Tracking solutions in tachyon cosmology

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(Received 27 February 2004; published 1 June 2004)

We perform a thorough phase-plane analysis of the flow defined by the equations of motion of a FRW Universe filled with a tachyonic fluid plus a barotropic one. The tachyon potential is assumed to be of inverse square form, thus allowing for a two-dimensional autonomous system of equations. The Friedmann constraint, combined with a convenient choice of coordinates, renders the physical state compact. We find the fixed-point solutions, and discuss whether or not they represent attractors. The way the two fluids contribute at late times to the fractional energy density depends on how fast the barotropic fluid redshifts. If it does it fast enough, the tachyonic fluid takes over at late times, but if the opposite happens, the situation will not be completely dominated by the barotropic fluid; instead there will be a residual non-negligible contribution from the tachyon subject to restrictions coming from nucleosynthesis.

DOI: 10.1103/PhysRevD.69.123502 PACS number(s): 98.80.Cq

I. INTRODUCTION

A phase of accelerated inflation in the early stages of our Universe is favored by first-year Wilkinson Microwave Anisotropy Probe (WMAP) data [1]. Thus, it seems inflation is here to stay as the dominant paradigm for structure formation. The quest for a string theory motivated explanation of cosmological inflation has resulted in the emergence of the proposal of inflation driven by a tachyon field. The idea strongly relies on the possibility of describing tachyon condensates in terms of perfect fluids within string theories [2]. A plethora of papers studying cosmological consequences of such fluids have appeared since, some in the framework of general relativity [3–6], some others in the brane-world scenario [7,8].

As happens with standard scalar fields, one's favorite inflationary behavior is tailored by *ad hoc* choices of the initial conditions and the shape of the potential, but it is important to investigate up to what extent the features of the model depend on those choices. One way to address that problem is to consider tachyon field dynamics, because for a given potential such an analysis will provide us with constraints on the initial conditions.

The stability of tachyonic inflation against changes in initial conditions has been studied for an exponential potential [5] and for the inverse power-law potential [6]. Exact solutions for a purely tachyonic matter content with an inverse square potential are known [3], but no solutions exist for cases which combine tachyonic and barotropic fluids, so a dynamical systems approach may be relevant. Interestingly, the inverse square potential plays the same role for tachyon fields as the exponential potential [21] does for standard scalar fields [9–12]. On the one hand, those are the potentials that give power-law solutions. On the other hand, only those potentials allow constructing a two-dimensional autonomous system [10] using the evolution equations, whereas for any other potential the number of dimensions will be higher if the system is to remain autonomous.

Earlier results concerning the combination of a self-interacting tachyon T and a barotropic fluid were obtained in Ref. [6], where potentials of the form $V\propto T^{-\alpha}$ with $0<\alpha<2$ were studied. On the contrary, here we precisely address the $V\propto T^{-2}$ case (see also Ref. [13] for tachyonic fluids combined with quantum matter). The presence of the fluid brings in the crucial consequence of the appearance of fixed-point solutions in which the two fluids redshift at the same rate (tracking behavior [9]), so that there is some sort of equilibrium. Tracking solutions are particularly interesting because their dynamical effects mimic a decaying cosmological constant (see Refs. [10,11,14] for seminal references). Now, the fine-tuning problems posed by a cosmological constant would be waived precisely because of the independence on the initial conditions. Nevertheless, the contribution of such relics to the fractional energy density are bounded by nucleosynthesis.

On top of those interesting features, tracking solutions act as attractors at large, which means that the system does not care which the initial conditions are.

In Sec. II we study the phase plane, find its fixed points and characterize them. In Sec. III we discuss the cosmological consequences of the attractor solutions: in Sec. III A we consider tachyon dominated solutions, whereas in Sec. III B we discuss the tracking ones. Finally, in Sec. IV we outline our main conclusions and future prospects.

