ and $\Delta k_j=2\pi 2^j/L$. The index $l=0,1,\ldots,2^j-1$ denotes the location of the spatial point within $Ll/2^j < x_l < L(l+1)/2^j$. These bases are complete, and they satisfy the orthogonal relations ${}_s\langle j,l'|j,l\rangle_s=\delta_{l,l'}$ and ${}_w\langle j',l'|j,l\rangle_w=\delta_{j,j'}\delta_{l,l'}$. For 3D samples, the DWT bases are given by the direct product of the 1D bases. Thus, a density field $|\delta\rangle$ can be decomposed into the phase-space modes $(\mathbf{j},\mathbf{l})$ as

$$\delta_{\mathbf{j},\mathbf{l}} \equiv {}_s\langle \mathbf{j},\mathbf{l}|\delta\rangle = \int d^3x\,\delta(\mathbf{x})\,\phi_{\mathbf{j},\mathbf{l}}(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\hat{\delta}(\mathbf{k})\,\hat{\phi}_{\mathbf{j},\mathbf{l}}(\mathbf{k}),$$

$$\tilde{\delta}_{\mathbf{j},\mathbf{l}} \equiv {}_w\langle \mathbf{j},\mathbf{l}|\delta\rangle = \int d^3x\,\delta(\mathbf{x})\,\psi_{\mathbf{j},\mathbf{l}}(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\,\delta(\mathbf{k})\,\hat{\psi}_{\mathbf{j},\mathbf{l}}(\mathbf{k}), \tag{50}$$

where $\phi_{\mathbf{j},\mathbf{l}}(\mathbf{x})=\langle x|\mathbf{j},\mathbf{l}\rangle_s$, $\hat{\phi}_{\mathbf{j},\mathbf{l}}(\mathbf{k})=\langle k|\mathbf{j},\mathbf{l}\rangle_s$, $\psi_{\mathbf{j},\mathbf{l}}(\mathbf{x})=\langle x|\mathbf{j},\mathbf{l}\rangle_w$, and $\hat{\phi}_{\mathbf{j},\mathbf{l}}(\mathbf{k})=\langle k|\mathbf{j},\mathbf{l}\rangle_w$; i.e., they are the scaling functions and wavelet functions in either the x representation or the k representation, respectively.

Subsequently, the two-point correlation functions in Eq. (46) can be rewritten in terms of the DWT bases as

$$\langle \tilde{\chi}_{\mathbf{j},\mathbf{l}}\tilde{\chi}_{\mathbf{j},\mathbf{l}'}\rangle = \int \frac{d^3k}{(2\pi)^3}|f_{\chi,\mathbf{k}}(t)|^2\hat{\psi}_{\mathbf{j},\mathbf{l}}(\mathbf{k})\,\hat{\psi}^*_{\mathbf{j},\mathbf{l}'}(\mathbf{k}),$$

$$\langle \chi_{\mathbf{j},\mathbf{l}}\chi_{\mathbf{j},\mathbf{l}'}\rangle = \int \frac{d^3k}{(2\pi)^3}|f_{\chi,\mathbf{k}}(t)|^2\hat{\phi}_{\mathbf{j},\mathbf{l}}(\mathbf{k})\,\hat{\phi}^*_{\mathbf{j},\mathbf{l}'}(\mathbf{k}),$$

$$\langle \tilde{\pi}_{i;\mathbf{j},\mathbf{l}}\tilde{\pi}_{i;\mathbf{j},\mathbf{l}'}\rangle = \int \frac{d^3k}{(2\pi)^3}|f_{\pi,\mathbf{k}}(t)|^2\hat{\psi}_{\mathbf{j},\mathbf{l}}(\mathbf{k})\,\hat{\psi}^*_{\mathbf{j},\mathbf{l}'}(\mathbf{k}),$$

$$\langle \pi_{i;\mathbf{j},\mathbf{l}}\pi_{i;\mathbf{j},\mathbf{l}'}\rangle = \int \frac{d^3k}{(2\pi)^3}|f_{\pi,\mathbf{k}}(t)|^2\hat{\phi}_{\mathbf{j},\mathbf{l}}(\mathbf{k})\,\hat{\phi}^*_{\mathbf{j},\mathbf{l}'}(\mathbf{k}), \tag{51}$$

where the time t is taken to be t_j specified by the relation $2\pi 2^j/L=k=aH$. Since $a=\exp(Ht)$, one has

$$t_j=\frac{1}{H}\left[j\ln 2+\ln\left(\frac{2\pi}{LH}\right)\right]. \tag{52}$$

Thus, Eq. (51) can be used to determine the correlations between fluctuations at different spatial points $\mathbf{l}$ and $\mathbf{l}'$, both crossing out of the Hubble horizon at the same time t_j during the inflationary epoch.

