Hamiltonian lattice quantum chromodynamics at finite density with Wilson fermions

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Quantum chromodynamics (QCD) at sufficiently high density is expected to undergo a chiral phase transition. Understanding such a transition is of particular importance for neutron star or quark star physics. In Lagrangian SU(3) lattice gauge theory, the standard approach breaks down at large chemical potential μ, due to the complex action problem. The Hamiltonian formulation of lattice QCD does not encounter such a problem. In a previous work, we developed a Hamiltonian approach at finite chemical potential μ and obtained reasonable results in the strong-coupling regime. In this paper, we extend the previous work to Wilson fermions. We study the chiral behavior and calculate the vacuum energy, chiral condensate, and quark number density, as well as the masses of light hadrons. There is a first-order chiral phase transition at zero temperature.

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I. INTRODUCTION

Quantum Chromodynamics (QCD) is the fundamental theory of strong interactions. It is an SU(3) gauge theory of quarks and gluons. The precise determination of the QCD phase diagram on the temperature T and chemical potential μ plane will provide valuable information for the experimental search for the quark-gluon plasma (QGP). The ultimate goal of machines such as the Relativistic Heavy Ion Collider (RHIC) at BNL and the Large Hadron Collider (LHC) at CERN is to create the QGP phase, and replay the birth and evolution of the Universe. Such a new state of matter may also exist in the core of neutron stars or quark stars at low temperature T and large chemical potential μ. Lattice gauge theory (LGT), proposed by Wilson [1], is a first principles nonperturbative method for QCD. Although it is the most reliable technique for investigating phase transitions in QCD, it is not free of problems: the complex action at finite chemical potential and species doubling with naive fermions.

In the Lagrangian formulation of LGT at finite chemical potential, the success is limited to SU(2) gauge theory [2,3], while in the physical SU(3) case, complex action [4,5] spoils numerical simulations with importance sampling. Even though much effort [6–8] has recently been made for SU(3) LGT, and some very interesting information on the phase diagram at large T and small μ has been obtained, it is still extremely difficult to do simulations at large chemical potential. QCD at large μ is of particular importance for neutron star or quark star physics. Hamiltonian formulation of LGT does not encounter the notorious ''complex action problem.'' Recently, we proposed a Hamiltonian approach to LGT with naive fermions at finite chemical potential [9,10] and solve it in the strong-coupling regime. We predicted that at zero temperature, there is a first-order chiral phase transition at critical chemical potential $\mu_C = m_{dyn}^{(0)} = M_N^{(0)}/3$, with $m_{dyn}^{(0)}$ and $M_N^{(0)}$ being the dynamical mass of quark and nucleon mass at $\mu = 0$, respectively. (We expect this is also true for Kogut-Susskind fermions.) By solving the gap and Bethe-Salpeter equations, the authors of Ref. [11] obtained the critical point the same as ours; but they concluded that the chiral transition is of second order, different from ours. Our order of transition is consistent with other lattice simulation results [12].

Wilson's approach to lattice fermions [1] has been extensively used in hadron spectrum calculations as well as in QCD at finite temperature. It avoids the species doubling and preserves the flavor symmetry, but it explicitly breaks the chiral symmetry [1,13–18], one of the most important symmetries of the original theory. Nonperturbative fine tuning of the bare fermion mass has to be done, in order to define the chiral limit [19,20].

In this paper, we study Hamiltonian lattice QCD with Wilson fermions at finite chemical potential. We derive the effective Hamiltonian in the strong-coupling regime and diagonalize it by Bogoliubov transformation. The vacuum energy, chiral condensate, and masses of pseudoscalar, vector meson, and nucleons are computed. In the nonperturbatively defined chiral limit, we obtain reasonable results for the critical point and some physical quantities in the large N_c limit, with N_c the number of colors.

To our knowledge, the only existing literature about the same system ($r \neq 0$ and $\mu \neq 0$) is Ref. [21], where the author used a very different approach: the solution to the gap equation. In contrary to the conventional predictions [19], the author found that even at $\mu = 0$, there is a critical value for the effective four fermion coupling K, below which dynamical mass of quark vanishes. He introduced the concept of total chemical potential and found that the transition order depends on the input parameters K and r as well as the momentum. In contrast, we find that in the chiral limit, dynamical mass of quark does not vanish for all values of K if $\mu < \mu_C$ (the chiral-symmetry broken phase); and at $\mu = \mu_C$, our order of chiral phase transition does not depend on the input parameter.

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The rest of the paper is organized as follows. In Sec. II, we derive the effective Hamiltonian at finite chemical potential. In Sec. III, we present the results for the vacuum energy, chiral condensate, and hadron masses. In Sec. IV, we estimate the critical chemical potential at zero temperature. The results are summarized in Sec. V.

II. EFFECTIVE HAMILTONIAN IN THE STRONG-COUPLING REGIME

A. $\mu=0$ case

We begin with QCD Hamiltonian[1] with Wilson fermions at chemical potential $\mu=0$ on one-dimensional continuum time and three-dimensional spatial discretized lattice,

$$H=M\sum_x \bar{\psi}(x)\psi(x)+\frac{1}{2a}\sum_x\sum_{k=\pm1}^{\pm d}\bar{\psi}(x)\gamma_k U(x,k)\psi(x+\hat{k}) - \frac{r}{2a}\sum_x\sum_{k=\pm1}^{\pm d}\bar{\psi}(x)U(x,k)\psi(x+\hat{k}) + \frac{g^2}{2a}\sum_x\sum_{j=1}^{d}E_j^\alpha(x)E_j^\alpha(x)-\frac{1}{ag^2}\sum_p \mathrm{Tr}(U_p+U_p^+-2), \tag{2.1}$$

where

$$M=m+\frac{rd}{a}, \tag{2.2}$$

$d=3$ is the spatial dimension and m, a, r, and g are, respectively, the bare fermion mass, spatial lattice spacing, Wilson parameter, and bare coupling constant. $U(x,k)$ is the gauge link variable at site x and direction $\hat{k}$. The Fermion field ψ on each lattice site carries spin (Dirac), color and flavor indexes; here and in the following, whenever there is a summation sign "Σ_x," summations over these indexes are implied. The convention $\gamma_{-k}=-\gamma_k$ is used. $E_j^\alpha(x)$ is the color-electric field at site x and direction j, and summation over $\alpha=1,2,\ldots,8$ is implied. U_p is the product of gauge link variables around an elementary spatial plaquette, and it represents the color magnetic interactions. In the continuum limit $a\to0$, Eq. (2.1) approaches the continuum QCD Hamiltonian in the temporal gauge $A_4=0$.

