Thermal heat kernel expansion and the one-loop effective action of QCD at finite temperature

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The heat kernel expansion for field theory at finite temperature is constructed. It is based on the imaginary time formalism and applies to generic Klein-Gordon operators in flat space-time. Full gauge invariance is manifest at each order of the expansion and the Polyakov loop plays an important role at any temperature. The expansion is explicitly worked out up to operators of dimension 6 included. The method is then applied to compute the one-loop effective action of QCD at finite temperature with massless quarks. The calculation is carried out within the background field method in the $\overline{\mathrm{MS}}$ scheme up to dimension-6 operators. Further, the action of the dimensionally reduced effective theory at high temperature is also computed to the same order. Existing calculations are reproduced and new results are obtained in the quark sector for which only partial results existed up to dimension 6.

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I. INTRODUCTION

The extension of field theory from zero to finite temperature and density is a natural step undertaken quite early [1–6]. The interest is both at a purely theoretical level and in the study of concrete physical theories. At the theoretical level one needs appropriate formulations of the thermal problem, for which there are several formalisms available [7], as well as mathematical tools to carry out the calculations. From the point of view of concrete theories a central point is the study of the different phases of the model and the nature of the phase transitions. That study applies not only to condensed matter theories but also to fundamental ones, such as the electroweak phase transition, of direct interest in early cosmology and baryogenesis [8], and quantum chromodynamics which displays a variety of phases in addition to the hadronic one [9–12]. Such new phases can presumably be probed at the laboratory in existing [BNL Relativistic Heavy Ion Collider (RHIC)] [13] and future (ALICE) facilities. Obviously one expects all these features of QCD at finite temperature to be fully consistent with manifest gauge invariance. As is well known Lorentz invariance is manifestly broken due to the privileged choice of the reference frame at rest with the heat bath; however, gauge invariance remains an exact symmetry. At zero temperature preservation of gauge invariance involves mixing of finite orders in perturbation theory. As will become clear below, compliance with gauge invariance requires mixing of infinite orders in perturbation theory at finite temperature.

The purpose of the present work is twofold. The first part (Sec. II) is devoted to introduce a systematic expansion for the one-loop effective action of generic gauge theories at finite temperature in such a way that gauge invariance is manifest at each order. In the second part this technique is applied to QCD in the high-temperature regime, first to compute its one-loop gluon and quark effective action (Sec. III) and then to derive the Lagrangian of the dimensionally reduced effective theory (Sec. IV). Further applications can and will be considered in other cases of interest [14].

The effective action, an extension to quantum field theory of the thermodynamical potentials of statistical mechanics, plays a prominent theoretical role, being directly related to quantities of physical interest. To one loop it takes the form $c\,\mathrm{Tr}\log(K)$, where K is the differential operator controlling the quadratic quantum fluctuations above a classical background. Unfortunately, this quantity is afflicted by mathematical pathologies, such as ultraviolet divergences or many-valuation (particularly in the fermionic case). For this reason, it has proved useful to express the effective action in terms of the diagonal matrix elements of the heat kernel (or simply the heat kernel, from now on) $\langle x|e^{-\tau K}|x\rangle$, by means of a proper time representation [see, e.g., Eq. (2.17) below] [15,16]. Unlike the one-loop effective action, the heat kernel is one-valued and ultraviolet finite for any positive proper time τ (we assume that the real part of K is positive). A further simplifying property is that, after computing the loop momentum integration implied by taking the diagonal matrix element, the result is independent of the space-time dimension, apart from a geometrical factor. In practice the computation of the heat kernel is through the so-called heat kernel expansion. This is an expansion which classifies the various contributions by their mass scale dimension, as carried by the background fields and their derivatives. This is equivalent to an expansion in the powers of the proper time τ. In this way the heat kernel is written as a sum of all local operators allowed by the symmetries with certain numerical coefficients known as Seeley-DeWitt or heat kernel coefficients. The perturbative and derivative expansions are two resummations of the heat kernel expansion. This expansion has been computed to high orders in flat and curved space-time in manifolds with or without boundary and in the presence of non-Abelian background fields [17–23].

In order to apply the heat kernel technique to the computation of the effective action at finite temperature it is necessary to extend the heat kernel expansion to the thermal case. This can be done within the imaginary time formalism, which amounts to a compactification of the Euclidean time

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coordinate. The space-time becomes a topological cylinder. (As usual in this context, we consider only flat space-times without boundary.) Now, the heat kernel describes how an initial Dirac delta function in the space-time manifold spreads out as the proper time passes, with the Klein-Gordon operator K acting as a Laplacian operator. As is known, the standard small-τ asymptotic expansion is insensitive to global properties of the space-time manifold. This means that the space-time compactification, and hence the temperature, will not be seen in the strict expansion in powers of τ. (As a consequence, the ultraviolet sector and hence the renormalization properties of the theory and the quantum anomalies are temperature independent, a well-known fact in finite-temperature field theory [24,25].) Within a path integral formulation of the propagation in proper time, this corresponds to an exponential suppression [namely, of order $e^{-\beta^2/4\tau}$; cf. Eq. (2.5)] of closed paths which wind around the space-time cylinder. The compactification is made manifest if instead of counting powers of τ one classifies the contributions by their mass dimension. The corresponding thermal Seeley-DeWitt coefficients will then be powers of τ but with exponentially suppressed τ-dependent corrections. As a result of the compactification, the new expansion will not be Lorentz invariant, although rotational invariance will be maintained. In addition, we find coefficients of half-integer order which at zero temperature can appear only for manifolds with boundary (as distributions with support on the boundary [26]). Such half-order terms vanish in a strict proper time expansion.

Another relevant issue is the preservation of gauge invariance. At zero temperature the only local gauge covariant quantities available are the matter fields, the field strength tensor and their covariant derivatives. However, at finite temperature there is a further gauge covariant quantity which plays a role: namely, the (untraced) thermal Wilson line or Polyakov loop. Since temperature effects in the imaginary time formalism come from the winding around the space-time cylinder, the Polyakov loop appears naturally in the thermal heat kernel. Our calculation, anticipated in [27], shows that the thermal heat kernel coefficients at a point x become functions of the untraced Polyakov loop that starts and ends at x. Although such a dependence is consistent with gauge invariance at finite temperature, it is not required by it either. Nevertheless, there is a simple argument which shows that the heat kernel expansion cannot be simply given by a sum of gauge covariant local operators (albeit with Lorentz symmetry broken down to rotational symmetry). For the Klein-Gordon operator describing a gas of identical particles free from any external fields other than a chemical potential (plus a possible mass term) it is obvious that such a chemical potential (which can be regarded as a constant c-number scalar potential A_0) has no effect through the covariant derivatives, and so it is invisible in the gauge covariant local operators. However, it is visible in the Polyakov loop, and it is only in this way that the effective action, or the grand-canonical potential, and hence the particle density, can depend on the chemical potential. The dependence of the thermal heat kernel coefficients on the Polyakov loop was overlooked in previous calculations [28,29], although it was made manifest in particular cases and configurations in [30]. Of course, the relevance of the Polyakov loop is well known in quarkless QCD at high temperature, where it is the order parameter signaling the presence of a deconfining phase [5]. The determination of the effective action of the Polyakov loop after integration of all others degrees of freedom has been pursued, e.g., in [31]. Our results imply that, because the formulas are quite general and should hold for any gauge group, the Polyakov loop must be accounted for, not only in the color degrees of freedom and at high temperature, but also in other cases such as the chiral flavor group with vector and axial-vector couplings and at any finite temperature [14]. The thermal heat kernel expansion is derived in Sec. II.

In Sec. III we apply the previous technique to the computation of the effective action of QCD at finite temperature to one loop. Here we refer to the effective action in the technical sense of generating function of one-particle irreducible diagrams. For the quark sector (we consider massless quarks for simplicity) the method applies directly by taking as Klein-Gordon operator the square of the Dirac operator and using an integral representation for the fermionic determinant. In the gluon sector, the fluctuation operator is of the Klein-Gordon type in the Feynman gauge, and so the technique applies too, but this time in the adjoint representation of the gauge group and including the ghost determinant. The calculation is carried out using the covariant background field method. To treat ultraviolet divergences dimensional regularization is applied, plus the modified minimal subtraction ($\overline{\text{MS}}$) scheme. We have also made the calculation using the Pauli-Villars scheme as a check. In this computation the background gauge fields are not stationary, and this allows us to write expressions which are manifestly invariant under all gauge transformations (recall that in the time-compactified space-time there are topologically large gauge transformations [32]). The result is expressed using gauge invariant local operators, including operators of up to dimension 6, and the Polyakov loop $\Omega(x)$. This is done for arbitrary SU(N) (N being the number of colors). For SU(2) and SU(3) the traces on the color group are worked out, to dimension 6 for SU(2) and to dimension 4 for SU(3). In our expansion the dependence on the Polyakov loop is treated exactly [we keep all orders in an expansion in powers of $\log(\Omega)$] but the expansion in covariant derivatives is truncated without spoiling gauge invariance at finite temperature. In particular the time covariant derivative is not kept to all orders. This is probably the best one can do for nonstationary backgrounds and general gauge groups. If one considers particular gauge groups and stationary backgrounds, one still has to truncate the expansion in the spatial covariant derivatives, but it is possible to add all orders in the temporal gluon component. This is the viewpoint adopted in the recent work [33,34] for SU(2) as a color gauge group. The calculation presented here and that of [33,34] are in a sense complementary, since neither of them can be deduced from each other; i.e., we find terms of the effective action functional which are missed by the stationarity condition, and there are terms of higher order in $A_0(x)$ which are not kept at a finite order of our expansion. Nevertheless, there are terms which can be compared in both approaches (see Sec. III).

As is known, the effective action of perturbative QCD at finite temperature contains infrared divergences due to the massless gluons in the chromomagnetic sector [35,36]. Such divergences come from stationary quantum fluctuations which are light even at high temperature, whereas the nonstationary modes become heavy, with an effective mass of the order of the temperature T, from the Matsubara frequency. So the procedure which has been devised to avoid the infrared problem is to integrate out the heavy, nonstationary modes to yield the action of an effective theory for the stationary modes—i.e., of gluons in three Euclidean dimensions [10,11,37–42]. In this way one obtains a dimensionally reduced theory $\mathcal{L}_{3D}$. (One can go further and integrate out the chromoelectric gluons which become massive through the Debye mechanism. We do not consider such further reduction here.) By construction, $\mathcal{L}_{3D}$ reproduces the static Green functions of the four-dimensional theory $\mathcal{L}_{4D}$. Of course, the infrared divergences will reappear now if this action is used in perturbation theory. However, residing in a lower dimension, $\mathcal{L}_{3D}$ is better behaved in the ultraviolet and also more amenable to nonperturbative techniques, such as lattice gauge theory. The parameters of $\mathcal{L}_{3D}$ (masses, coupling constants) can be computed in standard perturbative QCD since they are infrared finite, coming from integration of the heavy nonstationary modes, although they are scale dependent due to the standard ultraviolet divergences of four-dimensional QCD. Section IV is devoted to obtaining the action of the reduced theory. This is easily done from the calculation of the effective action in Sec. III by removing the static Matsubara mode in the gluonic loop integrations. This theory inherits the gauge invariance under stationary gauge transformations of the four-dimensional theory, but a larger gauge invariance is no longer an issue since more general gauge transformations would not preserve the stationarity of the fields. In addition, at high temperature fluctuations of the Polyakov loop far from unity (or from a center of the gauge group element in the quarkless case) are suppressed and so it is natural to expand the action in powers of A_0. We obtain the action up to operators of dimension 6 included (counting each gluon field as mass dimension 1) and compare with existing calculations to the same order quoted in the literature [10,11,40,43–45]. The relevant scales $\Lambda^T_{M,E}$ for the running coupling constant in the high-temperature regime are identified and reproduced [44]. For the dimension-6 terms, in the gluon sector we find agreement with [43] if the Polyakov loop is expanded in perturbation theory and in the quark sector we reproduce the results of [45] for the particular case considered there (no chromomagnetic gluons and no more than two spatial derivatives). We give the general result for SU(N) and simpler expressions for the cases of SU(2) and SU(3).

The heat kernel and the QCD parts of the paper may interest different audiences, the first one being more methodological and the second one more phenomenological, and to some extent they can be read independently. The QCD part does not require all the details of the derivation of the thermal heat kernel expansion but only the final formulas. In fact, one of the points of this paper is that the thermal coefficients need not be computed each time for each application, but only once, and then applied in a variety of situations.

II. HEAT KERNEL EXPANSION AT FINITE TEMPERATURE

A. Polyakov loop and the heat kernel

We will consider Klein-Gordon operators of the form

$$K=M(x)-D_\mu^2, \quad D_\mu=\partial_\mu+A_\mu(x). \tag{2.1}$$

$M(x)$ is a scalar field which is a Hermitian matrix in internal space (gauge group space), and the gauge fields $A_\mu(x)$ are anti-Hermitian matrices. K acts on the particle wave function in $d+1$ Euclidean dimensions and in the fundamental representation of the gauge group. At finite temperature in the imaginary time formalism the time coordinate is compactified to a circle; i.e., the space-time has topology $\mathcal{M}_{d+1}=S^1\times\mathcal{M}_d$. Correspondingly, the wave functions are periodic in the bosonic case, with period β (the inverse temperature), antiperiodic in the fermionic case, and the external fields M,A_μ are periodic.

In order to obtain the heat kernel $\langle x|e^{-\tau K}|x\rangle$ (a matrix in internal space) we use the symbols method, extended to finite temperature in [46,47]: For an operator $\hat{f}=f(M,D_\mu)$ constructed out of M and D_μ,

$$\langle x|f(M,D_\mu)|x\rangle=\frac{1}{\beta}\sum_{p_0}\int\frac{d^dp}{(2\pi)^d}\langle x|f(M,D_\mu+ip_\mu)|0\rangle. \tag{2.2}$$

Here p_0 are the Matsubara frequencies, $2\pi n/\beta$ for bosons and $2\pi(n+\frac{1}{2})/\beta$ for fermions, and the sum extends to all integers n. On the other hand, $|0\rangle$ is the zero-momentum wave function, $\langle x|0\rangle=1$. The matrix-valued function $\langle x|f(M,D_\mu+ip_\mu)|0\rangle$ is the symbol of $\hat{f}$. It is important to note that this wave function is periodic (in fact constant) and not antiperiodic, even for fermions. The antiperiodicity of the fermionic wave function is only reflected in the Matsubara frequencies in this formalism. Whenever the symbols method is used, ∂_μ acts on the periodic external fields. Ultimately ∂_μ acts on $|0\rangle$ giving zero (this means in practice a right-acting derivative operator).

In order to introduce the necessary concepts gradually and to provide the rationale for the occurrence of the Polyakov loop in the simplest case, in what remains of this subsection we will consider the case of no vector potential, space-independent scalar potential, and constant c-number mass term:

$$\boldsymbol{A}(x)=0, \quad A_0=A_0(x_0), \quad M(x)=m^2, \quad [m^2,\ ]=0. \tag{2.3}$$

This choice avoids complications coming from the spatial covariant derivatives and commutators at this point of the discussion. The result will be the zeroth-order term of an expansion in the number of commutators $[D_\mu,\ ]$ and $[M,\ ]$.

An application of the symbols method yields in this case

$$\langle x|e^{-\tau K}|x\rangle = \frac{1}{\beta}\sum_{p_0}\int \frac{d^dp}{(2\pi)^d}\langle x|e^{-\tau[m^2+\boldsymbol{p}^2-(D_0+ip_0)^2]}|0\rangle$$

$$= \frac{e^{-\tau m^2}}{(4\pi\tau)^{d/2}}\frac{1}{\beta}\sum_{p_0}\langle x|e^{\tau(D_0+ip_0)^2}|0\rangle. \tag{2.4}$$

[After the replacement $\boldsymbol{D}\to\boldsymbol{D}+\boldsymbol{p}$ dictated by Eq. (2.2), $D_i=\partial_i$ can be set to zero due to $|0\rangle$.]

The sum over the Matsubara frequencies implies that the operator $(1/\beta)\Sigma_{p_0}e^{\tau(D_0+ip_0)^2}$ is a periodic function of D_0 with period $2\pi i/\beta$; thus, it is actually a one-valued function of $e^{-\beta D_0}$. This can be made explicit by using Poisson's summation formula, which yields

$$\frac{1}{\beta}\sum_{p_0}e^{\tau(D_0+ip_0)^2}=\frac{1}{(4\pi\tau)^{1/2}}\sum_{k\in\mathbb{Z}}(\pm)^k e^{-k\beta D_0}e^{-k^2\beta^2/4\tau} \tag{2.5}$$

($\pm$ for bosons or fermions, respectively). This observation allows us to apply the operator identity [47]

$$e^{\beta\partial_0}e^{-\beta D_0}=\Omega(x), \tag{2.6}$$

where $\Omega(x)$ is the thermal Wilson line or untraced Polyakov loop:

$$\Omega(x)=T\exp\left(-\int_{x_0}^{x_0+\beta}A_0(x_0',\boldsymbol{x})dx_0'\right). \tag{2.7}$$

[T refers to temporal ordering and the definition is given for a general scalar potential $A_0(x)$.] The Polyakov loop appears here as the phase difference between gauge covariant and noncovariant time translations around the compactified Euclidean time. Physically, the Polyakov loop can be interpreted as the propagator of heavy particles in the gauge field background. The identity (2.6) is trivial if one chooses a gauge in which A_0 is time independent (which always exists globally) since in such a gauge $\Omega=e^{-\beta A_0}$, and D_0, A_0, and ∂_0 all commute. The identity itself is gauge covariant and holds in any gauge [47].

