Non-Abelian Stokes theorem for SU(2) gauge fields

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We derive a version of non-Abelian Stokes theorem for SU(2) gauge fields in which neither additional integration nor surface ordering are required. The path ordering is eliminated by introducing the instantaneous color orientation of the flux. We also derive the non-Abelian Stokes theorem on the lattice and discuss various terms contributing to the trace of the Wilson loop.

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I. INTRODUCTION

The usual Abelian Stokes theorem relates the integral along a closed curve $\mathcal{C}$ bounding some surface $S_{\mathcal{C}}$ and an integral defined on this surface. This version of the Stokes theorem is most relevant in physical applications since it allows us to express the holonomy of a gauge noninvariant electromagnetic potential via physically observable magnetic flux. The invention of non-Abelian gauge theories necessitated a non-Abelian generalization of the Stokes theorem. Nowadays, there are quite a few formulations of the non-Abelian Stokes theorem (NAST) available (for review see, e.g., Ref. [1] and references therein). Generically there exist two principal approaches: the operator [2–7] and the path-integral [8–13] approaches.

It is worth mentioning that the central issue of any formulation of the NAST is how to make sense of the path ordering prescription inherent to the non-Abelian holonomy (Wilson loop). In this respect neither the operator nor path-integral approach is helpful for concrete calculation of the Wilson loop. Indeed, while in the former case path ordering is traded for a more complicated surface ordering prescription, in the latter case an additional path integral over auxiliary variables is introduced which cannot be calculated even approximately.

In this paper we derive a new version of non-Abelian Stokes theorem focusing exclusively on SU(2)-valued Wilson loops in the fundamental representation. Moreover, the central object of our discussion is the phase of the Wilson loop φ_w. It determines the Wilson loop trace, $1/2\,\mathrm{Tr}\,W = \cos\varphi_w$, which is the only gauge invariant quantity associated with gauge holonomy.

The basic idea is to introduce the instantaneous color orientation of the chromomagnetic flux piercing the loop. Evidently this color orientation remains unknown until the Wilson loop is calculated by some other means. Nevertheless, it allows us to avoid the path ordering and represent the Wilson loop phase as an ordinary integral to which Abelian Stokes theorem applies. Furthermore, we relate the resulting surface integral with properties of gauge fields on this surface.

As we noted above, in order to get explicitly the color orientation of the flux one has to calculate the Wilson loop first. In this respect our formulation is well suited for lattice studies where the gauge holonomy is to be evaluated numerically. We derive the lattice version of the non-Abelian Stokes theorem. Finally, we discuss the physical meaning and origin of various terms contributing to the trace of the Wilson loop and present the results of our qualitative numerical simulations.

II. NON-ABELIAN STOKES THEOREM IN THE CONTINUUM LIMIT

Consider a Wilson loop operator in the fundamental representation, $W(T)$, evaluated on a closed contour $\mathcal{C} = \{x_\mu(t), t \in [0;T], x_\mu(0) = x_\mu(T)\}$, which is parametrized by differentiable functions $x_\mu(t)$. By definition the operator $W(T)$ provides a solution to the first-order differential equation

$$\langle \psi(t)|(i\partial_t + A) = 0, \tag{1}$$

$$\langle \psi(t)| = \langle \psi(0)|W(t), \tag{2}$$

$$W(t) = \mathrm{T}\exp\left\{ i\int_0^t A(\tau)d\tau \right\},$$

where $A = \frac{1}{2}\sigma^a A^a_\mu[x(t)]\dot{x}_\mu(t)$ is the tangential component of the gauge potential, σ^a are the Pauli matrices, differention is taken with respect to the parameter t, and $\langle\psi|$ is a vector in the spin-1/2 irreducible representation space (IRR) of the SU(2) group. Since Eq. (1) is nothing else but the time-dependent Schrödinger equation, the Wilson loop $W(t)$ can be interpreted as a quantum mechanical evolution operator with the time-dependent Hamiltonian $H = -A(t)$. Moreover, the corresponding state space coincides with spin-1/2 IRR, in which a convenient basis is provided by generalized (spin) coherent states [14] (see, e.g., Refs. [15–17] for reviews). The spin coherent states $\{\langle\vec{n}|\}$ are parametrized by a set of unit three-dimensional vectors $\vec{n}, \vec{n}^2 = 1$ and in this basis arbitrary state $\langle\psi|$ has a unique representation

$$\langle\psi| = e^{i\varphi}\langle\vec{n}|. \tag{3}$$

The action of SU(2) operator g reads

$$\langle\vec{n}|g = e^{i\varphi}\langle\vec{n}_g|, \tag{4}$$

where the phase factor φ depends on both $\vec{n}$ and g, and $\varphi = \varphi(g,\vec{n})$. Therefore, in the basis of coherent states Eq. (2) becomes

$$\langle \psi(t)| = e^{i\varphi(t)}\langle \vec{n}(t)| = \langle \vec{n}(0)|W(t), \tag{5}$$

where without loss of generality we have taken $\varphi(0)=0$ or equivalently $\langle \psi(0)| = \langle \vec{n}(0)|$.

