Energy-momentum tensor of cosmological fluctuations during inflation

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We study the renormalized energy-momentum tensor (EMT) of cosmological scalar fluctuations during the slow-rollover regime for chaotic inflation with a quadratic potential and find that it is characterized by a negative energy density which grows during slow rollover. We also approach the back-reaction problem as a second-order calculation in perturbation theory, finding no evidence that the back reaction of cosmological fluctuations is a gauge artifact. In agreement with the results for the EMT, the average expansion rate is decreased by the back reaction of cosmological fluctuations.

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I. INTRODUCTION

After the latest cosmic microwave background (CMB) data [1,2], inflation (see [3,4] for a textbook review) seems the most promising theory in explaining the large scale structure of the Universe. According to inflation, the large scale structure of the Universe exhibits the fingerprint of quantum fluctuations amplified during the accelerated era [5]. On modeling inflation and its transition to the standard big bang cosmology the constraints on the amplitude and spectrum of CMB fluctuations become constraints on the physics of inflation. All this modeling depends on the linear treatment of cosmological perturbations. Within the framework of inflationary cosmology, on using the most recent data, we are therefore close to measuring the details of the spectrum of fluctuations, while not having a definite idea on the energy content of these fluctuations and how they back react on the inflationary expansion responsible for their amplification.

Although cosmological linear perturbation theory during inflation is almost a textbook subject, the understanding of non-linear effects is still at the forefront of research. The non-linearity and gauge invariance of general relativity are tenacious obstacles both at technical and interpretational levels; nonetheless, the gauge issue has been solved to higher orders in perturbation theory [6,7]. Further, many interesting effects are appreciable only beyond linear order. From the theoretical point of view the back reaction of gravitational fluctuations on the geometry is one of the most interesting issues [8]. Within the inflationary context, this problem has been tackled by Abramo, Brandenberger and Mukhanov [7,9]. The intringuing result that the energy-momentum tensor (EMT) of fluctuations may slow down inflation [7,9] has subsequently generated renewed interest in the subject of back reaction [10–12]. The final answer to the physical significance of this back-reaction effect is still under debate [10–12]. The problem of gravitational back reaction for black holes has also been tackled for gravitational waves [13]. More recently these non-linear effects have also drawn attention in connection with observations, since non-linear cosmological perturbations introduce non-Gaussian signatures in the power spectrum [14,15].

The aim of this paper is to compute the renormalized EMT of cosmological fluctuations during inflation, according to the adiabatic regularization scheme [16] also used in our previous paper [17]. The model considered here is the slow rollover regime of inflation driven by a massive inflaton, but we believe that the results obtained here also hold for other inflationary models. We find that the averaged (with respect to the adiabatic vacuum) renormalized EMT of cosmological fluctuations during slow rollover is characterized by a negative energy density and a de Sitter–like equation of state (this result was found for long-wavelength modes in [7,9]). In a naive approach this would lead us to think that the EMT of cosmological fluctuations slows down inflation. We also evaluate the back reaction on the geometry in a systematic way by proceeding self-consistently to second order in perturbation theory. In order to do this we give a systematic treatment of second-order perturbation theory for single scalar field driven inflation (see also [15,18,19] for the second-order formalism). The gauge used for scalar perturbations in this paper is the uniform curvature gauge [20] (UCG) generalized to second order. In the UCG the spatial sections are not perturbed by scalar fluctuations. We believe that this gauge is more convenient for our problem than the more frequently used longitudinal gauge.

In this paper we focus on the EMT of scalar cosmological perturbations. This is indeed the relevant effect since vector perturbations decay and gravitational waves are described by an EMT which is equivalent to a massless field [21], whose main effect is non-leading with respect to scalar fluctuations (see however [8] for the two-loop calculation). The calculation of the EMT of gravitational waves will be the subject of a separate publication [22].

The plan of the paper is as follow. In Sec. II we present

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the linear cosmological perturbations in the UCG. In Sec. III we extend the UCG to second order and we give expressions for the Einstein and the energy-momentum tensors. In Sec. IV we present and illustrate the use of the Einstein equations to second order. In Secs. V and VII we give the approximate solutions for first and second order fluctuations, respectively, using the renormalized values computed in Sec. VI. We discuss the back reaction on the geometry in Sec. VIII and we give our conclusions in Sec. IX. In the three appendixes we (A) compare our analytical approximation with the WKB method, (B) exhibit the fourth order adiabatic expansion and (C) compare some of our results with those obtained in a different gauge.

II. LINEAR PERTURBATIONS IN THE UNIFORM CURVATURE GAUGE

We consider inflation in a flat universe driven by a classical minimally coupled scalar field with a general potential $V(\phi)$. The action is

$$S \equiv \int d^4x \mathcal{L} = \int d^4x \sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{1}$$

where $\mathcal{L}$ is the Lagrangian density.

Let us now study the fluctuations of the scalar field $\varphi(t,\mathbf{x})$ around its homogeneous classical[1] value $\phi(t)$ and include metric perturbations. For the homogeneous case we have

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0$$

$$H^2 = \frac{8\pi G}{3}\left[\frac{\dot{\phi}^2}{2} + V\right] \tag{2}$$

where $H = \dot{a}/a$ is the Hubble parameter and a is the scale factor.

The scalar perturbations around a flat Robertson-Walker metric are

$$ds^2 = -(1+2\alpha)dt^2 - a\beta_{,i}dt dx^i + a^2[\delta_{ij}(1-2\psi) + 2\gamma_{,ij}]dx^i dx^j, \tag{3}$$

where the symbol ",i" denotes the derivative with respect to the spatial coordinates. We choose to work in the uniform curvature gauge:

$$ds^2 = -(1+2\alpha)dt^2 - a\beta_{,i}dt dx^i + a^2\delta_{ij}dx^i dx^j. \tag{4}$$

We note that this choice fixes uniquely the gauge, just as the more frequently used longitudinal gauge (for a review of cosmological perturbations in this gauge see [25]). This can be seen by setting $\epsilon^0 = -\psi/H$, $\epsilon = \gamma$, where $\epsilon^\mu = (\epsilon^0, \epsilon^{,i})$ is an infinitesimal coordinate transformation ($x^\mu \to x^\mu + \epsilon^\mu$). In order to see the connection with the more known longitudinal gauge we write the metric [25] in that gauge,

$$ds^2 = -(1+2\Phi)dt^2 + a^2(1-2\Psi)\delta_{ij}dx^i dx^j, \tag{5}$$

and note that the transformation between the two gauges can be obtained through a time reparametrization $\epsilon^0 = -a\beta/2$, $\epsilon = 0$:

$$\Phi = -\alpha + \frac{d}{dt}\left(\frac{a}{2}\beta\right), \quad \Psi = -\frac{aH}{2}\beta. \tag{6}$$

Let us now derive the equation of motion in the uniform curvature gauge (see also [20][2]). The scalar field fluctuations obey the following equation of motion:

$$\begin{aligned} \ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{a^2}\nabla^2\varphi + V_{\phi\phi}\varphi &= 2\alpha\ddot{\phi} + \dot{\phi}\left(\dot{\alpha} + 6H\alpha - \frac{1}{2a}\nabla^2\beta\right) \\ &= \dot{\alpha}\dot{\phi} - 2\alpha V_\phi - \frac{\dot{\phi}}{2a}\nabla^2\beta, \end{aligned} \tag{7}$$

where an overdot denotes a derivative with respect to the time t. Starting from the Einstein equations, $G^\mu_\nu = 8\pi G T^\mu_\nu$, in order to obtain an equation for φ only one needs the energy and momentum constraints in their linearized version—i.e. the G^0_0 and G^0_i linear equations:

$$\begin{aligned} \frac{H}{a}\nabla^2\beta &= 8\pi G(\dot{\phi}\dot{\varphi} + V_\phi\varphi + 2V\alpha) \\ &= 8\pi G\frac{\dot{\phi}^2}{H}\frac{d}{dt}\left(\frac{H}{\dot{\phi}}\varphi\right), \end{aligned} \tag{8}$$

$$\alpha_{,i} = 4\pi G\frac{\dot{\phi}}{H}\varphi_{,i}. \tag{9}$$

Because of the absence of anisotropic stress, we also have $\dot{\beta} + 2H\beta = 2\alpha/a$ (this relation replaces the equality $\Phi = \Psi$ in the longitudinal gauge) [20]. On substituting these two latter equations in Eq. (7) one obtains

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{a^2}\nabla^2\varphi + \left[V_{\phi\phi} + 2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\right]\varphi = 0. \tag{10}$$

It is important to note that the effective potential for the fluctuations can be rewritten as

[1]For a quantum treatment of the homogeneous inflaton see [23,24].

[2]We observe that the equations in [20] are for a metric perturbation which has $g_{0i} = a\beta_{,i}$ and not $g_{0i} = a\beta_{,i}/2$ as is stated there.

$$V_{\phi\phi}+2\frac{\dot{H}}{H}\left(3H-\frac{\dot{H}}{H}+2\frac{\ddot{\phi}}{\dot{\phi}}\right)=V_{\phi\phi}-6H^2\left(\epsilon-\frac{1}{3}\epsilon^2+\frac{\dot{\epsilon}}{3H}\right), \tag{11}$$

where we have introduced the (positive) slow-rollover parameter $\epsilon\equiv-\dot{H}/H^2$. This means that the self-consistent inclusion of gravitational fluctuations changes the effective potential for the field fluctuations. In particular, gravitational fluctuations generally decrease the effective mass to first order in ϵ.

Equation (10) is generally seen written in conformal time. The Fourier transform modes of $v=a\varphi$ satisfy

$$v_k''+\left(k^2-\frac{z''}{z}\right)v_k=0, \quad z=a\frac{\dot{\phi}}{H}, \tag{12}$$

where a prime denotes a derivative with respect to the conformal time η, $d\eta=dt/a$. On comparing the last equation with Eq. (12) of [26] it is immediate to see that φ satisfies the same equation as the Mukhanov variable Q. Therefore, the uniform curvature gauge has the advantage of singling out the true dynamical degrees of freedom (the matter ones), even if it has the disadvantage of being non-diagonal in the metric perturbations.