The point of using Eq. (2.6) is that the translation operator in Euclidean time, $e^{\beta\partial_0}$, has no other effect than moving x_0 to $x_0+\beta$ and this operation is the identity in the compactified time,

$$e^{\beta\partial_0}=1 \tag{2.8}$$

(even in the fermionic case, recall that after applying the method of symbols the derivatives act on the external fields and not on the particle wave functions), so one obtains the remarkable result

$$e^{-\beta D_0}=\Omega(x). \tag{2.9}$$

That is, whenever the differential operator D_0 appears periodically (with period $2\pi i/\beta$), it can be replaced by the multiplicative operator (i.e., the ordinary function) $-(1/\beta)\log[\Omega(x)]$. The many-valuation of the logarithm is not effective due to the assumed periodic dependence. Another point to note is that D_0 (or any function of it) acts as a gauge covariant operator on the external fields $F(x_0,\boldsymbol{x})$ and so transforms according to the local gauge transformation at the point $(x_0,\boldsymbol{x})$. Correspondingly, the Polyakov loop, which is also gauge covariant, starts at time x_0 and not at time zero in Eq. (2.7); this difference would be irrelevant for the traced Polyakov loop, but not in the present context.

An application of the rule (2.9), yields, in particular,

$$\frac{1}{\beta}\sum_{p_0}e^{\tau(D_0+ip_0)^2}=\frac{1}{(4\pi\tau)^{1/2}}\sum_{k\in\mathbb{Z}}(\pm)^k\Omega^k e^{-k^2\beta^2/4\tau}. \tag{2.10}$$

More generally,

$$\sum_{p_0}f(ip_0+D_0)=\sum_{p_0}f\left(ip_0-\frac{1}{\beta}\log(\Omega)\right), \tag{2.11}$$

provided the sum is absolutely convergent, so that the sum is a periodic function of D_0. Thus it will prove useful to introduce the quantity Q defined as

$$Q=ip_0+D_0=ip_0-\frac{1}{\beta}\log(\Omega). \tag{2.12}$$

The second equality holds in expressions of the form (2.11). (Note that the two definitions of Q are not equivalent in other contexts—e.g., in $\Sigma_{p_0}f_1(Q)Xf_2(Q)$—unless $[D_0,X]=0$.)

The heat kernel in Eq. (2.4) becomes

$$\langle x|e^{-\tau K}|x\rangle=\frac{1}{(4\pi\tau)^{d/2}}e^{-\tau m^2}\frac{1}{\beta}\sum_{p_0}e^{\tau Q^2} \tag{2.13}$$

$$=\frac{1}{(4\pi\tau)^{(d+1)/2}}e^{-\tau m^2}\varphi_0(\Omega). \tag{2.14}$$

In the first equality we have removed the brackets $\langle x|\cdot|0\rangle$ since for multiplicative operators like $\Omega(x)$, these brackets just pick up the value of the function at x. In the last equality we have used the definition of the functions $\varphi_n(\Omega)$ which will appear frequently below:

$$\varphi_n(\Omega;\tau/\beta^2)=(4\pi\tau)^{1/2}\frac{1}{\beta}\sum_{p_0}\tau^{n/2}Q^n e^{\tau Q^2},$$

$$Q=ip_0-\frac{1}{\beta}\log(\Omega). \tag{2.15}$$

Note that there is a bosonic and a fermionic version of each such function, and the two versions are related by the replacement $\Omega\to-\Omega$. As indicated, these functions depend only on the combination τ/β^2. In the zero-temperature limit, the sum over p_0 becomes a Gaussian integral, yielding

$$\varphi_n(\Omega;0)=\begin{cases}\left(-\dfrac{1}{2}\right)^{n/2}(n-1)!! & (n \text{ even}),\\ 0 & (n \text{ odd}).\end{cases} \tag{2.16}$$

As can be seen, for instance, from Eq. (2.10), in this limit only the $k=0$ mode remains, whereas the other modes become exponentially suppressed, either at low temperature or low proper time τ.

The result in Eq. (2.14) is sufficient to derive the grand-canonical potential of a gas of relativistic free particles. For definiteness we consider the bosonic case [48]. The effective action (related to the grand-canonical potential through $W=\beta\Omega_{\rm gc}$) is obtained as

$$W=\mathrm{Tr}\log(K)=-\mathrm{Tr}\int_0^\infty\frac{d\tau}{\tau}\langle x|e^{-\tau K}|x\rangle. \tag{2.17}$$

K includes a chemical potential $A_0=-i\mu$ as unique external field, and the corresponding Polyakov loop is $\Omega=\exp(i\beta\mu)$. Using Eq. (2.14), subtracting the zero-temperature part (which corresponds to setting $\varphi_0\to 1$), and carrying out the integrations yields the standard result [24]

$$W=N\int\frac{d^dx\,d^dk}{(2\pi)^d}[\log(1-e^{-\beta(\omega_k-\mu)})+\log(1-e^{-\beta(\omega_k+\mu)})]. \tag{2.18}$$

N is the number of species and $\omega_k=\sqrt{k^2+m^2}$.

In next subsection, after the introduction of more general external fields, we will consider expansions in the number of spatial covariant derivatives and mass terms. At zero temperature, the derivative expansion involves temporal derivatives as well, as demanded by Lorentz invariance, but such an expansion is more subtle at finite temperature. The direct method would be to expand in powers of D_0 in Eq. (2.4); however, this procedure spoils gauge invariance (e.g., $D_0|0\rangle=A_0|0\rangle$ is not gauge covariant). As a rule, giving up the periodic dependence in D_0 breaks gauge invariance [47]. One can try to first fix the gauge so that A_0 is stationary and then expand in powers of A_0. This is equivalent to expanding in powers of $\log(\Omega)$. By construction this procedure preserves invariance under infinitesimal (or more generally, topologically small) gauge transformations; however, it does not preserve invariance under discrete gauge transformations ([47,49] and Sec. III D below). This is because $\log(\Omega)$ is many-valued under such transformations. An expansion in the number of temporal covariant derivatives which does not spoil one-valuation or gauge invariance is described next.

B. Diagonal thermal heat kernel coefficients

Here we will consider the heat kernel expansion at finite temperature in the completely general case of nontrivial and non-Abelian gauge and mass term fields $A_\mu(x)$ and $M(x)$.

First of all one has to specify the counting of the expansion. At zero temperature, the expansion is defined as one of $\langle x|e^{-\tau K}|x\rangle$ in powers of τ [after extracting the geometrical factor $(4\pi\tau)^{-(d+1)/2}$]. Each power of τ is tied to a local operator constructed with the covariant derivatives D_μ and $M(x)$ [cf. Eqs. (2.23) and (2.24)]. The heat kernel $e^{-\tau K}$ is dimensionless by assigning engineering mass dimensions $-2,+1$, and $+2$ to τ, D_μ, and M, respectively. So at zero temperature, the expansion in powers of τ is equivalent to counting the mass dimension carried by the local operators.

At finite temperature there is a further dimensional quantity β, the two countings are no longer equivalent, and one has to specify the concrete expansion to be used. It is well known that the finite-temperature corrections are negligible in the ultraviolet region, so that, for instance, the temperature does not modify the renormalization properties of a quantum field theory [24,25] and also the quantum anomalies are not affected [3,50]. The ultraviolet limit corresponds to the small τ limit in the heat kernel. As noted before and can be seen, e.g., in Eq. (2.10), the finite-β and small-τ corrections are of the order of $e^{-\beta^2/4\tau}$ or less, and so they are exponentially suppressed. Of course, the same exponential suppression applies to the low-temperature and finite-τ limit. This implies that a strict expansion of the heat kernel in powers of τ will yield precisely the same asymptotic expansion as at zero temperature. In order to pick up nontrivial finite-temperature corrections we arrange our expansion according to the mass dimension of the local operators. In this counting we take the Polyakov loop Ω, D_μ, and M as zeroth, first, and second order, respectively. In addition one has to specify that $\Omega(x)$ is at the left in all terms (equivalently, one could define a similar expansion with Ω always at the right). This is required because the commutator of Ω with other quantities generates commutators $[D_0,\ ]$ which are dimensionful in our counting. After these specifications the expansion of $\langle x|e^{-\tau K}|x\rangle$ for a generic gauge group is unique and well defined and full gauge invariance is manifest at each order.

The expansion just described, in which each term contains arbitrary functions of the Polyakov loop but only a finite number of covariant derivatives (including timelike ones), is the natural extension of the standard covariant derivative expansion at zero temperature. Its justification is given in great detail in [47]. For the reader's convenience we have summarized the main points in Appendix A.

In this expansion the terms are ordered by powers of τ but with coefficients which depend on β^2/τ and Ω:

$$\langle x|e^{-\tau(M-D_\mu^2)}|x\rangle=(4\pi\tau)^{-(d+1)/2}\sum_n a_n^T(x)\tau^n. \tag{2.19}$$

From the definition it is clear that the zeroth-order term for a general configuration is just

$$a_0^T(x)=\varphi_0(\Omega(x);\tau/\beta^2), \tag{2.20}$$

already computed in the previous subsection [cf. Eq. (2.14)]. This is because when the particular case (2.3) is inserted in the full expansion all terms of higher order, with one or more $[D_\mu,\ ]$ or m^2, vanish identically.

For subsequent reference we introduce the following notation. The field strength tensor is defined as $F_{\mu\nu}=[D_\mu,D_\nu]$ and, likewise, the electric field is $E_i=F_{0i}$. In addition, the notation $\hat{D}_\mu$ means the operation $[D_\mu, \ ]$. Finally we will use a notation of the type $X_{\mu\nu\alpha}$ to mean $\hat{D}_\mu\hat{D}_\nu\hat{D}_\alpha X=[D_\mu,[D_\nu,[D_\alpha,X]]]$—e.g., $M_{00}=\hat{D}_0^2M$, $F_{\alpha\mu\nu}=\hat{D}_\alpha F_{\mu\nu}$.

The method for expanding a generic function $\langle x|f(M,D_\mu)|x\rangle$ has been explained in detail in [47]. We have applied this procedure to compute the heat kernel coefficients to mass dimension 6. However, for the heat kernel there is an alternative approach which uses the well-known Seeley-DeWitt coefficients at zero temperature. This is the method that we explain in detail here. The idea is as follows. The symbols method formula (2.2) is applied to the temporal dimension only:

$$\langle x|e^{-\tau(M-D_\mu^2)}|x\rangle=\frac{1}{\beta}\sum_{p_0}\langle \boldsymbol{x}|e^{-\tau(M-Q^2-D_i^2)}|\boldsymbol{x}\rangle,$$

$$Q=ip_0+D_0. \tag{2.21}$$

(The brackets $\langle x_0|\ |0\rangle$, associated with the Hilbert space over x_0, are understood although not written explicitly.) This implies that we can use the standard zero-temperature expansion for the d-dimensional heat kernel with effective Klein-Gordon operator:

$$K_0=Y-D_i^2, \quad Y=M-Q^2. \tag{2.22}$$

In this context Y is the non-Abelian mass term, because, although it contains temporal derivatives (in Q), it does not contain spatial derivatives and so acts multiplicatively on the spatial Hilbert space. The standard heat kernel expansion gives then

$$\langle \boldsymbol{x}|e^{-\tau(Y-D_i^2)}|\boldsymbol{x}\rangle=(4\pi\tau)^{-d/2}\sum_{n=0}^{\infty}a_n(Y,\hat{D}_i)\tau^n, \tag{2.23}$$

where the coefficients $a_n(Y,\hat{D}_i)$ are polynomials of dimension $2n$ made out of Y and $\hat{D}_i=[D_i, \ ]$. To lowest orders [17,19],

$$a_0=1,$$

$$a_1=-Y,$$

$$a_2=\frac{1}{2}Y^2-\frac{1}{6}Y_{ii}+\frac{1}{12}F_{ij}^2,$$

$$a_3=-\frac{1}{6}Y^3+\frac{1}{12}\{Y,Y_{ii}\}+\frac{1}{12}Y_i^2-\frac{1}{60}Y_{iijj}-\frac{1}{60}[F_{iij},Y_j]$$
$$-\frac{1}{30}\{Y,F_{ij}^2\}-\frac{1}{60}F_{ij}YF_{ij}+\frac{1}{45}F_{ijk}^2-\frac{1}{30}F_{ij}F_{jk}F_{ki}$$
$$+\frac{1}{180}F_{iij}^2+\frac{1}{60}\{F_{ij},F_{kkij}\}. \tag{2.24}$$

(As noted before $Y_{ii}=\hat{D}_i^2Y$, $F_{ijk}=\hat{D}_iF_{jk}$, etc.)

Equation (2.23) inserted into Eq. (2.21) is of course correct but not very useful as it stands. For instance, for the zeroth order, the expansion in Eq. (2.23) would be needed to all orders to reproduce the simple result (2.20), since $e^{\tau Q^2}$ is not a polynomial in Q. In view of this, we consider instead

$$\langle \boldsymbol{x}|e^{-\tau(M-Q^2-D_i^2)}|\boldsymbol{x}\rangle=(4\pi\tau)^{-d/2}\sum_{n=0}^{\infty}e^{\tau Q^2}\tilde{a}_n(Q^2,M,\hat{D}_i)\tau^n, \tag{2.25}$$

which introduces a new set of polynomial coefficients $\tilde{a}_n(Q^2,M,\hat{D}_i)$. By their definition, it is clear that these coefficients are unchanged if "Q^2" is everywhere replaced by "Q^2+c number." This implies that in $\tilde{a}_n$ the quantity Q^2 appears only in the form $[Q^2, \ ]$. This is an essential improvement over the original coefficients a_n, since each $[Q^2, \ ]$ will yield at least one $\hat{D}_0$, and so higher orders in $[Q^2, \ ]$ appear only at higher orders in the heat kernel expansion.[1]

The calculation of the coefficients $\tilde{a}_n(Q^2,M,\hat{D}_i)$ follows easily from the relation

$$\sum_{n=0}^{\infty}a_n\tau^n=e^{\tau Q^2}\sum_{n=0}^{\infty}\tilde{a}_n\tau^n. \tag{2.26}$$

If one takes the expression on the left-hand side (LHS) and moves all Q^2 blocks to the left using the commutator $[Q^2, \ ]$, two types of terms will be generated: (i) terms with Q^2 only inside commutators and (ii) terms with one or more Q^2 blocks at the left. The terms of type (i) are those corresponding to $\Sigma_n\tilde{a}_n\tau^n$. To lowest orders one finds

$$\tilde{a}_0=1,$$

$$\tilde{a}_1=-M,$$

$$\tilde{a}_2=\frac{1}{2}M^2-\frac{1}{6}M_{ii}+\frac{1}{12}F_{ij}^2+\frac{1}{2}[Q^2,M]+\frac{1}{6}(Q^2)_{ii}. \tag{2.27}$$

Once the $\tilde{a}_n$ coefficients are so constructed one has to proceed to rearrange Eq. (2.25) as an expansion in powers of M, $\hat{D}_i$, and $\hat{D}_0$. The expansions in M and $\hat{D}_i$ are already inherited from Eq. (2.23). It remains to expand $[Q^2, \ ]$ in terms of $[Q, \ ]$ or, equivalently, in terms of $\hat{D}_0=[D_0, \ ]$ since the quantities Q and D_0 differ by a c number. To do this, in the $\tilde{a}_n$ coefficients Q is to be moved to the left, introducing

[1]This kind of resummations is standard also at zero temperature to move, e.g., the mass term $e^{-\tau M}$ to the left and leave only a $[M, \ ]$ dependence in the coefficients [17].

$\hat{D}_0$, until all the terms so generated are local operators made out of $\hat{D}_\mu$ and M and all uncommutated Q's are at the left: e.g.,

$$\tilde{a}_2 = \frac{1}{2}M^2 - \frac{1}{6}M_{ii} + \frac{1}{12}F_{ij}^2 - \frac{1}{2}M_{00} + \frac{1}{3}E_i^2 + \frac{1}{6}E_{0ii} + QM_0 - \frac{1}{3}QE_{ii}. \tag{2.28}$$

(Recall that E_i stands for the electric field F_{0i}.) We can see two types of contributions in $\tilde{a}_2$: namely, those without a Q at the left and those with one. If Q is assigned an engineering dimension of mass, all the terms are of the same dimension, mass to the fourth. However, in our counting only the dimension carried by $\hat{D}_\mu$ and M is computed, and so the two types of terms are of different order: namely, mass to the fourth and mass to the third, respectively. Indeed, when $\tilde{a}_2$ is introduced in Eq. (2.25) (i.e., it gets multiplied by $e^{\tau Q^2}$) and then in Eq. (2.21) (the sum over the Matsubara frequencies is carried out) we will obtain the contributions (using $\Sigma_{p_0} Q^n e^{\tau Q^2} \sim \varphi_n$)

$$\tilde{a}_2 \to \varphi_0(\Omega)\left(\frac{1}{2}M^2 - \frac{1}{6}M_{ii} + \frac{1}{12}F_{ij}^2 - \frac{1}{2}M_{00} + \frac{1}{3}E_i^2 + \frac{1}{6}E_{0ii}\right)\tau^2 + \varphi_1(\Omega)\left(M_0 - \frac{1}{3}E_{ii}\right)\tau^{3/2}. \tag{2.29}$$

These are contributions to the thermal heat kernel coefficients a_2^T and $a_{3/2}^T$, respectively, introduced in Eq. (2.19). Note the presence of half-integer order coefficients from terms with an odd number of Q's.