Equation (1) imposes no restrictions on the initial vector $\langle \vec{n}(0)|$, which therefore can be taken arbitrarily. However, there exists a distinguished initial state which is of particular importance for the discussion below. Namely, let us take $\langle \vec{n}(0)|$ to be the eigenstate of the full evolution operator

$$\langle \vec{n}(0)|W(T) = e^{i\varphi(T)}\langle \vec{n}(0)|. \tag{6}$$

Note that generically $1/2\mathrm{Tr}W(T) \neq \pm 1$ and we assume this from now on. The gauge invariant trace of the Wilson loop is given by $1/2\ \mathrm{Tr}\ W(T) = \cos\varphi(T)$. On the other hand, one gets from Eqs. (1), (3) the following equation for the Wilson loop phase factor [18]:

$$\varphi(T) = \int_0^T (\langle \vec{n}|A|\vec{n}\rangle - i\langle \vec{n}|\partial_t|\vec{n}\rangle)dt. \tag{7}$$

Using standard properties of the spin coherent states [15–17] one can represent Eq. (7) in vectorlike notation

$$\frac{1}{2}\mathrm{Tr}\,W(T) = \cos\left[\frac{1}{2}\int_C \vec{n}\vec{A}dt + \frac{1}{4}\int_{S_C} \vec{n}\cdot[\partial_\mu\vec{n}\times\partial_\nu\vec{n}]d^2\sigma^{\mu\nu}\right], \tag{8}$$

where the vector field $\vec{n}(t)$ has been smoothly extended from the contour $\mathcal{C}$ into an arbitrary surface $S_\mathcal{C}$ bounded by $\mathcal{C}$. Note that Eq. (8) may be identically rewritten in the gauge invariant form

$$\frac{1}{2}\mathrm{Tr}W(T) = \cos\left[\frac{1}{4}\int_{S_C} \vec{n}\vec{F}_{\mu\nu}d^2\sigma^{\mu\nu} + \frac{1}{4}\int_{S_C} \vec{n}\cdot[D_\mu\vec{n}\times D_\nu\vec{n}]d^2\sigma^{\mu\nu}\right], \tag{9}$$

where $D^{ab}_\mu = \delta^{ab}\partial_\mu - \varepsilon^{acb}A^c_\mu$ is the covariant derivative and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \varepsilon^{abc}A^b_\mu A^c_\nu$ is the non-Abelian field strength. Note that apparent surface dependence of Eqs. (8), (9) is only superficial since they are identical to Eq. (7) which is explicitly surface independent. However, the concrete way to extend $\mathcal{C}\ni\vec{n}(t)\to\vec{n}(\sigma)\in S_\mathcal{C}$ is not fixed yet and we discuss this point in details below.

Let us emphasize that Eqs.(8), (9) cannot be used to actually calculate the Wilson loop since construction of the evolving states $\vec{n}(t)$ requires knowledge of the Wilson loop itself. Nevertheless, Eq. (9) might be relevant for theoretical considerations since it represents the phase factor $\varphi(T)$ as an integral of the field strength introduced first by 'tHooft [19] and Polyakov [20] in connection with monopoles. Equation (9) is the NAST to be discussed in more detail below.

By construction the vector field $\vec{n}(t)$ is covariantly constant along the contour $\mathcal{C}, \dot{x}_\mu D_\mu\vec{n} = 0$. Equation (5) implies also that for any $t\in[0;T]$ the state $\langle \vec{n}(t)|$ is an eigenstate of the Wilson loop calculated on $\mathcal{C}$ starting from the point $x_\mu(t)$. In other words, $\langle \vec{n}(t)|$ is an eigenstate of $W^+(t)W(T)W(t)$. The interpretation of $\vec{n}(t)$ is then straightforward; it is the instantaneous color orientation of the flux piercing the loop $\mathcal{C}$. Since the contour considered is not infinitesimal, the color direction of the flux is different at various points on $\mathcal{C}$.

The assignment of the vector field $\vec{n}(t)$ to a given closed path $\mathcal{C}$ is unfortunately not unique. The ambiguity comes from Eq. (6) which possesses two solutions with opposite sign of $\vec{n}$:

$$\langle \pm\vec{n}(0)|W(T) = e^{\pm i\varphi(T)}\langle \pm\vec{n}(0)|. \tag{10}$$

However, this sign ambiguity is only global: if $\langle \vec{n}(0)|$ is an eigenstate with $\varphi(T)>0$, then for any $t\in[0;T]$ $\langle \vec{n}(t)|$ is an eigenstate of $W^+(t)W(T)W(t)$ with the same positive phase. Therefore there is only a global freedom to change $\vec{n}(t)\to-\vec{n}(t)$ for all t simultaneously.

Consider the infinitesimal closed contour $\delta\mathcal{C}_x$ located in (μ,ν) plane at point x, which bounds the elementary surface element $\delta\sigma^{\mu\nu}_x$. Since in this case $W(T)\equiv W(\delta\mathcal{C}_x) = 1 + (i/2)\vec{\sigma}\vec{F}_{\mu\nu}\delta\sigma^{\mu\nu}_x + o(\delta\sigma)$, the eigenvalue problem (6),(10) could easily be solved:

$$\vec{n}^{(\mu\nu)}_x = \pm\vec{F}\,\delta\sigma/|\vec{F}\,\delta\sigma|,$$

$$\varphi(\delta\mathcal{C}_x) = \pm|\vec{F}\,\delta\sigma| = \pm\sqrt{(\vec{F}\,\delta\sigma)^2}, \tag{11}$$

where we have explicitly indicated the $(\mu\nu)$ dependence of the eigenvector and evident Lorenz indices have been suppressed. Therefore the infinitesimal version of non-Abelian Stokes theorem (9) is (no summation over μ,ν)

$$\frac{1}{2}\mathrm{Tr}\,W(\delta\mathcal{C}_x) \approx \cos\left[\frac{1}{2}\vec{n}^{(\mu\nu)}_x\vec{F}_{\mu\nu}(x)\,\delta\sigma^{\mu\nu}_x + o(\delta\sigma)\right]$$

$$\approx 1 - \frac{1}{8}(\vec{F}_{\mu\nu}\delta\sigma^{\mu\nu}_x)^2 + o(\delta\sigma^2), \tag{12}$$

where we have used the covariant constancy of $\vec{n}$ on $\delta\mathcal{C}_x$.