II. PHASE PLANE

The evolution equations for a flat $(k=0)$ Friedmann-Robertson-Walker (FRW) cosmological model filled with a tachyon field T evolving in a potential $V(T)$ and a barotropic perfect fluid with equation of state $p_\gamma=(\gamma-1)\rho_\gamma$ are

$$-2\dot{H}=\frac{V\dot{T}^2}{\sqrt{1-\dot{T}^2}}+\gamma\rho_\gamma, \tag{1}$$

$$\frac{\ddot{T}}{1-\dot{T}^2}+3H\dot{T}+\frac{V_{,T}}{V}=0, \tag{2}$$

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$$\dot{\rho}_\gamma + 3\gamma H \rho_\gamma = 0, \tag{3}$$

which are, in turn, subject to the Friedmann constraint

$$3H^2 = \frac{V}{\sqrt{1-\dot{T}^2}} + \rho_\gamma. \tag{4}$$

Here and throughout overdots denote differentiation with respect to cosmic time t, $H \equiv \dot{a}/a$ is the Hubble parameter, and a is the synchronous scale factor.

One can also define an energy density ρ_T and a pressure p_T for the tachyon, so that it can be thought of as a perfect fluid. We have then

$$\rho_T = \frac{V}{\sqrt{1-\dot{T}^2}}, \tag{5}$$

$$p_T = -V\sqrt{1-\dot{T}^2}. \tag{6}$$

If we use $\log a^3$ as independent variable instead of the cosmological time, for any time dependent function f we get

$$f' = \frac{\dot{f}}{3H}. \tag{7}$$

As usual, we also introduce convenient variables:

$$x \equiv \dot{T}, \tag{8}$$

$$y \equiv \frac{V}{3H^2}, \tag{9}$$

$$z \equiv \frac{\rho_\gamma}{3H^2}. \tag{10}$$

Let us concentrate now on the inverse square potential $V = \beta T^{-2}$. The evolution of the model is described by the dynamical system

$$x' = (x^2-1)(x - \sqrt{\alpha y}), \tag{11}$$

$$y' = y[x(x-\sqrt{\alpha y}) + z(\gamma - x^2)], \tag{12}$$

$$z' = z(z-1)(\gamma - x^2), \tag{13}$$

along with the constraint

$$\frac{y}{\sqrt{1-x^2}} + z = 1, \tag{14}$$

which renders the phase space two dimensional so that we may speak of phase plane. For the sake of simplicity, we are using the following definition:

$$\alpha \equiv \frac{4}{3\beta} > 0. \tag{15}$$

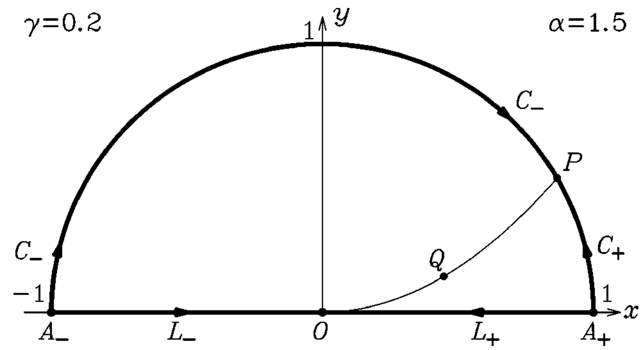

FIG. 1. Phase space and fixed points.

The physical constraints $V>0$ and $\rho_\gamma > 0$ set limits on the dependent variables: $-1 < x < 1$, $y > 0$ and $x^2 + y^2 \leqslant 1$. In consequence, the phase plane is restricted to the upper half of the unit disk centered at the origin, as depicted in Fig. 1. For convenience we will also include the whole segment $-1 \leqslant x \leqslant 1$, $y = 0$ in our phase plane.

In addition, one can define a barotropic index for the tachyon fluid:

$$\gamma_T \equiv \frac{\rho_T + p_T}{\rho_T}, \tag{16}$$

and, provided $V \neq 0$, one gets $\gamma_T = \dot{T}^2$.

The fixed points $(x^\star, y^\star)$ are given by the conditions

$$x'(x^*, y^*) = 0, \tag{17}$$

$$y'(x^*, y^*) = 0. \tag{18}$$

Depending on the values of γ and β there may be up to five fixed points (O, $A_\pm$, P, and Q) and up to six heteroclinic orbits that connect pairs of fixed points ($L_\pm$, $C_\pm$, $M_\pm$).