As we have just shown, each zero-temperature heat kernel coefficient a_k in Eq. (2.23) allows us to obtain a corresponding coefficient $\tilde{a}_k$ with the same engineering dimension $2k$. Such a coefficient in turn contributes, in general, to several heat thermal coefficients a_n^T (with mass dimension $2n$). Let us discuss in detail to which a_n^T contributes each $\tilde{a}_k$. The change from engineering to real dimension comes about because some terms in $\tilde{a}_k$ contain factors of Q at the left which do not act as $\hat{D}_0$ and so count as dimensionless. Therefore it is clear that for given k, the allowed n satisfy $n \leqslant k$, the equal sign corresponding to terms having all Q's in commutators. On the other hand, the maximum number of $[Q^2, \,]$'s in $\tilde{a}_k$ ($k>0$) is $k-1$, and from these, at most $k-1$ uncommutated Q's can reach the left of the term. This yields the further condition $k \leqslant 2n-1$. Note further that a factor Q^ℓ gives rise to a coefficient $\varphi_\ell(\Omega)$ in a_n^T. In summary, in the computation of the thermal coefficients a_n^T up to $n=3$ (mass dimension 6), we find the scheme

$$\begin{aligned}
&a_0 \sim \tilde{a}_0 \sim \varphi_0 a_0^T,\\
&a_1 \sim \tilde{a}_1 \sim \varphi_0 a_1^T,\\
&a_2 \sim \tilde{a}_2 \sim \varphi_0 a_2^T + \varphi_1 a_{3/2}^T,\\
&a_3 \sim \tilde{a}_3 \sim \varphi_0 a_3^T + \varphi_1 a_{5/2}^T + \varphi_2 a_2^T,\\
&a_4 \sim \tilde{a}_4 \sim \varphi_0 a_4^T + \varphi_1 a_{7/2}^T + \varphi_2 a_3^T + \varphi_3 a_{5/2}^T,\\
&a_5 \sim \tilde{a}_5 \sim \varphi_0 a_5^T + \varphi_1 a_{9/2}^T + \varphi_2 a_4^T + \varphi_3 a_{7/2}^T + \varphi_4 a_3^T.
\end{aligned} \tag{2.30}$$

The mixing of terms is a nuisance that does not occur at zero temperature; however, it cannot be avoided: Q contains p_0 and must count as zeroth order (otherwise, if Q were of order 1 the expansion would consist of polynomials in Q and the sum over p_0 would not converge). On the other hand, counting p_0 as zeroth order and D_0 as first order even when it is inside Q results in a breaking of gauge invariance, as we noted at the end of the previous subsection. The fact that Ω counts as dimensionless and $\hat{D}_0$ as dimension 1 is necessary to have an order by order gauge invariant expansion. This counting is well defined provided that all Ω's are at the left (for instance) of the local operators [cf. Eq. (2.36) and discussion below].

From Eq. (2.30) we can see that we do not need the complete zero-temperature coefficients a_4 and a_5. Here a_3^T requires only terms Y^n, with $n=2,3,4$ in $a_4(Y,\hat{D}_i)$ and $n=4,5$ in $a_5(Y,\hat{D}_i)$. We have extracted the zero-temperature coefficients from [18]. These authors actually provide the traced coefficients $b_n(x)$ defined by

$$\mathrm{Tr}(e^{-\tau(Y-D_i^2)}) = (4\pi\tau)^{-d/2}\sum_{n=0}^{\infty}\int d^dx\, \mathrm{tr}(b_n)\tau^n, \tag{2.31}$$

where Tr is the trace in the full Hilbert space of wave functions and tr is the trace over the internal space only. The coefficient a_n is obtained by means of a first order variation of b_{n+1} [cf. Eq. (2.41)]. The advantage of this procedure is that the traced coefficients are much more compact and better checked.

As we have said, we have computed the thermal heat kernel coefficients up to and including mass dimension 6 by the procedure just described and also by that detailed in [47]. This latter approach uses the symbols method for space and time coordinates and so computes the coefficients from scratch (in passing it yields the zero-temperature coefficients as well). We have verified that the two computations give identical results after using the appropriate Bianchi identities (in practice the method of [47] tends to give somewhat more compact expressions). The results are as follows:

$$a_0^T=\varphi_0,$$

$$a_{1/2}^T=0,$$

$$a_1^T=-\varphi_0 M,$$

$$a_{3/2}^T=\varphi_1\left(M_0-\frac{1}{3}E_{ii}\right),$$

$$a_2^T=\varphi_0 a_2^{T=0}+\frac{1}{6}\bar{\varphi}_2(E_i^2+E_{0ii}-2M_{00}),$$

$$a_{5/2}^T=\frac{1}{3}(2\varphi_1+\varphi_3)M_{000}+\frac{1}{6}\varphi_1 M_{0ii}-\frac{1}{3}\varphi_1(2M_0M+MM_0)+\frac{1}{6}\varphi_1(\{M_i,E_i\}+\{M,E_{ii}\})-\left(\frac{1}{3}\varphi_1+\frac{1}{5}\varphi_3\right)E_{00ii}-\frac{1}{30}\varphi_1 E_{iijj}$$
$$-\left(\frac{5}{6}\varphi_1+\frac{2}{5}\varphi_3\right)E_{0i}E_i-\left(\frac{1}{2}\varphi_1+\frac{4}{15}\varphi_3\right)E_iE_{0i}+\frac{1}{30}\varphi_1[E_j,F_{iij}]-\varphi_1\left(\frac{1}{10}F_{0ij}F_{ij}+\frac{1}{15}F_{ij}F_{0ij}\right),$$

$$a_3^T=\varphi_0 a_3^{T=0}-\left(\frac{1}{4}\bar{\varphi}_2-\frac{1}{10}\bar{\varphi}_4\right)M_{0000}-\frac{1}{60}\bar{\varphi}_2(3M_{00ii}-15M_{00}M-5MM_{00}-15M_0^2+4\{M,E_i^2\}+2E_iME_i+4ME_{0ii}+6E_{0ii}M$$
$$+4M_iE_{0i}+6E_{0i}M_i+7M_0E_{ii}+3E_{ii}M_0+6M_{0i}E_i+4E_iM_{0i})+\left(\frac{3}{20}\bar{\varphi}_2-\frac{1}{15}\bar{\varphi}_4\right)E_{000ii}+\frac{1}{60}\bar{\varphi}_2E_{0iijj}$$
$$+\left(\frac{1}{2}\bar{\varphi}_2-\frac{1}{5}\bar{\varphi}_4\right)E_{00i}E_i+\left(\frac{7}{30}\bar{\varphi}_2-\frac{1}{10}\bar{\varphi}_4\right)E_iE_{00i}+\left(\frac{19}{30}\bar{\varphi}_2-\frac{4}{15}\bar{\varphi}_4\right)E_{0i}^2+\frac{1}{180}\bar{\varphi}_2(2\{E_i,E_{jji}\}+4\{E_i,E_{ijj}\}+5E_{ii}^2+4E_{ij}^2$$
$$+4F_{0iij}E_j-2E_jF_{0iij}-2E_{0ij}F_{ij}-[E_{ij},F_{0ij}]-4E_{0i}F_{jji}+2F_{jji}E_{0i}+2E_iF_{ij}E_j+2\{E_iE_j,F_{ij}\}$$
$$+7F_{00ij}F_{ij}+3F_{ij}F_{00ij}+8F_{0ij}^2). \qquad (2.32)$$

In these formulas $a_n^{T=0}$ stands for the zero-temperature coefficient. These are the same as those in Eqs. (2.24) but using M instead of Y and space-time indices instead of space indices—e.g., $a_2^{T=0}=\frac{1}{2}M^2-\frac{1}{6}M_{\mu\mu}+\frac{1}{12}F_{\mu\nu}^2$. For convenience we have introduced the auxiliary functions

$$\bar{\varphi}_2=\varphi_0+2\varphi_2,\quad \bar{\varphi}_4=\varphi_0-\frac{4}{3}\varphi_4,$$

$$\bar{\varphi}_{2n}=\varphi_0-\frac{(-2)^n}{(2n-1)!!}\varphi_{2n}, \qquad (2.33)$$

which vanish at $\tau/\beta^2=0$. As a result of the Bianchi identity, there is some ambiguity in writing the terms. We have chosen to order the derivatives so that all spatial derivatives are done first and the temporal derivatives are the outer ones. This choice appears naturally in our approach and in addition is optimal to obtain the traced coefficients b_n^T since the zeroth derivative of the Polyakov loop vanishes [cf. Eq. (2.36) below], and so terms of the form $\varphi_n X_0$ do not contribute to the traced coefficients upon using integration by parts. The terms a_0^T, a_1^T, $a_{3/2}^T$, and a_2^T were given in [27].

C. Traced thermal heat kernel coefficients

The zero-temperature traced heat kernel coefficients have been introduced in Eq. (2.31) (for the d-dimensional operator $Y-D_i^2$). Of course, the choice $b_n=a_n$ would suffice, however, exploiting the trace cyclic property and integration by parts more compact choices are possible. At lowest orders the coefficients can be taken as (we give the formulas for $K=M-D_\mu^2$ at zero temperature; the heat kernel coefficients are dimension independent) [18,21]

$$b_0=1,$$

$$b_1=-M,$$

$$b_2=\frac{1}{2}M^2+\frac{1}{12}F_{\mu\nu}^2,$$

$$b_3=-\frac{1}{6}M^3-\frac{1}{12}M_\mu^2-\frac{1}{12}F_{\mu\nu}MF_{\mu\nu}-\frac{1}{60}F_{\mu\mu\nu}^2$$
$$+\frac{1}{90}F_{\mu\nu}F_{\nu\alpha}F_{\alpha\mu}. \qquad (2.34)$$

By construction $a_n - b_n$ is a commutator which vanishes inside Tr. Likewise, we can introduce the traced coefficients at finite temperature:

$$\mathrm{Tr}(e^{-\tau(M-D_\mu^2)}) = (4\pi\tau)^{-(d+1)/2}\sum_n \int d^{d+1}x\, \mathrm{tr}(b_n^T)\tau^n, \tag{2.35}$$

with b_n^T simpler than a_n^T. Once again we choose a canonical form for these coefficients where a function of Ω put at the left is multiplied by a local operator (i.e., an operator made out of M and $\hat{D}_\mu$). To simplify the traced coefficients and bring them to the canonical form we need to work out the commutators of the form $[X,f(\Omega)]$ [in particular $\hat{D}_\mu f(\Omega)$] as a combination of terms of the type function of Ω times local operator. As shown in Appendix B, the rules are as follows: let f denote a function of Ω [e.g., $\varphi_n(\Omega)$] and let $f^{(n)}$ be its nth derivative with respect to the variable $-\log(\Omega)/\beta$; then,

$$\hat{D}_0 f = 0,$$

$$\hat{D}_i f = -f' E_i + \frac{1}{2} f'' E_{0i} - \frac{1}{3!} f^{(3)} E_{00i} + \cdots,$$

$$[X,f] = -f' X_0 + \frac{1}{2} f'' X_{00} - \frac{1}{3!} f^{(3)} X_{000} + \cdots. \tag{2.36}$$

These formulas imply that, unlike the zero-temperature case, the cyclic property mixes terms of different order at finite temperature. This is because, as noted above, $\hat{D}_0$ has dimensions of mass whereas Ω counts as dimensionless. So, for instance, $\varphi_0(\Omega)$ is of order zero and $\hat{D}_i$ is of first order, yet $\hat{D}_i\varphi_0(\Omega)$ contains terms of all orders, starting with dimension 2. As we will discuss below, this implies that there is a certain amount of freedom in the choice of the traced coefficients. To apply these commutation rules to a_n^T we further need the relation

$$\varphi_n' = \sqrt{\tau}(2\varphi_{n+1} + n\varphi_{n-1}). \tag{2.37}$$

Using these rules we can apply integration by parts and the cyclic property to the previously computed coefficients a_n^T and choose a more compact form for them valid inside the trace. In this way we obtain, up to mass dimension 6,

$$b_0^T = \varphi_0,$$

$$b_{1/2}^T = 0,$$

$$b_1^T = -\varphi_0 M,$$

$$b_{3/2}^T = 0,$$

$$b_2^T = \varphi_0 b_2 - \frac{1}{6}\overline{\varphi}_2 E_i^2,$$

$$b_{5/2}^T = -\frac{1}{6}\varphi_1\{M_i, E_i\},$$

$$b_3^T = \varphi_0 b_3 + \frac{1}{6}\overline{\varphi}_2\left(\frac{1}{2}M_0^2 + E_i M E_i + \frac{1}{10}E_{ii}^2 + \frac{1}{10}F_{0ij}^2 - \frac{1}{5}E_i F_{ij} E_j\right) + \left(\frac{1}{10}\overline{\varphi}_4 - \frac{1}{6}\overline{\varphi}_2\right)E_{0i}^2. \tag{2.38}$$

This is the main result of this section, where the φ_n functions are given in Eqs. (2.15) and (2.33). In these formulas the b_n are the zero-temperature coefficients given in Eqs. (2.34). We note that the coefficient b_3^T above is not identical to that given in [27]. [The coefficient in [27] corresponds to replace $\varphi_0 b_3$ above by $\varphi_0 b_3'$, where b_3' differs from b_3 in Eqs. (2.34) by a cyclic permutation.] The two versions of b_3^T differ by higher-order terms. In what follows we use the coefficient in Eqs. (2.38).

Several remarks should be made about these expressions. Either at zero or finite temperature there is an ambiguity in the choice of the traced coefficients b_n^T; however, the ambiguity is essentially larger at finite temperature. Indeed, writing the expansion as

$$\mathrm{Tr}(e^{-\tau(M-D_\mu^2)}) = (4\pi\tau)^{-(d+1)/2}\sum_n B_n^T\tau^n, \tag{2.39}$$

we find that, although b_n is ambiguous, $B_n^{T=0}$ is not. This is because at zero temperature the expansion is tied to a series expansion in powers of a parameter (say, τ). At finite temperature the expansion is not tied to a parameter (it is rather a commutator expansion) and so the ambiguity exits not only for b_n^T but also for B_n^T. For instance, b_2^T above has been expressed in terms of the coefficient b_2 given in Eqs. (2.34). Nothing changes at zero temperature if we add $M_{\mu\mu}$ to b_2 since the addition is a pure commutator; however, in b_2^T it would mean to add $\varphi_0 M_{\mu\mu}$ which is no longer a pure commutator, thereby changing the functional B_2^T. In fact, $\varphi_0 M_{\mu\mu}$, which is formally of dimension 4, can be expressed as a sum of terms of dimension 5 and higher, using integration by parts and the commutation rules (2.36). So the concrete choice of b_2^T affects the form of the higher orders, $b_{5/2}^T$, b_3^T, etc.

Taking into account this ambiguity, our criterion for choosing the traced coefficients has been to recursively bring the b_n^T to a compact form. We observe that inside the trace (upon applying the commutation rules) $a_{3/2}^T$ is a sum of terms of dimension 4 and higher, so we choose $b_{3/2}^T = 0$. Then a_2^T, augmented with the terms generated from $a_{3/2}^T$, is brought to the most compact form. This in turn produces higher-order terms which are added to $a_{5/2}^T$, and so on. Of course, this is not the only possibility, since taken a b_n^T to be simplest may imply a greater complication in the higher-order coefficients. For instance, as can be shown, it is possible to arrange the expansion so that all half-order traced coefficients vanish: e.g., $b_{5/2}^T$ can be removed at the cost of complicating b_2^T.

It should be clear that the ambiguity in the expansion B_n^T in Eq. (2.39) does not affect its sum but only amounts to a reorganization of the series. On the other hand, the untraced coefficients a_n^T are not ambiguous: once brought to their canonical form they are unique functionals of M and A_μ.

The heat kernel is symmetric under transposition of operators, the b_n^T have been chosen so that this mirror symmetry holds at each order.

As is well known [17], not only the a_n^T allow one to obtain the b_n^T but also the converse is true. By their definition,

$$\langle x|e^{-\tau(M-D_\mu^2)}|x\rangle = -\frac{1}{\tau}\frac{\delta}{\delta M(x)}\mathrm{Tr}(e^{-\tau(M-D_\mu^2)}). \tag{2.40}$$

Using the expansions in both sides, one finds, at zero temperature [using Eq. (2.39)],

$$a_n^{T=0}(x) = -\frac{\delta B_{n+1}^{T=0}}{\delta M(x)}. \tag{2.41}$$

At finite temperature, the variation of b_k^T contributes not only to a_{k-1}^T but also to all higher-order coefficients, in general. So we have, instead,

$$a_n^T(x) \simeq -\frac{\delta}{\delta M(x)}\sum_{1\le k\le n+1} B_k^T \tau^{k-n-1}, \tag{2.42}$$

where on the RHS only the terms of dimension $2n$ are to be retained and k takes integer as well as half-integer values. We have checked our results by verifying that this relation holds for our coefficients.