It is amusing to note that Eqs. (8), (9) look similar to the non-Abelian Stokes theorem of Ref. [8] apart from the absence of the path integral over $\vec{n}(t)$ in Eq. (9). We conclude therefore that the path integral of Ref. [8] is exactly saturated by the two particular trajectories $\pm\vec{n}(t)$. It is worth mentioning, however, that the construction of $\vec{n}(t)$, Eq. (5), makes no reference to the classical equations of motion and therefore Eqs. (8), (9) do not correspond in general to any semi classical approximation.

Notice that Eq. (9) is still not uniquely defined. The point is that the vector field $\vec{n}(t)$ may be extended arbitrarily from $\mathcal{C}$ to $S_{\mathcal{C}}$. The only requirement is that the extension $\mathcal{C} \ni \vec{n}(t) \to \vec{n}(\sigma) \in S_{\mathcal{C}}$ must be continuous and the distribution $\vec{n}(\sigma)$ must agree with $\vec{n}(t)$ at the boundary $\delta S_{\mathcal{C}} = \mathcal{C}$. On the other hand, Eq. (9) is applicable to any closed contour, in particular to any infinitesimal area element of $S_{\mathcal{C}}$. Therefore, at every point $x(\sigma) \in S_{\mathcal{C}}$ we have a naturally defined direction $\vec{n}(\sigma)$ which can be used to make Eq. (9) unambiguous. Then the only remaining problem is the choice of sign of $\vec{n}(\sigma)$ since at every point $x(\sigma) \in S_{\mathcal{C}}$ considered separately there is no distinction between $\pm\vec{n}(\sigma)$. Here the continuity requirement for the extension $\mathcal{C} \ni \vec{n}(t) \to \vec{n}(\sigma) \in S_{\mathcal{C}}$ becomes crucial. Indeed, since the surface $S_{\mathcal{C}}$ is assumed to be regular (e.g., smooth and without self-intersections), the field $\vec{F}\,\delta\sigma$ is continuous on $S_{\mathcal{C}}$ and therefore Eq. (11) allows one to define $\vec{n}(\sigma)$ continuously as well (these arguments may fail for exceptional configurations which are not generic and which we do not consider for that reason). This way one could provide a well-defined and unambiguous meaning to Eqs. (8), (9) since for any spanning surface $S_{\mathcal{C}}$ and for any boundary distribution $\vec{n} \in \mathcal{C}$ there exists a unique extension $\mathcal{C} \ni \vec{n}(t) \to \vec{n}(\sigma) \in S_{\mathcal{C}}$. Moreover, following the analogy with Abelian case one could try to apply Eq. (9) to closed (infinitesimal) surface S and obtain the non-Abelian Bianchi identities written in Abelian-like form in terms of $\vec{n}, \vec{A}$. However, this would require one to extend the field $\vec{n} \in S$ into the volume bounded by S which is not a straightforward procedure. The point is that the field $\vec{n}(\sigma)$ is essentially a tensorial-like quantity [see Eq. (12)] and is only defined on the infinitesimal surface elements $\delta\sigma \in S$. The extension of $\vec{n} \in S$ into the enclosed volume and the corresponding form of non-Abelian Bianchi identities go beyond the scope of present publication and will be discussed elsewhere.

The sign ambiguity in the definition of $\vec{n}(\sigma)$ is reminiscent to the model of Refs. [21,22] (Alice electrodynamics). Indeed, the key feature of Alice electrodynamics is that the U(1) generator [$\vec{n}(\sigma)$ in our case] is known only up to the sign. Moreover, the definition of the central object of the model, Alice string, is based on the continuity arguments similar to the above reasoning. However, the relevance of Alice electrodynamics to the SU(2) gauge theory is still unclear and we will not dwell on this issue.

In the next section we consider the non-Abelian Stokes theorem on the lattice. As a by-product we also illustrate the appearance of various terms in Eq. (9).