III. BEYOND LINEAR ORDER

To second order we consider a metric having the coefficients

$$g_{00}=-1-2\alpha-2\alpha^{(2)}$$

$$g_{0i}=-\frac{a}{2}(\beta_{,i}+\beta_{,i}^{(2)})$$

$$g_{ij}=a^2\left[\delta_{ij}+\frac{1}{2}(\partial_i\chi_j^{(2)}+\partial_j\chi_i^{(2)}+h_{ij}^{(2)})\right]. \tag{13}$$

The above metric is the extension of the uniform curvature gauge to second order: $\alpha^{(2)}$ and $\beta^{(2)}$ are scalar perturbations to second order. To second order, scalar, vector and tensor perturbations do not evolve independently as is the case in first order. For this reason we take into account second order vector and tensor perturbations, represented by the divergenceless vector $\chi_j^{(2)}$ and by the transverse and traceless tensor $h_{ij}^{(2)}$, respectively. In the above we have omitted first vector perturbations (which die away kinematically) and tensor perturbations (which satisfy the usual equation $\ddot{h}+3H\dot{h}-\nabla^2h/a^2=0$). With this approximation we are neglecting the EMT of vector and tensor perturbations, and their correlations with the scalar perturbations. We finally note that the choice in Eq. (13) (including vector and tensor metric elements to first order) fixes the gauge completely to second order.

The Einstein tensor expanded to second order is

$$G_0^0=G_0^{0(0)}+\delta G_0^{0(1)}+\delta G_0^{0(2)}=-3H^2-\frac{H}{a}\nabla^2\beta+6H^2\alpha-\frac{H}{a}\nabla^2\beta^{(2)}+6H^2\alpha^{(2)}-12H^2\alpha^2$$
$$+\frac{3}{4}H^2|\vec{\nabla}\beta|^2+\frac{H}{a}(\vec{\nabla}\alpha\cdot\vec{\nabla}\beta+2\alpha\nabla^2\beta)+\frac{1}{8a^2}[\beta_{,ij}\beta^{,ij}-(\nabla^2\beta)^2], \tag{14}$$

$$G_i^0=G_i^{0(0)}+\delta G_i^{0(1)}+\delta G_i^{0(2)}=-2H\alpha_{,i}-2H\alpha_{,i}^{(2)}+8H\alpha\alpha_{,i}-\frac{1}{2a}H\alpha_{,i}\nabla^2\beta+\frac{1}{2a}\vec{\nabla}\alpha\cdot\vec{\nabla}\beta_{,i}-\frac{H}{2}\vec{\nabla}\beta_{,i}\cdot\vec{\nabla}\beta-\frac{1}{4}\nabla^2\dot{\chi}_i^{(2)}, \tag{15}$$

$$G_j^i=G_j^{i(0)}+\delta G_j^{i(1)}+\delta G_j^{i(2)}$$
$$=\delta_j^i\Bigg\{-(3H^2+2\dot{H})+2\alpha(3H^2+2\dot{H})+2H\dot{\alpha}+\frac{1}{a^2}\nabla^2\alpha-\frac{H}{a}\nabla^2\beta-\frac{1}{2a}\nabla^2\dot{\beta}+2\alpha^{(2)}(3H^2+2\dot{H})+2H\dot{\alpha}^{(2)}+\frac{1}{a^2}\nabla^2\alpha^{(2)}$$
$$-\frac{H}{a}\nabla^2\beta^{(2)}-\frac{1}{2a}\nabla^2\dot{\beta}^{(2)}+\frac{H}{a}\vec{\nabla}\alpha\cdot\vec{\nabla}\beta+\frac{H}{2}\vec{\nabla}\beta\cdot\vec{\nabla}\dot{\beta}+\left(\frac{1}{4}|\vec{\nabla}\beta|^2-4\alpha^2\right)(3H^2+2\dot{H})-8H\alpha\dot{\alpha}+\left(\frac{\dot{\alpha}}{2a}+2\frac{H}{a}\alpha\right)\nabla^2\beta$$
$$-\frac{2}{a^2}\alpha\nabla^2\alpha+\frac{\alpha}{a}\nabla^2\dot{\beta}-\frac{1}{a^2}|\vec{\nabla}\alpha|^2+\frac{1}{8a^2}[\beta^{,\ell m}\beta_{,\ell m}-(\nabla^2\beta)^2]\Bigg\}+\Bigg\{\frac{1}{2a}\dot{\beta}^{,i}_{,j}+\frac{H}{a}\beta^{,i}_{,j}-\frac{1}{a^2}\alpha^{,i}_{,j}+\frac{1}{2a}\dot{\beta}^{(2),i}_{,j}+\frac{H}{a}\beta^{(2),i}_{,j}-\frac{1}{a^2}\alpha^{(2),i}_{,j}$$
$$+\frac{1}{a^2}\alpha^{,i}\alpha_{,j}-\frac{H}{a}\beta^{,i}\alpha_{,j}+\frac{2}{a^2}\alpha\alpha^{,i}_{,j}-\frac{2}{a}H\alpha\beta^{,i}_{,j}-\frac{1}{2a}\dot{\alpha}\beta^{,i}_{,j}-\frac{1}{a}\alpha\dot{\beta}^{,i}_{,j}+\frac{1}{4a^2}(\nabla^2\beta\beta^{,i}_{,j}-\beta^{,i}_{,k}\beta^{,k}_{,j})+\frac{3}{4}H(\dot{\chi}_j^{(2),i}+\dot{\chi}_{,j}^{(2)i}+\dot{h}_j^{(2)i})$$
$$+\frac{1}{4}(\ddot{\chi}_j^{(2),i}+\ddot{\chi}_{,j}^{(2)i}+\ddot{h}_j^{(2)i})-\frac{1}{4a^2}\nabla^2h_j^{(2)i}\Bigg\}. \tag{16}$$

To second order the scalar field is expanded as

$$\phi(t,\mathbf{x}) = \phi(t) + \varphi(t,\mathbf{x}) + \varphi^{(2)}(t,\mathbf{x}). \tag{17}$$

The EMT of inflaton fluctuations

$$T^{\mu}_{\nu} = \partial^{\mu}\phi\partial_{\nu}\phi + \delta^{\mu}_{\nu}\mathcal{L} \tag{18}$$

to second order is

$$\begin{aligned} T^0_0 &= T^{0(0)}_0 + \delta T^{0(1)}_0 + \delta T^{0(2)}_0 \\ &= -\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] + \dot{\phi}^2\alpha - \dot{\phi}\dot{\varphi} - V_{\phi}\varphi + \dot{\phi}^2\alpha^{(2)} - \dot{\phi}\dot{\varphi}^{(2)} - V_{\phi}\varphi^{(2)} - \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2a^2}|\vec{\nabla}\varphi|^2 \\ &\quad - \frac{1}{2}V_{\phi\phi}\varphi^2 - 2\dot{\phi}^2\alpha^2 + 2\dot{\phi}\alpha\dot{\varphi} + \frac{1}{8}\dot{\phi}^2|\vec{\nabla}\beta|^2, \end{aligned} \tag{19}$$

$$T^0_i = T^{0(0)}_i + \delta T^{0(1)}_i + \delta T^{0(2)}_i = -\dot{\phi}\varphi_{,i} - \dot{\phi}\varphi^{(2)}_{,i} - \dot{\varphi}\varphi_{,i} + 2\dot{\phi}\alpha\varphi_{,i}, \tag{20}$$

$$\begin{aligned} T^i_j &= T^{i(0)}_j + \delta T^{(1)}_{ii} + \delta T^{(2)}_{ii} \\ &= \left[\frac{1}{2}\dot{\phi}^2 - V\right]\delta^i_j + [\dot{\phi}\dot{\varphi} - \dot{\phi}^2\alpha - V_{\phi}\varphi]\delta^i_j + \Bigg[\dot{\phi}\dot{\varphi}^{(2)} - \dot{\phi}^2\alpha^{(2)} - V_{\phi}\varphi^{(2)} \\ &\quad + \frac{1}{2}\left(\dot{\varphi}^2 - 4\dot{\phi}\alpha\dot{\varphi} - \frac{1}{4}\dot{\phi}^2|\vec{\nabla}\beta|^2 + 4\dot{\phi}^2\alpha^2 + \frac{1}{a}\dot{\phi}\vec{\nabla}\varphi\cdot\vec{\nabla}\beta - \frac{1}{a^2}|\vec{\nabla}\varphi|^2 - V_{,\phi\phi}\varphi^2\right)\Bigg]\delta^i_j + \frac{1}{a^2}\varphi^{,i}\varphi_{,j} - \frac{\dot{\phi}}{2a}\beta^{,i}\varphi_{,j}. \end{aligned} \tag{21}$$

IV. EQUATIONS TO SECOND ORDER

In this section we exhibit the second order equations corresponding to Eqs. (8), (9), (10), using, when convenient, the homogeneous and first order equations. We first give the momentum constraint to second order, which is obtained from the expressions given in the Eqs. (15) and (20):

$$\alpha^{(2)}_{,i} + \frac{1}{8H}\nabla^2\dot{\chi}^2_i = 4\pi G\frac{\dot{\phi}}{H}\varphi^{(2)}_{,i} + S_i \tag{22}$$

where χ_i are vector metric perturbations and

$$\begin{aligned} S_i &= \frac{4\pi G}{H}(\dot{\varphi} - 2\dot{\phi}\alpha)\varphi_{,i} + 4\alpha\alpha_{,i} - \frac{1}{4aH}(\alpha_{,i}\nabla^2\beta - \alpha^{,j}\beta_{,ij}) \\ &\quad - \frac{1}{4}\beta^{,j}\beta_{,ij}. \end{aligned} \tag{23}$$