III. ONE-LOOP EFFECTIVE ACTION OF CHIRAL QCD AT HIGH TEMPERATURE

Here we will apply the thermal heat kernel expansion just derived to obtain the one-loop effective action of QCD with massless quarks in the high-temperature region. We remark that the effective action we are referring to is the standard one in quantum field theory: namely, the classical generator of the one-particle irreducible diagrams. As a consequence our classical fields may be time dependent. The quantum effective action in the sense of dimensional reduction [38], as an effective field theory for the static modes, is of great relevance in high-temperature QCD and is also discussed below, in Sec. IV. We will use the background field method, which preserves gauge invariance [51]. The Euclidean action is

$$S = -\frac{1}{2g^2}\int d^4x\,\mathrm{tr}(F_{\mu\nu}^2) + \int d^4x\,\bar{q}\not{D}q. \tag{3.1}$$

Here $D_\mu = \partial_\mu + A_\mu$, with A_μ and $F_{\mu\nu} = [D_\mu, D_\nu]$ anti-Hermitian matrices of dimension N. They belong to the fundamental representation of the Lie algebra of the gauge group $\mathrm{SU}(N)$.[2]

A. Quark sector

In this subsection we work out the quark contribution which is somewhat simpler than the gluon contribution. (The latter requires the use of the adjoint representation, introduction of ghost fields, and teatment of the infrared divergences.) Upon functional integration of the quark fields, the partition function of the system picks up the following factor from the quark sector:

$$Z_q[A] = \mathrm{Det}(\not{D})^{N_f} = \mathrm{Det}(\not{D}^2)^{N_f/2}, \tag{3.2}$$

where N_f denotes the number of quark flavors. (As usual, we have squared the Dirac operator to obtain a Klein-Gordon operator.) The corresponding contribution to the effective action is (we use the convention $Z = e^{-\Gamma[A]}$)

$$\Gamma_q[A] = -\frac{N_f}{2}\mathrm{Tr}\log(\not{D}^2) = \frac{N_f}{2}\int_0^\infty \frac{d\tau}{\tau}\mathrm{Tr}\exp(\tau\not{D}^2)$$
$$=: \int d^4x\,\mathcal{L}_q(x), \tag{3.3}$$

$$\mathcal{L}_q(x) = \frac{N_f}{2}\int_0^\infty \frac{d\tau}{\tau}\frac{\mu^{2\epsilon}}{(4\pi\tau)^{D/2}}\sum_n \tau^n\,\mathrm{tr}(b_{n,q}^T). \tag{3.4}$$

In this formula the Dirac trace is included in the $b_{n,q}^T$ and "tr" refers to color trace (in the fundamental representation). The ultraviolet divergences at $\tau=0$ are regulated using dimensional regularization, with the convention $D = 4-2\epsilon$. As is standard in dimensional regularization, the factor $\mu^{2\epsilon}$ is introduced in order to deal with an effective Lagrangian of mass dimension 4 rather than $4-2\epsilon$.

To apply our thermal heat kernel expansion we need only to identify the corresponding Klein-Gordon operator. We use

$$\gamma_\mu = \gamma_\mu^\dagger, \quad \gamma_\mu\gamma_\nu = \delta_{\mu\nu} + \sigma_{\mu\nu}, \quad \mathrm{tr}_{\mathrm{Dirac}}(1) = 4. \tag{3.5}$$

The expression

$$-\not{D}^2 = -D_\mu^2 - \frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu} \tag{3.6}$$

identifies $-\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}$ as the (square) mass term M of the Klein-Gordon operator in this case. A direct application of Eqs. (2.38) shows that b_1^T and $b_{5/2}^T$ cannot contribute (they

[2]Our point of view will be that A_μ itself is the quantum field, independently of any particular choice of basis in $\mathrm{su}(N)$. So the coupling constant g is also independent of that choice. As a result of gauge invariance, A_μ is not renormalized.

have a single M and this cancels due to the trace over Dirac space). The other coefficients give, to mass dimension 6 included,

$$b_{0,q}^T=4\varphi_0,$$

$$b_{2,q}^T=-\frac{2}{3}(\varphi_0 F_{\mu\nu}^2+\overline{\varphi}_2 E_i^2),$$

$$\begin{aligned}b_{3,q}^T=&\varphi_0\left(\frac{32}{45}F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}+\frac{1}{6}F_{\lambda\mu\nu}^2-\frac{1}{15}F_{\mu\mu\nu}^2\right)\\&+\overline{\varphi}_2\left(\frac{1}{15}E_{ii}^2-\frac{1}{10}F_{0ij}^2-\frac{2}{15}E_iF_{ij}E_j\right)\\&+\left(\frac{2}{5}\overline{\varphi}_4-\overline{\varphi}_2\right)E_{0i}^2.\end{aligned}\tag{3.7}$$

In these formulas the functions φ_n [defined in Eqs. (2.15) and (2.33)] correspond to their fermionic versions. All fields are in the fundamental representation.

The required integrals over τ in Eq. (3.4) are of the form

$$I_{\ell,n}^{\pm}(\omega):=\int_0^\infty\frac{d\tau}{\tau}(4\pi\mu^2\tau)^\epsilon\tau^\ell\varphi_n^{\pm}(\omega),\quad |\omega|=1,\tag{3.8}$$

where $\varphi_n^{\pm}$ refers to the bosonic or fermionic version, respectively. In the quark sector the argument ω will be the Polyakov loop in the fundamental representation or, in practice, any of its eigenvalues. These integrals can be done in closed form (see Appendix C). In particular,

$$\begin{aligned}&I_{\ell,2n}^{-}(e^{2\pi i\nu})\\&=(-1)^n(4\pi)^\epsilon\left(\frac{\mu\beta}{2\pi}\right)^{2\epsilon}\left(\frac{\beta}{2\pi}\right)^{2\ell}\frac{\Gamma\left(\ell+n+\epsilon+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\\&\quad\times\left[\zeta\left(1+2\ell+2\epsilon,\frac{1}{2}+\nu\right)+\zeta\left(1+2\ell+2\epsilon,\frac{1}{2}-\nu\right)\right],\\&-\frac{1}{2}<\nu<\frac{1}{2}.\end{aligned}\tag{3.9}$$

The integrals $I_{\ell,n}^{\pm}(\omega)$ are one-valued functions of ω—i.e., periodic in terms of ν; however, to apply the explicit formula (3.9), ν has to be taken in the interval $-\frac{1}{2}<\nu<\frac{1}{2}$. The generalized Riemann ζ function $\zeta(z,q)=\Sigma_{n=0}^{\infty}(n+q)^{-z}$ has only a single pole at $z=1$ [52], so the dimensionally regulated integrals yield the standard pole of the type $1/\epsilon$ solely for the integrals $I_{0,2n}^{-}$, which appear in $b_{2,q}^T$.

We can now proceed to compute the contributions to the effective Lagrangian. The zeroth order requires $I_{-2,0}^{-}$. Using the relation $\zeta(1-n,q)=-B_n(q)/n$, $n=1,2,\ldots$, with $B_n(q)$ the Bernoulli polynomial of order n [52], one finds

$$I_{-2,0}^{-}=-\frac{2}{3}\left(\frac{2\pi}{\beta}\right)^4B_4\left(\frac{1}{2}+\nu\right)+O(\epsilon),\tag{3.10}$$

so the effective potential is

$$\mathcal{L}_{0,q}(x)=\pi^2N_fT^4\left(\frac{2N}{45}-\frac{1}{12}\mathrm{tr}[(1-4\overline{\nu}^2)^2]\right),$$

$$\Omega(x)=e^{2\pi i\overline{\nu}},\quad -\frac{1}{2}<\overline{\nu}<\frac{1}{2}.\tag{3.11}$$

Here N is the number of colors, tr is taken in the fundamental representation of the gauge group, and $\overline{\nu}$ is the matrix $\log(\Omega)/(2\pi i)$ with eigenvalues in the branch $|\overline{\nu}|<1/2$. This is the well-known result [9].

The terms of mass dimension 4 have a pole at $\epsilon=0$. Using the relation

$$\zeta(1+z,q)=\frac{1}{z}-\psi(q)+O(z)\tag{3.12}$$

[where $\psi(q)$ is the digamma function], one finds

$$\begin{aligned}I_{0,0}^{-}=&\frac{1}{\epsilon}+\log(4\pi)-\gamma_E+2\log(\mu\beta/4\pi)-\psi\left(\frac{1}{2}+\nu\right)\\&-\psi\left(\frac{1}{2}-\nu\right)+O(\epsilon),\end{aligned}$$

$$I_{0,\overline{2}}^{-}:=I_{0,0}^{-}+2I_{0,2}^{-}=-2+O(\epsilon).\tag{3.13}$$

[For convenience, we have introduced the integrals $I_{\ell,\overline{2n}}^{\pm}$ analogous to $I_{\ell,2n}^{\pm}$ in Eq. (3.8) but using $\overline{\varphi}_{2n}$ instead of φ_{2n}.]

The terms $[\epsilon^{-1}+\log(4\pi)-\gamma_E]$ in $I_{0,0}^{-}$ come with $\mathrm{tr}(F_{\mu\nu}^2)$ and are removed by adopting the $\overline{\mathrm{MS}}$ scheme. We will discuss this in conjunction with gluon sector. After renormalization,

$$\begin{aligned}\mathcal{L}_{2,q}(x)=&-\frac{1}{3}\frac{1}{(4\pi)^2}N_f\,\mathrm{tr}\left\{\left[2\log(\mu/4\pi T)-\psi\left(\frac{1}{2}+\overline{\nu}\right)\right.\right.\\&\left.\left.-\psi\left(\frac{1}{2}-\overline{\nu}\right)\right]F_{\mu\nu}^2-2E_i^2\right\}.\end{aligned}\tag{3.14}$$

Finally, the terms of mass dimension 6 in four space-time dimensions require $I_{1,0}^{-}$, $I_{1,\overline{2}}^{-}$, and $I_{1,\overline{4}}^{-}$. Using the relation $\psi^{(n)}(q)=(-1)^{n+1}n!\zeta(n+1,q)$ [52], one obtains

$$I_{1,0}^{-}=-\left(\frac{\beta}{4\pi}\right)^2\left[\psi''\left(\frac{1}{2}+\nu\right)+\psi''\left(\frac{1}{2}-\nu\right)\right]+O(\epsilon),$$

$$I_{1,\overline{2}}^{-}=-2I_{1,0}^{-}+O(\epsilon),\quad I_{1,\overline{4}}^{-}=-4I_{1,0}^{-}+O(\epsilon).\tag{3.15}$$

[All these integrals are related through simple proportionality factors, as follows from Eq. (C6).] This yields

$$\mathcal{L}_{3,q}(x) = -\frac{2}{(4\pi)^4}\frac{N_f}{T^2}\mathrm{tr}\Bigg[\Bigg[\psi''\left(\frac{1}{2}+\overline{\nu}\right)+\psi''\left(\frac{1}{2}-\overline{\nu}\right)\Bigg]\times\left(\frac{8}{45}F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}+\frac{1}{24}F^2_{\lambda\mu\nu}-\frac{1}{60}F^2_{\mu\mu\nu}+\frac{1}{20}F^2_{0\mu\nu}-\frac{1}{30}E^2_{ii}+\frac{1}{15}E_iF_{ij}E_j\right)\Bigg]. \quad (3.16)$$

In all these formulas $\overline{\nu}$ is the matrix $\log(\Omega)/(2\pi i)$ in the fundamental representation and in the branch $|\overline{\nu}|<\frac{1}{2}$ in the eigenvalue sense. Note the hierarchy in powers of temperature, $\mathcal{L}_0\sim T^4$, $\mathcal{L}_2\sim T^0$, $\mathcal{L}_3\sim T^{-2}$, implying that the heat kernel expansion at finite temperature is essentially an expansion on k^2/T^2 with k the typical gluon momentum. Terms of order T^2 are forbidden since there is no available gauge invariant operator of dimension 2.

B. Gluon sector

In the background field approach [51] the gluon field is split into a classical field plus a quantum fluctuation—i.e., $A_\mu\to A_\mu+a_\mu$ in the action (3.1). As is standard in the effective action formalism, the appropriate currents are added so that the classical field A_μ is a solution of the equations of motion (and so no terms linear in the fluctuation remain). The one-loop effective action corresponds then to neglect contributions beyond the quadratic terms in the quantum fluctuations and integrate over a_μ. (The quark fields are taken as pure fluctuation, so a_μ does not change the quark sector at one loop.)

The quadratic piece of the gluon action is

$$S^{(2)} = -\frac{1}{g^2}\int d^4x\,\mathrm{tr}[-a_\nu\hat{D}^2_\mu a_\nu-2a_\mu[F_{\mu\nu},a_\nu]-(\hat{D}_\mu a_\mu)^2]. \quad (3.17)$$

Here all covariant derivatives are those associated to the classical gluon field A_μ. Note that the first two terms are of the standard Klein-Gordon form, but the last one is not. Before doing the functional integration over a_μ one has to fix the gauge of these fields. This implies adding a gauge fixing term and the corresponding Faddeev-Popov term [53] in the action. We take the covariant Feynman gauge $\hat{D}_\mu a_\mu=f(x)$, since the associated gauge fixing action precisely cancels the offending term $(\hat{D}_\mu a_\mu)^2$ in Eq. (3.17). After adding the ghost term one has

$$S^{(2)} = -\frac{1}{g^2}\int d^4x\,\mathrm{tr}[-a_\nu\hat{D}^2_\mu a_\nu-2a_\mu[F_{\mu\nu},a_\nu]-\overline{C}\hat{D}^2_\mu C]. \quad (3.18)$$

The coupling constant has no effect here since it can be absorbed in the normalization of the fields. The ghost fields C and $\overline{C}$ are anticommuting (although periodic in Euclidean time) and are matrices in the fundamental representation of su(N).

The full effective action (to one loop) is

$$\Gamma[A] = -\frac{\mu^{-2\epsilon}}{2g_0^2}\int d^Dx\,\mathrm{tr}(F^2_{\mu\nu})+\Gamma_q[A]+\Gamma_g[A], \quad (3.19)$$

where the first piece is the tree level action (accounting for renormalization; g_0 is dimensionless), the second one is the quark contribution, obtained in the previous subsection, and the last term follows from functional integration over a_μ and C, $\overline{C}$ in Eq. (3.18):

$$\Gamma_g[A] = \frac{1}{2}\mathrm{Tr}\log(-\hat{D}^2_\mu-2\hat{F}_{\mu\nu})-\mathrm{Tr}\log(-\hat{D}^2_\mu) =: \int d^4x\,\mathcal{L}_g(x), \quad (3.20)$$

where $\hat{D}_\mu=[D_\mu,\ ]$ and $\hat{F}_{\mu\nu}=[F_{\mu\nu},\ ]$. From Eq. (3.18), we can see that the Klein-Gordon operator over the gluon field a_μ acts on an internal space of dimension $D\times(N^2-1)$, where $D=4-2\epsilon$ is the number of gluon polarizations (including the two unphysical ones) and corresponds to the Lorentz index μ, and N^2-1 is the dimension of the adjoint representation of the group. $\hat{D}_\mu$ and $\hat{F}_{\mu\nu}$ act in the adjoint representation. The covariant derivative of the Klein-Gordon operator is the identity in the Lorentz space whereas the "mass term" is a matrix in that space: namely, $(M)_{\mu\nu}=-2\hat{F}_{\mu\nu}$. Similarly, the space of the Klein-Gordon operator over the ghost fields has dimension N^2-1, the mass term is zero, and the corresponding covariant derivative is just D_μ but in the adjoint representation.

Applying once again the heat kernel representation, we have

$$\mathcal{L}_g(x) = -\frac{1}{2}\int_0^\infty\frac{d\tau}{\tau}\frac{\mu^{2\epsilon}}{(4\pi\tau)^{D/2}}\sum_n \tau^n\widehat{\mathrm{tr}}(b^T_{n,g}), \quad (3.21)$$

where, for convenience, the Lorentz trace over gluons as well as the ghost contribution are included in the coefficient $b^T_{n,g}$. Here $\widehat{\mathrm{tr}}$ denotes the color trace in the adjoint representation. A straightforward calculation yields

$$b_{0,g}^T=(D-2)\varphi_0(\hat{\Omega}),$$

$$b_{2,g}^T=\left(-2+\frac{D-2}{12}\right)\varphi_0(\hat{\Omega})\hat{F}_{\mu\nu}^2-\frac{D-2}{6}\overline{\varphi}_2(\hat{\Omega})\hat{E}_i^2,$$

$$b_{3,g}^T=\varphi_0(\hat{\Omega})\left[\left(\frac{4}{3}+\frac{D-2}{90}\right)\hat{F}_{\mu\nu}\hat{F}_{\nu\lambda}\hat{F}_{\lambda\mu}+\frac{1}{3}\hat{F}_{\lambda\mu\nu}^2-\frac{D-2}{60}\hat{F}_{\mu\mu\nu}^2\right]+\frac{1}{6}\overline{\varphi}_2(\hat{\Omega})\left(-2\hat{F}_{0\mu\nu}^2+\frac{D-2}{10}(\hat{E}_{ii}^2+\hat{F}_{0ij}^2-2\hat{E}_i\hat{F}_{ij}\hat{E}_j)\right)+(D-2)\left(\frac{1}{10}\overline{\varphi}_4(\hat{\Omega})-\frac{1}{6}\overline{\varphi}_2(\hat{\Omega})\right)\hat{E}_{0i}^2. \quad (3.22)$$

The coefficients $b_{1,g}^T$ and $b_{5/2,g}^T$ vanish, as do all terms with a single M, due to the Lorentz trace. The contributions with $D-2$ come from pieces without M in Eqs. (2.38) and (2.34). The effect of the ghost is to remove two gluon polarizations, $D\to D-2$. Unlike the fermionic case, the thermal heat kernel coefficients depend explicitly on the space-time dimension through these polarization factors. In these formulas the functions φ_n correspond to their bosonic versions. In addition, its argument $\hat{\Omega}$ and all field strength tensor and covariant derivatives are in the adjoint representation.

We can now proceed to the calculation of the effective Lagrangian. We note that the integrals over τ are no different to those for the quark sector [see Eq. (3.8) and Appendix E], after the replacement $\nu\to\nu-\frac{1}{2}$ [coming from $\varphi_n^+(\omega)=\varphi_n^-(-\omega)$] and so $0<\nu<1$ now:

$$I_{\ell,2n}^+(e^{2\pi i\nu})=(-1)^n(4\pi)^\epsilon\left(\frac{\mu\beta}{2\pi}\right)^{2\epsilon}\left(\frac{\beta}{2\pi}\right)^{2\ell}\times\frac{\Gamma\left(\ell+n+\epsilon+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}[\zeta(1+2\ell+2\epsilon,\nu)+\zeta(1+2\ell+2\epsilon,1-\nu)],\quad 0<\nu<1. \quad (3.23)$$

In this way, for the effective potential one obtains

$$\mathcal{L}_{0,g}(x)=\frac{\pi^2}{3}T^4\widehat{\mathrm{tr}}[B_4(\hat{\nu})+B_4(1-\hat{\nu})] \quad (3.24)$$

$$=-\frac{\pi^2}{45}T^4(N^2-1)+\frac{2\pi^2}{3}T^4\,\widehat{\mathrm{tr}}[\hat{\nu}^2(1-\hat{\nu})^2],$$

$$\hat{\nu}=\log(\hat{\Omega})/(2\pi i),\quad 0<\hat{\nu}<1. \quad (3.25)$$

This is also in agreement with the well-known result [9]. We emphasize that $\hat{\Omega}$ and $\hat{\nu}$ are now in the adjoint representation as indicated by the notation $\widehat{\mathrm{tr}}$.