III. NON-ABELIAN STOKES THEOREM ON THE LATTICE

The derivation of the lattice NAST begins from consideration of the fundamental representation Wilson loop on the lattice:

$$W = \prod_{i=0}^{N-1} U_i = U_0 \times U_1 \times \cdots \times U_{N-1}, \tag{13}$$

where $U_i \in \mathrm{SU}(2)$ are the link matrices parametrized as

$$U_i = \cos\nu_i + i \sin\nu_i \vec{\sigma}\vec{u}_{i,+} = \cos\nu_i - i \sin\nu_i \vec{\sigma}\vec{u}_{i,-},$$

$$\vec{u}^2_{i,\pm} = 1, \tag{14}$$

and the corresponding lattice path piercing the sites $s_0, s_1, \ldots, s_{N-1}$ is assumed to be closed and non-self-intersecting. There is an analogous parametrization of the Wilson loop,

$$W = \cos\varphi_w + i \sin\varphi_w \vec{\sigma}\vec{w}_+ = \cos\varphi_w - i \sin\varphi_w \vec{\sigma}\vec{w}_-,$$

$$\vec{w}^2_{\pm} = 1,$$

and therefore the state $\langle \vec{w}_+(s_0)|$ to be ascribed to the site s_0 is

$$\langle \vec{w}_+(s_0)| \equiv \langle \vec{w}_+|,$$

$$\langle \vec{w}_+(s_0)|W = e^{i\varphi_w}\langle \vec{w}_+(s_0)|. \tag{15}$$

Note that one could equally take the eigenstate $\langle \vec{w}_-(s_0)|$ instead (see the discussion in the previous section). As far as only a single Wilson loop is concerned, there is not much difference between the two choices and it is sufficient to take either of them.

Starting from $\langle \vec{w}_+(s_0)|$ one constructs the corresponding eigenstates $\langle \vec{w}_+(s_i)|$, $i = 1, \ldots, N-1$, in all other sites $s_1, \ldots, s_{N-1}$ using Eqs. (3),(5):

$$\langle \vec{w}_+(s_i)|U_i = e^{i\gamma_i}\langle \vec{w}_+(s_{i+1})|, \; i = 0, \ldots, N-1. \tag{16}$$

Since the initial state was taken to be an eigenstate of W, the chain (16) is closed, $s_N \equiv s_0, \langle \vec{w}_+(s_N)| \equiv \langle \vec{w}_+(s_0)|$, and one gets the following relation between the Wilson loop phase φ_w and the phases γ_i coming from links U_i [18]:

$$\varphi_w = \sum_{i=0}^{N-1} \gamma_i. \tag{17}$$

We are in haste to add that, strictly speaking, Eq. (17) is not entirely correct. The point is that the left hand side (LHS) is always bounded, $|\varphi_w| \leqslant \pi$, while the sum on the right can take values outside the interval $[-\pi; \pi]$. To be precise, Eq. (17) should express the equality of the phase factors $e^{i\varphi_w}$, not the angles φ_w by themselves. Therefore, there are terms $2\pi k, k \in Z$ missing in Eq. (17). From now on in all equations like Eq. (17) the mod 2π operation is always assumed and will not be indicated explicitly.

In order to make one step further we need to consider in more detail Eq. (16). Consider an SU(2) operator U which upon acting on some initial state $\langle \vec{n}_1|$ brings it to another state $\langle \vec{n}_2|$:

$$\langle \vec{n}_1|U = e^{i\gamma}\langle \vec{n}_2|, \quad U = \cos\nu + i \sin\nu \vec{\sigma}\vec{u}_+, \tag{18}$$

where γ is an additional phase which depends on both $\langle\vec{n}_1|$ and U. There are two eigenstates $\langle\vec{u}_\pm|$ of the operator U which form a complete basis in the spin-1/2 IRR:

$$\langle\vec{u}_\pm|U=e^{\pm i\upsilon}\langle\vec{u}_\pm|,$$

$$1=|\vec{u}_+\rangle\langle\vec{u}_+|+|\vec{u}_-\rangle\langle\vec{u}_-|. \tag{19}$$

Using the resolution of unity (19) in Eq. (18) one gets

$$e^{i\upsilon}\langle\vec{n}_1|\vec{u}_+\rangle\langle\vec{u}_+|+e^{-i\upsilon}\langle\vec{n}_1|\vec{u}_-\rangle\langle\vec{u}_-|=e^{i\gamma}\langle\vec{n}_2|,$$

$$\gamma=\upsilon+\Omega_0(\vec{n}_1,\vec{u}_+,\vec{n}_2)=-\upsilon+\Omega_0(\vec{n}_1,\vec{u}_-,\vec{n}_2), \tag{20}$$

where $\Omega_0(\vec{n}_1,\ldots,\vec{n}_N)$ is the oriented area of the spherical polygon on unit two-dimensional sphere S^2 with corners at the north pole of S^2 and at $\vec{n}_1,\ldots,\vec{n}_N$ (in that order). In deriving Eq. (20) standard properties of the spin coherent states have been used (see, e.g., Refs. [15–17]):

$$\langle\vec{n}_1|\vec{n}_2\rangle/|\langle\vec{n}_1|\vec{n}_2\rangle|=e^{i\Omega_0(\vec{n}_1,\vec{n}_2)}.$$

Note that in our normalization Area$(S^2)=2\pi$. In particular, there is no additional 1/2 factor in front of Ω_0.