The effective Hamiltonian, obtained by strong-coupling expansion up to the second order, is [20]

$$\begin{aligned}H_{eff}=&M\sum_x\bar{\psi}_f(x)\psi_f(x)-\frac{K(r^2+1)d}{a}\sum_x\psi_f^\dagger(x)\psi_f(x)\\&+\frac{K}{8aN_c}\sum_x\sum_{k=\pm j}\{(r^2+1)\psi_{f_1}^\dagger(x)\psi_{f_2}(x)\psi_{f_2}^\dagger(x+\hat{k})\psi_{f_1}(x+\hat{k})\\&+(r^2-1)\psi_{f_1}^\dagger(x)\gamma_4\psi_{f_2}(x)\psi_{f_2}^\dagger(x+\hat{k})\gamma_4\psi_{f_1}(x+\hat{k})\\&-(r^2-1)\psi_{f_1}^\dagger(x)\gamma_5\psi_{f_2}(x)\psi_{f_2}^\dagger(x+\hat{k})\gamma_5\psi_{f_1}(x+\hat{k})\\&+(r^2+1)\psi_{f_1}^\dagger(x)\gamma_4\gamma_5\psi_{f_2}(x)\psi_{f_2}^\dagger(x+\hat{k})\gamma_4\gamma_5\psi_{f_1}(x+\hat{k})\\&+[r^2+(1-2\delta_{|k|,j})]\psi_{f_1}^\dagger(x)\gamma_4\gamma_j\psi_{f_2}(x)\psi_{f_2}^\dagger(x+\hat{k})\gamma_4\gamma_j\psi_{f_1}(x+\hat{k})\\&-[r^2-(1-2\delta_{|k|,j})]\psi_{f_1}^\dagger(x)\gamma_j\psi_{f_2}(x)\psi_{f_2}^\dagger(x+\hat{k})\gamma_j\psi_{f_1}(x+\hat{k})\\&-[r^2+(1-2\delta_{|k|,j})]\psi_{f_1}^\dagger(x)\gamma_4\sigma_j\psi_{f_2}(x)\psi_{f_2}^\dagger(x+\hat{k})\gamma_4\sigma_j\psi_{f_1}(x+\hat{k})\\&-[r^2-(1-2\delta_{|k|,j})]\psi_{f_1}^\dagger(x)\sigma_j\psi_{f_2}(x)\psi_{f_2}^\dagger(x+\hat{k})\sigma_j\psi_{f_1}(x+\hat{k})\}.\end{aligned} \tag{2.3}$$

[1]We use a representation of γ matrices described in [33].

This effective Hamiltonian is equivalent to that in Ref. [19], though different representations of γ matrices are used. Here $\sigma_j=\epsilon_{jj_1j_2}\gamma_{j_1}\gamma_{j_2}$. The flavor indexes are explicitly written. This effective Hamiltonian describes the nearest-neighbor four fermion interactions, with

$$K=\frac{1}{g^2C_N} \tag{2.4}$$

being the effective four fermion coupling constant. Here $C_N=(N_c^2-1)/(2N_c)$ is the Casimir invariant of the SU(N_c) gauge group.

B. $\mu\neq 0$ case

In the continuum, the grand canonical partition function of QCD at finite temperature T and chemical potential μ is

$$Z=\mathrm{Tr}\, e^{-\beta(H-\mu N)},\quad \beta=(k_BT)^{-1}, \tag{2.5}$$

where k_B is the Boltzmann constant and N is the particle number operator,

$$N=\sum_x \psi^\dagger(x)\psi(x). \tag{2.6}$$

According to Eq. (2.5) and following the procedure in Sec. II A, the role of the Hamiltonian at strong coupling is now played by

$$H^\mu_{eff}=H_{eff}-\mu N. \tag{2.7}$$

In this Hamiltonian, there are three input parameters: r, m, and μ. Suppose we study the phase structure of the system in the chiral limit. Such a limit can be reached by fine tuning the bare quark mass m so that the pion becomes massless. In such a case, there are only two free parameters left: r and μ.

The vacuum energy is the expectation value of $H-\mu N$ in its ground state $|\Omega\rangle$, and also the expectation value of H^μ_{eff} in its ground state $|\Omega_{eff}\rangle$, given by

$$E_\Omega=\langle\Omega|H-\mu N|\Omega\rangle=\langle\Omega_{eff}|H^\mu_{eff}|\Omega_{eff}\rangle. \tag{2.8}$$

III. PHYSICAL QUANTITIES AT $\mu\neq 0$ AND $T=0$

A. Meson masses

One way to compute the masses of mesons as well as the contributions of the mesons to the vacuum energy is to bosonize the effective Hamiltonian Eq. (2.3). We introduce the following operators [9,22,23]:

$$\Pi_{f_1f_2}(x)=\frac{1}{2\sqrt{-\bar v}}\psi^\dagger_{f_1}(x)(1-\gamma_4)\gamma_5\psi_{f_2}(x),$$

$$\Pi^\dagger_{f2f1}(x)=\frac{1}{2\sqrt{-\bar v}}\psi^\dagger_{f_2}(x)(1+\gamma_4)\gamma_5\psi_{f_1}(x),$$

$$V_{jf_1f_2}(x)=\frac{1}{2\sqrt{-\bar v}}\psi^\dagger_{f_1}(x)(1-\gamma_4)\gamma_j\psi_{f_2}(x),$$

$$V^\dagger_{jf_2f_1}(x)=\frac{1}{2\sqrt{-\bar v}}\psi^\dagger_{f_2}(x)(1+\gamma_4)\gamma_j\psi_{f_1}(x). \tag{3.1}$$

j stands for the positive spatial direction, and $\bar v$ and $v^\dagger$ denote, respectively, the expectation value of $\bar\psi\psi$ and $\psi^\dagger\psi$ in the vacuum state $|\Omega_{eff}\rangle$ of H_{eff}, i.e.,

$$\bar v=\langle\bar\psi(x)\psi(x)\rangle_{eff}=\frac{1}{N_fN_s}\langle\Omega_{eff}|\sum_x\ \bar\psi(x)\psi(x)|\Omega_{eff}\rangle,$$

$$v^\dagger=\langle\psi^\dagger(x)\psi(x)\rangle_{eff}=\frac{1}{N_fN_s}\Omega_{eff}|\sum_x\ \psi^\dagger(x)\psi(x)|\Omega_{eff}\rangle. \tag{3.2}$$

Here N_s is the total number of lattice sites and N_f the number of flavors. It is shown in Appendix A that under the linearization prescription [23], operators defined in Eq. (3.1) satisfy the canonical commutation relations for bosons and then the effective Hamiltonian H^μ_{eff} in Eq. (2.7) can be expressed in terms of these operators in the following way:

$$H^\mu_{eff}\sim H^\mu_{Linear}=E^{(0)}_\Omega+H_\Pi+H_V, \tag{3.3}$$

where

$$E^{(0)}_\Omega=N_fN_s\left[M\bar v-\left(\frac{Kd(1+r^2)}{a}+\mu\right)v^\dagger+\frac{Kdr^2}{4aN_c}(v_2^\dagger+\bar v_2)\right.$$
$$\left.-\frac{Kd}{4aN_c}(\bar v_2-v_2^\dagger+v^\dagger_{2\sigma_j}+\bar v_{2\sigma_j})\right], \tag{3.4}$$

and

$$H_\Pi=\left(2M-\frac{Kd(1-r^2)}{aN_c}\bar v\right)\sum_{x,f_1,f_2}\Pi^\dagger_{f_2f_1}(x)\Pi_{f_1f_2}(x)$$
$$+\frac{Kr^2}{4aN_c}\bar v\sum_{x,f_1,f_2,k}[\Pi^\dagger_{f_1f_2}(x)\Pi_{f_2f_1}(x+\hat k)$$
$$+\Pi_{f_1f_2}(x)\Pi^\dagger_{f_2f_1}(x+\hat k)]$$
$$-\frac{K}{4aN_c}\bar v\sum_{x,f_1,f_2,k}[\Pi^\dagger_{f_1f_2}(x)\Pi^\dagger_{f_2f_1}(x+\hat k)$$
$$+\Pi_{f_1f_2}(x)\Pi_{f_2f_1}(x+\hat k)],$$

$$H_V=\left(2M-\frac{Kd(1-r^2)}{aN_c}\overline{v}\right)\sum_{x,f_1,f_2,j}V^\dagger_{jf_2f_1}(x)V_{jf_1f_2}(x)$$
$$+\frac{Kr^2}{4aN_c}\overline{v}\sum_{x,f_1,f_2,k,j}[V^\dagger_{jf_1f_2}(x)V_{jf_2f_1}(x+\hat{k})$$
$$+V_{jf_1f_2}(x)V^\dagger_{jf_2f_1}(x+\hat{k})]$$
$$-\frac{K}{4aN_c}\overline{v}\sum_{x,f_1,f_2,k,j}[V^\dagger_{jf_1f_2}(x)V^\dagger_{jf_2f_1}(x+\hat{k})$$
$$+V_{jf_1f_2}(x)V_{jf_2f_1}(x+\hat{k})](1-2\delta_{|k|,j}). \tag{3.5}$$