In order to analyze the stability of fixed points $(x^\star, y^\star)$ one studies the linearized dynamical system obtained by expanding Eqs. (11)–(12) about the fixed points (see, e.g., Ref. [16]). Then one tries solutions in the form $(x,y) = (b,c)e^{\lambda t}$ in the linear approximation, and finds that the characteristic exponent λ and the constant vector (b,c) must be respectively an eigenvalue and an eigenvector of the matrix

$$\left(\begin{array}{cc} \dfrac{\partial x'}{\partial x} & \dfrac{\partial x'}{\partial y} \\ \dfrac{\partial y'}{\partial x} & \dfrac{\partial y'}{\partial y} \end{array} \right)_{(x,y)=(x^*,y^*)} . \tag{19}$$

Clearly if both characteristic exponents have negative (positive) real parts, solutions near the fixed point will converge towards (move away from) it: the fixed point is asymptotically stable (unstable). In both cases, if the characteristic exponents are real, nearby solutions enter or exit the fixed point in the direction of the eigenvector with the eigenvalue of maximum real part: the fixed point is a node. If the characteristic exponents are complex conjugates the solutions near the fixed point move along spirals: the fixed point is a spiral point. If the signs of the two characteristic exponents

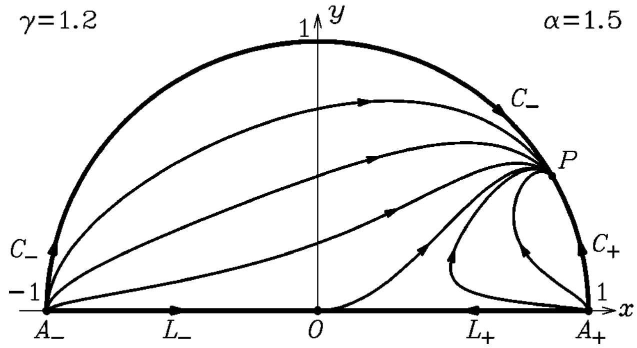

FIG. 2. Phase space for $\alpha=1.5$ and $\gamma=1.2$.

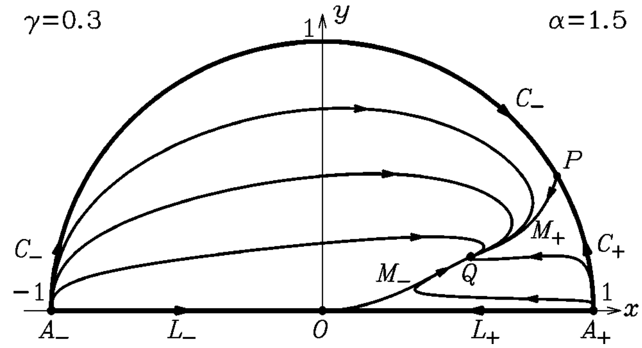

FIG. 3. Phase space for $\alpha=1.5$ and $\gamma=0.3$.

are different, the fixed point is an unstable saddle: most solutions will move away from it, but, apart from the fixed point itself, there is a set of initial conditions which lead to solutions converging towards the fixed point: the stable manifold, which is tangent at the fixed point to the stable space of eigenvectors corresponding to the negative characteristic exponents. There is also a set of points in the solutions that exit from the fixed point: the unstable manifold, which is tangent at the fixed point to the unstable space of eigenvectors corresponding to the positive characteristic exponent.

The fixed point O located at the origin $(x,y)=(0,0)$ corresponds to $z=1$ and is an unstable saddle (except in the very particular case in which $\gamma=0$, which will be discussed below). The orbits $L_\pm$ in the stable manifold correspond to the characteristic exponent $\lambda_1=-1$ while the exponent for the unstable space is $\lambda_2=\gamma/2$.

The fractional densities of the two fluids are respectively defined as

$$\Omega_\gamma \equiv \frac{\rho_\gamma}{3H^2}=z, \tag{20}$$

$$\Omega_T \equiv \frac{\rho_T}{3H^2}=\frac{y}{\sqrt{1-x^2}}. \tag{21}$$

The fixed points $A_\pm$ located at $(x,y)=(\pm 1,0)$ correspond to $z=0$, and are unstable nodes. The orbits $L_\pm$ in the unstable manifold correspond to the characteristic exponent $\lambda_1=2$ and orbits $C_\pm$ to $\lambda_2=\gamma/2$. The fixed point P at $(x,y)=(\sqrt{\alpha y_1},y_1)$, with

$$y_1 \equiv \frac{\sqrt{\alpha^2+4}-\alpha}{2}, \tag{22}$$

always admits the arcs $C_\pm$ as the orbits in its stable manifold corresponding to the characteristic exponent $\lambda_1=-1+\alpha y_1/2<0$, while the second exponent is $\lambda_2=\alpha y_1-\gamma$. In consequence, P is an asymptotically stable node for $\gamma>\gamma_1\equiv\alpha y_1$, in which case the phase space looks as depicted in Fig. 2: P is a global attractor, nearly all solutions end there.