From very general arguments one expects that Eq. (18) has an interpretation of ordinary rotation of the vector $\vec{n}_1$ around the axis $\vec{u}_+$ by the angle υ. However, as far as only initial and final states are taken into account the rotation operator remains in fact undetermined. Indeed, there are infinitely many rotations which connect two given states. On the other hand, among various SU(2) operators there is a distinguished unique $G_{n_1\to n_2}$ which describes the motion $\vec{n}_1\to\vec{n}_2$ along the shortest geodesic line connecting $\vec{n}_1,\vec{n}_2$:

$$G_{n_1\to n_2}=(\vec{n}_1\vec{n}_2)+i\vec{\sigma}\cdot[\vec{n}_1\times\vec{n}_2],$$

$$\langle\vec{n}_1|G_{n_1\to n_2}=e^{i\Omega_0(\vec{n}_1,\vec{n}_2)}\langle\vec{n}_2|.$$

The physical relevance of geodesic curves is widely known. The importance of the geodesic matrices $G_{n_1\to n_2}$ in the present context comes from the consideration of the diagram

$$\begin{array}{ccc}\vec{n}_1 & \overset{U}{\to} & \vec{n}_2\\ G_{m_1\to n_1}\uparrow & & \uparrow G_{m_2\to n_2}\\ \vec{m}_1 & \overset{U}{\to} & \vec{m}_2\end{array}.$$

Here the same operator U corresponds to the rotations $\vec{n}_1\to\vec{n}_2$ and $\vec{m}_1\to\vec{m}_2$. The diagram is closed by two geodesic matrices $G_{m_1\to n_1}$ and $G_{m_2\to n_2}$. From the analysis of the diagram one obtains the relation

$$G_{m_1\to n_1}U=UG_{m_2\to n_2}, \tag{21}$$

which follows from the fact that adjoint SU(2) action is equivalent to SO(3) rotation. Therefore the matrix

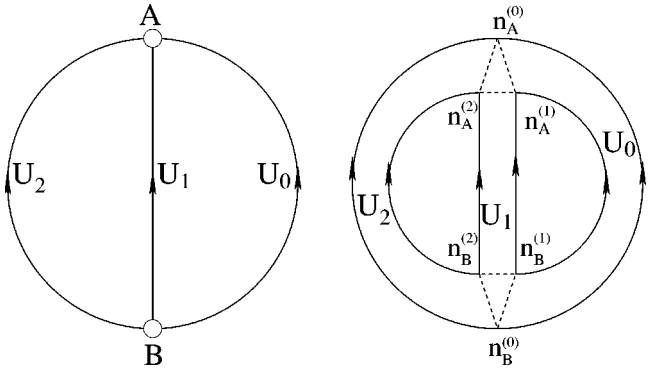

FIG. 1. NAST in the simplest case of three links (see text).

$UG_{m_2\to n_2}U^+$ is again geodesic with initial and final states being $\vec{m}_1$ and $\vec{n}_1$ and thus equals to $G_{m_1\to n_1}$. Equation (20) allows to rewrite Eq. (21) in the form

$$\Omega_0(\vec{n}_1,\vec{u}_\pm,\vec{n}_2)+\Omega_0(\vec{m}_2,\vec{u}_\pm,\vec{m}_1)$$
$$=\Omega_0(\vec{m}_2,\vec{n}_2)+\Omega_0(\vec{n}_1,\vec{m}_1). \tag{22}$$

Equations (17),(20),(22) are sufficient to derive the NAST on the lattice. We illustrate the derivation on the simplest example. Generalization to usual cubic geometry is straightforward but technically notorious. Thus we will only discuss the final result.

Consider the simplest nontrivial configuration of three links, Fig. 1, from which one could construct three different Wilson loops

$$W_0=U_0U_2^+,\quad W_1=U_0U_1^+,\quad W_2=U_1U_2^+.$$

Let φ_i be the phase angle of the corresponding Wilson loop, $1/2\,\mathrm{Tr}\,W_i=\cos\varphi_i$. Applying the procedure of the previous section to each Wilson loop separately, one gets six states $\{\vec{n}_A^{(i)},\vec{n}_B^{(i)}\}, i=0,1,2$, sitting at points A and B; a pair of states with fixed i is assigned to the corresponding Wilson loops W_i. In particular,

$$\langle\vec{n}_A^{(i)}|W_i=e^{i\varphi_i}\langle\vec{n}_A^{(i)}|,\quad i=0,1,2. \tag{23}$$

Let us evaluate the phase angle φ_0. According to Eqs. (17), (20),

$$\varphi_0=\upsilon_0+\Omega_0(\vec{n}_A^{(0)},\vec{u}_{0,+},\vec{n}_B^{(0)})-\upsilon_2+\Omega_0(\vec{n}_B^{(0)},\vec{u}_{2,+},\vec{n}_A^{(0)}),$$

where υ_2 enters with minus sign because U_2 is conjugated in the definition of W_0. Using Eq. (22) we can write

$$\varphi_0=\upsilon_0+\Omega_0(\vec{n}_A^{(1)},\vec{u}_{0,+},\vec{n}_B^{(1)})+\Omega_0(\vec{n}_A^{(0)},\vec{n}_A^{(1)})$$
$$+\Omega_0(\vec{n}_B^{(1)},\vec{n}_B^{(0)})-\upsilon_2+\Omega_0(\vec{n}_B^{(2)},\vec{u}_{2,+},\vec{n}_A^{(2)})$$
$$+\Omega_0(\vec{n}_B^{(0)},\vec{n}_B^{(2)})+\Omega_0(\vec{n}_A^{(2)},\vec{n}_A^{(0)}). \tag{24}$$