$v_2^\dagger$, $\overline{v}_2$, $v^\dagger_{2\sigma_j}$, and $\overline{v}_{2\sigma_j}$ are the expectation values of four fermion operators,

$$\overline{v}_2=\frac{1}{2dN_fN_s}\langle\Omega_{eff}|\sum_{x,f_1,f_2,k}\overline{\psi}_{f_1}(x)\psi_{f_2}(x)\overline{\psi}_{f_2}(x+\hat{k})$$
$$\times\psi_{f_1}(x+\hat{k})|\Omega_{eff}\rangle,$$

$$v_2^\dagger=\frac{1}{2dN_fN_s}\langle\Omega_{eff}|\sum_{x,f_1,f_2,k}\psi^\dagger_{f_1}(x)\psi_{f_2}(x)\psi^\dagger_{f_2}(x+\hat{k})$$
$$\times\psi_{f_1}(x+\hat{k})|\Omega_{eff}\rangle,$$

$$\overline{v}_{2\sigma_j}=\frac{1}{2dN_fN_s}\langle\Omega_{eff}|\sum_{x,f_1,f_2,k,j}[r^2+(1-2\delta_{|k|,j})]$$
$$\times\overline{\psi}_{f_1}(x)\sigma_j\psi_{f_2}(x)\overline{\psi}_{f_2}(x+\hat{k})\sigma_j\psi_{f_1}(x+\hat{k})|\Omega_{eff}\rangle,$$

$$v^\dagger_{2\sigma_j}=\frac{1}{2dN_fN_s}\langle\Omega_{eff}|\sum_{x,f_1,f_2,k,j}[r^2-(1-2\delta_{|k|,j})]$$
$$\times\psi^\dagger_{f_1}(x)\sigma_j\psi_{f_2}(x)\psi^\dagger_{f_2}(x+\hat{k})\sigma_j\psi_{f_1}(x+\hat{k})|\Omega_{eff}\rangle. \tag{3.6}$$

After a Fourier transformation,

$$\Pi_{f_1f_2}(x)=\sum_p e^{ipx}\tilde{\Pi}_{f_1f_2}(p), \tag{3.7}$$

H_Π in Eq. (3.5) becomes

$$H_\Pi=\left(2M-\frac{Kd}{aN_c}(1-r^2)\right)\sum_{p,f_1,f_2}\tilde{\Pi}^\dagger_{f_1f_2}(p)\tilde{\Pi}_{f_2f_1}(p)$$
$$+\frac{Kr^2}{2aN_c}\overline{v}\sum_{f_1f_2}\sum_p[\tilde{\Pi}^\dagger_{f_1f_2}(p)\tilde{\Pi}_{f_2f_1}(p)$$
$$+\tilde{\Pi}(p)_{f_1f_2}\tilde{\Pi}^\dagger_{f_2f_1}(p)]\sum_{j=1}^d\cos p_ja$$
$$-\frac{K}{2aN_c}\overline{v}\sum_{p,f_1,f_2}[\tilde{\Pi}^\dagger_{f_1f_2}(p)\tilde{\Pi}^\dagger_{f_2f_1}(-p)$$
$$+\tilde{\Pi}_{f_1f_2}(-p)\tilde{\Pi}_{f_2f_1}(p)]\sum_{j=1}^d\cos p_ja. \tag{3.8}$$

The Bogoliubov transformation [20],

$$\tilde{\Pi}(p)\to\tilde{\Pi}(p)\cosh u_p+\tilde{\Pi}^\dagger(-p)\sinh u_p,$$
$$\tilde{\Pi}^\dagger(p)\to\tilde{\Pi}^\dagger(p)\cosh u_p+\tilde{\Pi}(-p)\sinh u_p, \tag{3.9}$$

diagonalizes H_Π if

$$\tanh 2u_p=-\frac{2G_2}{G_1}\sum_{l=1}^d\cos p_la,$$
$$G_1=2M-\frac{Kd}{aN_c}\overline{v}(1-r^2)+\frac{Kr^2}{aN_c}\overline{v}\sum_{l=1}^d\cos p_la,$$
$$G_2=-\frac{K}{2aN_c}\overline{v}. \tag{3.10}$$

The resulting H_Π is

$$H_\Pi=G_1\sum_{p,f_1,f_2}(1-\tanh^2 2u_p)^{1/2}\tilde{\Pi}^\dagger_{f_1f_2}(p)\tilde{\Pi}_{f_2f_1}(p)$$
$$-\frac{G_1}{2}N_f^2\sum_p\left([1-(1-\tanh^2 2u_p)^{1/2}]\right.$$
$$\left.+\frac{2G_2r^2}{G_1}\sum_{l=1}^d\cos p_la\right). \tag{3.11}$$

Now $\tilde{\Pi}^\dagger_{f_1f_2}(p)$ stands for the pseudo-scalar creation operator in the momentum space. According to Eq. (3.11), the difference between the pseudo-scalar meson energy and vacuum energy is

$$E_{\Pi}=G_1(1-\tanh^2 2u_p)^{1/2}$$
$$=\left(2M-\frac{Kd}{aN_c}\bar{v}(1-r^2)-\frac{K}{aN_c}(1-r^2)\bar{v}\sum_{l=1}^{d}\cos p_l a\right)^{1/2}$$
$$\times\left(2M-\frac{Kd}{aN_c}\bar{v}(1-r^2)+\frac{K}{aN_c}(1+r^2)\bar{v}\sum_{l=1}^{d}\cos p_l a\right)^{1/2}, \quad (3.12)$$

which gives the pseudo-scalar mass when $p_j=0$. The pseudoscalar mass square is

$$M_{\Pi}^2=E_{\Pi}^2|_{p_j=0}=4\left(M+\frac{Kdr^2}{aN_c}\bar{v}\right)\left(M+\frac{Kdr^2}{aN_c}\bar{v}-\frac{Kd}{aN_c}\bar{v}\right). \quad (3.13)$$

In order to define the chiral limit, one has to fine tune $M\to M_{chiral}$ so that the pion becomes massless. From Eq. (3.13), we get

$$M_{chiral}=-\frac{Kdr^2}{aN_c}\bar{v}. \quad (3.14)$$

In this limit, the pseudoscalar mass square behaves as $M_{\Pi}^2\propto M-M_{chiral}$, which is the PCAC relation.