For example, in the case of a free scalar field χ with mass m, the DWT mode-mode correlations can be expressed as

$$\langle \tilde{\chi}_{\mathbf{j},\mathbf{l}}\tilde{\chi}_{i;\mathbf{j},\mathbf{l}'}\rangle = \int \frac{d^3k}{(2\pi)^3 2(k^2+m^2)^{1/2}}\hat{\psi}^*_{\mathbf{j},\mathbf{l}}(\mathbf{k})\,\hat{\psi}^*_{\mathbf{j},\mathbf{l}'}(\mathbf{k}),$$

$$\langle \chi_{\mathbf{j},\mathbf{l}}\chi_{i;\mathbf{j},\mathbf{l}'}\rangle = \int \frac{d^3k}{(2\pi)^3 2(k^2+m^2)^{1/2}}\hat{\phi}^*_{\mathbf{j},\mathbf{l}}(\mathbf{k})\,\hat{\phi}^*_{\mathbf{j},\mathbf{l}'}(\mathbf{k}). \tag{53}$$

Since $\hat{\psi}_{\mathbf{j},\mathbf{l}}(\mathbf{k})$ is localized at $k\simeq 2\pi 2^j/L$ and $\hat{\phi}^*_{\mathbf{j},\mathbf{l}}(\mathbf{k})$ is nonzero at $k\leqslant 2\pi 2^j/L$, one can approximate $(k^2+m^2)^{-1/2}\approx m^{-1}$ if the $\mathbf{j}$ scales are greater than the Compton wavelength of mass m. Thus,

$$\langle \tilde{\chi}_{\mathbf{j},\mathbf{l}}\tilde{\chi}_{\mathbf{j},\mathbf{l}'}\rangle \simeq \frac{1}{(2\pi)^3 2m}\int d^3k\,\hat{\psi}^*_{\mathbf{j},\mathbf{l}}(\mathbf{k})\,\hat{\psi}^*_{\mathbf{j},\mathbf{l}'}(\mathbf{k})=0 \quad \text{if } l\neq l',$$

$$\langle \chi_{\mathbf{j},\mathbf{l}}\chi_{\mathbf{j},\mathbf{l}'}\rangle \simeq \frac{1}{(2\pi)^3 2m}\int d^3k\,\hat{\phi}^*_{\mathbf{j},\mathbf{l}}(\mathbf{k})\,\hat{\phi}^*_{\mathbf{j},\mathbf{l}'}(\mathbf{k})=0 \quad \text{if } l\neq l', \tag{54}$$

where the orthogonality of the DWT bases with respect to the indices $\mathbf{l},\mathbf{l}'$ has been applied. Similar results hold for the field $\pi_{i;\mathbf{j},\mathbf{l}}$. Hence, for the free field case the DWT modes are not correlated and, therefore, the correlation function $\langle \delta_{\mathbf{k}}(\mathbf{x})\delta_{\mathbf{k}}(\mathbf{x}')\rangle$ is trivial, as long as the scale under consideration is larger than the Compton wavelength of the scalar field.

However, for the self-interacting $O(N)$ model the DWT mode-mode correlations [Eq. (51)] are generally nonzero. Accordingly, the two-point space-scale correlation $\langle \delta_{\mathbf{k}}(\mathbf{x})\delta_{\mathbf{k}}(\mathbf{x}')\rangle$ may become nontrivial and initial condition dependent. Using Eqs. (39)–(43), it is straightforward to calculate the normalized two-point space-scale correlation function defined as $\langle \delta_{\mathbf{j},\mathbf{l}}\delta_{\mathbf{j},\mathbf{l}'}\rangle/\langle \delta^2_{\mathbf{j},\mathbf{l}}\rangle$.

As a numerical example, the normalized two-point space-scale correlations of density perturbations under different initial conditions are plotted in Fig. 7, in which the parameters are taken to be $H=2m$ and $L=2m^{-1}$. From Eq. (52), one has

$$t_j \simeq (0.35j+0.23)m^{-1}. \tag{55}$$

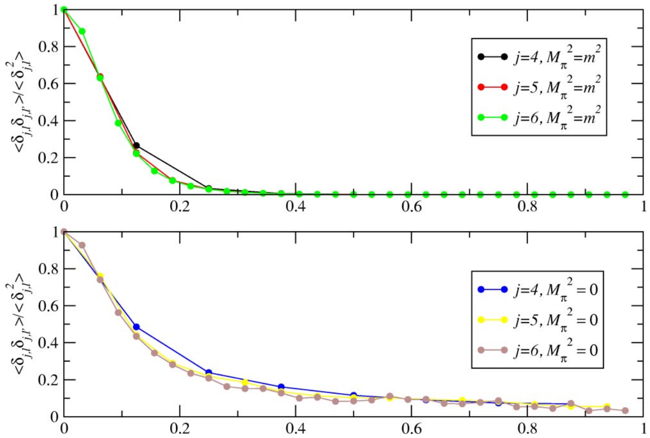

FIG. 7. The normalized mode-mode correlation functions of density fields in phase space are plotted against the separation r (in units of m^{-1}) of two perturbation modes with respect to various scales j under two different sets of initial conditions. The parameter L is taken to be $2m^{-1}$.