The term S_i can be written as $\partial_i s + v_i$; further,

$$\alpha^{(2)} = 4\pi G\frac{\dot{\phi}}{H}\varphi^{(2)} + s \tag{24}$$

where

$$\begin{aligned} s &\equiv -4\pi G\epsilon\varphi^2 + 2\alpha^2 - \frac{1}{8}|\vec{\nabla}\beta|^2 + \frac{1}{\nabla^2}\Bigg[\frac{4\pi G}{H}\vec{\nabla}\cdot(\dot{\varphi}\vec{\nabla}\varphi) \\ &\quad + \frac{1}{4aH}(\alpha^{,kj}\beta_{,kj} - \alpha^{,k}_{,k}\beta^{,j}_{,j})\Bigg] \end{aligned} \tag{25}$$

contains the quadratic contribution of first order perturbations. We note that s includes non-local spatial contributions which nevertheless to leading order in ϵ and for long wavelength have an ordinary behavior on large scales. One can in fact approximate in such a limit, for the isotropic case, the first term in the second line of Eq. (25) as $(2\pi G)/H\varphi\dot{\varphi}$. On combining Eqs. (14), (19), the energy constraint to second order, upon using Eq. (24) and the lower order constraints, is given by

$$\frac{H}{a}\nabla^2\beta^{(2)} = 8\pi G\frac{\dot{\phi}^2}{H}\frac{d}{dt}\left(\frac{H}{\dot{\phi}}\varphi^{(2)}\right) - Q + 16\pi G V s, \tag{26}$$

where Q is defined as

$$\begin{aligned} Q &= 12H^2\alpha^2 - \frac{3}{4}H^2|\vec{\nabla}\beta|^2 - \frac{H}{a}(\alpha_{,i}\beta^{,i} + 2\alpha\nabla^2\beta) \\ &\quad - \frac{1}{8a^2}[\beta_{,ij}\beta^{,ij} - (\nabla^2\beta)^2] - 8\pi G\Bigg[\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2a^2}|\vec{\nabla}\varphi|^2 \\ &\quad + \frac{1}{2}V_{\phi\phi}\varphi^2 + 2\dot{\phi}^2\alpha^2 - 2\dot{\phi}\alpha\dot{\varphi} - \frac{1}{8}\dot{\phi}^2|\vec{\nabla}\beta|^2\Bigg]. \end{aligned} \tag{27}$$

We observe that Eq. (26) for $\beta^{(2)}$ is reminiscent of a universe filled with two components, with the term $Q-16\pi GVs$ playing the role of a non-adiabatic pressure term [25], upon comparing with the first order equation (8). From this analogy one may guess that the terms on the right hand side of Eq. (26) should approximatively cancel. Explicit calculations confirm this property.

The equation of motion for the scalar field to second order, after using all the previous constraints, is given by

$$\ddot{\varphi}^{(2)}+3H\dot{\varphi}^{(2)}-\frac{1}{a^2}\nabla^2\varphi^{(2)} +\left[V_{\phi\phi}+2\frac{\dot{H}}{H}\left(3H-\frac{\dot{H}}{H}+2\frac{\ddot{\phi}}{\dot{\phi}}\right)\right]\varphi^{(2)}=D, \tag{28}$$

with an homogeneous part which is the same as found [see Eq. (10)] for the first order fluctuations and an inhomogeneous term D which is purely quadratic in terms of the first order fluctuations. In particular one obtains

$$D=R+\dot{\phi}\dot{s}-2V_{\phi}s+\frac{\dot{\phi}}{2H}(Q-16\pi GVs) \tag{29}$$

and

$$R=-\frac{1}{2}V_{\phi\phi\phi}\varphi^2-2\dot{\phi}\alpha\dot{\alpha}+\dot{\alpha}\dot{\varphi}+\frac{2}{a^2}\alpha\nabla^2\varphi-2V_{\phi\phi}\alpha\varphi +\frac{\dot{\phi}}{2a}\alpha_{,i}\beta^{,i}-\frac{1}{4}V_{\phi}|\vec{\nabla}\beta|^2+\frac{1}{a^2}\alpha_{,i}\varphi^{,i}-\frac{H}{a}\beta_{,i}\varphi^{,i} -\frac{1}{2a}\dot{\varphi}\nabla^2\beta+\frac{\dot{\phi}}{4}\beta_{,i}\dot{\beta}^{,i}-\frac{1}{2a}\varphi_{,i}\dot{\beta}^{,i}-\frac{1}{a}\beta_{,i}\dot{\varphi}^{,i}. \tag{30}$$

We have therefore obtained the perturbative equation for the scalar field fluctuations to second order, which is a novel result.

V. APPROXIMATE SOLUTION FOR LINEAR PERTURBATION

The equation for the Mukhanov variable (12) does not have an exact solution, except for the known case of an exponential potential [27]. Approximate schemes to obtain the long wavelength solution, such as the slow-rollover technique [28], exist. Any approximation must agree with the solution to Eq. (12) with $k=0$:

$$v_{|k=0}=Cz+Dz\int\frac{dt}{az^2} \tag{31}$$

where C and D are constants [when the first term in Eq. (31) dominates; the curvature perturbation $\zeta=H\varphi/\dot{\phi}$ is constant to leading order]. For small wavelengths ($k\sim 0$) C and D include the k dependence of the modes. However, in order to perform an adiabatic regularization, we not only need the solution for small k, but for the whole spectrum. The slow rollover approximation applied to Eq. (12) leads to a Bessel function whose infrared limit agrees with Eq. (31) only on freezing a spurious time dependence of the Bessel solution to the value on horizon crossing. This problem is the same as was encountered in [17] on trying to extend the approximate solution for the inflaton fluctuations in a rigid space-time to fields with different mass. Therefore, the slow rollover paradigm is not useful in order to obtain an approximate solution over the whole k spectrum.

In order to consider the slow rollover of massive chaotic inflation we take $\dot{H}\simeq -m^2/3$, $\ddot{H}\simeq 0$ and neglect terms $\mathcal{O}(\dot{H}^2/H^4)$ [17]. During slow rollover the equation for φ is then

$$\ddot{\varphi}_k+3H\dot{\varphi}_k+\left[\frac{k^2}{a^2}+m^2+6\dot{H}\right]\varphi_k=0. \tag{32}$$

On comparing with the fluctuations in a rigid spacetime considered in [17], we see that the gauge-invariant fluctuations in Eq. (10) have a negative mass since $\dot{H}\simeq -m^2/3$: this is due to the proper inclusion of gravitational fluctuations. This result is true for many inflationary models and it is related to the red spectrum of the curvature perturbations.

In order to have an approximate solution for the whole k spectrum we proceed in analogy with [17] and we choose, for large k,

$$\varphi_k=\frac{1}{a^{3/2}}\left(\frac{\pi\lambda}{4H}\right)^{1/2}H_{\nu}^{(1)}(\lambda\xi) \tag{33}$$

where $\xi=k/(aH)$ and

$$\lambda=1-\frac{\dot{H}}{H^2}=1+\epsilon$$

$$\nu=\frac{3}{2}-\frac{1}{3}\frac{m^2}{H^2}-3\frac{\dot{H}}{H^2}=\frac{3}{2}+\frac{2}{3}\frac{m^2}{H^2}=\frac{3}{2}+2\epsilon. \tag{34}$$

We note that ν for the gauge-invariant fluctuations differs from the corresponding quantity for inflaton fluctuations in rigid space-times [17].

The reason the procedure followed in [17] does not succeed in producing an approximate solution, valid over the whole range of k, is the dependence of ν on H. In fact, among the terms which are apparently $\mathcal{O}(\dot{H}^2/H^4)$, the term

$$\sim\dot{H}H^2\frac{\partial\nu}{\partial H}\ln\zeta \tag{35}$$

leads to a term which is of order $\mathcal{O}(\dot{H}/H^2)$,

$$\sim\mathcal{O}\left(\frac{\dot{H}^2}{H^4}\right)\left[\ln\frac{k}{H}+\frac{1}{2}\left(\frac{H_0^2-H^2}{\dot{H}}\right)\right], \tag{36}$$

making the approximation not valid for small k and at large times. For this reason we consider Eq. (33) only as an approximation for modes which are still inside the Hubble radius. We also note that the solution for the ultraviolet regime in Eq. (33) is the same as that of the slow-rollover approximation, although here $\dot{H}\simeq$const and not $\epsilon\simeq$const.

On considering the infrared limit for the gauge-invariant fluctuations in Eq. (31) we know that $\varphi_k\sim\mathcal{O}(1/H)$, since $\dot{\phi}\sim$const. For $k\leqslant aH$ we must to replace the solution in Eq. (33) in order to reproduce the correct behavior in the infrared by

$$\varphi_{\mathbf{k}}=\frac{1}{a^{3/2}}\left(\frac{\pi\lambda}{4H}\right)^{1/2}\left(\frac{H(t_k)}{H(t)}\right)^2 H^{(1)}_{3/2}(\lambda\xi), \qquad (37)$$

where t_k is a time related to the instant for which the mode k crosses the Hubble radius. We leave the details of estimating the time t_k to Appendix A and give the Hubble parameter for this time value:

$$H(t_k)=H_0\sqrt{1+2\frac{\dot{H}_0}{H_0^2}\log\frac{\lambda_0 k}{H_0\nu_0}}. \qquad (38)$$

We note that we have a nearly scale invariant spectrum due to the dependence on t_k.

We also note that the amplitude of the curvature perturbations, $\zeta_k=H\varphi_k/\dot{\phi}$, associated with the solution in Eq. (37), need not obey the observational constraints ($k^{3/2}|\zeta_k|\sim10^{-5}$). The reason is that the spectrum is red tilted and we are considering modes which could have exited the Hubble radius much earlier than the ones relevant for observations. Only if the duration of inflation is minimal in order to solve the horizon problem do the curvature perturbations associated with Eq. (37) satisfy the observational constraints.

Cosmological fluctuations are canonically quantized as usual and we shall consider the vacuum defined by Eqs. (33) and (37):

$$\hat{\varphi}(t,\mathbf{x})=\frac{1}{(2\pi)^3}\sum_{\mathbf{k}}\left[\varphi_k(t)e^{i\mathbf{k}\cdot\mathbf{x}}\hat{b}_{\mathbf{k}}+\varphi_k^*(t)e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{b}^\dagger_{\mathbf{k}}\right] \qquad (39)$$

where the $\hat{b}_k$ are time-independent Heisenberg operators.

VI. ADIABATIC SUBTRACTION

We now compute the integrals obtained on taking the expectation values of the relevant operators. The strategy is slightly different from that of our previous calculation [17] since the mode function is obtained by matching two approximate solutions. Therefore we split the integral:

$$\int_{\ell}^{+\infty}dk\to\int_{\ell}^{\nu aH/\lambda}dk+\int_{\nu aH/\lambda}^{+\infty}dk \qquad (40)$$

where $\ell=CH_0$ is an infrared cutoff related to the beginning of inflation[3] [29,30] and $k=\nu aH/\lambda$ is the turning point of the Bessel function in Eq. (33). We substitute the infrared solution (37) in the first integral and the ultraviolet solution (33) in the second integral. For the ultraviolet part we employ, as in the previous article [17], dimensional regularization using a d dimensional space measure. Subsequently an adiabatic subtraction is performed in order to obtain the renormalized quantities.