The mass dimension-4 piece of the effective Lagrangian, coming from $b_{2,g}^T$, requires $I_{0,0}^+$ which is ultraviolet divergent and $I_{0,\underline{2}}^+$ which is UV finite [cf. Eq. (3.13)]. The finite pieces, in the $\overline{\mathrm{MS}}$ scheme, are found to be

$$\mathcal{L}_{2,g}(x)=\frac{1}{(4\pi)^2}\widehat{\mathrm{tr}}\left[\frac{11}{12}\left(2\log(\mu/4\pi T)+\frac{1}{11}-\psi(\hat{\nu})-\psi(1-\hat{\nu})\right)\hat{F}_{\mu\nu}^2-\frac{1}{3}\hat{E}_i^2\right],\quad 0<\hat{\nu}<1. \quad (3.26)$$

On the other hand, the divergent contribution in the gluon sector, combined with that in the quark sector and the tree level Lagrangian, yields (all terms have been multiplied by the factor $\mu^{2\epsilon}$ to restore dimensions)

$$\mathcal{L}_{\mathrm{tree}}(x)+\mathcal{L}_q^{\mathrm{div}}(x)+\mathcal{L}_g^{\mathrm{div}}(x)=-\frac{1}{2g_0^2}\mathrm{tr}(F_{\mu\nu}^2)+\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}+\log(4\pi)-\gamma_E\right)\times\left(\frac{11}{12}\widehat{\mathrm{tr}}(\hat{F}_{\mu\nu}^2)-\frac{N_f}{3}\mathrm{tr}(F_{\mu\nu}^2)\right). \quad (3.27)$$

Use of the SU(N) identity (E5) yields the renormalized tree level Lagrangian

$$\mathcal{L}_{\mathrm{tree}}(x)+\mathcal{L}_q^{\mathrm{div}}(x)+\mathcal{L}_g^{\mathrm{div}}(x)=-\frac{1}{2g^2(\mu)}\mathrm{tr}(F_{\mu\nu}^2), \quad (3.28)$$

with the standard one-loop renormalization group improved in the $\overline{\mathrm{MS}}$ scheme,

$$\frac{1}{g^2(\mu)}=\frac{1}{g_0^2}-\beta_0\left(\frac{1}{\epsilon}+\log(4\pi)-\gamma_E\right),$$

$$\beta_0=\frac{1}{(4\pi)^2}\left(\frac{11}{3}N-\frac{2}{3}N_f\right), \quad (3.29)$$

guaranteeing the scale independence of Eq. (3.27). Note that, due to gauge invariance, the classical fields A_μ do not need ultraviolet renormalization. (In the context of the dimensionally reduced effective theory, finite, temperature-dependent, renormalization has been found to be useful in practice [40,43]. See Sec. IV.)

Putting together all terms of mass dimension 4 (renormalized tree level plus one loop), we find

$$\mathcal{L}_2(x)=\left(-\frac{1}{2g^2(\mu)}+\beta_0\,\log(\mu/4\pi T)+\frac{1}{6}\frac{1}{(4\pi)^2}N\right)\times\mathrm{tr}(F_{\mu\nu}^2)-\frac{11}{12}\frac{1}{(4\pi)^2}\widehat{\mathrm{tr}}\{[\psi(\hat\nu)+\psi(1-\hat\nu)]\hat F_{\mu\nu}^2\}+\frac{1}{3}\frac{1}{(4\pi)^2}N_f\,\mathrm{tr}\left\{\left[\psi\left(\frac{1}{2}+\overline{\nu}\right)+\psi\left(\frac{1}{2}-\overline{\nu}\right)\right]F_{\mu\nu}^2\right\}-\frac{2}{3}(N-N_f)\frac{1}{(4\pi)^2}\mathrm{tr}[E_i^2],$$

$$-\frac{1}{2}<\overline{\nu}<\frac{1}{2},\ 0<\hat\nu<1. \tag{3.30}$$

The terms of mass dimension 6 are easily obtained from the coefficient $b_{3,g}^T$ and the integrals $I_{1,0}^+$, $I_{1,\overline{2}}^+$, and $I_{1,\overline{4}}^+$:

$$\mathcal{L}_{3,g}(x)=\frac{1}{2}\frac{1}{(4\pi)^4}\frac{1}{T^2}\widehat{\mathrm{tr}}\Bigg[[\psi''(\hat\nu)+\psi''(1-\hat\nu)]\times\left(\frac{61}{45}\hat F_{\mu\nu}\hat F_{\nu\lambda}\hat F_{\lambda\mu}+\frac{1}{3}\hat F_{\lambda\mu\nu}^2-\frac{1}{30}\hat F_{\mu\mu\nu}^2+\frac{3}{5}\hat F_{0\mu\nu}^2-\frac{1}{15}\hat E_{ii}^2+\frac{2}{15}\hat E_i\hat F_{ij}\hat E_j\right)\Bigg]. \tag{3.31}$$

Note again the hierarchy in powers of temperature, $\mathcal{L}_0\sim T^4$, $\mathcal{L}_2\sim T^0$, $\mathcal{L}_3\sim T^{-2}$.

C. Infrared divergence and other renormalization schemes

The integrals $I^{\pm}\ell,n$ may contain not only ultraviolet divergences but also infrared ones (corresponding to the large τ region). Specifically, this happens if $\ell\geqslant 0$, $n=0$, and $e^{2\pi i\nu}=\pm 1$ (see Appendix C). In the quark sector (i.e., in the fundamental representation) and for a generic configuration of $A_0(x)$, no eigenvalue of Ω will be -1 in the bulk and so such divergence can be disregarded. Unfortunately, in the gluon sector the situation is different since for any gauge configuration at least $N-1$ eigenvalues of $\hat\Omega(x)$ are necessarily unity. Therefore, the singular value $\nu=$integer always appears when evaluating the adjoint trace in $\mathcal{L}_{2,g}$ and $\mathcal{L}_{3,g}$. The infrared divergences are characteristic of massless theories at finite temperature [35,36].

For $\nu=0$, the infrared divergence comes solely from the static Matsubara mode, $p_0=0$, in φ_0. The corresponding integral over τ has no natural scale and so the point of view can be taken that such divergences are automatically removed by dimensional regularization [54]. As explained in Appendix C, the integrals $I_{\ell,2n}^+$ without the static mode are given by the same expressions (3.23) after the replacement $\nu\to 1+\nu$ in the first ζ function. The resulting prescription is then to use the formulas of $\mathcal{L}_{2,g}$ and $\mathcal{L}_{3,g}$ with the replacements

$$\psi(\hat\nu)+\psi(1-\hat\nu)|_{\hat\nu=0}\to\psi(1+\hat\nu)+\psi(1-\hat\nu)|_{\hat\nu=0}=-2\gamma_E,$$

$$\psi''(\hat\nu)+\psi''(1-\hat\nu)|_{\hat\nu=0}\to\psi''(1+\hat\nu)+\psi''(1-\hat\nu)|_{\hat\nu=0}=-4\zeta(3), \tag{3.32}$$

to be made in the subspace $\hat\Omega=1$ only, when taking the trace in the adjoint representation. One may worry that subtracting this subspace is not consistent with gauge invariance. This is not so. As will be discussed below, the periodicity of the effective action as a function of $\log(\hat\Omega)$ is an important requirement. This property is not spoiled by the previous prescriptions.

Alternatively, one can regulate the infrared divergence by including a cutoff function $e^{-m^2\tau}$ in the τ integral. The infrared finite modes are unaffected in the limit of small m. The static mode in φ_0 develops powerlike divergences to be added to the result obtained through dimensional regularization. These terms are easily computed and are[3]

$$\mathcal{L}_{2,\mathrm{IR}}=\frac{1}{48\pi}\frac{T}{m}\mathrm{tr}[11F_{\mu\nu\perp}^2+2E_{i\perp}^2],$$

$$\mathcal{L}_{3,\mathrm{IR}}=\frac{1}{240\pi}\frac{T}{m^3}\mathrm{tr}\Bigg[-\frac{61}{3}F_{\mu\nu\perp}F_{\nu\alpha}F_{\alpha\mu}+E_{i\perp}F_{ij}E_{j\perp}+E_iF_{ij\parallel}E_j-5F_{\mu\nu\lambda\perp}^2+\frac{1}{2}F_{\mu\mu\nu\perp}^2+\frac{9}{2}F_{0\mu\nu\perp}^2+3E_{0i\perp}^2-\frac{1}{2}E_{ii\perp}^2\Bigg]. \tag{3.33}$$

Even though this is a gluonic term, the result has been expressed in the fundamental representation, which is often preferable. [Unfortunately this is not so easily done for the other gluonic contributions, for a general SU(N) group, due to the presence of the Polyakov loop in the formulas.] In these expressions we have used the notation $F_{\mu\nu\parallel}$ to denote the pieces of $F_{\mu\nu}$ which commute with Ω and $F_{\mu\nu\perp}$ for the remainder. Specifically, in the gauge in which Ω is diagonal, $F_{\mu\nu\parallel}$ is the diagonal part of $F_{\mu\nu}$. As shown in Appendix E, only terms involving at least one perpendicular component may be infrared divergent, and this is verified in Eqs. (3.33).

We have used here the $\overline{\mathrm{MS}}$ scheme in dimensional regularization. Alternatively one can use Pauli-Villars regularization which amounts to inserting a regulating factor $(1-e^{-\tau M^2})$ in the τ integration [33]. All convergent integrals (including $I_{0,\overline{2}}^{\pm}$) are unchanged in the limit of large M, whereas

[3]Note that $I_{\ell,\overline{2n}}^+$ also contains φ_0 and so is also afflicted by the divergence. This implies that introducing $e^{-m^2\tau}$ is not equivalent to a regularization of the digamma function (and its derivatives) in the final formulas, since simple scaling relations of the type (3.15) or (C6) no longer hold.

$$I_{0,0}^{+,\mathrm{PV}}=2\log(M/\mu)+2\log(\mu\beta/4\pi)-\psi(\nu)-\psi(1-\nu)+O(M^{-1}),\quad 0<\nu<1,$$

$$I_{0,0}^{-,\mathrm{PV}}=2\log(M/\mu)+2\log(\mu\beta/4\pi)-\psi\left(\frac{1}{2}+\nu\right)-\psi\left(\frac{1}{2}-\nu\right)+O(M^{-1}),\quad -\frac{1}{2}<\nu<\frac{1}{2}. \tag{3.34}$$

(Note that these formulas do not actually depend on the scale μ.) The Pauli-Villars-renormalized result is obtained by combining $\log(M^2/\mu^2)$ with the bare coupling constant in the tree level Lagrangian to yield the renormalized coupling constant $g_{\mathrm{PV}}(\mu)$. If, as usual, the Λ_R parameter in the scheme R is defined as the scale $\mu=\Lambda_R$ for which $1/g_R^2(\mu)$ vanishes, it is found that the Pauli-Villars and $\overline{\mathrm{MS}}$ schemes give identical renormalized results, at one loop, when

$$\log(\Lambda_{\mathrm{PV}}^2/\Lambda_{\overline{\mathrm{MS}}}^2)=\frac{1}{11-2N_f/N}. \tag{3.35}$$

The difference between both scales comes from the $\frac{1}{11}$ in Eq. (3.26), which is due to the -2ϵ extra gluon polarizations in the dimensional regularization scheme [55].

D. Results for SU(2) and SU(3)

We can particularize our formulas for SU(2) by working out the color traces explicitly. We use the anti-Hermitian su(2) basis $\vec{\sigma}/2i$, so

$$A_0=-\frac{i}{2}\vec{\sigma}\cdot\vec{A}_0,\quad F_{\mu\nu}=-\frac{i}{2}\vec{\sigma}\cdot\vec{F}_{\mu\nu},\quad \text{etc.} \tag{3.36}$$

It is convenient to choose the "Polyakov gauge," in which A_0 is time independent and diagonal [46]. In SU(2), $A_0=-\frac{1}{2}i\sigma_3\phi$. In this case the eigenvalues of the Polyakov loop in the fundamental representation are $\exp(\pm i\beta\phi/2)$, and in the adjoint representation are $\exp(\pm i\beta\phi)$ and 1. Full results for $\mathcal{L}_{0,2,3}(x)$ in both sectors are given in Appendix D. Here we quote the results for $\mathcal{L}_2(x)$ from the gluon and quark loops,

$$\mathcal{L}_{2,q}(x)=\frac{N_f}{48\pi^2}\left\{\left[2\log\left(\frac{\mu}{4\pi T}\right)-\psi\left(\frac{1}{2}+\overline{\nu}\right)-\psi\left(\frac{1}{2}-\overline{\nu}\right)-1\right]\vec{E}_i^2+\left[2\log\left(\frac{\mu}{4\pi T}\right)-\psi\left(\frac{1}{2}+\overline{\nu}\right)-\psi\left(\frac{1}{2}-\overline{\nu}\right)\right]\vec{B}_i^2\right\}, \tag{3.37}$$

with $\overline{\nu}=(\beta\phi/4\pi+1/2)(\mathrm{mod}\,1)-1/2$ and $B_i=\frac{1}{2}\epsilon_{ijk}F_{jk}$ is the magnetic field:

$$\begin{aligned}\mathcal{L}_{2,g}(x)=-\frac{11}{48\pi^2}\Bigg[&\left(2\log(\mu/4\pi T)-\frac{1}{11}-\psi(\hat{\nu})-\psi(1-\hat{\nu})\right)\vec{E}_{i\parallel}^2+\left(\frac{12}{11}\frac{\pi T}{m}+2\log(\mu/4\pi T)-\frac{1}{11}+\gamma_E-\frac{1}{2}\psi(\hat{\nu})-\frac{1}{2}\psi(1-\hat{\nu})\right)\vec{E}_{i\perp}^2\\&+\left(2\log(\mu/4\pi T)+\frac{1}{11}-\psi(\hat{\nu})-\psi(1-\hat{\nu})\right)\vec{B}_{i\parallel}^2\\&+\left(\frac{\pi T}{m}+2\log(\mu/4\pi T)+\frac{1}{11}+\gamma_E-\frac{1}{2}\psi(\hat{\nu})-\frac{1}{2}\psi(1-\hat{\nu})\right)\vec{B}_{i\perp}^2\Bigg].\end{aligned} \tag{3.38}$$

Here $\hat{\nu}=\beta\phi/2\pi\ (\mathrm{mod}\,1)$ and

$$\vec{E}_i=\vec{E}_{i\parallel}+\vec{E}_{i\perp},\quad \vec{B}_i=\vec{B}_{i\parallel}+\vec{B}_{i\perp} \tag{3.39}$$

are the decompositions of the electric and magnetic fields in the directions parallel and perpendicular to $\vec{A}_0$. This decomposition is gauge invariant provided that in a general gauge the parallel direction is that marked by the Polyakov loop vector.

The quark and gluon sector contributions are periodic in ϕ with periods $4\pi T$ and $2\pi T$, respectively. This periodicity in A_0 of the coefficients multiplying the local operators is a consequence of gauge invariance. Indeed, after choosing the Polyakov gauge there is still freedom to make further nonstationary gauge transformations within this gauge. Such transformations (named discrete transformations in [46]) are of the form $U(x_0)=\exp(x_0\Lambda)$, where Λ is a constant diagonal matrix. Its eigenvalues λ_j, $j=1,\dots,N$ [we consider a general SU(N) group in this discussion], are quantized by the requirement of periodicity in x_0. For quarks, $U(x_0)$ must be strictly periodic and hence $\lambda_j=2\pi i n_j/\beta$, $n_j\in\mathbb{Z}$ (the integers n_j are $\boldsymbol{x}$ independent by continuity). Since under a discrete transformation $A_0(\boldsymbol{x})\to A_0(\boldsymbol{x})+\Lambda$, the eigenvalues of $\log(\Omega)/(2\pi i)$ change as $\nu_j\to\nu_j-n_j$. In SU(2) this implies that the effective action in the quark sector must be periodic in ϕ with period $4\pi T$. In the gluon sector, periodicity of $A_\mu(x)$ in x_0 only requires that $U(x_0+\beta)=e^{2\pi i k/N}U(x_0)$, $k=1,\dots,N$, and there is an additional symmetry associated with the center of the gauge group [6,9,56]. That is, $\lambda_j=2\pi i(n_j+k/N)/\beta$ in the absence of quarks (note that k is both $\boldsymbol{x}$ independent and j independent). The eigenvalues of $\log(\hat{\Omega})/(2\pi i)$ change as $\nu_{j\ell}:=\nu_j-\nu_\ell\to\nu_{j\ell}-n_j+n_\ell$ and the effective action in the gluon sector must be invariant under such replacement. In SU(2) it corresponds to periodicity in ϕ

with period $2\pi T$. From this discussion it follows that an expansion in powers of $\log(\Omega)$ breaks gauge invariance under discrete gauge transformations. The local operators $\vec{E}_{i\parallel}^2$, $\vec{B}_{i\parallel}^2$, $\vec{E}_{i\perp}^2$, and $\vec{B}_{i\perp}^2$ are directly gauge invariant.