Thus for $j=4$–6 shown in Fig. 7, we have $t_j \simeq (1.63$–$2.33)m^{-1}$ which corresponds to the number of e-foldings $Ht_j \simeq 3.3$–4.7. Since the inflation under consideration lasts from $t=0$ to about $t=70m^{-1}$, the correlations in Fig. 7 actually probe the inflaton dynamics at the beginning of inflation. With $M_\chi^2(0)=M_\pi^2(0)\simeq m^2>0$, the space-scale correlation function approaches zero drastically as the distance r between the two modes increases, while the correlation deviates from zero and lasts for a long range for perturbations with $M_\chi^2=M_\pi^2=0$ initially.

Figure 8 plots the same correlations as in Fig. 7 but with respect to $j=8,9$ or $t_j \simeq (2.75$–$3.38)m^{-1}$, which corresponds to the number of e-foldings $Ht_j \simeq 5.52$–6.8. The behavior of correlations under the initial condition $M_\chi^2(0)=M_\pi^2(0) \simeq m^2$ remains the same. However, the correlations with $M_\chi^2(0)=M_\pi^2(0)=0$ do not coincide with those shown in Fig. 7. That is, the space-scale correlation of density perturbations in the model with $M_\pi^2>0$ initially does not experience a significant evolution during the first few e-foldings of the inflationary expansion, while the correlation in the model with initial $M_\pi^2=0$ does. Therefore, the j dependence of the mode-mode correlation is practical to test the dynamical predictions about the evolution (t_e dependence) of the primordial perturbations. Since the resolution of DWT basically can be refined as required by the problem at hand, one is capable of exploring the physics of the very early Universe by means of mode-mode correlations in phase space.

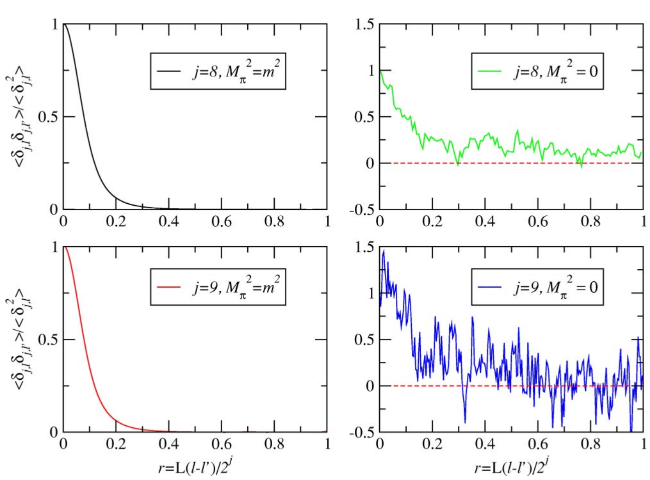

FIG. 8. The space-scale correlations as a function of r (in units of m^{-1}) under two sets of initial conditions as those in Fig. 7 are plotted with respect to finer scales $j=8, 9$.

VI. CONCLUSION

Using the $O(N)$ model as an example, we have investigated the self-interacting effects on the primordial perturbations using a self-consistent off-equilibrium approach. It is known that most information of the dynamics and initial conditions of the system will be lost during the nonequilibrium evolution. The nonequilibrium evolution can only be probed via the observable remains produced from such a process. In this paper, we have shown that the off-equilibrium evolution of cosmological inflation may have two such observable ''remains.''

The first one is the running spectral index of primordial perturbations induced by the scale dependence due to the self-interaction of the inflaton. We found that the running spectral index depends essentially on the rate of particle creation and the energy transfer from the background to the inflaton fluctuations as well as the evolution of the background field. It is a signature of the energy transferring dynamics during the inflation. Although the running index of the $O(N)$ model is small compared to current data, the negative running (Fig. 4) does coincide with the WMAP observation.

The second remain is the correlation function between phase-space modes of the density perturbation. Under the influence of the self-interaction, fluctuations created from the background field are no longer white noises. Although the one-point distribution function of fluctuations in each Hartree mode is Gaussian, the power of the fluctuations is scale dependent, which gives rise to the correlation between the phase-space modes of the energy density perturbation. Moreover, since the dynamical evolution of the correlation depends upon the initial conditions of the inflation, the mode-mode correlation of density perturbations also provides a window to study the dynamics of the self-interaction *as well as* the initial conditions of the inflation. Here, we would like to emphasize two important results. First, the initial-condition dependence of correlation functions is irrelevant to the number of fields in action. Second, the inflationary density perturbations after the superhorizon evolution are not fluctuations of the thermal equilibrium state. Therefore, the dependence of the correlation function upon the initial conditions does not contradict with the reheating of the Universe, which is generally produced by interactions between ϕ and other fields. Thus, we may expect that the nontrivial mode-mode correlation in phase space is detectable via a

DWT analysis on the cosmic microwave background (CMB) temperature map or other observations of the large-scale structure relevant to the density perturbations [18].

Although the $O(N)$ model is just an example to illuminate off-equilibrium effects from the self-interaction of the inflaton field, we believe the implication from what we have found in this paper is also useful to other interacting inflation models. For instance, the scale dependence of perturbations drawn from the self-consistent off-equilibrium dynamics is generally different from that obtained by the third or higher derivative of a classical effective potential. Therefore, the running spectral index determined by the formalism of various effective potentials would be questionable if the interaction or self-interaction of the inflaton is substantial.

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