From our previous calculation we know that the leading contributions come from terms which contain $\langle\varphi^2\rangle$, while terms such as $\langle\nabla_\mu\varphi\nabla^\mu\varphi\rangle$ are more ultraviolet and therefore non-leading. By the symbol $\langle\cdots\rangle$ we denote the average over the quantum state defined by Eqs. (33) and (37). It is important to note that the standard fourth order adiabatic subtraction is sufficient to regularize the EMT of cosmological fluctuations as well. In fact, the terms which apparently would not be regularized by a fourth order expansion [see the term $\beta_{,ij}\beta^{,ij}-(\nabla^2\beta)^2$ in the G^0_0 equation (14), for instance] vanish on averaging over an homogeneous state. As an example of the calculations we exhibit the details of the calculation of $\langle\varphi^2\rangle_{\rm REN}$.

The ultraviolet and infrared integrals are respectively

$$\langle\varphi^2\rangle^{UV}=\frac{\hbar}{16\pi^2}H^2\left\{A+B\frac{m^2}{H^2}+C\frac{H^2}{m^2}-\left(2+\frac{2}{3}\frac{m^2}{H^2}\right)\times\left(\frac{1}{2\pi^{1/2}}\right)^{d-3}\left(\nu\frac{aH}{\lambda}\right)^{d-3}\Gamma\left(\frac{1}{2}-\frac{d}{2}\right)\right\}+\mathcal{O}(d-3), \qquad (41)$$

$$\langle\varphi^2\rangle^{IR}=\frac{\hbar}{4\pi^2}\frac{1}{\lambda^2}\left(\frac{H_0}{H(t)}\right)^4H^2\frac{1}{6\epsilon_0}\left\{\left(1-2\epsilon_0\log\frac{\lambda l}{H\nu}\right)^3-(1-2\epsilon_0\log a)^3\right\}, \qquad (42)$$

where $\epsilon_0=-\dot{H}/H_0^2$, A, B and C are constants with a complicated dependence on hypergeometric functions and we have taken, for the calculation of Eq. (42), $\lambda_0/H_0\nu_0\simeq\lambda/H\nu$ as is done for the case of the t_k derivation in Appendix A. At the end of inflation $\log a\to1/(2\epsilon_0)$ which is the inverse of the slow-rollover perturbation parameter. Therefore, in the above expression, in order to be as accurate as possible towards the end of inflation we have included all the leading logarithmic terms which are multiplied by the small parameter ϵ_0. The adiabatic fourth order is (see Appendix B and [17]):

$$\langle\varphi^2\rangle_{(4)}=\frac{\hbar}{16\pi^2}H^2\left\{\frac{127}{45}-\frac{43}{90}\frac{m^2}{H^2}+\frac{29}{15}\frac{H^2}{m^2}-\left(2+\frac{2}{3}\frac{m^2}{H^2}\right)\times\left(\frac{1}{2\pi^{1/2}}\right)^{d-3}(am)^{d-3}\Gamma\left(\frac{1}{2}-\frac{d}{2}\right)+\mathcal{O}\left(\frac{1}{a^3}\right)\right\}+\mathcal{O}(d-3), \qquad (43)$$

and the resulting renormalized quantity is

[3] See also [17] for numerical considerations on the initial states.

$$\langle\varphi^2\rangle_{\rm REN}=\lim_{d\to 3}(\langle\varphi^2\rangle^{IR}+\langle\varphi^2\rangle^{UV}-\langle\varphi^2\rangle_{(4)})$$
$$=\frac{\hbar}{4\pi^2}\frac{1}{\lambda^2}\left(\frac{H_0}{H(t)}\right)^4 H^2\frac{1}{6\epsilon_0}\left\{\left(1-2\epsilon_0\log\frac{\lambda l}{H\nu}\right)^3-(1-2\epsilon_0\log a)^3\right\}+\frac{\hbar}{16\pi^2}H^2\left\{A+B\frac{m^2}{H^2}+C\frac{H^2}{m^2}-4\left(1+\frac{1}{3}\frac{m^2}{H^2}\right)\ln\left(\frac{\nu H}{\lambda m}\right)-\frac{127}{45}+\frac{43}{90}\frac{m^2}{H^2}-\frac{29}{15}\frac{H^2}{m^2}+\mathcal{O}\left(\frac{1}{a^3}\right)\right\}. \tag{44}$$

The leading behavior of the renormalized correlator at the beginning of inflation, which actually originates in the infrared region, grows with $\log a$:

$$\langle\varphi^2\rangle_{\rm REN}\sim\frac{\hbar}{4\pi^2}H^2\left(\frac{H_0}{H}\right)^4\ln a. \tag{45}$$

Its time derivative is

$$\frac{d}{dt}\langle\varphi^2\rangle_{\rm REN}\sim\frac{\hbar}{4\pi^2}\frac{H_0^4}{H}+2\epsilon H\langle\varphi^2\rangle_{\rm REN}=\frac{\hbar}{4\pi^2}\frac{H_0^4}{H}+\frac{2}{3}\frac{m^2}{H}\langle\varphi^2\rangle_{\rm REN}. \tag{46}$$

We note that this result is very different from the de Sitter result (H constant in time and $\epsilon=0$) since the second term on the right hand side may dominate for large times.

The above result in Eq. (45) is reminiscent[4] of the usual Hartree-Fock term formulas for stochastic inflation [31] [see Eq. (5) of [32]]:

$$\frac{d}{dt}\langle\varphi^2\rangle_{\rm REN}=\frac{\hbar}{4\pi^2}H^3-2\frac{m^2}{3H}\langle\varphi^2\rangle_{\rm REN}. \tag{47}$$

However, Eqs. (46) and (47) differ not only in the driving terms, but in particular in the mass term: in our result there is a negative mass term, whereas in the stochastic approach in rigid de Sitter space time there is a positive mass term. Our result (46) is completely consistent with Eq. (32) where the gravitational terms change the sign of the mass term in the evolution equation for the fluctuations. The difference is therefore due to the inclusion of gravitational fluctuations which are not properly taken into account in stochastic inflation.

At the end of inflation, for the characteristic time scale of slow rollover $\delta t\sim 3H_0/m^2$, the correlator in the expression in Eq. (44) saturates to

$$\langle\varphi^2\rangle_{\rm REN}^{\rm MAX}\sim\frac{\hbar}{4\pi^2}H^2\left(\frac{H_0}{H}\right)^4\frac{1}{6\epsilon_0}=\frac{\hbar}{8\pi^2}\frac{H_0^6}{H^2}\frac{1}{m^2}. \tag{48}$$

It is instructive to also compute $\langle\varphi\dot\varphi\rangle_{\rm REN}$ and $\langle\dot\varphi^2\rangle_{\rm REN}$ by the same method, using the approximate relation for the infrared modes $\dot\varphi_k\simeq-(\dot H/H)\varphi_k$. The final result [omitting all the steps in Eqs. (41), (42), (43), (44)] is

$$H\langle\varphi\dot\varphi\rangle_{\rm REN}=H\langle\varphi\dot\varphi\rangle^{IR}+\frac{\hbar}{16\pi^2}H^4\left[D+E\frac{m^2}{H^2}+\frac{8}{3}\frac{m^2}{H^2}\ln\left(\frac{\nu H}{\lambda m}\right)+4+\frac{4}{9}\frac{m^2}{H^2}+\mathcal{O}\left(\frac{1}{a^3}\right)\right], \tag{49}$$

$$\langle\dot\varphi^2\rangle_{\rm REN}=\langle\dot\varphi^2\rangle^{IR}+\frac{\hbar}{16\pi^2}H^4\left[F+G\frac{m^2}{H^2}+2\frac{m^2}{H^2}\ln\left(\frac{\nu H}{\lambda m}\right)+\frac{61}{60}-\frac{27}{10}\frac{m^2}{H^2}+\mathcal{O}\left(\frac{1}{a^3}\right)\right], \tag{50}$$

where D, E, F and G may be written in terms of generalized hypergeometric functions and for the infrared contribution one has[5]

$$\langle\varphi\dot\varphi\rangle^{IR}\simeq H\epsilon\langle\varphi^2\rangle^{IR}, \tag{51}$$

$$\langle\dot\varphi^2\rangle^{IR}\simeq H^2\epsilon^2\langle\varphi^2\rangle^{IR}. \tag{52}$$

VII. APPROXIMATE SOLUTION FOR SECOND ORDER SCALAR PERTURBATIONS

An approximate solution for the leading quantities to second order can be obtained for large scales where the infrared modes dominate. Let us consider $\varphi^{(2)}$ which satisfies Eq. (28). On neglecting terms $\mathcal{O}(\ddot H/H^3,\dot H^2/H^4)$ and the second order derivative we obtain

$$3H\dot\varphi^{(2)}-m^2\varphi^{(2)}\simeq-V_\phi\frac{m^2}{6H^2}\frac{\varphi^2}{M_{\rm pl}^2}, \tag{53}$$

which can be rewritten, on also using the homogeneous equation of motion for the inflaton, as

$$3\frac{d}{dt}(H\varphi^{(2)})\simeq\dot\phi\frac{m^2}{2H}\frac{\varphi^2}{M_{\rm pl}^2}. \tag{54}$$

Let us note that we use the notation $M_{\rm pl}^2=1/(8\pi G)$ for the (reduced) Planck mass definition. In the same approximation the energy constraint is

[4] We note that this resemblance holds on only retaining the linear terms in $\log a$.

[5] More precisely, one may obtain the infrared contribution from the behavior of the Mukhanov variable, $\langle\varphi\dot\varphi\rangle^{IR}\simeq[H\epsilon+\ddot H/(2\dot H)]\langle\varphi^2\rangle^{IR}$, where the second term is of order $\mathcal{O}(\epsilon^2)$.

$$2\alpha^{(2)}(3H^2+\dot H)\simeq -6\frac{\dot H}{M_{\rm pl}^2}\varphi^2+\frac{1}{M_{\rm pl}^2}\left[-\dot\phi\dot\varphi^{(2)}-V_\phi\varphi^{(2)}-\frac{1}{2}V_{\phi\phi}\varphi^2\right]. \tag{55}$$

The expression we want is $\langle\varphi^{(2)}\rangle$, which is averaged over the initial vacuum state, and can be obtained by using the leading contribution in Eq. (45) to the quadratic correlator of the first order fluctuations $\langle\varphi^2\rangle_{\rm REN}$. We obtain, directly from the last equation,

$$\langle\varphi^{(2)}\rangle\simeq\frac{\hbar}{24\pi^2}\dot\phi\frac{H_0^2m^2}{M_{\rm pl}^2\epsilon_0^2}\left[\frac{1}{4H^3}+\frac{1}{2HH_0^2}\ln\frac{H}{M}\right] \tag{56}$$

where M is a constant with the dimensions of a mass. It is also convenient to exhibit a slightly different form, which is of course equivalent in the large a limit at the end of inflation since the infrared contributions are then a maximum.