We can compare these results with those in [33,34]. That work goes beyond ours in that we compute the lowest terms in an expansion in $\hat{D}_0$ whereas in [33,34] all orders in A_0 are retained in the electric sector. On the other hand, unlike [33,34], we treat groups other than SU(2), our gauge field configurations are not stationary, and we consider higher-order terms in the spatial covariant derivatives.[4]

Let us restrict ourselves to stationary gauge configurations and the gluon sector in SU(2), as in [33]. In a notation close to that in [33], the terms of the effective Lagrangian which are quadratic in $F_{\mu\nu}$, but of any order in A_0, are of the form

$$-f_3(\phi)\vec{E}_{i\parallel}^2-f_1(\phi)\vec{E}_{i\perp}^2-h_3(\phi)\vec{B}_{i\parallel}^2-h_1(\phi)\vec{B}_{i\perp}^2 . \tag{3.40}$$

To obtain these SU(2) group structure functions in our expansion we would need to retain terms with two or four spatial indices but any number of commutators $[A_0, \ ]$. Nevertheless, in the parallel space our calculation is complete since all terms of the form $(\hat{D}_0^n F_{\mu\nu})_\parallel$, $n\geqslant 1$, vanish identically in the stationary case. This implies that $f_3(\phi)$ and $h_3(\phi)$ do not get any further contribution beyond those in $\mathcal{L}_{2,g}(x)$, and indeed, after passing to the Pauli-Villars scheme with $\Lambda_{\rm PV}=e^{1/22}\Lambda_{\overline{\rm MS}}$, one verifies that f_3 and h_3 of [33] are reproduced. f_1 is not reproduced to mass dimension 6, but h_1 is reproduced when we retain mass dimension 4 terms only, since in the magnetic sector the calculation in [33] introduces *ad hoc* simplifying approximations which in practice are equivalent to using $\mathcal{L}_{2,g}(x)$.

An important point is that of the periodicity of the structure functions, also emphasized in [33]. In our calculation, the coefficients of the local operators will always be periodic in ϕ due to gauge invariance. Yet this does not imply that the structure functions themselves should be periodic. The ones in the parallel sector, which coincide to all orders with the coefficients in $\mathcal{L}_{2,g}(x)$, will certainly be periodic, but f_1 and h_1 will not be periodic in ϕ. For instance, h_1 receives a contribution from $\mathcal{L}_{3,g}(x)$ of the form $f(\phi)\vec{B}_{0i}^2$ (see Appendix D). The function $f(\phi)$ is periodic and so this contribution is fully gauge invariant. However, the operator $f(\phi)\vec{B}_{0i}^2$ has still to be brought to the standard form in the Eq. (3.40). Using $\vec{B}_{0i}=\vec{A}_0\times\vec{B}_{i\perp}$, it follows that h_1 picks up a gauge invariant but nonperiodic contribution $\phi^2 f(\phi)$. (At this point we disagree with [33] which notes that f_1 needs not be periodic but requires periodicity of h_1.) We also note that in our calculation, f_1 and h_1 are both infrared divergent, whereas in the calculation of [33] only h_1 is divergent. This should indicate that a resummation to all orders in $\hat{D}_0$ of our expansion may remove spurious infrared divergences.

For SU(3) we present explicit results for the effective Lagrangian up to mass dimension 4 included. We use the convention

$$A_0=-\frac{i}{2}\lambda_s A_0^s=-\frac{i}{2}\vec{\lambda}\cdot\vec{A}_0, \quad F_{\mu\nu}=-\frac{i}{2}\lambda_s F_{\mu\nu}^s, \quad \text{etc.,} \tag{3.41}$$

where λ_s, $s=1,\ldots,8$, are the Gell-Mann matrices. In the Polyakov gauge,

$$A_0=-i\frac{\lambda_3}{2}\phi_3-i\frac{\sqrt{3}}{2}\lambda_8\phi_8 . \tag{3.42}$$

The effective Lagrangian from the quark sector can be expressed in terms of the quantities

$$\nu_1=\frac{1}{4\pi T}(\phi_3+\phi_8), \quad \nu_2=\frac{1}{4\pi T}(-\phi_3+\phi_8),$$

$$\nu_3=-\frac{1}{2\pi T}\phi_8 \tag{3.43}$$

as

$$\mathcal{L}_{0,q}=-\frac{\pi^2T^4N_f}{12}\left(-\frac{8}{5}+(1-4\overline{\nu_1^2})^2+(1-4\overline{\nu_2^2})^2 +(1-4\overline{\nu_3^2})^2\right), \tag{3.44}$$

and

[4]The stationarity condition *is* a restriction; there are gauge invariant terms of the effective action functional which are not reconstructible from the stationary case. One might think that starting from the stationary case, all commutators involving A_0 can be promoted to temporal covariant derivatives, with the prescription $[A_0, \ ]\to[D_0, \ ]$. This is consistent with gauge invariance but does not account all possible terms which may appear in the nonstationary case. For instance, $\mathrm{tr}([D_0,E_{i\parallel}]^2)$, which is equivalent to $\mathrm{tr}[(\partial_0 E_{i\parallel})^2]$, is obviously nonidentically zero, but cannot be recovered by the above prescription since $[A_0,E_{i\parallel}]=0$. This argument substantiates our claim that our expansion and that of [33,34] are in fact complementary to each other.

$$\mathcal{L}_{2,q}=\frac{N_f}{24\pi^2}\left[\log\left(\frac{\mu}{4\pi T}\right)-\frac{1}{2}\right]\vec{E}_i^2+\frac{N_f}{24\pi^2}\log\left(\frac{\mu}{4\pi T}\right)\vec{B}_i^2-\frac{N_f}{12(4\pi)^2}[f^-(\nu_1)+f^-(\nu_2)][(F^1_{\mu\nu})^2+(F^2_{\mu\nu})^2+(F^3_{\mu\nu})^2]$$
$$-\frac{N_f}{12(4\pi)^2}[f^-(\nu_1)+f^-(\nu_3)][(F^4_{\mu\nu})^2+(F^5_{\mu\nu})^2]-\frac{N_f}{12(4\pi)^2}[f^-(\nu_2)+f^-(\nu_3)][(F^6_{\mu\nu})^2+(F^7_{\mu\nu})^2]$$
$$-\frac{N_f}{36(4\pi)^2}[f^-(\nu_1)+f^-(\nu_2)+4f^-(\nu_3)](F^8_{\mu\nu})^2-\frac{N_f}{6\sqrt{3}(4\pi)^2}[f^-(\nu_1)-f^-(\nu_2)]F^3_{\mu\nu}F^8_{\mu\nu}, \tag{3.45}$$

where we have defined

$$f^-(\nu)=\psi\left(\frac{1}{2}+\overline{\nu}\right)+\psi\left(\frac{1}{2}-\overline{\nu}\right), \quad \overline{\nu}=\left(\nu+\frac{1}{2}\right)(\mathrm{mod}\,1)-\frac{1}{2}. \tag{3.46}$$

In the gluon sector, we introduce the invariants

$$\nu_{12}=\frac{1}{2\pi T}\phi_3, \quad \nu_{31}=-\frac{1}{4\pi T}(\phi_3+3\phi_8), \quad \nu_{23}=\frac{1}{4\pi T}(-\phi_3+3\phi_8), \tag{3.47}$$

in terms of which the effective Lagrangian is

$$\mathcal{L}_{0,g}(x)=\frac{4}{3}\pi^2T^4\left(-\frac{2}{15}+\hat{\nu}_{12}^2(1-\hat{\nu}_{12})^2+\hat{\nu}_{31}^2(1-\hat{\nu}_{31})^2+\hat{\nu}_{23}^2(1-\hat{\nu}_{23})^2\right) \tag{3.48}$$

and

$$\mathcal{L}_{2,g}(x)=-\frac{1}{(4\pi)^2}\left[11\log\left(\frac{\mu}{4\pi T}\right)-\frac{1}{2}\right]\vec{E}_i^2-\frac{1}{(4\pi)^2}\left[11\log\left(\frac{\mu}{4\pi T}\right)+\frac{1}{2}\right]\vec{B}_i^2-\frac{T}{4\pi m}\left(\vec{E}_{i\perp}^2+\frac{11}{12}\vec{B}_{i\perp}^2\right)$$
$$+\frac{1}{(4\pi)^2}\frac{11}{12}\left(f^+(0)+f^+(\nu_{12})+\frac{1}{2}f^+(\nu_{31})+\frac{1}{2}f^+(\nu_{23})\right)[(F^1_{\mu\nu})^2+(F^2_{\mu\nu})^2]$$
$$+\frac{1}{(4\pi)^2}\frac{11}{12}\left(f^+(0)+\frac{1}{2}f^+(\nu_{12})+f^+(\nu_{31})+\frac{1}{2}f^+(\nu_{23})\right)[(F^4_{\mu\nu})^2+(F^5_{\mu\nu})^2]$$
$$+\frac{1}{(4\pi)^2}\frac{11}{12}\left(f^+(0)+\frac{1}{2}f^+(\nu_{12})+\frac{1}{2}f^+(\nu_{31})+f^+(\nu_{23})\right)[(F^6_{\mu\nu})^2+(F^7_{\mu\nu})^2]+\frac{1}{(4\pi)^2}\frac{11}{12}\left(2f^+(\nu_{12})+\frac{1}{2}f^+(\nu_{31})\right.$$
$$\left.+\frac{1}{2}f^+(\nu_{23})\right)(F^3_{\mu\nu})^2+\frac{1}{(4\pi)^2}\frac{11}{8}[f^+(\nu_{31})+f^+(\nu_{23})](F^8_{\mu\nu})^2+\frac{1}{(4\pi)^2}\frac{11}{4\sqrt{3}}[f^+(\nu_{31})-f^+(\nu_{23})]F^3_{\mu\nu}F^8_{\mu\nu}, \tag{3.49}$$

with

$$f^+(\nu)=\psi(\hat{\nu})+\psi(1-\hat{\nu}) \quad (\nu\notin\mathbb{Z}), \quad \hat{\nu}=\nu \ (\mathrm{mod}\,1),$$
$$f^+(0)=-2\gamma_E. \tag{3.50}$$

Finally, the renormalized tree level is

$$\mathcal{L}_{\mathrm{tree}}(x)=\frac{1}{4g^2(\mu)}\vec{F}^2_{\mu\nu}. \tag{3.51}$$

In the stationary case, the most general structure compatible with SU(3) symmetry, constructed with two E_i's and any number of A_0's, contains six structure functions (see Appendix E)

$$f_{12}(\phi_3,\phi_8)[(E_i^1)^2+(E_i^2)^2]+f_{45}(\phi_3,\phi_8)[(E_i^4)^2+(E_i^5)^2]+f_{67}(\phi_3,\phi_8)[(E_i^6)^2+(E_i^7)^2]+f_{33}(\phi_3,\phi_8)(E_i^3)^2 + f_{88}(\phi_3,\phi_8)(E_i^8)^2+f_{38}(\phi_3,\phi_8)(E_i^3E_i^8) \tag{3.52}$$

(and similarly for B_iB_i, etc.). Our results for $\mathcal{L}_2$ are of this form. Our expressions corresponding to f_{33}, f_{88}, and f_{38} are already correct to all orders in A_0, since all $\hat{D}_0$ operators cancel in the directions 3 and 8 of the adjoint space. More generally for any SU(N) and any structure function, A_0 decomposes $F_{\mu\nu}$ into a parallel component (which commutes with A_0) and a perpendicular component (fully off diagonal in the gauge in which A_0 is diagonal). The structure functions not involving perpendicular components depend periodically on A_0 and can be computed exactly using the appropriate finite order of our expansion (that is, the lowest order at which the corresponding local operator appears in $\mathcal{L}$).

In Appendix E we give further details on the calculation for SU(3) and SU(N).

IV. DIMENSIONALLY REDUCED EFFECTIVE THEORY

As is well known, in the high-temperature limit nonstationary fluctuations become heavy and are therefore suppressed, and one expects QCD to behave as an effective three-dimensional theory for the stationary configurations only [10,11,37–42]. Our previous calculation of the effective action was obtained by separating background from fluctuation and integrating the latter to one loop. Clearly, we can adapt that procedure to obtain the action of the dimensionally reduced effective theory, to be denoted $\mathcal{L}'(\boldsymbol{x})$, by (i) using stationary backgrounds and (ii) taking purely nonstationary fluctuations only—that is, removing the static Matsubara mode in all frequency summations. In addition, there is a further factor β in $\mathcal{L}'(\boldsymbol{x})$ from the time integration. Note that $\mathcal{L}'(\boldsymbol{x})$ is not the effective action (or Lagrangian) of the dimensionally reduced theory but its true action (within the one-loop approximation), in the sense that functional integration over the stationary configurations with $\mathcal{L}'(\boldsymbol{x})$ yields the partition function. Besides taking A_μ stationary, we will assume that A_0 is small (in particular $|\nu|<1$), which is correct in the high-temperature regime. We will come back to this point later.

The static Matsubara mode is not present in the quark sector, so for that sector we simply find $\mathcal{L}'_q(\boldsymbol{x})=\beta\mathcal{L}_q(\boldsymbol{x})$. Likewise, the removal of the static mode is irrelevant in the ultraviolet region; hence, $\mathcal{L}'_{\rm tree}(\boldsymbol{x})=\beta\mathcal{L}_{\rm tree}(\boldsymbol{x})$ for the renormalized tree level.

As discussed in Appendix C, the removal of the static mode in the one-loop gluon sector (and for $|\nu|<1$) corresponds to replacing $\zeta(1+2\ell+2\epsilon,\nu)\to\zeta(1+2\ell+2\epsilon,1+\nu)$ in Eq. (3.23). For the effective potential this means $B_4(\hat{\nu})\to B_4(1+\hat{\nu})$ in Eq. (3.24), and so (dropping an A_0-independent term)

$$\mathcal{L}'_{0,g}(\boldsymbol{x})=\frac{2\pi^2}{3}T^3\,\widehat{\rm tr}[\hat{\nu}^2(1+\hat{\nu}^2)],\quad \hat{\nu}=\log(\hat{\Omega})/(2\pi i), \quad -1\le\hat{\nu}\le 1. \tag{4.1}$$

The analogous replacement in the mass dimension four and six terms gives [using the identity $\psi(1+\hat{\nu})+\psi(1-\hat{\nu})=\psi(\hat{\nu})+\psi(-\hat{\nu})$]

$$\mathcal{L}'_{2,g}(\boldsymbol{x})=\frac{1}{(4\pi)^2T}\widehat{\rm tr}\left[\frac{11}{12}\left(2\log(\mu/4\pi T)+\frac{1}{11}-\psi(\hat{\nu})-\psi(-\hat{\nu})\right)\hat{F}_{\mu\nu}^2-\frac{1}{3}\hat{E}_i^2\right], \tag{4.2}$$

$$\mathcal{L}'_{3,g}(\boldsymbol{x})=\frac{1}{2}\frac{1}{(4\pi)^4}\frac{1}{T^3}\widehat{\rm tr}\Big[(\psi''(\hat{\nu})+\psi''(-\hat{\nu}))\times\Big(\frac{61}{45}\hat{F}_{\mu\nu}\hat{F}_{\nu\lambda}\hat{F}_{\lambda\mu}+\frac{1}{3}\hat{F}_{\lambda\mu\nu}^2-\frac{1}{30}\hat{F}_{\mu\mu\nu}^2+\frac{3}{5}\hat{F}_{0\mu\nu}^2-\frac{1}{15}\hat{E}_{ii}^2+\frac{2}{15}\hat{E}_i\hat{F}_{ij}\hat{E}_j\Big)\Big]. \tag{4.3}$$

In these expressions $\hat{D}_0$ stands for $[A_0,\ ]$. Note that, having removed the static mode, $\mathcal{L}'(\boldsymbol{x})$ is free from infrared divergences.

At high temperature the effective potential suppresses configurations with $\Omega(\boldsymbol{x})$ far from unity, so by means of a suitable gauge transformation we can assume that $A_0(\boldsymbol{x})$ is small.[5] In the absence of quarks, the situation is similar although in this case $\Omega(\boldsymbol{x})$ lies near a center of the group element; the center symmetry is spontaneously broken signaling the deconfining phase [5,11,56]. After a suitable generalized (many-valued) gauge transformation the configuration can be brought to the small $A_0(\boldsymbol{x})$ region. It can be noted that only when A_0 is small ($|\nu|<1$) the non static fluctuations are the heavy ones. If we were to choose the gauge so that ν is near some other integer value n, the light mode would be the nth Matsubara mode and integrating out this light mode would yield a nonlocal (and so nonuseful) action for the effective theory.

[5]To bring A_0 to the $|\nu|<1$ basin it will be necessary to use a discrete gauge transformation, as described in the paragraph after Eq. (3.39). Because such transformations are global ($\boldsymbol{x}$ independent), this will be only possible if the original $A_0(\boldsymbol{x})$ lies in the same basin (i.e., near the same integer ν) for all $\boldsymbol{x}$. We assume this, since otherwise $\Omega(\boldsymbol{x})$ would be far from unity in the crossover region, thereby increasing the energy [57].

Because A_0 is small, it is standard to expand the $\mathcal{L}'(\boldsymbol{x})$ in powers of A_0, using the relation $\nu=-A_0/(2\pi i T)$ either in the fundamental or the adjoint representations. We expand up to and including terms of dimension 6, where now A_0 coming from Ω counts as dimension 1. Note that this new counting is free from any ambiguity (although it is in conflict with explicit gauge invariance).

The effective potential is already a polynomial in A_0. From Eqs. (3.11) and (4.1), we obtain

$$\mathcal{L}'_0(\boldsymbol{x})=-\left(\frac{N}{3}+\frac{N_f}{6}\right)T\langle A_0^2\rangle+\frac{1}{4\pi^2 T}\langle A_0^2\rangle^2 + \frac{1}{12\pi^2 T}(N-N_f)\langle A_0^4\rangle. \tag{4.4}$$

We have introduced the shorthand notation $\langle X\rangle := \mathrm{tr}(X)$ (trace in the fundamental representation) and used the SU(N) identity (E6). This result agrees with [40,43] (there written in the adjoint representation).