The H_V sector in Eq. (3.5) can be considered in a similar way. After a Fourier transformation,

$$V_j(x)=\sum_p e^{ipx}\tilde{V}_j(p), \quad (3.15)$$

and a Bogoliubov transformation,

$$\tilde{V}_j(p)\to\tilde{V}_j(p)\cosh w_p^{(j)}+\tilde{V}_j^{\dagger}(-p)\sinh w_p^{(j)},$$
$$\tilde{V}_j^{\dagger}(p)\to\tilde{V}_j^{\dagger}(p)\cosh w_j(p)+\tilde{V}_j(-p)\sinh w_p^{(j)}, \quad (3.16)$$

H_V becomes a diagonalized one,

$$H_V=G_1\sum_{p,j,f_1,f_2}(1-\tanh^2 2w_p^{(j)})^{1/2}\tilde{V}_{jf_1f_2}^{\dagger}(p)V_{jf_2f_1}(p)$$
$$-\frac{G_1}{2}N_f^2\sum_{p,j}\left([1-(1-\tanh^2 2w_p^{(j)})^{1/2}]+\frac{2G_2}{G_1}r^2\sum_{l=1}^{d}\cos p_l a\right), \quad (3.17)$$

if

$$\tanh 2w_p^{(j)}=-\frac{2G_2}{G_1}\left(\sum_{l=1}^{d}\cos p_l a-2\cos p_j a\right). \quad (3.18)$$

Now $\tilde{V}_{jf_1f_2}^{\dagger}(p)$ stands for the vector creation operator in the momentum space. The vector mass is

$$M_V=G_1(1-\tanh^2 2w_0^{(j)})^{1/2}\to^{M\to M_c}-2K\sqrt{\frac{d-1}{aN_c}}\bar{v}. \quad (3.19)$$

According to Eqs. (3.3), (3.11), and (3.17), the vacuum energy reads

$$E_{\Omega}=\langle\Omega|H^{\mu}|\Omega\rangle$$
$$=E_{\Omega}^{(0)}-\frac{G_1}{2}N_f^2\sum_p\left([1-(1-\tanh^2 2u_p)^{1/2}]+\frac{2G_2r^2}{G_1}\sum_{l=1}^{d}\cos p_l a\right)$$
$$-\frac{G_1}{2}N_f^2\sum_{p,j}\left([1-(1-\tanh^2 2w_p^{(j)})^{1/2}]+\frac{2G_2}{G_1}r^2\sum_{l=1}^{d}\cos p_l a\right). \quad (3.20)$$

This also gives the thermodynamic potential (grand potential) at $T=0$.

As shown in Ref. [20], at $\mu=0$ the results above are consistent with those of Smit in Ref. [19] where the $1/N_c$ expansion was used.

B. Results in the large N_c limit

As shown in Appendix B, in the chiral limit, the dominant contributions to the vacuum energy for large N_c is

$$E_{\Omega}\to N_fN_s\left[M_{chiral}\bar{v}-\left(\frac{Kd(1+r^2)}{a}+\mu\right)v^{\dagger}+\frac{Kdr^2}{4aN_c}[(v^{\dagger})^2+\bar{v^2}]-\frac{Kd}{4aN_c}[\bar{v^2}-(v^{\dagger})^2]\right]. \quad (3.21)$$

As shown Appendix C, under the mean-field approximation, i.e., by Wick contracting a pair of fermion fields in the four fermion terms in Eq. (2.3), one can obtain a bilinear Hamiltonian in the large N_c limit,

$$H_{eff}\sim H_{MFA}=A\sum_x\bar{\psi}(x)\psi(x)+B\sum_x\psi^{\dagger}(x)\psi(x)+C, \quad (3.22)$$

where

$$A=M_{chiral}-\frac{Kd}{2aN_c}(1-r^2)\bar{v},$$
$$B=\frac{Kd(1+r^2)}{a}\left(\frac{v^{\dagger}}{2N_c}-1\right),$$
$$C=-\frac{Kd}{4aN_c}[(1+r^2)v^{\dagger 2}-(1-r^2)\bar{v^2}]N_sN_f. \quad (3.23)$$

The coefficient A plays the role of dynamical mass of quark. According to Eqs. (2.8) and (3.22), the vacuum energy in presence of the chemical potential μ is now $E_\Omega = \langle \Omega_{eff} | H_{MFA} - \mu N | \Omega_{eff} \rangle$, which agrees with Eq. (3.21), derived from the large N_c limit of a Bosonized Hamiltonian Eq. (3.3).

In the large N_c limit, the fermion field ψ can be expressed as

$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta^\dagger(x) \end{pmatrix}. \tag{3.24}$$

The two-spinors ξ and $\eta^\dagger$ are the annihilation operator of positive energy fermion and creation operator of negative energy fermion, respectively. Let us define the state $|n_p, \bar{n}_p\rangle$ in the momentum space by

$$\begin{aligned} &\xi_p |0_p, \bar{n}_p\rangle = 0, \quad \xi_p^\dagger |0_p, \bar{n}_p\rangle = |1_p, \bar{n}_p\rangle, \\ &\xi_p |1_p, \bar{n}_p\rangle = |0_p, \bar{n}_p\rangle, \quad \xi_p^\dagger |1_p, \bar{n}_p\rangle = 0, \\ &\eta_p |n_p, 0_p\rangle = 0, \quad \eta_p^\dagger |n_p, 0_p\rangle = |n_p, 1_p\rangle, \\ &\eta_p |n_p, 1_p\rangle = |n_p, 0_p\rangle, \quad \eta_p^\dagger |n_p, 1_p\rangle = 0. \end{aligned} \tag{3.25}$$

The numbers n_p and $\bar{n}_p$ take the values 0 or 1 due to the Pauli principle. By definition, the up and down components of the fermion field are decoupled in such a state $|n_p, \bar{n}_p\rangle$. For the vacuum state of H_{eff}^μ, we make an ansatz

$$|\Omega_{eff}\rangle = \sum_{n_p, \bar{n}_p, p} f_{n_p, \bar{n}_p} |n_p, \bar{n}_p\rangle. \tag{3.26}$$

In the leading N_c limit, the dominant contributions to the chiral condensate and quark number density are

$$\begin{aligned} \langle \bar{\psi}\psi \rangle &= \frac{\langle \Omega | \Sigma_x \bar{\psi}(x)\psi(x) | \Omega \rangle}{N_f N_s} \to \bar{v} \\ &= \frac{1}{N_f N_s} \sum_{n_p, \bar{n}_p, p} C_{n_p, \bar{n}_p} \langle n_p, \bar{n}_p | \bar{\psi}\psi | n_p, \bar{n}_p \rangle \\ &= 2N_c (n + \bar{n} - 1), \\ n_q &= \frac{\langle \Omega | \sum_x \psi^\dagger(x)\psi(x) | \Omega \rangle}{2N_c N_f N_s} - 1 \to \frac{v^\dagger}{2N_c} - 1 \\ &= \frac{1}{2N_c N_f N_s} \sum_{n_p, \bar{n}_p, p} C_{n_p, \bar{n}_p} \langle n_p, \bar{n}_p | \psi^\dagger \psi | n_p, \bar{n}_p \rangle - 1 \\ &= n - \bar{n}. \end{aligned} \tag{3.27}$$

Here we denote $C_{n_p, \bar{n}_p} = f^2_{n_p, \bar{n}_p}$. Using Eqs. (3.21) and (3.27), in the large N_c limit we obtain the normalized vacuum energy,

$$\begin{aligned} \epsilon_\Omega &= \frac{E_\Omega}{2N_c N_f N_s} \\ &= M_{chiral}(n + \bar{n} - 1) - \frac{Kdr^2}{a}(n - \bar{n} + 1) \\ &\quad + \frac{Kdr^2}{a}(n^2 + \bar{n}^2 + 1 - 2\bar{n}) + \frac{Kd}{a}(n + \bar{n} - 2n\bar{n} - 1) \\ &\quad - \mu(n - \bar{n} + 1), \end{aligned} \tag{3.28}$$

where the quark number n and antiquark number $\bar{n}$,

$$\begin{aligned} n &= \langle n \rangle = \sum_{n_p, \bar{n}_p, p} C_{n_p, \bar{n}_p} n_p, \\ \bar{n} &= \langle \bar{n} \rangle = \sum_{n_p, \bar{n}_p, p} C_{n_p, \bar{n}_p} \bar{n}_p, \end{aligned} \tag{3.29}$$

are constrained in the range of $[0,1]$ and determined by minimizing the vacuum energy.