When $\gamma=\gamma_1$ a bifurcation arises: P turns into an unstable saddle and at the same time there appears a new attractor Q, which moves from P to O along an arc of the parabola $(x,y)=(\sqrt{\gamma},\gamma/\alpha)$ as γ decreases from γ_1 to 0 (see Fig. 1). The characteristic exponents are

$$\lambda=\frac{\alpha(\gamma-2)\pm\sqrt{16\alpha\gamma^2\sqrt{1-\gamma}+\alpha^2(4-20\gamma+17\gamma^2)}}{4\alpha}, \tag{23}$$

so that $\Re\alpha<0$ for all $0\leqslant\gamma<\gamma_1$, Q is always asymptotically stable and is a node (spiral point) when the argument of the square root is positive (negative) (see Table I). A particular case is shown in Fig. 3.

In the limit case in which $\gamma=0$, fixed points Q and O coincide and are the attractor in the system, as depicted in Fig. 4.

TABLE I. The properties of the critical points.

Name	x	y	Existence	Stability	Ω_T	γ_T
O	0	0	All γ and β	Unstable saddle for $\gamma\neq 0$ Stable node for $\gamma=0$	0	Undefined
A_+	1	0	All γ and β	Unstable node	1	1
A_-	-1	0	All γ and β	Unstable node	1	1
P	$\sqrt{\alpha y_1}$	y_1	All γ and β	Stable node for $\gamma<\gamma_1$ Unstable saddle for $\gamma\geqslant\gamma_1$	1	αy_1
Q	$\sqrt{\gamma}$	$\frac{\gamma}{\alpha}$	$\gamma<\gamma_1$	Stable node	$\frac{\gamma}{\alpha\sqrt{1-\gamma}}$	γ

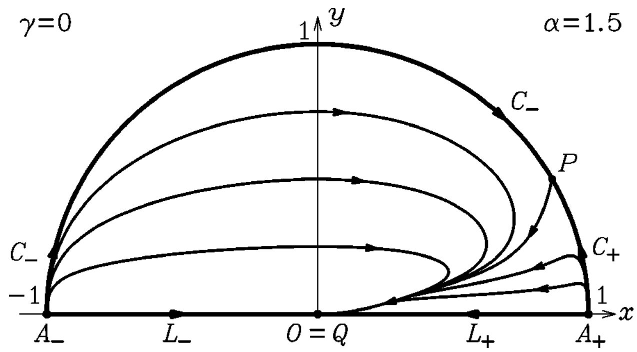

FIG. 4. Phase space for $\alpha=1.5$ and $\gamma=0$.

III. COSMOLOGICAL FEATURES OF THE ATTRACTOR SOLUTIONS

Now we turn our attention to the effective equation for the tachyon fluid in the attractor solutions, so that we can discuss in broad terms what sort of evolution they give rise to.

A. Tachyon dominated solutions

The attractor solutions $(x,y)=(\sqrt{\alpha y_1},y_1)$ depict a situation in which the energy density of the fluid vanishes, so it will be referred to as the tachyon dominated solution. It is straightforward to see that it corresponds to

$$\gamma_T=\alpha y_1 \tag{24}$$

and

$$T=\sqrt{\alpha y_1}t+T_0, \tag{25}$$

with T_0 an arbitrary integration constant. For the scale factor a we can set

$$a\propto t^{2/3\gamma_T}, \tag{26}$$

where the value of an integration constant has been fixed so that $\lim_{t\to 0}a=0$.

The solution will be inflationary if $\rho_T+3p_T<0$, and in terms of the tachyon field such condition is equivalent to

$$\dot{T}^2<\frac{2}{3}, \tag{27}$$

which holds provided $\beta>2/\sqrt{3}$.

Recently, tachyonic inflation has been cast in doubt [15], but it seems that the situation is not so clear cut because potentials have been found which seem to circumvent the problem [7].