Since $\dot\phi$ is almost constant, one may also obtain, from Eq. (54),

$$\langle\varphi^{(2)}\rangle^{\rm MAX}\simeq\frac{\dot\phi}{4HM_{\rm pl}^2}\langle\varphi^2\rangle_{\rm REN}^{\rm MAX}. \tag{57}$$

It is also useful to write the contribution to $\langle\alpha^{(2)}\rangle$ which we shall later use to study the back-reaction effects on some scalar observables. One obtains, for its leading behavior,

$$\langle\alpha^{(2)}\rangle^{\rm MAX}\simeq\frac{\dot\phi}{2HM_{\rm pl}^2}\langle\varphi^{(2)}\rangle^{\rm MAX}-\frac{3}{4}\frac{1}{M_{\rm pl}^2}\frac{\dot H}{H^2}\langle\varphi^2\rangle_{\rm REN}^{\rm MAX}\simeq\frac{\epsilon}{M_{\rm pl}^2}\langle\varphi^2\rangle_{\rm REN}^{\rm MAX}. \tag{58}$$

VIII. APPROACHES TO THE BACK REACTION

One may follow different approaches in order to study the back-reaction effects due to cosmological fluctuations. In the following we shall consider two methods in order to tackle this issue.

One consists of considering only the first order perturbations, then imposing to first order the energy and momentum constraints and finally defining an effective EMT by including all the quadratic terms present in the Einstein equations. Subsequently one averages everything over the quantum vacuum and first order quantities disappear. One finds that the effective EMT which appears in the averaged Einstein equations is modified by the back reaction. The results obtained in Secs. II, V and VI are sufficient for this purpose. However, one still has to study how any observable averaged over the quantum vacuum is then affected.

A second approach is related to the standard perturbation analysis of the Einstein equations up to second order. In this case we impose the energy and momentum constraints and study the inflaton equation of motion perturbatively up to second order. One does not define any modified EMT but directly studies in this framework any observable averaged over the quantum vacuum. In order to pursue this approach results given in Secs. III, IV and VII are also necessary.

A. Energy-momentum tensor of cosmological fluctuations

By the EMT of cosmological fluctuations we mean the second order part both of the scalar field EMT *and* of the Einstein tensor which is quadratic in first order fluctuations:

$$\tau^\mu_\nu\equiv T^\mu_\nu{}^{\rm quadratic}-M_{\rm pl}^2G^\mu_\nu{}^{\rm quadratic}. \tag{59}$$

This method of considering the EMT of gravitational fluctuations is treated in textbooks [33] and has also been previously used in [7,9,13]. In this scheme one considers the modified Einstein equations $M_{\rm pl}^2G^{\mu(0)}_\nu=T^{\mu(0)}_\nu+\langle\tau^\mu_\nu\rangle$ which therefore include back-reaction effects.

Let us consider the EMT of cosmological fluctuations averaged over the state annihilated by the operator $\hat b$ defined in Eq. (39).

For a generic potential the leading terms in the energy density are

$$\langle\tau^0_0\rangle\sim-\frac{V_{\phi\phi}}{2}\langle\varphi^2\rangle+12H^2M_{\rm pl}^2\langle\alpha^2\rangle=-\frac{V_{\phi\phi}}{2}\langle\varphi^2\rangle-6\dot H\langle\varphi^2\rangle. \tag{60}$$

It is important to note that the second term on the right hand side is in general *positive* and *larger* than the first during inflation. This second term is the contribution of metric perturbations. On using the slow-rollover parameters ϵ and η $(\equiv M_{\rm pl}^2V_{\phi\phi}/V)$ we can rewrite the leading terms in Eq. (60) during slow rollover as

$$\langle\tau^0_0\rangle\equiv-\varepsilon\sim-\frac{V_{\phi\phi}}{2}\langle\varphi^2\rangle\left(1-4\frac{\epsilon}{\eta}\right). \tag{61}$$

Analogously the average pressure is

$$\langle\tau^i_j\rangle\equiv p\,\delta^i_j\sim\delta^i_j\left(-\frac{V_{\phi\phi}}{2}\langle\varphi^2\rangle+12H^2M_{\rm pl}^2\langle\alpha^2\rangle\right)\sim-\varepsilon\,\delta^i_j. \tag{62}$$

We now restrict the analysis to a quadratic potential with $\eta\simeq\epsilon$. On using Eq. (45) the leading behavior for the initial time of the energy and pressure density is

$$\varepsilon_{\rm REN}\sim-p_{\rm REN}\sim-\frac{3}{2}m^2\langle\varphi^2\rangle_{\rm REN}\sim-\frac{3\hbar}{8\pi^2}m^2H^2\left(\frac{H_0}{H}\right)^4\ln a=-\frac{9\hbar}{8\pi^2}\epsilon H_0^4\ln a. \tag{63}$$

Further, on using Eq. (48), we have the maximum value

$$\varepsilon_{\rm REN}^{\rm MAX}\sim-p_{\rm REN}^{\rm MAX}\sim-\frac{3\hbar}{16\pi^2}\frac{H_0^6}{H^2}. \tag{64}$$

We then obtain that for

$$H_0 \sim (16\pi^2/\hbar)^{1/6} H^{2/3} M_{\rm pl}^{1/3} \tag{65}$$

the EMT tensor of cosmological fluctuations cannot be neglected with respect to the energy content of the homogeneous classical inflaton.

As a final remark we note that the amplitude and time dependence of the EMT depends on the approximation used for linear perturbations. The slightly different results in the amplitude of the EMT with respect to [7,9] are explained in Appendix C.

B. Back reaction on the geometry

In the perturbative approach to the Einstein equations any back-reaction effect is analyzed by evaluating perturbatively quantities which characterize the geometry.

The expansion scalar is defined as

$$\Theta = \nabla_\mu u^\mu, \tag{66}$$

where u^μ is a four-vector field defining the comoving frame. The four-vector is normalized, $u_\mu u^\mu = -1$. To second order the four vector is

$$u^0 = 1 - \alpha + \frac{3}{2}\alpha^2 - \alpha^{(2)}, \quad u^i = \frac{\beta^{,i} + \beta^{(2),i}}{a} \tag{67}$$

and the expansion scalar reads

$$\begin{aligned}\Theta &= \Theta^{(0)} + \Theta^{(1)} + \Theta^{(2)}\\ &= 3H - 3H\alpha + \frac{1}{a}\nabla^2\beta - 3H\alpha^{(2)} + \frac{1}{a}\nabla^2\beta^{(2)} + \frac{9}{2}H\alpha^2\\ &\quad + \frac{1}{a}\vec{\nabla}\alpha\cdot\vec{\nabla}\beta + \frac{1}{4}\vec{\nabla}\beta\cdot\vec{\nabla}\dot{\beta}.\end{aligned} \tag{68}$$

The leading terms, once we average, are

$$\begin{aligned}\langle\Theta\rangle &= \Theta^{(0)} + \langle\Theta^{(2)}\rangle\\ &= 3H\left(1 + \frac{3}{2}\langle\alpha^2\rangle - \langle\alpha^{(2)}\rangle\right)\\ &\simeq 3H\left(1 - \epsilon\frac{\langle\varphi^2\rangle^{\rm MAX}}{4M_{\rm pl}^2}\right)\end{aligned} \tag{69}$$

where the results given in Eqs. (45) and (58) have been used, together with the relations $\alpha = \dot{\phi}/(2HM_{\rm pl}^2)\varphi$ and $\dot{\phi}^2 = -2\dot{H}M_{\rm pl}^2$, in order to calculate the only two non-negligible second order contributions. We note that by leading terms we mean the first corrections in ϵ which are also leading in $\log a$, thus essentially including the infrared contribution present in the renormalized quantities, as already discussed. We observe that the result in Eq. (69) is completely consistent with the considerations on the EMT made in the previous subsection. When the magnitude of the EMT of cosmological fluctuations is of the same order as that of the background $\langle\Theta\rangle$ changes significantly.

The result in Eq. (69) is in contrast with [12], where a vanishing result was obtained for the back reaction on the expansion rate $\langle\Theta\rangle$ and where only first order classical perturbations are considered. We note that also in our gauge choice Θ is related to the Einstein tensor for long wavelengths and leading order in ϵ,

$$\Theta \simeq 3H\left(1 - \alpha - \alpha^{(2)} + \frac{3}{2}\alpha^2\right) \simeq \sqrt{-3G^0_0}, \tag{70}$$

and therefore the result in Eq. (69) could also be obtained by expanding $\sqrt{-24\pi G T^0_0}$ to second order and averaging over the quantum vacuum. The difference with respect to [12] may be due to the absence of second order fluctuations in [12] and/or to a peculiarity of inflation driven by a quadratic potential.

We now consider another geometric scalar observable which is associated with the rate of change of the expansion scalar: i.e.,

$$\Omega = u^\nu \nabla_\nu \nabla_\mu u^\mu. \tag{71}$$

Its expansion up to second order is given by

$$\begin{aligned}\Omega &= \Omega^{(0)} + \Omega^{(1)} + \Omega^{(2)}\\ &= 3\dot{H} - 6\dot{H}\alpha - 3H\dot{\alpha} - \frac{H}{a}\nabla^2\beta + \frac{1}{a}\nabla^2\dot{\beta} - 6\dot{H}\alpha^{(2)} - 3H\dot{\alpha}^{(2)}\\ &\quad - \frac{H}{a}\nabla^2\beta^{(2)} + \frac{1}{a}\nabla^2\dot{\beta}^{(2)} + 12\dot{H}\alpha^2 + 12H\alpha\dot{\alpha}\\ &\quad - 4\frac{H}{a}\vec{\nabla}\alpha\cdot\vec{\nabla}\beta + \frac{H}{a}\alpha\nabla^2\beta - \frac{1}{a}\alpha\nabla^2\dot{\beta} + \frac{1}{4}\frac{d}{dt}(\vec{\nabla}\beta\cdot\vec{\nabla}\dot{\beta})\\ &\quad + \frac{1}{a}\frac{d}{dt}(\vec{\nabla}\alpha\cdot\vec{\nabla}\beta) + \frac{1}{a^2}\vec{\nabla}\beta\cdot\vec{\nabla}(\nabla^2\beta).\end{aligned} \tag{72}$$

For the homogeneous case $\Theta = 3H$ and therefore $\Omega = 3\dot{H}$. During slow rollover for a massive inflaton $\dot{H}$ ($\simeq -m^2/3$) is constant in time, and therefore is a gauge-invariant quantity up to second order [6,34].