For a generic nucleon operator O_{Nucl} consisting of three quarks, the thermo mass is

$$M_{Nucl} = \langle \Omega_{eff} | O_{Nucl} H_{eff}^\mu O_{Nucl}^\dagger | \Omega_{eff} \rangle - E_\Omega. \tag{3.30}$$

Under the mean-field approximation in the large N_c limit, it becomes (see Appendix D)

$$M_{Nucl} = 3(A + B) - 3\mu. \tag{3.31}$$

IV. PHASE STRUCTURE AT $T=0$ AND $\mu \neq 0$

We now consider in the larger N_c limit, the critical behavior of the system at $T=0$ and $\mu \neq 0$. The ground state of the system corresponds to the lowest value of the vacuum energy. At some given inputs of Wilson parameter r and chemical potential μ, we can find the value of n and $\bar{n}$ when ϵ_Ω Eq. (3.28) is minimized. The result is

$$\begin{aligned} n &= \Theta(\mu - \mu_C), \\ \bar{n} &= 0. \end{aligned} \tag{4.1}$$

Here $\Theta(\mu - \mu_C)$ is the step function: it is 0 for $\mu < \mu_C$ and 1 for $\mu > \mu_C$, where μ_C is the critical chemical potential,

$$\mu_C = \frac{Kd}{a}(1 + 2r^2). \tag{4.2}$$

Substituting Eq. (4.1) into Eq. (3.27), we obtain the chiral condensate and quark number density,

$$\langle \bar{\psi}\psi \rangle = 2N_c[\Theta(\mu - \mu_C) - 1],$$

$$n_q = \Theta(\mu - \mu_C). \tag{4.3}$$

There is clearly is a first-order chiral phase transition. For $\mu < \mu_C$, the system is in the confinement phase with chiral-symmetry breaking. For $\mu > \mu_C$, chiral symmetry is restored.

According to Eqs. (3.23), (3.31), and (4.1), the thermo mass of the nucleon is

$$M_{Nucl} = M^{(0)}_{Nucl} - 3\mu, \tag{4.4}$$

where

$$M^{(0)}_{Nucl} = 3m^{(0)}_{dyn} \tag{4.5}$$

is the nucleon mass at $\mu = 0$, and

$$m^{(0)}_{dyn} = \frac{Kd(1+r^2)}{a} \tag{4.6}$$

is the dynamical mass of quark at $\mu = 0$. From Eq. (4.4), one sees that the nucleon thermo mass vanishes at $\mu = M^{(0)}_{Nucl}/3 = m^{(0)}_{dyn}$, before the chiral phase transition takes place. This is not surprising because Wilson fermions break explicitly the chiral symmetry. The value of μ_C should coincide with $M^{(0)}_{Nucl}/3$ when r is very small, i.e., as in the case of naive or Kogut-Susskind fermions [9].

V. DISCUSSIONS

In the preceding sections, we have investigated $(d+1)$-dimensional Hamiltonian lattice QCD at finite density with Wilson fermions in the strong-coupling regime. We compute the vacuum energy, meson and nucleon masses, chiral condensate, and quark number density. At finite chemical potential, there is an interplay between the bare fermion mass in the chiral limit and the chiral condensate, which has to be determined self-consistently. The critical behavior of the system in the large N_c limit is considered: a first-order chiral phase transition is found at $\mu = \mu_C$; the nucleon thermo mass vanishes before the chiral transition takes place, which is due to the explicit breakdown of chiral symmetry by Wilson fermions.

We have not yet specified the nature of the chiral-symmetric phase for $\mu > \mu_C$. Is it a QGP phase or a color-superconducting phase [24,25]? Up to now, there has been no first-principles investigation of such a phase in SU(3) gauge theory. The answer to this question might be very important to our understanding of the formation of the neutron star or quark star.

We also know that the strong-coupling regime is far from the continuum limit. One has to develop a new numerical method to study the continuum physics. The Monte Carlo Hamiltonian method developed recently [26,27] might eventually be useful for such a purpose. We hope to discuss these interesting issues in the future.

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APPENDIX A: BOSONIZATION AND LINEARIZATION

In Sec. III, to bosonize the Hamiltonian Eq. (2.3), we introduce the operators Π, $\Pi^\dagger$, V, and $V^\dagger$, defined by Eq. (3.1). In Eq. (2.3), there are terms having direct correspondence to these operators. In addition, there are also terms irrelevant to these operators.

In order that the operators in Eq. (3.1) represent the appropriate mesons, they must satisfy the commutation relations for boson operators. However, a direct calculation shows

$$\begin{aligned}
&[\Pi_{f_1,f_2}(x), \Pi^\dagger_{f_2,f_1}(x)] = \frac{1}{2\bar{v}}[\bar{\psi}_{f_1}(x)\psi_{f_1}(x) + \bar{\psi}_{f_2}(x)\psi_{f_2}(x)], \\
&[V_{j,f_1,f_2}(x), V^\dagger_{l,f_2,f_1}(x)] \\
&\quad = \delta_{jl}\frac{1}{2\bar{v}}[\bar{\psi}_{f_1}(x)\psi_{f_1}(x) + \bar{\psi}_{f_2}(x)\psi_{f_2}(x)] - \frac{1}{4\bar{v}}[\psi^\dagger_{f_1}(x) \\
&\qquad \times(\gamma_j\gamma_l - \gamma_l\gamma_j)\psi_{f_1}(x) + \psi^\dagger_{f_2}(x)(\gamma_j\gamma_l - \gamma_l\gamma_j)\psi_{f_2}(x)],
\end{aligned} \tag{A1}$$

which are not consistent with the commutation relations between the annihilation and creation operators for bosons. A similar situation also appears in quantum theory of magnetization [28] and many-particle systems [29], where a linearization prescription is used to simplify the theory. Using such a procedure, i.e., with the fermion bilinears on the right-hand side of the commutation relations replaced by the vacuum expectation value $\bar{\psi}(x)\psi(x) \to \bar{v}$ and for degenerate quarks, Eq. (A1) becomes

$$\begin{aligned}
&[\Pi(x), \Pi^\dagger(x)] \approx \frac{1}{\bar{v}}\frac{\langle\Omega_{eff}|\sum_x \bar{\psi}\psi|\Omega_{eff}\rangle}{N_f N_s} = 1, \\
&[V_j(x), V^\dagger_l(x)] \approx \delta_{jl}\frac{\langle\Omega_{eff}|\sum_x \bar{\psi}\psi|\Omega_{eff}\rangle}{N_f N_s \bar{v}} \\
&\qquad - \frac{\langle\Omega_{eff}|\sum_x \psi^\dagger(x)(\gamma_j\gamma_l - \gamma_l\gamma_j)\psi(x)|\Omega_{eff}\rangle}{2N_f N_s \bar{v}} \\
&\qquad = \delta_{jl},
\end{aligned} \tag{A2}$$

which are the correct commutation relations for bosons. As shown in Refs. [20,23], the linearization prescription leads to results consistent with Ref. [19], where systematic $1/N_c$ expansion was used.