B. Tracking solutions

We move on now to the most interesting case, which is represented by the attractor solutions with $(x,y)=(\sqrt{\gamma},\gamma/\alpha)$. They satisfy

$$\Omega_T=1-\Omega_\gamma=\frac{\gamma}{\alpha\sqrt{1-\gamma}}, \tag{28}$$

so that the energy density of the tachyon and of the fluid scale exactly as the same power of the scale factor, namely $\rho_\gamma\propto\rho_T\propto a^{-2/3\gamma}$. For that reason, according to the definition by Liddle and Scherrer [9], these solutions display a tracking behavior. Now, as in the case of the tachyon dominated solutions the expansion factor obeys a power law, $a\propto t^{2/3\gamma}$, and inflation will proceed if $\gamma<\min\{2/3,\alpha y_1\}$.

Nevertheless, apart from the considerations above, there exist restrictions on the values of α (or β) that come from observations. Using standard nucleosynthesis and the observed abundances of primordial nuclides, the strong constraint that the fractional energy density of scalar matter cannot exceed 0.05 at temperatures near 1 MeV was set [17]. If we restrict the discussion to tracking inflationary solutions ($\gamma<2/3$), and set that $\Omega_T<0.05$, we see that $\alpha>23.09$ is required. Such a bound, though, can be evaded if the solution begins its history away from the tracking solution and in a region where $\Omega_T\ll 1$ (close to O), and only reaches Q in the course of the evolution. In addition, one must take into account that our tracking solution requires $\gamma<1$, which is a condition known to be noncompatible with the equation of state of the Universe at epochs from redshifts of order ten billion to nearly the present [18].

IV. CONCLUSIONS

The evolution equations of a spatially flat FRW Universe containing a barotropic fluid plus a tachyon T with an inverse-square potential $V(T)=\beta T^{-2}$ define a two-dimensional flow. The evolution of such models has been investigated by studying the orbits of that flow in the physical state, which is in this case a subset of the Euclidean plane. The Friedmann constraint, combined with a careful choice of coordinates, renders this subset compact.

We have shown that the energy density of the tachyon dominates at late times for $\gamma>\alpha(\sqrt{\alpha^2+4}-\alpha)/2$, where γ is the barotropic index of the fluid and $\alpha=4/3\beta$. In contrast, for $\gamma<\alpha(\sqrt{\alpha^2+4}-\alpha)/2$, the barotropic fluid does not dominate completely and the contribution of tachyonic energy density to the total one is not negligible. Nucleosynthesis imposes, then, tight bounds on the admissible values of α, but such restrictions can be relaxed if the locus of the initial solution is far from that of the tracking one and in a region where $\Omega_T\ll 1$ (close to O), and only reaches Q in the course of the evolution. Note also that our tracking solution will only exist if $\gamma<1$, but the barotropic index of the Universe does not satisfy that constraint between $z\sim 10^{10}$ and $z\sim 1$ [18].

Finally, a possible generalization of this work would be considering generalized tachyon cosmologies like those presented in Ref. [19].

ACKNOWLEDGMENTS

We thank B. Bassett and G. Shiu for comments. J.M.A. and R.L. are supported by the University of the Basque Country through research grant UPV00172.310-14456/2002. J.M.A. also acknowledges support from the Spanish Ministry of Science and Technology through research grant BFM2000-0018. R.L. is also supported by the Basque Government through grant BFI01.412, the Spanish Ministry of Science and Technology jointly with FEDER funds through research grant BFM2001-0988.

[1] C.L. Bennett *et al.*, Astrophys. J., Suppl. Ser. **148**, 1 (2003); E. Komatsu *et al.*, *ibid.* **148**, 119 (2003); G. Hinshaw *et al.*, *ibid.* **148**, 135 (2003); D.N. Spergel *et al.*, *ibid.* **148**, 175 (2003); H.V. Peiris *et al.*, *ibid.* **148**, 213 (2003).

[2] A. Mazumdar, S. Panda, and A. Pérez-Lorenzana, Nucl. Phys. **B614**, 101 (2001); A. Sen, J. High Energy Phys. **0204**, 048 (2002); **0207**, 065 (2002).

[3] T. Padmanabhan, Phys. Rev. D **66**, 021301 (2002); A. Feinstein, *ibid.* **66**, 063511 (2002).