In particular for SU(2) and SU(3), using the identity (E7) valid for those groups, we find

$$\mathcal{L}'_0(\boldsymbol{x})=-\left(\frac{N}{3}+\frac{N_f}{6}\right)T\langle A_0^2\rangle+\frac{1}{24\pi^2 T}(6+N-N_f)\langle A_0^2\rangle^2, \quad N=2,3, \tag{4.5}$$

which reproduces the result quoted in [10] and [11] for $N=3$. We note that consistency requires to include up to two-loop contributions in the effective potential [58].

The terms of dimension four with derivatives come from $\mathcal{L}'_2(\boldsymbol{x})$, given essentially in Eq. (3.30) [with $\psi(1-\hat{\nu})\to\psi(-\hat{\nu})$ and an extra factor β], and setting $\overline{\nu}$ and $\hat{\nu}$ to zero. The result can be written as (the subindex 4 indicates operators of dimension 4, and all gluon fields count as mass dimension 1)

$$\mathcal{L}'_{(4)}(\boldsymbol{x})=-\frac{1}{T g_E^2(T)}\langle E_i^2\rangle-\frac{1}{T g_M^2(T)}\langle B_i^2\rangle \tag{4.6}$$

(once again in the fundamental representation). For the (chromo)electric and magnetic effective couplings we find

$$\frac{1}{g_E^2(T)}=\frac{1}{g^2(\mu)}-2\beta_0[\log(\mu/4\pi T)+\gamma_E] + \frac{1}{3(4\pi)^2}\left[N+8N_f\left(\log 2-\frac{1}{4}\right)\right],$$

$$\frac{1}{g_M^2(T)}=\frac{1}{g^2(\mu)}-2\beta_0[\log(\mu/4\pi T)+\gamma_E] + \frac{1}{3(4\pi)^2}(-N+8N_f\log 2). \tag{4.7}$$

It is possible to rescale A_i and A_0 (with different renormalization factors) so that $\mathcal{L}'_{(4)}(\boldsymbol{x})$ looks like the zero-temperature renormalized tree level (3.28) [39,40,43]. However, we will work with the original variables.

The result for $g_M^2(T)$ coincides with [10] for $N=3$. It also agrees with [43] (setting $N_f=0$) assuming a suitable N-dependent factor between the scales Λ there and μ here. The scale-independent ratio

$$\frac{g_E^2(T)}{g_M^2(T)}=1-\frac{2}{3}\frac{g^2(\mu)}{(4\pi)^2}(N-N_f)+O(g^4) \tag{4.8}$$

found here differs from that reference. On the other hand, in analogy with

$$\frac{1}{g^2(\mu)}=2\beta_0\log(\mu/\Lambda_{\overline{\mathrm{MS}}}), \tag{4.9}$$

magnetic and electric thermal Λ parameters can be introduced [44]:

$$\frac{1}{g_{E,M}^2(T)}=2\beta_0\log(T/\Lambda_{E,M}^T), \tag{4.10}$$

which set the scale of high temperatures for both coupling constants. For the magnetic sector we find

$$\log(\Lambda_M^T/\Lambda_{\overline{\mathrm{MS}}})=\gamma_E-\log(4\pi)+\frac{N-8N_f\log 2}{22N-4N_f}, \tag{4.11}$$

in agreement with [44].

Next, we consider terms of dimension 6. They come from $\mathcal{L}'_2(\boldsymbol{x})$ expanding the digamma functions to second order in ν and from $\mathcal{L}'_3(\boldsymbol{x})$ to zeroth order. From the quark sector we obtain

$$\mathcal{L}'_{(6),q}(\boldsymbol{x})=\frac{28}{45}\zeta(3)\frac{\beta^3}{(4\pi)^4}N_f\Big\langle F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}+6F_{\mu\mu\nu}^2 + \frac{9}{2}F_{0\mu\nu}^2+30A_0^2F_{\mu\nu}^2-3E_{ii}^2+6E_iF_{ij}E_j\Big\rangle, \tag{4.12}$$

where we have made use of the identity $F_{\lambda\mu\nu}^2=2F_{\mu\mu\nu}^2-4F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}$, valid inside the functional trace [43]. For gluons we have, instead,

$$\mathcal{L}'_{(6),g}(\boldsymbol{x})=-\frac{2}{45}\zeta(3)\frac{\beta^3}{(4\pi)^4}\widehat{\mathrm{tr}}\Big(\hat{F}_{\mu\nu}\hat{F}_{\nu\lambda}\hat{F}_{\lambda\mu}+\frac{57}{2}\hat{F}_{\mu\mu\nu}^2 + 27\hat{F}_{0\mu\nu}^2+165\hat{A}_0^2\hat{F}_{\mu\nu}^2-3\hat{E}_{ii}^2+6\hat{E}_i\hat{F}_{ij}\hat{E}_j\Big). \tag{4.13}$$

Using Eqs. (E5) and (E6), this gives, for the full result,

$$\mathcal{L}'_{(6)}(\boldsymbol{x}) = -\frac{2}{15}\frac{\zeta(3)}{(4\pi)^4T^3}\Bigg[\left(\frac{2}{3}N-\frac{14}{3}N_f\right)\langle F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}\rangle + (19N-28N_f)\langle F^2_{\mu\mu\nu}\rangle + (18N-21N_f)\langle F^2_{0\mu\nu}\rangle + (110N-140N_f)\langle A_0^2F^2_{\mu\nu}\rangle - (2N-14N_f)\langle E^2_{ii}\rangle + (4N-28N_f)\langle E_iF_{ij}E_j\rangle + 110\langle A_0^2\rangle\langle F^2_{\mu\nu}\rangle + 220\langle A_0F_{\mu\nu}\rangle^2\Bigg]. \tag{4.14}$$

For SU(2) and SU(3) the term with $\langle A_0^2F^2_{\mu\nu}\rangle$ can be eliminated by using the identity (E7). In addition, in SU(2) the term with $\langle F^2_{0\mu\nu}\rangle$ can also be removed using Eq. (E8). This produces

$$\mathcal{L}'_{(6)}(\boldsymbol{x}) = -\frac{2}{15}\frac{\zeta(3)}{(4\pi)^4T^3}\Bigg[(3-7N_f)\left\langle\frac{2}{3}F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}-\frac{1}{3}F^2_{0\mu\nu} - 2E^2_{ii}+4E_iF_{ij}E_j\right\rangle + (57-28N_f)\langle F^2_{\mu\mu\nu}\rangle + \left(165-\frac{70}{3}N_f\right)(\langle A_0^2\rangle\langle F^2_{\mu\nu}\rangle + 2\langle A_0F_{\mu\nu}\rangle^2)\Bigg], \quad \text{for} \quad N=3, \tag{4.15}$$

$$\mathcal{L}'_{(6)}(\boldsymbol{x}) = -\frac{4}{15}\frac{\zeta(3)}{(4\pi)^4T^3}\Bigg[(2-7N_f)\left\langle\frac{1}{3}F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}-E^2_{ii} + 2E_iF_{ij}E_j\right\rangle + (19-14N_f)\langle F^2_{\mu\mu\nu}\rangle + (74-14N_f) \times\langle A_0^2\rangle\langle F^2_{\mu\nu}\rangle + (146-21N_f)\langle A_0F_{\mu\nu}\rangle^2\Bigg], \quad \text{for} \quad N=2. \tag{4.16}$$

$\mathcal{L}'_{(6)}(\boldsymbol{x})$ has been computed previously in [43] for the gluon sector and arbitrary number of colors. Our result agrees with that calculation (and disagrees with [45]). The dimension-6 Lagrangian in the quark sector has been computed in [45] for SU(3), in the absence of chromomagnetic field ($A_i=0$) and neglecting terms with more than two spatial derivatives (i.e., neglecting E^2_{ii}). Our result reproduces that calculation in that limit as well.

V. CONCLUSIONS

In the present work we have developed in full detail the heat kernel expansion at finite temperature introduced in [27]. We have paid special attention to the role played by the untraced Polyakov loop or thermal Wilson line in maintaining manifest gauge invariance. This is a highly nontrivial problem since preserving gauge invariance at finite temperature requires infinite orders in perturbation theory. The conflict between finite-order perturbation theory and finite-temperature gauge invariance has been previously illustrated, e.g., in the radiatively induced Chern-Simons action of $(2+1)$-dimensional fermionic theories [49]. In the case where the heat bath is chosen to be at rest the Polyakov loop is generated by the imaginary time component of the gauge field and can be regarded as a non-Abelian generalization of the well-known chemical potential. Actually, we have provided arguments supporting this interpretation; if the Polyakov loop was absent or represented in perturbation theory, the particle number could not be fixed, as one expects from standard thermodynamics requirements. The new ingredient of our technique is that a certain combination of the Polyakov loop and the temperature has to be treated as an independent variable, in order to guarantee manifest gauge invariance. This can be done without fixing the gauge.

An immediate application of our method can be found in QCD at finite temperature in the region of phenomenological interest corresponding to the quark-gluon plasma phase. In fact, the heat kernel expansion corresponds in this case to a high-temperature derivative expansion organized in a very efficient way. In the case of QCD the finite-temperature heat kernel expansion can be applied to compute the one-loop effective action stemming from the fermion determinant and from the bosonic determinant corresponding to gluonic fluctuations around a given background field. As a result we have been able to reproduce previous partial calculations and to extend them up to terms of order T^{-2} including the Polyakov loop effects, for a general gauge group SU(N). As a by-product we have computed the action of the dimensionally reduced effective theory to the same order. Further we have studied the emerging group structures in the case of two and three colors.

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APPENDIX A

In this appendix we explain and justify the definition of covariant derivative expansion at finite temperature introduced in [47]. This expansion has been applied in [46,49,59].

When the symbols method (2.2) is used, one starts with a given operator $f(M,D_\mu)$ acting on the space of particle wave functions. Because M and D_μ transform covariantly (homogeneously) under gauge transformations, so do $f(M,D_\mu)$ and $f(M,D_\mu+ip_\mu)$ (since p_μ is just a c number). However, the function $\langle x|f(M,D_\mu+ip_\mu)|0\rangle$ is not gauge covariant in general. For instance, $\langle x|D_\mu|0\rangle=\langle x|(\partial_\mu+A_\mu)|0\rangle=\langle x|A_\mu|0\rangle=A_\mu(x)\langle x|0\rangle=A_\mu(x)$. Gauge invariance is broken by the zero four-momentum state $|0\rangle$ but is recovered in Eq. (2.2) after integration over spatial momenta and summation over the Matsubara frequencies. This is as it should be, since $\langle x|f(M,D_\mu)|x\rangle$ is manifestly covariant (we are assuming ultraviolet convergence of f—e.g., the heat kernel for $\tau>0$). In the spatial case, gauge covariance is recovered because, after integration over momenta, all spatial covariant derivatives appear only in the form of commutators $[D_i,\ ]$. That is, if one drags all D_i to the right (generating commutators), all the noncovariant terms cancel after integration. There is a simple mechanism for this cancellation—namely, if D_i on the right-hand side of Eq. (2.2) is replaced by D_i+ia_i, a_i being a c number, this shift can be absorbed by a redefinition of p_i and nothing changes. Certainly, all terms with D_i in a commutator are manifestly invariant under the shift, but not those with D_i at the right and outside commutators which would develop and a_i-dependent spurious contribution. One concludes that no such noncovariant terms can survive the momentum integration. Indeed, the only way in which D_i can appear gauge covariantly in the effective action functional is through commutators $[D_i,\ ]$. At zero temperature the same holds for D_0; however, at finite temperature, D_0 can appear in two different ways without spoiling gauge invariance—namely, through the commutator $[D_0,\ ]$ and through the Polyakov loop $\Omega(x)$—and in general both are realized on the right-hand side of Eq. (2.2). To see how this comes about in detail, assume we have already carried out the momentum integration and all D_i are in commutators (so the operator is multiplicative regarding $\boldsymbol{x}$ space). This will produce a typical term of the form

$$\mathrm{TT}=\sum_{p_0}\langle x|h_1(D_0+ip_0)Xh_2(D_0+ip_0)Y\cdots|0\rangle, \tag{A1}$$

where $X,Y,\ldots$ are gauge covariant operators constructed with M, F_{ij}, E_i and their spatial covariant derivatives. If we now move the D_0+ip_0 to the left, also generating commutators, we will obtain typical terms of the form

$$\mathrm{TT}=\sum_{p_0}\langle x|h(D_0+ip_0)\hat{D}_0^nX\hat{D}_0^mY\cdots|0\rangle. \tag{A2}$$

(As always, $\hat{D}_0=[D_0,\ ]$.) As we know from Sec. II A, the sum over p_0 produces a one-valued function of $\Omega(x)$:

$$\mathrm{TT}=\tilde{h}(\Omega)\hat{D}_0^nX\hat{D}_0^mY\cdots. \tag{A3}$$

(We can remove $\langle x|\cdot|0\rangle$ since no nonmultiplicative operator remains in the expression, which is to be evaluated at x.) The shift mechanism $D_0\to D_0+ia_0$ does not work in the temporal direction since p_0 is discrete rather than continuous; however, it still implies that the D_0 at the left can appear only through a periodic dependence under $D_0\to D_0+2\pi i/\beta$. This restricts the D_0 not in commutators to appear through the Polyakov loop.

In Eq. (A3) gauge covariance is manifest. If we were to expand $\tilde{h}(\Omega)$ in powers of D_0 [recall Eq. (2.9)] the abovementioned periodicity, and thus gauge invariance, would be spoiled. So our counting is to assign zeroth order to the Polyakov loop and first order to each covariant derivative, either temporal or spatial. In this way, we obtain a natural generalization of the standard expansion in covariant derivatives used at zero temperature, with manifest gauge invariance order by order in the expansion.

APPENDIX B

Let us establish the commutation rules (2.36). It is sufficient to consider the case $[X,f]$ since $\hat{D}_\mu f$ is a particular case. Because f is a function of Ω, it is also a function of D_0 through the relationship $\Omega=e^{-\beta D_0}$. In fact, it is better to prove the relation for a general $f(D_0)$ (not necessarily periodic in its argument). No special property of D_0 is required, so the statement is that, for any two operators X and Y and for any function f,

$$[X,f(Y)]=-f'(Y)[Y,X]+\frac{1}{2}f''(Y)[Y,[Y,X]]+\cdots$$
$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}f^{(n)}(Y)D_Y^n(X),\qquad D_Y:=[Y,\ ]. \tag{B1}$$

It is sufficient to prove this identity for functions of the type $f(Y)=e^{-\lambda Y}$, where λ is a c number, since the general case is then obtained through Fourier decomposition. The RHS of Eq. (B1) is

$$\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}e^{-\lambda Y}D_Y^n(X)=e^{-\lambda Y}(e^{\lambda D_Y}-1)X$$
$$=e^{-\lambda Y}(e^{\lambda Y}Xe^{-\lambda Y}-X)=[X,e^{-\lambda Y}], \tag{B2}$$

which coincides with the LHS of Eq. (B1). We have used the well-known identity $e^{D_Y}(X)=e^YXe^{-Y}$.