The next step is to add zero in the form [see Eqs. (21),(22)]

$$0=\nu_1+\Omega_0(\vec{n}_A^{(2)},\vec{u}_{1,+},\vec{n}_B^{(2)})+\Omega_0(\vec{n}_A^{(1)},\vec{n}_A^{(2)})+\Omega_0(\vec{n}_B^{(2)},\vec{n}_B^{(1)})$$
$$-\nu_1+\Omega_0(\vec{n}_B^{(1)},\vec{u}_{1,+},\vec{n}_A^{(1)})$$

to Eq. (24) and collect various terms together using the relations like

$$\varphi_1=\nu_0+\Omega_0(\vec{n}_A^{(1)},\vec{u}_{0,+},\vec{n}_B^{(1)})-\nu_1+\Omega_0(\vec{n}_B^{(1)},\vec{u}_{1,+},\vec{n}_A^{(1)}),$$

$$\Omega(A)=\Omega_0(\vec{n}_A^{(0)},\vec{n}_A^{(1)})+\Omega_0(\vec{n}_A^{(1)},\vec{n}_A^{(2)})+\Omega_0(\vec{n}_A^{(2)},\vec{n}_A^{(0)}),$$

where $\Omega(A)=\Omega(\vec{n}_A^{(0)},\vec{n}_A^{(1)},\vec{n}_A^{(2)})$ is the area of spherical triangle constructed on the indicated three unit vectors at point A. In other words, $\Omega(A)$ is just the oriented solid angle between the triple $\{\vec{n}_A^{(0)},\vec{n}_A^{(1)},\vec{n}_A^{(2)}\}$.

Therefore the final equation which relates the phase angles $\varphi_i, i=0,1,2$ is

$$\varphi_0=\varphi_1+\varphi_2+\Omega(A)+\Omega(B), \tag{25}$$

where $\Omega(A)$ and $\Omega(B)$ are the oriented solid angles between the triads $\{\vec{n}_A^{(0)},\vec{n}_A^{(1)},\vec{n}_A^{(2)}\}$ and $\{\vec{n}_B^{(0)},\vec{n}_B^{(2)},\vec{n}_B^{(1)}\}$. Note the different ordering of states in $\Omega(A)$ and $\Omega(B)$, which corresponds to counting the outgoing flux at both points A and B.

Let us emphasize that Eq. (25) is valid irrespectively of the particular choice of the states $\{\vec{n}_A^{(i)},\vec{n}_B^{(i)}\}$ provided that the phases φ_i are calculated according to Eq. (17) or (23). It does not matter which particular solution of Eq. (23) was taken to construct $\{\vec{n}_A^{(i)},\vec{n}_B^{(i)}\}$ on each Wilson loop, Eq. (25) remains formally the same with either choice. But this means that Eq. (25) is ambiguous because φ_i changes sign when $\{\vec{n}_A^{(i)},\vec{n}_B^{(i)}\}$ are replaced by $\{-\vec{n}_A^{(i)},-\vec{n}_B^{(i)}\}$. In fact this is the same sign problem discussed previously. We will fix it after considering the continuum limit of Eq. (25).

The generalization of Eq. (25) to the case of usual cubic geometry is straightforward but technically involved. The final result

$$\varphi_w=\sum_{x\in S_\mathcal{C}}\varphi_x+\sum_{x\in S_\mathcal{C}}\Omega_x+\sum_{x\in\mathcal{C}}\alpha_x \tag{26}$$

is illustrated on Fig. 2. It is understood that phases φ_w,φ_x are calculated via Eq. (17). Here φ_w is the phase of the large Wilson loop, $1/2\,\mathrm{Tr}\,W(\mathcal{C})=\cos\varphi_w$, where $\mathcal{C}$ is the planar 2×2 closed contour (see Fig. 2), which bounds the surface $S_\mathcal{C}$. The first term on the RHS ("dynamical part") is the sum of contributions coming from four "internal" plaquettes belonging to $S_\mathcal{C}$. In particular,

$$\frac{1}{2}\mathrm{Tr}\,U_{p_x}=\cos\varphi_x,\quad p_x=0,1,2,3,$$

where U_{p_x} is the corresponding plaquette matrix. The second term ("solid angle") comes from the points common to four different "internal" plaquettes. We recall that application of the NAST, Eq. (25), requires construction of four color vectors per plaquette situated at plaquette's corners. Therefore there are four unit vectors at the point x (see Fig. 2), and Ω_x is just the oriented solid angle between them. The third term ("perimeter contribution") is analogous to the second one. It accounts for the difference in color direction between the states on the nearest to the loop "internal" plaquettes and the states on the loop itself. Technically α_i is an oriented solid angle between the corresponding three vectors.

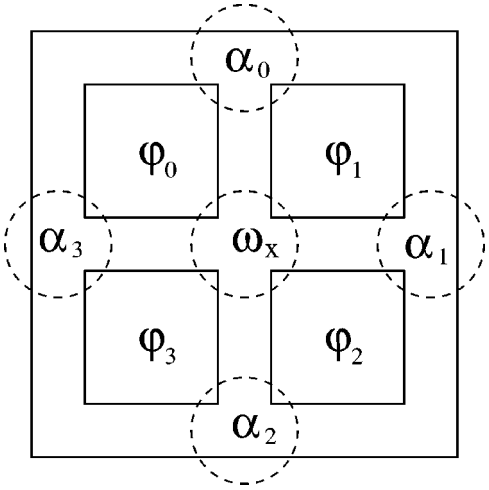

FIG. 2. NAST on square lattice (see text for details).