[4] G.W. Gibbons, Phys. Lett. B **537**, 1 (2002); M. Fairbairn and M.H.G. Tytgat, *ibid.* **546**, 1 (2002); T. Padmanabhan and T.R. Choudhury, Phys. Rev. D **66**, 081301(R) (2002); M. Sami, P. Chingangbam, and T. Qureshi, *ibid.* **66**, 043530 (2002); J.C. Hwang and H. Noh, *ibid.* **66**, 084009 (2002); G. Shiu and I. Wasserman, Phys. Lett. B **541**, 6 (2002); J.G. Hao and X.Z. Li, Phys. Rev. D **66**, 087301 (2002); Y.-S. Piao, R.-G. Cai, X. Zhang, and Y.-Z. Zhang, *ibid.* **66**, 121301 (2002); B. Chen, M. Li, and F.-L. Lin, J. High Energy Phys. **0211**, 050 (2002); J.S. Bagla, H.K. Jassal, and T. Padmanabhan, Phys. Rev. D **67**, 063504 (2003); X.Z. Li and X.H. Zhai, *ibid.* **67**, 067501 (2003); D.A. Steer and F. Vernizzi, hep-th/0310139; G.W. Gibbons, Class. Quantum Grav. **20**, S321 (2003); V. Gorini, A.Yu. Kamenshchik, U. Moschella, and V. Pasquier, hep-th/0311111, Phys. Rev. D (to be published).

[5] Z.-K. Guo, Y.-S. Piao, R.-G. Cai, and Y.-Z. Zhang, Phys. Rev. D **68**, 043508 (2003).

[6] L.R.W. Abramo and F. Finelli, Phys. Lett. B **575**, 165 (2003).

[7] S. Mukohyama, Phys. Rev. D **66**, 024009 (2002); M.C. Bento, O. Bertolami, and A.A. Sen, *ibid.* **67**, 063511 (2003).

[8] M. Sami, Mod. Phys. Lett. A **18**, 691 (2003); M.C. Bento, N.M.C. Santos, and A.A. Sen, astro-ph/0307292; G. Calcagni, Phys. Rev. D **69**, 103508 (2004).

[9] A.R. Liddle and R.J. Scherrer, Phys. Rev. D **59**, 023509 (1999).

[10] E.J. Copeland, A.R. Liddle, and D. Wands, Phys. Rev. D **57**, 4686 (1998).

[11] E.J. Copeland, A.R. Liddle, and D. Wands, Phys. Rev. D **57**, 4686 (1998); A.P. Billyard, A.A. Coley, and R.J. van den Hoogen, *ibid.* **58**, 123501 (1998); R.J. van den Hoogen, A.A. Coley, and D. Wands, Class. Quantum Grav. **16**, 1843 (1999); L.P. Chimento, A.S. Jakubi, and D. Pavón, Phys. Rev. D **62**, 063508 (2000).

[12] A. Nunes and J.P. Mimoso, Phys. Lett. B **488**, 423 (2000); A. Kehagias and G. Kofinas, gr-qc/0402059.

[13] S. Nojiri and S. Odintsov, Phys. Lett. B **571**, 1 (2003).

[14] P.J. Steinhardt, L. Wang, and I. Zlatev, Phys. Rev. D **59**, 123504 (1999); L. Amendola, *ibid.* **60**, 043501 (1999); J.-P. Uzan, *ibid.* **59**, 123510 (1999).

[15] A. Frolov, L. Kofman, and A. Starobinsky, Phys. Lett. B **545**, 8 (2002); L. Kofman and A. Linde, J. High Energy Phys. **0207**, 004 (2002).

[16] M. Hirsch and S. Smale, *Differential Equations*, *Dynamical Systems*, *and Linear Algebra*, 2nd ed. (Academic Press, New York, 1974).

[17] R. Bean, S.H. Hansen, and A. Melchiorri, Phys. Rev. D **64**, 103508 (2001).

[18] R.R. Caldwell, M. Doran, C.M. Mueller, G. Schaefer, and C. Wetterich, Astrophys. J. Lett. **591**, 75 (2003); J.P. Kneller and G. Steigman, Phys. Rev. D **67**, 063501 (2003).

[19] L.P. Chimento, astro-ph/0311613, Phys. Rev. D (to be published).

[20] L.P. Chimento, Class. Quantum Grav. **15**, 965 (1998).

[21] The general solution to the Einstein equations for a FRW spacetime with an exponential potential was given in Ref. [20].