The four fermion terms in Eq. (2.3), which have direct correspondence to the boson operators, are

$$\begin{aligned}
&\sum_{x,k}\ [\psi^\dagger_{f_1}(x)\gamma_5\psi_{f_2}(x)\psi^\dagger_{f_2}(x+\hat{k})\gamma_5\psi_{f_1}(x+\hat{k})\\
&\quad+\psi^\dagger_{f_1}(x)\gamma_4\gamma_5\psi_{f_2}(x)\psi^\dagger_{f_2}(x+\hat{k})\gamma_4\gamma_5\psi_{f_1}(x+\hat{k})]\\
&\quad=-2\overline{v}\sum_{x,k}\ [\Pi^\dagger_{f_1f_2}(x)\Pi^\dagger_{f_2f_1}(x+\hat{k})\\
&\qquad+\Pi_{f_1f_2}(x)\Pi_{f_2f_1}(x+\hat{k})],\\
&\sum_{x,k}\ [\psi^\dagger_{f_1}(x)\gamma_5\psi_{f_2}(x)\psi^\dagger_{f_2}(x+\hat{k})\gamma_5\psi_{f_1}(x+\hat{k})\\
&\quad-\psi^\dagger_{f_1}(x)\gamma_4\gamma_5\psi_{f_2}(x)\psi^\dagger_{f_2}(x+\hat{k})\gamma_4\gamma_5\psi_{f_1}(x+\hat{k})]\\
&\quad=-2\overline{v}\sum_{x,k}\ [\Pi^\dagger_{f_1f_2}(x)\Pi_{f_2f_1}(x+\hat{k})\\
&\qquad+\Pi_{f_1f_2}(x)\Pi^\dagger_{f_2f_1}(x+\hat{k})],\\
&\sum_{x,k,j}\ [\psi^\dagger_{f_1}(x)\gamma_4\gamma_j\psi_{f_2}(x)\psi^\dagger_{f_2}(x+\hat{k})\gamma_4\gamma_j\psi_{f_1}(x+\hat{k})\\
&\quad+\psi^\dagger_{f_1}(x)\gamma_j\psi_{f_2}(x)\psi^\dagger_{f_2}(x+\hat{k})\gamma_j\psi_{f_1}(x+\hat{k})](1-2\delta_{|k|,j})\\
&\quad=-2\overline{v}\sum_{x,k,j}\ [V^\dagger_{jf_1f_2}(x)V^\dagger_{jf_2f_1}(x+\hat{k})\\
&\qquad+V_{jf_1f_2}(x)V_{jf_2f_1}(x+\hat{k})](1-2\delta_{|k|,j}),\\
&\sum_{x,k,j}\ [\psi^\dagger_{f_1}(x)\gamma_4\gamma_j\psi_{f_2}(x)\psi^\dagger_{f_2}(x+\hat{k})\gamma_4\gamma_j\psi_{f_1}(x+\hat{k})\\
&\quad-\psi^\dagger_{f_1}(x)\gamma_j\psi_{f_2}(x)\psi^\dagger_{f_2}(x+\hat{k})\gamma_j\psi_{f_1}(x+\hat{k})]\\
&\quad=2\overline{v}\sum_{x,k,j}\ [V^\dagger_{jf_1f_2}(x)V_{jf_2f_1}(x+\hat{k})\\
&\qquad+V_{jf_1f_2}(x)V^\dagger_{jf_2f_1}(x+\hat{k})]. \qquad \text{(A3)}
\end{aligned}$$

For the bilinear $\Sigma_x\overline{\psi}(x)\psi(x)$ in Eq. (2.3), a direct calculation shows

$$\begin{aligned}
&\left[\frac{1}{2}\sum_{x'}\ \overline{\psi}(x')\psi(x'),\Pi_{f_1f_2}(x)\right]\\
&\quad=\frac{1}{4\sqrt{-\overline{v}}}[\psi^\dagger_{f_1}(x)\gamma_4(1-\gamma_4)\gamma_5\psi_{f_2}(x)\\
&\qquad-\psi^\dagger_{f_1}(x)\gamma_4(1-\gamma_4)\gamma_5\gamma_4\psi_{f_2}(x)]\\
&\quad=-\frac{1}{2\sqrt{-\overline{v}}}\psi^\dagger_{f_1}(x)(1-\gamma_4)\gamma_5\psi_{f_2}(x)\\
&\quad=-\Pi_{f_1f_2}(x). \qquad \text{(A4)}
\end{aligned}$$

Similarly, one can also show

$$\begin{aligned}
&\left[\frac{1}{2}\sum_{x'}\ \overline{\psi}(x')\psi(x'),\Pi^\dagger_{f_2f_1}(x)\right]=\Pi^\dagger_{f_2f_1}(x),\\
&\left[\frac{1}{2}\sum_{x'}\ \overline{\psi}(x')\psi(x'),V_{jf_1f_2}(x)\right]=-V_{jf_1f_2}(x),\\
&\left[\frac{1}{2}\sum_{x'}\ \overline{\psi}(x')\psi(x'),V^\dagger_{jf_2f_1}(x)\right]=V^\dagger_{jf_2f_1}(x). \qquad \text{(A5)}
\end{aligned}$$

Equations (A4) and (A5) imply that the bilinear $\Sigma_x\overline{\psi}(x)\psi(x)$ can be bosonized as

$$\begin{aligned}
\sum_x\ \overline{\psi}(x)\psi(x)\sim\sum_x\ \Big(&\overline{v}+2\Pi^\dagger_{f_2f_1}(x)\Pi_{f_1f_2}(x)\\
&+2\sum_j\ V^\dagger_{jf_2f_1}(x)V_{jf_1f_2}(x)\Big). \qquad \text{(A6)}
\end{aligned}$$

For the four fermion term $\Sigma_x\overline{\psi}_{f_1}(x)\psi_{f_2}(x)\overline{\psi}_{f_2}(x+\hat{k})\psi_{f_1}(x+\hat{k})$ in Eq. (2.3), we have

$$\begin{aligned}
&\left[\frac{1}{2}\sum_{x'}\ \overline{\psi}_{f_1'}(x')\psi_{f_2'}(x')\overline{\psi}_{f_2'}(x'+\hat{k})\psi_{f_1'}(x'+\hat{k}),\Pi_{f_1f_2}(x)\right]\\
&\quad=\sum_{x'}\ \overline{\psi}_{f_1'}(x')\psi_{f_2'}(x')\left[\frac{1}{2}\overline{\psi}_{f_2'}(x'+\hat{k})\psi_{f_1'}(x'+\hat{k}),\Pi_{f_1f_2}(x)\right]\\
&\qquad+\sum_{x'}\ \left[\frac{1}{2}\overline{\psi}_{f_1'}(x')\psi_{f_2'}(x'),\Pi_{f_1f_2}(x)\right]\\
&\qquad\times\overline{\psi}_{f_2'}(x'+\hat{k})\psi_{f_1'}(x'+\hat{k})\\
&\quad\simeq-2\overline{v}\Pi_{f_1f_2}(x), \qquad \text{(A7)}
\end{aligned}$$

where on the right-hand side Eq. (A4) and the linearization prescription have been used. In addition, we also have

$$\begin{aligned}
&\left[\frac{1}{2}\sum_{x'}\ \overline{\psi}_{f_1'}(x')\psi_{f_2'}(x')\overline{\psi}_{f_2'}(x'+\hat{k})\psi_{f_1'}(x'+\hat{k}),\Pi^\dagger_{f_2f_1}(x)\right]\\
&\quad\simeq2\overline{v}\Pi^\dagger_{f_2f_1}(x),\\
&\left[\frac{1}{2}\sum_{x'}\ \overline{\psi}_{f_1'}(x')\psi_{f_2'}(x')\overline{\psi}_{f_2'}(x'+\hat{k})\psi_{f_1'}(x'+\hat{k}),V_{jf_1f_2}(x)\right]\\
&\quad\simeq-2\overline{v}V_{jf_1f_2}(x),
\end{aligned}$$

$$\left[\frac{1}{2}\sum_{x'}\ \overline{\psi}_{f_1'}(x')\psi_{f_2'}(x')\overline{\psi}_{f_2'}(x'+\hat{k})\psi_{f_1'}(x'+\hat{k}), V^{\dagger}_{jf_2f_1}(x)\right]$$
$$\simeq 2\overline{v}V^{\dagger}_{jf_2f_1}(x). \tag{A8}$$