The leading terms, once we average, are

$$\begin{aligned}\langle\Omega\rangle &= \Omega^{(0)} + \langle\Omega^{(2)}\rangle\\ &= 3\dot{H} - 6\dot{H}\langle\alpha^{(2)}\rangle - 3H\langle\dot{\alpha}^{(2)}\rangle + 12\dot{H}\langle\alpha^2\rangle + 12H\langle\alpha\dot{\alpha}\rangle\\ &\simeq 3\dot{H}[1 + F(\epsilon, \log a)]\end{aligned} \tag{73}$$

where $F(\epsilon, \log a) \simeq \epsilon F_1(H) + \epsilon^2 \log a F_2(H) + \cdots$. Therefore one observes that in this case the leading corrections to order ϵ which come from the infrared region cancel, while these were present in $\langle\Theta\rangle$. Let us again note that at the end of inflation driven by a massive inflaton $\log a \simeq 1/(2\epsilon_0)$ so that any $\log a$ correction effectively reduces the perturbative order in powers of ϵ of the correction.

The result on $\langle\Omega\rangle$ should be interpreted with care since we are not able to go to higher order self-consistently within our approximation. We also note that this vanishing result may also be a peculiarity of inflation driven by a quadratic potential. In any case the rate of change of the Hubble parameter can also be studied more carefully at the background level, without considering any back reaction. Including the first non-trivial corrections one finds

$$\dot{H} \simeq -\frac{m^2}{3}\left(1-\frac{1}{9}\frac{m^2}{H^2}\right). \tag{74}$$

To conclude this section we stress that in both the results (63) and (69) the corrections to the background values, which grow with time, are $\mathcal{O}(\epsilon)$. In the de Sitter limit ($\epsilon=0$) these corrections vanish, consistently with the absence of physical scalar perturbations to first order for a universe driven by a cosmological constant.

IX. DISCUSSION AND CONCLUSIONS

The renormalized EMT of cosmological fluctuations during inflation with a quadratic potential is studied. With respect to our previous work [17] we have self-consistently taken into account the gravitational fluctuations accompanying the scalar field fluctuations.

We find that the renormalized EMT of cosmological fluctuations during slow rollover carries negative energy density and has a de Sitter–like equation of state to leading order. The negative sign for the renormalized energy density is due to the inclusion of gravitational fluctuations (see also [7,9] for the same claim regarding long-wavelength modes) and did not appear in the previous calculation in rigid space-time [17]. This effect is generally true for single field inflationary models with $\epsilon > \eta/4$, as is seen from Eq. (61).

The back-reaction problem was also treated up to second order in perturbation theory. For this purpose we extended the UCG gauge beyond linear order by taking into account vector and tensor perturbations to second order. The inclusion of vector and tensor perturbations to second order is important in order to obtain the correct equations for the second order scalar perturbations. We derived the equation for the second order scalar field fluctuations and approximatively solved the system of second order equations for single field inflation during slow rollover. We computed the expansion rate Θ and its coordinate invariant derivative Ω to second order, and we found that the expansion rate is decreased by the EMT of cosmological fluctuations, while Ω is not affected to leading order. A non-vanishing correction for Ω will probably appear in the next order.

We do not find evidence that the back reaction of cosmological fluctuations is a gauge artifact as is claimed in [10]. The second order corrections to the local expansion rate Θ are non-vanishing and are computed to lowest order in Eq. (69). We note that the back reaction of cosmological scalar fluctuations vanishes in the de Sitter limit ($\epsilon=0$), as it should. Since Θ is not a gauge-invariant quantity in the model under consideration [6,34], we also computed Θ in the gauge wherein the inflaton is a clock: such a check was done by a completely new calculation (illustrated in Appendix C) in order to also study the transformation properties of the metric elements. We confirm that back reaction is an $\mathcal{O}(\epsilon)$ effect, but its time behavior is different (there is no $\log a$ enhancement from the infrared region) because of the differing times in the two frames. In this new gauge calculations are also more involved since a higher accuracy in the solution for the Mukhanov variable would be required.

It is interesting to note that the inclusion of gravitational fluctuations systematically contributes negatively. To first order gravitational fluctuations contribute negatively to the effective mass for the fluctuations, and to second order carry negative energy density. The self-consistent inclusion of gravity also changes the stochastic picture in an inflationary background.

The approach we have used is mean field theory on a curved space-time. The procedure of averaging over a quantum state with a cutoff related to the patch which undergoes inflation can be seen as a spatial average over the particle horizon. This average is completely different from the average over the Hubble region during inflation used in the stochastic approach [31] and in which gravitational fluctuations were neglected. We think that the differences between the results of stochastic inflation and those found here are not just due to a different coarse graining, but are also due to our inclusion of gravitational fluctuations. We think that the relation between these two different approaches should be further investigated.

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APPENDIX A: COMPARISON WITH WKB METHODS

In this appendix we show how the procedure used in this paper to match solutions which are good approximations in different ranges of k space includes the recent application of the WKB method in the context of cosmological perturbations [35,36]. In this paper we have chosen a Bessel form in Eq. (33) for modes which are still inside the Hubble radius. The scaled equation ($\psi_k = a^{3/2}\varphi_k$) associated with Eq. (32) is given by

$$\ddot{\psi}_k + \left[\left(\frac{k}{aH\nu'}\right)^2 - 1\right] H^2 \nu'^2 \psi_k = 0, \tag{A1}$$

where $\nu' = \nu/\lambda$. The turning point of this equation is obtained for

$$a(t) = \frac{k}{H(t)\nu'(t)}. \tag{A2}$$

So, for our calculation, we can neglect the weak time dependence on the right side of Eq. (A2) with respect the exponential dependence on the left side and obtain the following result for the turning point:

$$t_k \simeq -\frac{H_0}{\dot{H}_0}\left(1-\sqrt{1+2\frac{\dot{H}_0}{H_0^2}\log\frac{\lambda_0 k}{H_0\nu_0}}\right) \simeq \frac{1}{H_0}\log\frac{\lambda_0 k}{H_0\nu_0}. \tag{A3}$$

The solution near the turning point of Eq. (32), on using the WKB method, is given by

$$\varphi_k = \frac{1}{a^{3/2}}[B_1 \mathrm{Ai}(\alpha_k^{1/3}(t-t_k)) + B_2 \mathrm{Bi}(\alpha_k^{1/3}(t-t_k))]. \tag{A4}$$

Around the turning point we can consider the following approximate expression [37] in Eq. (33):

$$H_\nu^{(1)}(x) = \frac{w}{\sqrt{3}} e^{i[\pi/6+\nu(w-w^3/3-\arctan w)]} H_{1/3}^{(1)}\left(\frac{\nu}{3}w^3\right) + \mathcal{O}\left(\frac{1}{\nu}\right) \tag{A5}$$

or

$$H_\nu^{(1)}(x) \approx \left(\frac{2}{\nu}\right)^{1/3} e^{i\nu(w-w^3/3-\arctan w)} \left\{ \mathrm{Ai}\left[-\left(\frac{\nu}{2}\right)^{2/3} w^2\right] - i\,\mathrm{Bi}\left[-\left(\frac{\nu}{2}\right)^{2/3} w^2\right]\right\}, \tag{A6}$$

where $w=(1/\nu)\sqrt{x^2-\nu^2}$. We have checked numerically that the above approximations are remarkably good. The Airy functions are precisely the functions used to match the WKB solution, Eq. (A4), inside and outside the Hubble radius [36]. In particular around the turning point the match is specified by

$$-\left(\frac{\nu(t_k)}{2}\right)^{2/3} w^2 \approx \alpha_k^{1/3}(t-t_k). \tag{A7}$$

Therefore, by choosing a Bessel form inside the Hubble radius one already has the correct form in the vicinity of the turning point. This cannot be achieved with the WKB approximation, since the WKB solution is singular in correspondence of the turning point.

APPENDIX B: THE FOURTH ORDER ADIABATIC EXPANSION

In order to remove the divergences which appear in the integrated quantities as poles in the Γ functions, we shall employ the method of *adiabatic subtraction* [38]. Such a method consists of replacing our functions with an expansion in powers of derivatives of the logarithm of the scale factor. This expansion coincides with the adiabatic expansion introduced by Lewis in [39] for a time dependent oscillator.

Usually it is more convenient to formulate the adiabatic expansion by using the modulus of mode functions $x_k = |\varphi_{\mathbf{k}}/\sqrt{2}|$ and the conformal time η [38] ($d\eta = dt/a$). We follow this procedure and write an expansion in derivatives with respect to the conformal time (denoted by a prime) for x_k. We then go back to cosmic time and insert the expansion in the expectation values we wish to compute. Adiabatic expansions in cosmic time and conformal time lead to equivalent results, because of the explicit covariance under time reparametrization [40].