Equation (26) has a simple physical interpretation. The magnitude of the total flux, φ_w, piercing large closed contour $\mathcal{C}$ is the sum of a few terms. The first term sums up the magnitudes of elementary fluxes penetrating the surface $S_\mathcal{C}$. Since the theory is non-Abelian, each elementary flux has its own color orientation which is no less important than the flux magnitude (for flux piercing finite contour $\mathcal{C}$ the color direction of the flux varies along $\mathcal{C}$). The other terms in Eq. (26) take into account the difference in color orientation of various fluxes on $S_\mathcal{C}$ as well as of the total flux piercing $\mathcal{C}$. It is worth mentioning that for pure Abelian fields [or for a SU(2) gauge copy of Abelian configurations] the second and third contributions in Eq. (26) vanish identically and one gets the usual Abelian Stokes theorem.

Let us consider Eq. (26) in the limit of vanishing lattice spacing, $a\to0$. The contribution of the first term was in fact already calculated in Eqs. (11),(12):

$$\text{"dynamicalpart"}=a^2\sum_{x\in S_\mathcal{C}}\frac{1}{2}\vec{n}_x\vec{F}_{\mu\nu}(x)+o(a^2)$$
$$\approx\frac{1}{4}\int_{S_C}\vec{n}\vec{F}_{\mu\nu}d^2\sigma^{\mu\nu}, \tag{27}$$

where $\vec{n}_x, x\in S_\mathcal{C}$ is given by Eq. (11). In order to get the continuum limit of the second term consider the point $x\in S_\mathcal{C}$ and let $(\mu\nu)$ be a plane tangential to $S_\mathcal{C}$ at x. Then Ω_x is the oriented solid angle between the four vectors

$$\vec{n}_1=\vec{n}_x-a\cdot(D_\mu\vec{n}_x+D_\nu\vec{n}_x),$$

$$\vec{n}_2=\vec{n}_x-a\cdot D_\nu\vec{n}_x,$$

$$\vec{n}_3=\vec{n}_x,$$

$$\vec{n}_4=\vec{n}_x-a\cdot D_\mu\vec{n}_x, \tag{28}$$

where $\vec{n}_x$ is again given by Eq. (11) and D is the covariant derivative. It is straightforward then to evaluate Ω_x:

$$\Omega_x=\frac{1}{2}a^2(\vec{n}\cdot[D_\mu\vec{n}\times D_\nu\vec{n}])+o(a^2). \tag{29}$$

Therefore,

$$\text{``solid angle''}=\frac{1}{4}\int_{S_C}\vec{n}\cdot[D_\mu\vec{n}\times D_\nu\vec{n}]d^2\sigma^{\mu\nu}. \tag{30}$$

Unfortunately, there exists no simple expression for the third term, Eq. (26), in the continuum limit. However, this is to be expected. Indeed, one can readily convince oneself that the meaning of the ''perimeter contribution'' is to provide correct boundary conditions in Eq. (26). In other words, the third term, Eq. (26), guarantees that the vector field $\vec{n}(\sigma)\in S_\mathcal{C}$ agrees with $\vec{n}(t)\in\mathcal{C}$ on the boundary $\delta S_\mathcal{C}=\mathcal{C}$.

Combining Eqs. (27),(30) one formally reproduces Eq. (9), confirming that Eq. (26) is indeed the lattice formulation of the non-Abelian Stokes theorem (9). However, this conclusion relies heavily on Eq. (29) which is only valid if $\vec{n}_x, x\in S_\mathcal{C}$, is continuous across the plaquette boundaries. This suggests a natural way to fix the relative sign of eigenstates on neighboring plaquettes analogously to the continuum considerations above. Namely, we propose to fix the particular distribution of eigenstates by the requirement that

$$R=\sum_{x\in S_\mathcal{C}}|\Omega_x|+\sum_{x\in\mathcal{C}}|\alpha_x| \tag{31}$$

take the minimal possible value [it is assumed, of course, that eigenvectors at the boundary $\vec{n}(t)\in\mathcal{C}$ are held fixed from the very beginning]. This prescription fixes completely and unambiguously all the states $\vec{n}(\sigma)\in S_\mathcal{C}$ provided that the functional R has a unique minimum. The uniqueness of the minimum of R is a separate issue and we have no analytical methods to investigate it. However, at least numerically the minimum of Eq. (31) might be approximated with high accuracy.

IV. NUMERICAL SIMULATIONS

In this section we describe simple lattice experiments with Eq. (26) which we performed in pure SU(2) lattice gauge theory considered on 12^4 lattice at $\beta=2.4$ using the standard Wilson action.

Since the decomposition (26) is gauge invariant (see discussion in previous section), it is legitimate to ask what the contribution is of each term into the Wilson loop expectation value

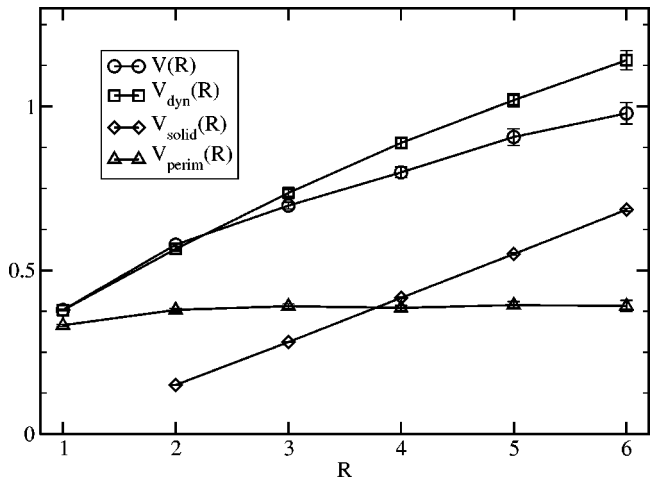

FIG. 3. Various terms contributing to the heavy quark potential. Lines are drawn to guide the eye.