Equations (A7) and (A8) also imply that the four fermion term $\Sigma_x\overline{\psi}_{f_1}(x)\psi_{f_2}(x)\overline{\psi}_{f_2}(x+\hat{k})\psi_{f_1}(x+\hat{k})$ can be bosonized as

$$\sum_x\ \overline{\psi}_{f_1}(x)\psi_{f_2}(x)\overline{\psi}_{f_2}(x+\hat{k})\psi_{f_1}(x+\hat{k})$$
$$\sim\sum_x\ \Big(\overline{v}_2+4\overline{v}\Pi^{\dagger}_{f_2f_1}(x)\Pi_{f_1f_2}(x)$$
$$+4\overline{v}\sum_j\ V^{\dagger}_{jf_2f_1}(x)V_{jf_1f_2}(x)\Big). \tag{A9}$$

In the linear prescription, the four fermion operators in Eq. (2.3), which are irrelevant to the operators in Eq. (3.1), are replaced by their vacuum expectation values. For example,

$$\psi^{\dagger}(x)\psi(x)\sim v^{\dagger},$$

$$\sum_{x,k}\ \psi^{\dagger}_{f_1}(x)\psi_{f_2}(x)\psi^{\dagger}_{f_2}(x+\hat{k})\psi_{f_1}(x+\hat{k})\sim 2dN_fN_sv^{\dagger}_2,$$

$$\sum_{x,k,j}\ [r^2+(1-2\delta_{|k|,j})]\overline{\psi}_{f_1}(x)\sigma_j\psi_{f_2}(x)\overline{\psi}_{f_2}(x+\hat{k})$$
$$\times\sigma_j\psi_{f_1}(x+\hat{k})\sim 2dN_fN_s\overline{v}_{2\sigma_j},$$

$$\sum_{x,k,j}\ [r^2-(1-2\delta_{|k|,j})]\psi^{\dagger}_{f_1}(x)\sigma_j\psi_{f_2}(x)\psi^{\dagger}_{f_2}(x+\hat{k})$$
$$\times\sigma_j\psi_{f_1}(x+\hat{k})\sim 2dN_fN_sv^{\dagger}_{2\sigma_j}. \tag{A10}$$

Collecting the above results, we obtain Eqs. (3.3), (3.4), and (3.5).

APPENDIX B: VACUUM ENERGY IN THE LARGE N_c LIMIT

According to Eq. (3.6),

$$\overline{v}_2=\frac{1}{2dN_fN_s}\sum_{x,k,f_1,f_2}\ (\langle\overline{\psi}_{f_1,c_1}(x)\psi_{f_2,c_1}(x)\rangle_{eff}$$
$$\times\langle\overline{\psi}_{f_2,c_2}(x+\hat{k})\psi_{f_1,c_2}(x+\hat{k})\rangle_{eff}$$
$$+\langle\psi_{f_2}(x)\overline{\psi}_{f_2,c_2}(x+\hat{k})\rangle_{eff}\langle\psi_{f_1}(x+\hat{k})\overline{\psi}_{f_1,c_1}(x)\rangle_{eff}),$$

$$v^{\dagger}_2=\frac{1}{2dN_fN_s}\sum_{x,k,f_1,f_2}\ (\langle\psi^{\dagger}_{f_1,c_1}(x)\psi_{f_2,c_1}(x)\rangle_{eff}$$
$$\times\langle\psi^{\dagger}_{f_2,c_2}(x+\hat{k})\psi_{f_1,c_2}(x+\hat{k})\rangle_{eff}$$
$$+\langle\psi_{f_2}(x)\psi^{\dagger}_{f_2,c_2}(x+\hat{k})\rangle_{eff}\langle\psi_{f_1}(x+\hat{k})\psi^{\dagger}_{f_1,c_1}(x)\rangle_{eff}), \tag{B1}$$

where the color and flavor indexes for the fermion fields are explicitly specified and summation over the repeated color index is implied. $\langle\cdots\rangle_{eff}$ stands for the expectation value taken in the vacuum state $|\Omega_{eff}\rangle$. Because

$$\langle\overline{\psi}_{f_1,c_1}(x)\psi_{f_2,c_1}(x)\rangle_{eff}=\overline{v}\,\delta_{f_1,f_2}\propto N_c,$$
$$\langle\psi_{f_2,c_1}(x)\overline{\psi}_{f_2,c_2}(x+\hat{k})\rangle_{eff}\propto\delta_{c_1,c_2},$$
$$\langle\psi^{\dagger}_{f_1,c_1}(x)\psi_{f_2,c_1}(x)\rangle_{eff}=v^{\dagger}\delta_{f_1,f_2}\propto N_c,$$
$$\langle\psi_{f_2,c_1}(x)\psi^{\dagger}_{f_2,c_2}(x+\hat{k})\rangle_{eff}\propto\delta_{c_1,c_2}, \tag{B2}$$

the terms from Wick contraction of bilinear at different lattice sites are subdominant for large N_c. Therefore we have

$$\overline{v}_2\to\overline{v}^2,\quad v^{\dagger}_2\to(v^{\dagger})^2, \tag{B3}$$

with $\overline{v}$ and $v^{\dagger}$ expectation value of fermion bilinears defined in Eq. (3.2). Expectation values of other four fermion operators in Eq. (3.6) can be considered in a similar way,

$$\overline{v}_{2\sigma_j}\propto(\mathrm{Tr}\,\sigma_j)^2=0,\quad v^{\dagger}_{2\sigma_j}\propto(\mathrm{Tr}\,\sigma_j)^2=0, \tag{B4}$$

in the large N_c limit.

In the chiral limit Eq. (3.14), G_1 defined in Eq. (3.10) scales as $\overline{v}K/N_c\propto K$ and G_1/G_2 does not depend on N_c. The N_f^2 terms in Eq. (3.20) are subdominant because they are proportional to K, while other nonvanishing terms of $E^{(0)}_{\Omega}$ in Eq. (3.20), defined in Eq. (3.4), are dominant because they behave as $\overline{v}K\propto KN_c$ or $v^{\dagger}K\propto KN_c$. Using this fact and substituting Eqs. (B3) and Eq. (B4) into Eq. (3.4), we obtain Eq. (3.21) from Eq. (3.20), i.e., the vacuum energy in the large N_c limit.

APPENDIX C: MEAN-FIELD APPROXIMATION IN THE LARGE N_c LIMIT

The mean-field approximation is a popular technique widely used in quantum field theory with four fermion interactions (for example, to bilinearize the Nambu-Jona-Lasino model [30,31]) and quantum theory of many-particle systems [29].