The variable x_k satisfies the Pinney equation

$$\ddot{x}_k + 3H\dot{x}_k + \left[\frac{k^2}{a^2} + m^2 + 6\dot{H}\right] x_k = \frac{1}{a^6 x_k^3}. \tag{B1}$$

Following [17] we rewrite Eq. (B1) in conformal time in the following way:

$$(a x_k)'' + \Omega_k^2 (a x_k) = \frac{1}{(a x_k)^3} \tag{B2}$$

where

$$\Omega_k^2 = k^2 + m^2 a^2 - \frac{1}{6} a^2 \tilde{R} \tag{B3}$$

and $\tilde{R}$ is

$$\tilde{R} = R - 36\dot{H} \tag{B4}$$

with $R = 6a''/a^3$ the Ricci curvature. From Eqs. (B2), (B3), (B4) one obtains the expansion for x_k up to the fourth adiabatic order:

$$x_k^{(4)} = \frac{1}{a}\frac{1}{\Omega_k^{1/2}}\left(1 - \frac{1}{4}\epsilon_2 + \frac{5}{32}\epsilon_2^2 - \frac{1}{4}\epsilon_4\right) \tag{B5}$$

where Ω_k is defined in Eq. (B3) and ϵ_2, ϵ_4 are given by

$$\epsilon_2 = -\frac{1}{2}\frac{\Omega_k''}{\Omega_k^3} + \frac{3}{4}\frac{\Omega_k'^2}{\Omega_k^4}$$

$$\epsilon_4 = \frac{1}{4}\frac{\Omega_k'}{\Omega_k^3}\epsilon_2' - \frac{1}{4}\frac{1}{\Omega_k^2}\epsilon_2''. \tag{B6}$$

The solution in Eq. (B5) must be expanded again since $\tilde{R}$ is of adiabatic order 2. Therefore $x_k^{(4)}$ is

$$\begin{aligned} x_{k(4)} = \frac{1}{c^{1/2}}\frac{1}{\Sigma_k^{1/2}}\Bigg\{ & 1 + \frac{1}{4}c\frac{\tilde{R}}{6}\frac{1}{\Sigma_k^2} + \frac{5}{32}c^2\frac{\tilde{R}^2}{36}\frac{1}{\Sigma_k^4} \\ & + \frac{1}{16}\frac{1}{\Sigma_k^4}\left[c''\left(m^2 - \frac{\tilde{R}}{6}\right) - 2c'\frac{\tilde{R}'}{6} - c\frac{\tilde{R}''}{6}\right] \\ & - \frac{5}{64}\frac{1}{\Sigma_k^6}\left[c'^2 m^4 - 2c'^2 m^2 \frac{\tilde{R}}{6} - 2c' m^2 c\frac{\tilde{R}'}{6}\right] \\ & + \frac{9}{64}\frac{1}{\Sigma_k^6} c\frac{\tilde{R}}{6} c'' m^2 - \frac{65}{256}\frac{1}{\Sigma_k^8} c\frac{\tilde{R}}{6} c'^2 m^4 \\ & + \frac{5}{32}\epsilon_{2*}^2 - \frac{1}{4}\epsilon_{4*}\Bigg\} \end{aligned} \tag{B7}$$

where $c=a^2$ and

$$\Sigma_k=(k^2+a^2m^2)^{1/2}$$

$$\epsilon_{2*}=-\frac{1}{2}\frac{\Sigma_k''}{\Sigma_k^3}+\frac{3}{4}\frac{\Sigma_k'^2}{\Sigma_k^4}$$

$$\epsilon_{4*}=\frac{1}{4}\frac{\Sigma_k'}{\Sigma_k^3}\epsilon_2'-\frac{1}{4}\frac{1}{\Sigma_k^2}\epsilon_2''. \tag{B8}$$

APPENDIX C: COMPARISON WITH CALCULATIONS IN DIFFERENT GAUGES

In [7,9] a different procedure was followed. The calculation was performed in the Newtonian gauge [see Eq. (5)] and the regularization of the metric perturbation $\langle\Phi^2\rangle$ was considered. We can compare our results by using the relation between field fluctuations φ in the uniform curvature gauge and field fluctuations φ_N in the Newtonian gauge:

$$\varphi=\varphi_N+\frac{\dot\phi}{H}\Phi. \tag{C1}$$

Assuming that $\dot\Phi$ in negligible on large scales, we obtain [25]

$$\varphi\simeq\frac{\dot\phi}{H}\left(\frac{\epsilon-1}{\epsilon}\right)\Phi, \tag{C2}$$

which, on using the approximation used in [7,9,25], leads to

$$\varphi_k\simeq\frac{\dot\phi}{H}\left(\frac{\epsilon-1}{\epsilon}\right)k^{-3/2}\frac{m}{2\sqrt{6}M_{\rm pl}}\left[1-\frac{\log[k/(aH)]}{1+\log[a_{\rm end}/a]}\right], \tag{C3}$$

where $a_{\rm end}$ is the scale factor at the end of inflation. This approximation for the gauge-invariant field fluctuations is different from our Eq. (37).

After this paper appeared in the electronic form, it was suggested that corrections to $\langle\Theta\rangle$ should disappear in the gauge wherein the inflaton is a clock, homogeneous in space. Let us note that the inflaton can work as a clock only when it is not oscillating, as is the case in the slow-rollover regime.

In order to check this claim we performed a new calculation, since the gauge where the inflaton is homogeneous is different to the one used in this paper. The gauge in which the inflaton is unperturbed is described by a metric with three scalar degrees of freedom and, again, this is a choice which fixes the gauge completely because of the presence of the scalar field. As before we do not pay attention to the vector and tensor degrees of freedom and therefore we consider the following second order line element depending only on three scalars:

$$ds^2=-(1+2\tilde\alpha+2\tilde\alpha^{(2)})d\tilde t^2-a(\tilde\beta_{,i}+\tilde\beta^{(2)}_{,i})d\tilde t d\tilde x^i + a^2[\delta_{ij}(1-2\tilde\psi-2\tilde\psi^{(2)})]d\tilde x^i d\tilde x^j, \tag{C4}$$

where we have taken $\tilde a(\tilde t)=a(\tilde t)$, since in the perturbative approach $a(\tilde t)$ and $\tilde a(\tilde t)$ satisfy the same equation in t and $\tilde t$, respectively (however the scale factor is not observable for the case under consideration). We expect that $\tilde\psi$ will satisfy a dynamical equation of motion in this gauge.

Instead of solving the system in this gauge, we find solutions for the metric perturbations in Eq. (C4) by using its gauge relation with the metric in Eq. (13). An infinitesimal coordinate transformation up to second order [6],

$$x^\mu\to\tilde x^\mu=x^\mu+\epsilon^\mu_{(1)}+\frac{1}{2}(\epsilon^\mu_{(1),\nu}\epsilon^\nu_{(1)}+\epsilon^\mu_{(2)}) \tag{C5}$$

(where $\epsilon_{(1)}$ and $\epsilon_{(2)}$ are the coordinate changes to first an second order, respectively), induces the most generic change in a geometric object $T=T^{(0)}+T^{(1)}+T^{(2)}$:

$$T^{(1)}\to\tilde T^{(1)}=T^{(1)}-\mathcal{L}_{\epsilon_{(1)}}T^{(0)} \tag{C6}$$

$$T^{(2)}\to\tilde T^{(2)}=T^{(2)}-\mathcal{L}_{\epsilon_{(1)}}T^{(1)}+\frac{1}{2}(\mathcal{L}^2_{\epsilon_{(1)}}T_0-\mathcal{L}_{\epsilon_{(2)}}T_0). \tag{C7}$$

The time reparametrization which relates the two gauges can be found by imposing that the field perturbation be zero to first and second order in the metric (C4). According to Eqs. (C6), (C7) the field transforms as

$$\varphi\to\tilde\varphi=\varphi-\epsilon^0_{(1)}\dot\phi \tag{C8}$$

$$\varphi^{(2)}\to\tilde\varphi^{(2)}=\varphi^{(2)}-\epsilon^0_{(1)}\dot\varphi+\frac{1}{2}[\epsilon^0_{(1)}(\epsilon^0_{(1)}\dot\phi)^{\cdot}-\epsilon^0_{(2)}\dot\phi], \tag{C9}$$

which leads to

$$\epsilon^0_{(1)}=\frac{\varphi}{\dot\phi}=\frac{\zeta}{H} \tag{C10}$$

$$\epsilon^0_{(2)}=\frac{2}{\dot\phi}\varphi^{(2)}-\frac{1}{\dot\phi^2}\varphi\dot\varphi\simeq-\frac{\zeta\dot\zeta}{H^2}-\frac{1}{2}\frac{\ddot H}{\dot H H^2}\zeta^2, \tag{C11}$$

where we have used the gauge invariant curvature perturbation $\zeta=H\varphi/\dot\phi$, which is constant on large scales and the expression in Eq. (57). From the above we understand that we need a large time reparametrization in order to keep the scalar field homogeneous in space.

The above transformation, up to second order, is not the one required since it would lead to the presence of a scalar second order contribution in the traceless part of the metric which depends on a quadratic form in $\partial_i\beta$ and $\partial_i\epsilon^0_{(1)}$. In order to destroy this contribution a scalar second order transformation in the spatial part of the coordinates is also required, $\epsilon^i_{(2)}=\partial^i\epsilon^s_{(2)}$, while still having $\epsilon^i_{(1)}=0$. We find

$$\epsilon^s_{(2)} = \frac{1}{2a}\beta\epsilon^0_{(1)} - \frac{1}{2a^2}(\epsilon^0_{(1)})^2 + \frac{3}{2}\frac{\partial^i\partial^j}{(\nabla^2)^2}\left[\frac{1}{a^2}\epsilon^0_{(1)}D_{ij}\epsilon^0_{(1)} - \frac{1}{2a}(\epsilon^0_{(1)}D_{ij}\beta + \beta D_{ij}\epsilon^0_{(1)})\right], \tag{C12}$$

where $D_{ij} = \partial_i\partial_j - 1/3\nabla^2\delta_{ij}$.