$$\langle\exp\{i\varphi_w\}\rangle\sim e^{-TV(R)}, \tag{32}$$

where we have restricted ourselves to the consideration of rectangular $T\times R, T\gg R$ loops only. Therefore the problem is to calculate

$$\left\langle\exp\left\{i\sum_{x\in S_\mathcal{C}}\varphi_x\right\}\right\rangle\sim e^{-TV_{dyn}(R)}, \tag{33}$$

$$\left\langle\exp\left\{i\sum_{x\in S_\mathcal{C}}\Omega_x\right\}\right\rangle\sim e^{-TV_{solid}(R)}, \tag{34}$$

$$\left\langle\exp\left\{i\sum_{x\in\mathcal{C}}\alpha_x\right\}\right\rangle\sim e^{-TV_{perim}(R)}. \tag{35}$$

Notice that the T,R dependence of the expectation values (33)– (35) is an *ad hoc* assumption which has to be checked separately. However, we have found that Eqs. (33)–(35) indeed accurately describe numerical data.

We calculated the expectation values (32)–(35) on 50 statistically independent configurations using the spatial smearing algorithm (see, e.g., Ref. [23] for details). For each rectangular loop $\mathcal{C}=\{T\times R\}$ Eqs. (15),(16) were applied to construct the eigenstates on $\mathcal{C}$. The same procedure was used to build the eigenvectors $\{\vec{n}_p^{(i)}\}, i=0,\ldots,3$, on each ''internal'' plaquette $p\in S_\mathcal{C}$ (only surfaces with minimal area were considered). Finally, the functional (31) was minimized with respect to the inversions $\{\vec{n}_p^{(i)}\}\to\{-\vec{n}_p^{(i)}\}, p\in S_\mathcal{C}$, using a variant of the simulated annealing algorithm [24] and keeping the boundary conditions $\vec{n}\in\mathcal{C}$ fixed.

The results of our simulations are presented on Fig. 3, where circles represent the full heavy quark potential (32), squares correspond the ''dynamical'' part (33), and finally diamonds and triangles stand for ''solid'' (34) and ''perimeter'' (35) contributions, respectively. Note that the solid curves on Fig. 3 are drawn to guide the eye.

There are few striking features of the expectation values (33)–(35) to be mentioned here. First of all, the ''perimeter'' potential, Eq. (35), turns out to be practically R independent:

$$V_{perim}(R) \approx \text{const}, \tag{36}$$

which might be an indication that the perimeter contribution drops out in the expectation value of the full Wilson loop (32). Second, both $V_{dyn}(R)$ and $V_{solid}(R)$ appear to be linear at large distances, $R \gtrsim 3$, albeit with somewhat larger slope than the full potential $V(R) \approx \sigma_{SU(2)} R$:

$$V_{dyn}(R) \approx \sigma_{dyn} R, \quad V_{solid}(R) \approx \sigma_{solid} R, \tag{37}$$

$$\frac{\sigma_{dyn}}{\sigma_{SU(2)}} \approx \frac{\sigma_{solid}}{\sigma_{SU(2)}} \approx 1.6. \tag{38}$$

Although $V_{dyn}(R)$ deviates from linear behavior at distances $R \lesssim 3$, $V_{solid}(R)$ is rising strictly linear starting from the smallest possible distance $R=2$. The existence of a linearly rising term in the heavy quark potential at short distances has been widely discussed in the literature see, e.g., Refs. [25–29] and references therein.

Finally, we emphasize that the expectation value of the full Wilson loop (32) is not factorizable into the terms (33)–(35). It is clearly seen from Fig. 3 that

$$V(R) \neq V_{dyn}(R) + V_{solid}(R) + V_{perim}(R) + \text{const}, \tag{39}$$

and therefore there are various interference terms contributing to $V(R)$. The point, however, is that the linear piece coming from V_{solid} might survive at small distances since the ''solid angle'' contribution (30),(34),(37) is formally not suppressed by the action even at very large β.

V. CONCLUSIONS

We have derived a new version of the non-Abelian Stokes theorem for the Wilson loop in the fundamental representation of the SU(2) gauge group. By considering the instantaneous color direction of the flux piercing the loop we were able to avoid the path ordering in the conventional definition of the Wilson loop operator. Moreover, this approach allows one to represent the phase angle of the Wilson loop as an ordinary one-dimensional integral to which the usual Stokes theorem applies. Furthermore, we were able to relate the resulting surface integral with properties of non-Abelian gauge fields on that surface.

Unfortunately, our formulation can hardly be called a ''theorem'' because it does not help to calculate the Wilson loop itself. However, this drawback is not specific to this paper since other known variants of the non-Abelian Stokes theorem are also not very useful for Wilson loop calculations. At the same time our construction is well suited for numerical investigations. To achieve this goal we have also derived the non-Abelian Stokes theorem on the lattice and illustrated the origin and physical meaning of various terms contributing to the trace of the Wilson loop.

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