In order to show how to derive Eq. (3.22) from Eq. (2.3), let us look at the first and last of the four fermion terms in Eq. (2.3) as an example. Under the mean-field approximation, i.e., replacing a pair of fermion fields by their vacuum expectation value, the first four fermion term in Eq. (2.3) becomes

$$\sum_{x,k,f_1,f_2} \psi^\dagger_{f_1,c_1}(x)\psi_{f_2,c_1}(x)\psi^\dagger_{f_2,c_2}(x+\hat{k})\psi_{f_1,c_2}(x+\hat{k})$$

$$\sim \sum_{x,k,f_1,f_2} (\langle \psi^\dagger_{f_1,c_1}(x)\psi_{f_2,c_1}(x)\rangle_{eff}\psi^\dagger_{f_2,c_2}(x+\hat{k})\psi_{f_1,c_2}(x+\hat{k})$$
$$+\psi^\dagger_{f_1,c_1}(x)\psi_{f_2,c_1}(x)\langle \psi^\dagger_{f_2,c_2}(x+\hat{k})\psi_{f_1,c_2}(x+\hat{k})\rangle_{eff}$$
$$+\psi^\dagger_{f_1,c_1}(x)\langle \psi_{f_2,c_1}(x)\psi^\dagger_{f_2,c_2}(x+\hat{k})\rangle_{eff}\psi_{f_1,c_2}(x+\hat{k})$$
$$+\psi_{f_2,c_1}(x)\psi^\dagger_{f_2,c_2}(x+\hat{k})\langle \psi^\dagger_{f_1,c_1}(x)\psi_{f_1,c_2}(x+\hat{k})\rangle_{eff})$$
$$-2dN_fN_sv_2^\dagger. \tag{C1}$$

Again, the color and flavor indexes for the fermion fields are explicitly specified and summation over the repeated color index is implied. According to Eq. (B2), for large N_c, Wick contractions of bilinear at different lattice sites give subdominant contribution, therefore

$$\sum_{x,k,f_1,f_2} \psi^\dagger_{f_1,c_1}(x)\psi_{f_2,c_1}(x)\psi^\dagger_{f_2,c_2}(x+\hat{k})\psi_{f_1,c_2}(x+\hat{k})$$
$$\approx v^\dagger \sum_{x,k,f_1,f_2} \delta_{f_1,f_2}[\psi^\dagger_{f_2,c}(x+\hat{k})\psi_{f_1,c}(x+\hat{k})$$
$$+\psi^\dagger_{f_1,c}(x)\psi_{f_2,c}(x)]-2dN_fN_s(v^\dagger)^2$$
$$=v^\dagger\sum_{x,k,f}[\psi^\dagger_{f,c}(x)\psi_{f,c}(x)+\psi^\dagger_{f,c}(x+\hat{k})\psi_{f,c}(x+\hat{k})]$$
$$-2dN_fN_s(v^\dagger)^2$$
$$=4dv^\dagger\sum_x \psi^\dagger(x)\psi(x)-2dN_fN_s(v^\dagger)^2. \tag{C2}$$

Under the mean-field approximation in the large N_c limit, the dominant contribution to the last four fermion term in Eq. (2.3) is

$$\sum_{x,k,f_1,f_2} \psi^\dagger_{f_1,c_1}(x)\sigma_j\psi_{f_2,c_1}(x)\psi^\dagger_{f_2,c_2}(x+\hat{k})\sigma_j\psi_{f_1,c_2}(x+\hat{k})$$

$$\sim \sum_{x,k,f_1,f_2} (\langle \psi^\dagger_{f_1,c_1}(x)\sigma_j\psi_{f_2,c_1}(x)\rangle_{eff}\psi^\dagger_{f_2,c_2}(x+\hat{k})$$
$$\times\sigma_j\psi_{f_1,c_2}(x+\hat{k})+\psi^\dagger_{f_1,c_1}(x)\sigma_j\psi_{f_2,c_1}(x)\langle\psi^\dagger_{f_2,c_2}(x+\hat{k})$$
$$\times\sigma_j\psi_{f_1,c_2}(x+\hat{k})\rangle_{eff})-2dN_fN_sv^\dagger_{2\sigma_j}$$
$$\propto \mathrm{Tr}(\sigma_j)=0. \tag{C3}$$

Treating other four fermion terms in the same way, one obtains Eq. (3.22), i.e., the bilinear Hamiltonian H_{MFA}.

APPENDIX D: NUCLEON MASS

Under the mean-field approximation in the large N_c limit, Eq. (3.30) becomes

$$M_{Nucl}=\langle\Omega_{eff}|O_{Nucl}(H_{MFA}-\mu N)O^\dagger_{Nucl}|\Omega_{eff}\rangle-E_\Omega. \tag{D1}$$

Substituting the bilinear Hamiltonian Eq. (3.22), the fermion field Eq. (3.24), and the particle number operator Eq. (2.6) into Eq. (D1), we have

$$M_{Nucl}=\langle\Omega_{eff}|O_{Nucl}\Big(A\sum_x \bar{\psi}(x)\psi(x)$$
$$+(B-\mu)\sum_x \psi^\dagger(x)\psi(x)+C\Big)O^\dagger_{Nucl}|\Omega_{eff}\rangle-E_\Omega$$
$$=\langle\Omega_{eff}|O_{Nucl}\Big(A\sum_x (\xi^\dagger\xi-\eta\eta^\dagger)$$
$$+(B-\mu)\sum_x (\xi^\dagger\xi+\eta\eta^\dagger)+C\Big)O^\dagger_{Nucl}|\Omega_{eff}\rangle-E_\Omega$$
$$=\langle\Omega_{eff}|O_{Nucl}\Big(A\sum_x \xi^\dagger\xi+(B-\mu)\sum_x \xi^\dagger\xi\Big)$$
$$\times O^\dagger_{Nucl}|\Omega_{eff}\rangle, \tag{D2}$$

where we have used the fact that the terms with $\eta^\dagger\eta$ and C are canceled by E_Ω.

Let us take the proton as an example. One can write the operator $O^\dagger_{Nucl}$ explicitly as

$$O^\dagger_{Nucl}=\frac{1}{\sqrt{18N_s}}\sum_x \epsilon_{c_1c_2c_3}\xi^\dagger_{c_1,u,1}(x)[\xi^\dagger_{c_2,u,1}(x)\xi^\dagger_{c_3,d,2}(x)$$
$$-\xi^\dagger_{c_2,u,2}(x)\xi^\dagger_{c_3,d,1}(x)], \tag{D3}$$

where c_1, c_2, and c_3 are the color indexes, u and d stand, respectively, for the u quark and d quark, and 1 and 2 are the spin-up and -down indexes. Using the anticommutation relationship for fermions,

$$\xi_{c,f,s}(x)\xi^\dagger_{c',f',s'}(x')$$
$$=\delta_{c,c'}\delta_{f,f'}\delta_{s,s'}\delta_{x,x'}-\xi^\dagger_{c',f',s'}(x')\xi_{c,f,s}(x), \tag{D4}$$

one obtains

$$\sum_x \xi^\dagger(x)\xi(x)O^\dagger_{Nucl}|\Omega_{eff}\rangle=3O^\dagger_{Nucl}|\Omega_{eff}\rangle. \tag{D5}$$

Equation (3.31) is a consequence of Eqs. (D2) and (D5).

At $\mu=0$, the nucleon mass equals $3m^{(0)}_{dyn}$ [see Eq. (4.5)], which agrees with Refs. [19,32].

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