On using the gauge transformation (C6) we have the following relation for the metric fluctuations to first order:

$$\tilde{\alpha} = \alpha - \dot{\epsilon}^0_{(1)}, \quad \tilde{\beta} = \beta - \frac{2}{a}\epsilon^0_{(1)}, \quad \tilde{\psi} = H\epsilon^0_{(1)}. \tag{C13}$$

On solving for the metric fluctuations in the UCG gauge, using Eqs. (8), (9) we have

$$\alpha = \epsilon\zeta, \quad \frac{\nabla^2\beta}{a} = 2\epsilon\dot{\zeta}, \tag{C14}$$

whereas in the unperturbed scalar field gauge we have

$$\tilde{\alpha} = -\frac{\dot{\zeta}}{H}, \quad \frac{\nabla^2\tilde{\beta}}{a} = 2\epsilon\dot{\zeta} - 2\frac{\nabla^2\zeta}{aH}, \quad \tilde{\psi} = \zeta. \tag{C15}$$

From the last relations we see that the cost of keeping the scalar field homogeneous in space is to squeeze a large fluctuation in the metric, i.e. $\tilde{\psi}$. In the UCG gauge metric fluctuations are suppressed with respect to ζ, i.e. $\mathcal{O}(\epsilon\dot{\zeta}, \epsilon\zeta)$, whereas in the gauge (C4) they are not. To second order we have

$$\tilde{\alpha}^{(2)} = \alpha^{(2)} - \epsilon^0_{(1)}\dot{\alpha} - 2\alpha\dot{\epsilon}^0_{(1)} - \frac{\dot{\epsilon}^0_{(2)}}{2} + \frac{1}{2}\epsilon^0_{(1)}\ddot{\epsilon}^0_{(1)} + \dot{\epsilon}^{02}_{(1)}$$
$$= \alpha^{(2)} - 2\epsilon^2\zeta^2 - \frac{\dot{\epsilon}^0_{(2)}}{2} + \frac{1}{2}\frac{\ddot{H}}{H^3}\zeta^2 + \frac{\dot{\zeta}^2}{H^2} + \frac{1}{2}\frac{\zeta\ddot{\zeta}}{H^2}, \tag{C16}$$

where, from Eq. (58), one has $\alpha^{(2)} \simeq 2\epsilon^2\zeta^2$. Moreover, we find for $\beta^{(2)}$ a transformation such that

$$\tilde{\beta}^{(2)} = \beta^{(2)} - \frac{1}{a}\epsilon^0_{(2)} + a\dot{\epsilon}^s_{(2)} + \frac{1}{2a}\frac{d}{dt}(\epsilon^0_{(1)})^2 + \frac{\partial^i}{\nabla^2}\left[\frac{2}{a}\partial_i\epsilon^0_{(1)}\dot{\epsilon}^0_{(1)} - \epsilon^0_{(1)}\partial_i\dot{\beta} - H\epsilon^0_{(1)}\partial_i\beta - \dot{\epsilon}^0_{(1)}\partial_i\beta - \frac{4}{a}\alpha\partial_i\epsilon^0_{(1)}\right]. \tag{C17}$$

Let us remember that we chose not to exhibit in detail the vector and tensor degrees of freedom, but it is straightforward to do so. It is suffices to say that to destroy the possible vector degrees of freedom arising in $\tilde{g}_{0i}$, a non-vanishing vector component in the coordinate transformation to second order, $\epsilon^{\perp i}_{(2)}$, has to be introduced, but this does not affect the scalars to second order, $\tilde{\alpha}^{(2)}$, $\tilde{\beta}^{(2)}$ and $\tilde{\psi}^{(2)}$.

Finally, for this last scalar we find

$$\tilde{\psi}^{(2)} = \frac{H}{2}\epsilon^0_{(2)} - \frac{H}{2}\epsilon^0_{(1)}\dot{\epsilon}^0_{(1)} - \frac{\epsilon^{02}_{(1)}}{2}(\dot{H} + 2H^2) + \frac{1}{6a^2}\partial^i\epsilon^0_{(1)}\partial_i\epsilon^0_{(1)} - \frac{1}{6a}\partial^i\beta\partial_i\epsilon^0_{(1)} + \frac{1}{6}\nabla^2\epsilon^s_{(2)}$$
$$= \frac{H}{2}\epsilon^0_{(2)} - \zeta^2 - \frac{1}{2H}\zeta\dot{\zeta} + \frac{1}{6a^2}\partial^i\epsilon^0_{(1)}\partial_i\epsilon^0_{(1)} - \frac{1}{6a}\partial^i\beta\partial_i\epsilon^0_{(1)} + \frac{1}{6}\nabla^2\epsilon^s_{(2)}. \tag{C18}$$

We are now ready to compute the expansion rate $\tilde{\Theta}$, which is a gauge dependent observable, in the unperturbed (uniform) field gauge (UFG), while neglecting vector and tensor contributions and only keeping terms up to second order:

$$\tilde{\Theta} = 3\tilde{H} - 3\tilde{H}\tilde{\alpha} + \frac{1}{a}\tilde{\nabla}^2\tilde{\beta} - 3\tilde{\psi}' - 3\tilde{H}\tilde{\alpha}^{(2)} + \frac{1}{a}\tilde{\nabla}^2\tilde{\beta}^{(2)} - 3\tilde{\psi}^{(2)\prime} + \frac{9}{2}\tilde{H}\tilde{\alpha}^2 + 3\tilde{\alpha}\tilde{\psi}' - 6\tilde{\psi}\tilde{\psi}' + \frac{1}{a}\vec{\tilde{\nabla}}\tilde{\alpha}\cdot\vec{\tilde{\nabla}}\tilde{\beta} - \frac{3}{a}\vec{\tilde{\nabla}}\tilde{\beta}\cdot\vec{\tilde{\nabla}}\tilde{\psi} + \frac{1}{4}\vec{\tilde{\nabla}}\tilde{\beta}\cdot\vec{\tilde{\nabla}}\tilde{\beta}'. \tag{C19}$$

Let us observe that the prime denotes the derivative with respect to $\tilde{t}$, and the relation between the two time derivatives may be written as

$$\frac{\partial}{\partial\tilde{t}} = \left[1 - \frac{\dot{\zeta}}{H} - \epsilon\zeta + \left(-\frac{1}{2}\epsilon^2 - \frac{1}{4}\frac{\ddot{H}^2}{\dot{H}^2H^2} + \frac{1}{4\dot{H}H^2}\frac{d^3}{dt^3}(H)\right)\zeta^2 + \left(\frac{\epsilon}{H} + \frac{1}{2}\frac{\ddot{H}}{\dot{H}H^2}\right)\zeta\dot{\zeta} + \frac{1}{H^2}\dot{\zeta}^2\right]\frac{\partial}{\partial t} - \frac{1}{2}\partial_i\dot{\epsilon}^s_{(2)}\frac{\partial}{\partial x_i}. \tag{C20}$$

Clearly the two time derivatives coincide on acting on second order terms if one does not go beyond such an approximation; however, one must exercise care for terms of lower order on going from one description to the other.

Further the difference between the spatial derivatives is given by

$$\frac{\partial}{\partial\tilde{x}^i} = \frac{\partial}{\partial x^i} - \partial_i\epsilon^0_{(1)}\frac{\partial}{\partial t} \tag{C21}$$

which, however, is not important since we shall neglect spatial derivatives.

One may wish to write the corrections to the unperturbed $\tilde{\Theta} = 3\tilde{H}$ in the $\tilde{x}$ frame, while expressing the corrections in terms of the already known quantities (in the x frame). For this case one must be careful in analyzing to second order the fourth term of the above expression, $-3\partial\tilde{\psi}/\partial\tilde{t}$.

To leading order, on averaging and neglecting terms with spatial derivatives, we have

$$\langle\tilde{\Theta}\rangle = 3\tilde{H}\left[1-\langle\tilde{\alpha}^{(2)}\rangle+\frac{3}{2}\langle\tilde{\alpha}^2\rangle+\frac{1}{\tilde{H}}\langle\tilde{\alpha}\dot{\tilde{\psi}}\rangle-\frac{1}{\tilde{H}}\langle\dot{\tilde{\psi}}^{(2)}\rangle-\frac{2}{\tilde{H}}\langle\tilde{\psi}\dot{\tilde{\psi}}\rangle-\frac{1}{\tilde{H}}\left\langle\frac{\partial t}{\partial\tilde{t}}\dot{\tilde{\psi}}\right\rangle\right]$$

$$\simeq 3\tilde{H}\left[1+\frac{\epsilon}{H}\langle\zeta\dot{\zeta}\rangle+\frac{1}{H^2}\langle\dot{\zeta}^2\rangle-\frac{1}{4}\frac{\ddot{H}}{H^3}\langle\zeta^2\rangle\right]$$

$$=3\tilde{H}\left[1+\frac{1}{2M_{pl}^2}\left(-\frac{1}{H}\langle\varphi\dot{\varphi}\rangle-\frac{1}{\dot{H}}\langle\dot{\varphi}^2\rangle+\mathcal{O}(\epsilon^2\langle\varphi^2\rangle)\right)\right]=3\tilde{H}\left[1+\frac{1}{2M_{pl}^2}\mathcal{O}(\epsilon,\epsilon^2\langle\varphi^2\rangle)\right], \tag{C22}$$

where on going from the first to the second line the terms $\mathcal{O}(\zeta\dot{\zeta})$ cancel. The infrared contributions to the terms in the last two lines vanish, leading to corrections to the expansion rate in the UFG gauge of order $\mathcal{O}(\epsilon)$, which are of the same order as the neglected gradient terms. The corrections to the expansion rate in the two frames are therefore of the same order in the slow-roll parameters.

If we choose to rewrite the averaged expression for $\tilde{\Theta}$ in terms of t, starting from Eq. (C19)[6] and using the relation in Eq. (C20) (omitting negligible spatial derivatives) to also compute $\tilde{H}(\tilde{t})=H(t)\partial t/\partial\tilde{t}$, one finds

$$\langle\tilde{\Theta}\rangle = 3H\left(1-\epsilon\frac{\langle\varphi^2\rangle}{4M_{\rm pl}^2}\right), \tag{C23}$$

which is completely consistent with our result in Eq. (69) and with the fact that Θ is a scalar (but not gauge invariant), which, therefore, transforms according to

$$\tilde{\Theta}(\tilde{x}^\mu)=\Theta(x^\mu). \tag{C24}$$

We have then checked that the back reaction of scalar cosmological fluctuations is $\mathcal{O}(\epsilon)$ in the expansion rate: such a correction indeed vanishes in the de Sitter limit, where scalar perturbations are absent. One may find a suitable gauge—for instance the UFG, which is valid only during slow rollover—in which the back reaction does not grow in time (because of cancellations of the infrared pieces). However, such a gauge seems inconvenient since the inflaton cannot be used as a clock in the subsequent stage of coherent oscillations. We feel that it is more proper to ask what the time behavior of the back reaction is in a frame which can be regularly continued rather than the opposite. Indeed, one of the advantages of the UCG is the possibility of continuing the calculation through the oscillatory phase of the inflaton [41], which is not easy to do in the UFG. It would be interesting to check if the same conclusions are obtained for the non-local quantity used in [42].

[6]Since the term $-3\tilde{H}\tilde{\alpha}$ also has to be taken in to account. Nonetheless, one finds that such a term is, again, negligible and that the leading result can be obtained starting directly from Eq. (C22).

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