APPENDIX C

The basic integrals are

$$I_n^{\pm}(\nu,\alpha):=\int_0^\infty d\tau\ \tau^{\alpha-1}\varphi_n^{\pm}(e^{2\pi i\nu}),$$
$$\nu,\alpha\in\mathbb{R},\quad n=0,1,2,\ldots, \tag{C1}$$

where the functions φ_n are defined in Eq. (2.15) and $\pm$ refers to the bosonic and fermionic versions, respectively. For the bosonic version,

$$I_n^{+}(\nu,\alpha)=\frac{\sqrt{4\pi}}{\beta}\left(\frac{2\pi i}{\beta}\right)^n\sum_{k\in\mathbb{Z}}(k-\nu)^n$$
$$\times\int_0^\infty d\tau\ \tau^{\alpha+(n-1)/2}e^{-(2\pi/\beta)^2(k-\nu)^2\tau},\qquad \nu\notin\mathbb{Z}. \tag{C2}$$

We have excluded the case $e^{2\pi i\nu}=1$ which is discussed below. Integration over τ gives

$$I_n^+(\nu,\alpha)=i^n\frac{\Gamma\left(\alpha+n/2+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{\beta}{2\pi}\right)^{2\alpha}\times\sum_{k\in\mathbb{Z}}\frac{(k-\nu)^n}{|k-\nu|^n}\frac{1}{|k-\nu|^{2\alpha+1}}. \tag{C3}$$

Defining $\nu=k_0+\hat{\nu}$, $0<\hat{\nu}<1$, the sum over k can be split into the sum for $k\leqslant k_0$ and another for $k>k_0$. In terms of the generalized Riemann ζ function [52] this gives

$$I_n^+(\nu,\alpha)=\frac{\Gamma\left(\alpha+n/2+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{\beta}{2\pi}\right)^{2\alpha}[(-i)^n\zeta(2\alpha+1,\hat{\nu})+i^n\zeta(2\alpha+1,1-\hat{\nu})],$$
$$0<\hat{\nu}<1,\quad \nu=k_0+\hat{\nu},\quad k_0\in\mathbb{Z}. \tag{C4}$$

For the fermionic version, using $\varphi_n^-(\omega)=\varphi_n^+(-\omega)$ (and so $\nu\to\nu+\frac{1}{2}$), one obtains

$$I_n^-(\nu,\alpha)=\frac{\Gamma\left(\alpha+n/2+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{\beta}{2\pi}\right)^{2\alpha}\left[(-i)^n\zeta\left(2\alpha+1,\frac{1}{2}+\bar{\nu}\right)+i^n\zeta\left(2\alpha+1,\frac{1}{2}-\bar{\nu}\right)\right],\quad -\frac{1}{2}<\bar{\nu}<\frac{1}{2}. \tag{C5}$$

Note that

$$I_{2n}^\pm(\nu,\alpha)=(-1)^n\frac{\Gamma\left(\alpha+n+\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)}I_0^\pm(\nu,\alpha). \tag{C6}$$

The formulas are consistent with periodicity and parity

$$I_n^\pm(\nu,\alpha)=I_n^\pm(\nu+1,\alpha)=(-1)^nI_n^\pm(-\nu,\alpha). \tag{C7}$$

As discussed in Sec. IV, the dimensionally reduced effective theory for the stationary configurations requires us to remove the static mode from the summation over Matsubara frequencies in the bosonic integrals. This prescription breaks periodicity in ν but this is not relevant for the effective theory, since it only describes the small A_0 (or ν) region. (A prescription that preserves periodicity would be to remove the frequency $k=k_0$ when $\hat{\nu}<\frac{1}{2}$ and $k=k_0+1$ when $\hat{\nu}>\frac{1}{2}$.) The result for the $|\nu|<1$ is

$$I_n'^{\,+}(\nu,\alpha)=\frac{\Gamma\left(\alpha+n/2+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{\beta}{2\pi}\right)^{2\alpha}[(-i)^n\zeta(2\alpha+1,1+\nu)+i^n\zeta(2\alpha+1,1-\nu)],\quad -1<\nu<1. \tag{C8}$$

A related issue is that of the infrared divergences for integer ν. As a result of periodicity, we can restrict the discussion to the case $\nu=0$. For $n\neq0$, the static Matsubara mode does not contribute to $I_n^+(\nu,\alpha)$, and so there is no infrared divergence in this case. On the other hand, in $I_0^+(\nu,\alpha)$, the static mode is either infrared or ultraviolet divergent. In dimensional regularization such an integral [$\nu=k=n=0$ in Eq. (C2)] is defined as zero since it has no natural scale [54]. So for all n the result is equivalent to removing the static mode

$$I_n^+(\nu,\alpha)=I_n'^{\,+}(0,\alpha)=\begin{cases}(-1)^{n/2}2\pi^{-1/2}\Gamma\left(\alpha+\frac{n}{2}+\frac{1}{2}\right)(\beta/2\pi)^{2\alpha}\zeta(2\alpha+1), & \text{even } n\\ 0, & \text{odd } n\end{cases}\quad \text{for } \nu\in\mathbb{Z}. \tag{C9}$$

Alternatively one can regulate the infrared divergence by adding a cutoff function $e^{-m^2\tau}$ $(m\to0)$ in the τ integral. This amounts to adding a contribution $\sqrt{4\pi}\Gamma(\alpha+1/2)/(\beta m^{2\alpha+1})$ in $I_0^+(\nu,\alpha)$ for integer ν.

APPENDIX D

In this appendix we present results for SU(2) in both sectors, including all terms of mass dimension 6. All results are given in the $\overline{\text{MS}}$ scheme. In these formulas we have allowed for an explicit infrared cut off m, as commented at the end of Appendix C. The results with strict dimensional regularization are recovered by removing all infrared divergent terms from the formulas. The conventions are those of Sec. III D:

$$\mathcal{L}_{\rm tree}(x)=\frac{1}{4g^2(\mu)}\vec{F}^2_{\mu\nu}, \tag{D1}$$

$$\mathcal{L}_{0,g}(x)=\frac{\pi^2T^4}{3}\left(-\frac{1}{5}+4\hat{\nu}^2(1-\hat{\nu})^2\right), \tag{D2}$$

$$\begin{aligned}\mathcal{L}_{2,g}(x)=&-\frac{11}{96\pi^2}\left[\frac{1}{11}+2\log\left(\frac{\mu}{4\pi T}\right)-\psi(\hat{\nu})-\psi(1-\hat{\nu})\right]\vec{F}^2_{\mu\nu\parallel}-\frac{11}{96\pi^2}\left[\frac{\pi T}{m}+\frac{1}{11}+2\log\left(\frac{\mu}{4\pi T}\right)+\gamma_E-\frac{1}{2}\psi(\hat{\nu})\right.\\&\left.-\frac{1}{2}\psi(1-\hat{\nu})\right]\vec{F}^2_{\mu\nu\perp}+\frac{1}{24\pi^2}\vec{E}_i^2-\frac{1}{48\pi^2}\left(\frac{\pi T}{m}\right)\vec{E}^2_{i\perp},\end{aligned} \tag{D3}$$

$$\begin{aligned}\mathcal{L}_{3,g}(x)=&\frac{61}{2160\pi^2}\left(\frac{1}{4\pi T}\right)^2\left[8\left(\frac{\pi T}{m}\right)^3+2\zeta(3)-\psi''(\hat{\nu})-\psi''(1-\hat{\nu})\right](\vec{F}_{\mu\nu}\times\vec{F}_{\nu\alpha})\cdot\vec{F}_{\alpha\mu}\\&-\frac{1}{48\pi^2}\left(\frac{1}{4\pi T}\right)^2[\psi''(\hat{\nu})+\psi''(1-\hat{\nu})]\vec{F}^2_{\lambda\mu\nu\parallel}+\frac{1}{96\pi^2}\left(\frac{1}{4\pi T}\right)^2\left[16\left(\frac{\pi T}{m}\right)^3+4\zeta(3)-\psi''(\hat{\nu})-\psi''(1-\hat{\nu})\right]\vec{F}^2_{\lambda\mu\nu\perp}\\&+\frac{1}{480\pi^2}\left(\frac{1}{4\pi T}\right)^2[\psi''(\hat{\nu})+\psi''(1-\hat{\nu})]\vec{F}^2_{\mu\mu\nu\parallel}-\frac{1}{960\pi^2}\left(\frac{1}{4\pi T}\right)^2\left[16\left(\frac{\pi T}{m}\right)^3+4\zeta(3)-\psi''(\hat{\nu})-\psi''(1-\hat{\nu})\right]\vec{F}^2_{\mu\mu\nu\perp}\\&-\frac{3}{80\pi^2}\left(\frac{1}{4\pi T}\right)^2[\psi''(\hat{\nu})+\psi''(1-\hat{\nu})]\vec{F}^2_{0\mu\nu\parallel}+\frac{3}{160\pi^2}\left(\frac{1}{4\pi T}\right)^2\left[-8\left(\frac{\pi T}{m}\right)^3+4\zeta(3)-\psi''(\hat{\nu})-\psi''(1-\hat{\nu})\right]\vec{F}^2_{0\mu\nu\perp}\\&-\frac{1}{10\pi^2}\left(\frac{1}{4\pi T}\right)^2\left(\frac{\pi T}{m}\right)^3\vec{E}^2_{0i\perp}+\frac{1}{240\pi^2}\left(\frac{1}{4\pi T}\right)^2[\psi''(\hat{\nu})+\psi''(1-\hat{\nu})]\vec{E}^2_{ii\parallel}\\&-\frac{1}{480\pi^2}\left(\frac{1}{4\pi T}\right)^2\left[-8\left(\frac{\pi T}{m}\right)^3+4\zeta(3)-\psi''(\hat{\nu})-\psi''(1-\hat{\nu})\right]\vec{E}^2_{ii\perp}+\frac{1}{240\pi^2}\left(\frac{1}{4\pi T}\right)^2[\psi''(\hat{\nu})\\&+\psi''(1-\hat{\nu})]\varepsilon_{ijk}(\vec{E}_i\times\vec{E}_j)\cdot\vec{B}_k+\frac{1}{240\pi^2}\left(\frac{1}{4\pi T}\right)^2\left[8\left(\frac{\pi T}{m}\right)^3-4\zeta(3)-\psi''(\hat{\nu})-\psi''(1-\hat{\nu})\right]\varepsilon_{ijk}(\vec{E}_{i\perp}\times\vec{E}_{j\perp})\cdot\vec{B}_{k\parallel},\end{aligned} \tag{D4}$$

$$\mathcal{L}_{0,q}(x)=\frac{2}{3}\pi^2T^4N_f\left(\frac{2}{15}-\frac{1}{4}(1-4\overline{\nu^2})^2\right), \tag{D5}$$

$$\mathcal{L}_{2,q}(x)=\frac{N_f}{96\pi^2}\left[2\log\left(\frac{\mu}{4\pi T}\right)-\psi\left(\frac{1}{2}+\overline{\nu}\right)-\psi\left(\frac{1}{2}-\overline{\nu}\right)\right]\vec{F}^2_{\mu\nu}-\frac{N_f}{48\pi^2}\vec{E}_i^2, \tag{D6}$$

$$\begin{aligned}\mathcal{L}_{3,q}(x)=&\frac{N_f}{960\pi^2}\left(\frac{1}{4\pi T}\right)^2\left[\psi''\left(\frac{1}{2}+\overline{\nu}\right)+\psi''\left(\frac{1}{2}-\overline{\nu}\right)\right]\left(\frac{16}{3}(\vec{F}_{\mu\nu}\times\vec{F}_{\nu\alpha})\cdot\vec{F}_{\alpha\mu}+\frac{5}{2}\vec{F}^2_{\lambda\mu\nu}-\vec{F}^2_{\mu\mu\nu}\right.\\&\left.-2\varepsilon_{ijk}(\vec{E}_i\times\vec{E}_j)\cdot\vec{B}_k+3\vec{F}^2_{0\mu\nu}-2\vec{E}^2_{ii}\right).\end{aligned} \tag{D7}$$

It can be noted that the quark terms do not distinguish between parallel and perpendicular components. This is due to the fact that in SU(2) an even function of $\overline{\nu}$ [or any other element of su(2)] in the fundamental representation is necessarily a c number. Since the φ_n functions involved to mass dimension 6 are all even, the $\overline{\nu}$ dependence gets out of the trace in Eqs. (3.14) and (3.16) and A_0 is no longer a privileged direction in color space. This mechanism does not act in the adjoint representation—i.e., in the gluon sector—or for other SU(N) groups [cf. Eq. (3.45)].

The infrared divergence is tied to ν integer, so it does not exist for fermions, and also cancels in all gluon terms involving only parallel components.

APPENDIX E

In SU(N) the gauge can be chosen so that A_0 is diagonal. This form is unique (up to permutation of eigenvalues) and produces $N-1$ quantities invariant under SU(N) [ϕ_3, ϕ_8 for SU(3)]. If X represents $F_{\mu\nu}$ or any other element of su(N) [$X^\dagger=-X$, tr$(X)=0$] with N^2-1 independent components, we can use the remaining gauge freedom (the $N-1$ gauge transformations which leave A_0 diagonal) to fix $N-1$ of these components. This adds $(N^2-1)-(N-1)$ new invariants involving X (and A_0). Of these, $N-1$ are linear in X (the diagonal components of X), $N(N-1)/2$ are quadratic, and $(N-1)(N-2)/2$ are cubic. For instance, in SU(3), under a diagonal gauge transformation

$$X=\begin{pmatrix} x & a & b \\ -a^* & y & c \\ -b^* & -c^* & -x-y \end{pmatrix} \to \begin{pmatrix} x & e^{i(\alpha-\beta)}a & e^{i(2\alpha+\beta)}b \\ -e^{-i(\alpha-\beta)}a^* & y & e^{i(\alpha+2\beta)}c \\ -e^{-i(2\alpha+\beta)}b^* & -e^{-i(\alpha+2\beta)}c^* & -x-y \end{pmatrix}, \tag{E1}$$

the invariants are x, y, aa^*, bb^*, cc^*, and ab^*c (the last one is complex but its modulus is not independent). For $X=E_i$ this gives the six structure functions in Eq. (3.52). Each further vector $Y\in$ su(N) produces new N^2-1 invariants.

For computing the traces in the adjoint representation one possibility is to use the adjoint basis $(T^s)_{rt}=f_{rst}$, such that to $F_{\mu\nu}=F^s_{\mu\nu}t_s$ ($t_s=\lambda_s/2i$) in the fundamental representation it corresponds $\hat{F}_{\mu\nu}=F^s_{\mu\nu}T_s$ in the adjoint one. We have also used an alternative approach, as follows. The elements of su(N), such as the gluon quantum fluctuation a_μ, are $N\times N$ matrices, $(a_\mu)_{a\dot{a}}$. From the action $\hat{F}_{\mu\nu}(a_\lambda)=[F_{\mu\nu},a_\lambda]$, it follows

$$(\hat{F}_{\mu\nu})_{a\dot{a},b\dot{b}}=(F_{\mu\nu})_{ab}\delta_{\dot{a}\dot{b}}-\delta_{ab}(F_{\mu\nu})_{\dot{b}\dot{a}},\quad a,b,\dot{a},\dot{b}=1,\dots,N. \tag{E2}$$

In matrix notation this can be written as $\hat{F}_{\mu\nu}=F_{\mu\nu}\otimes 1-1\otimes F^T_{\mu\nu}=F_{\mu\nu}\otimes 1+1\otimes F^*_{\mu\nu}$ or even, in shorter form,

$$\hat{F}_{\mu\nu}=F_{\mu\nu}-F^T_{\mu\nu}=F_{\mu\nu}+F^*_{\mu\nu}, \tag{E3}$$

understanding that $F^T_{\mu\nu}$ or $F^*_{\mu\nu}$ always refer to the dotted space. Similarly, $\hat{A}_\mu=A_\mu-A^T_\mu=A_\mu+A^*_\mu$. Since dotted and undotted operators commute, it follows that the $\hat{\Omega}=\Omega\otimes\Omega^*=\Omega\otimes\Omega^{-1T}$ for the Polyakov loop. In the Polyakov gauge (A_0 stationary and diagonal) Ω is diagonal $(\Omega)_{ab}=\omega_a\delta_{ab}$ and $\hat{\Omega}$ is also diagonal in that basis, $(\hat{\Omega})_{a\dot{a},b\dot{b}}=\omega_{a\dot{a}}\delta_{ab}\delta_{\dot{a}\dot{b}}$, with $\omega_{a\dot{a}}=\omega_a\omega^{-1}_{\dot{a}}$.

From the point of view of the gauge group, the computation of the trace in the adjoint space involves only four different structures appearing in $b^T_{0,g}$, $b^T_{2,g}$, and $b^T_{3,g}$. These are

$$\widehat{\mathrm{tr}}[f(\hat{\Omega})]=\sum_{a\dot{a}}{}' f(\omega_{a\dot{a}}),$$

$$\widehat{\mathrm{tr}}[f(\hat{\Omega})\hat{F}^2_{\mu\nu}]=\sum_{a\dot{a}} f(\omega_{a\dot{a}})[(F^2_{\mu\nu})_{aa}+(F^2_{\mu\nu})_{\dot{a}\dot{a}}-2(F_{\mu\nu})_{aa}(F_{\mu\nu})_{\dot{a}\dot{a}}],$$

$$\widehat{\mathrm{tr}}[f(\hat{\Omega})\hat{F}_{\mu\nu}\hat{F}_{\nu\lambda}\hat{F}_{\lambda\mu}]=\sum_{a\dot{a}} f(\omega_{a\dot{a}})[(F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu})_{aa}+(F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu})_{\dot{a}\dot{a}}-(F_{\mu\nu})_{aa}(F_{\nu\lambda}F_{\lambda\mu})_{\dot{a}\dot{a}}-(F_{\mu\nu}F_{\nu\lambda})_{aa}(F_{\lambda\mu})_{\dot{a}\dot{a}}],$$

$$\widehat{\mathrm{tr}}[f(\hat{\Omega})\hat{E}_i\hat{F}_{ij}\hat{E}_j]=\sum_{a\dot{a}} f(\omega_{a\dot{a}})[(E_iF_{ij}E_j)_{aa}+(E_iF_{ij}E_j)_{\dot{a}\dot{a}}-(E_i)_{aa}([F_{ij},E_j])_{\dot{a}\dot{a}}-([E_i,F_{ij}])_{aa}(E_j)_{\dot{a}\dot{a}}-(E_iE_j)_{aa}(F_{ij})_{\dot{a}\dot{a}}-(F_{ij})_{aa}(E_iE_j)_{\dot{a}\dot{a}}]. \tag{E4}$$

[$\Sigma'_{a\dot{a}}$ in the first equation indicates that one of the N modes with $a=\dot{a}$ should not be included. This removes the singlet mode present in U(N) but not SU(N). The singlet mode does not contribute in the other formulas.] Often, $f(\omega)=f(\omega^{-1})$ [i.e., $f(\omega_{a\dot{a}})$ is symmetric in $a,\dot{a}$], but this property has been not used here. It can be observed that the contributions $a=\dot{a}$, which correspond to $\hat{\Omega}=1$ and are afflicted by infrared divergences, cancel in the subspace parallel (i.e., for $F_{\mu\nu}$ diagonal in the Polyakov gauge).

Useful SU(N) identities are ($\langle\ \rangle$ stands for trace in the fundamental representation)

$$\widehat{\mathrm{tr}}(\hat{X}^2)=2N\langle X^2\rangle,\quad X\in \mathrm{su}(N), \tag{E5}$$

$$\widehat{\mathrm{tr}}(\hat{X}^2\hat{Y}^2)=2N\langle X^2Y^2\rangle+2\langle X^2\rangle\langle Y^2\rangle+4\langle XY\rangle^2,\quad X,Y\in \mathrm{su}(N), \tag{E6}$$

$$\langle X^2Y^2\rangle=-\frac{1}{6}\langle [X,Y]^2\rangle+\frac{1}{6}\langle X^2\rangle\langle Y^2\rangle+\frac{1}{3}\langle XY\rangle^2,\quad X,Y\in \mathrm{su}(3), \tag{E7}$$

$$\langle [X,Y]^2\rangle=-2\langle X^2\rangle\langle Y^2\rangle+2\langle XY\rangle^2,\quad X,Y\in \mathrm{su}(2). \tag{E8}$$

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