

New twist on the dS/CFT correspondence

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We stress that the dS/CFT correspondence should be formulated using unitary principal series representations of the de Sitter isometry group or conformal group, rather than highest-weight representations as originally proposed. These representations, however, are infinite dimensional, and so do not account for the finite gravitational entropy of de Sitter space in a natural way. We then propose to replace the classical isometry group by a q -deformed version. This is carried out in detail for two-dimensional de Sitter space and we find that the unitary principal series representations deform to finite-dimensional unitary representations of the quantum group. We believe this provides a promising microscopic framework to account for the Bekenstein-Hawking entropy of de Sitter space.

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I. INTRODUCTION

A holographic formulation of quantum gravity in de Sitter space has been proposed in [1] (and anticipated in [2,3]), and the details of this correspondence have been elaborated further in [4–19]. The basic picture conjectures that quantum gravity in de Sitter is holographically dual to a boundary conformal field theory that can be viewed as residing at the spacelike past (and/or future) infinity. The isometry group of de Sitter space is identified with the conformal group on the boundary.

In another set of developments, holographic bounds on the entropy in de Sitter space have been formulated, with the conclusion that $S \leq A/4$, where A is the area of the cosmological horizon [20–30]. For eternal de Sitter space, one can view this relation as bounding the number of states accessible to a local observer. Exactly what one means by accessible is then open to debate. Banks *et al.* [23,24,27] have argued for the strongest interpretation of the bound, where the number of states in the Hilbert space is $e^{A/4}$. Susskind and co-workers [28,31] have instead argued the number of states should be countably infinite, with the number of states below a certain energy being bounded by $e^{A/4}$. Another possible interpretation compatible with semiclassical physics is that the spectrum of states is continuous and that the gravitational entropy $A/4$ is simply a finite contribution to an otherwise infinite number of accessible states.

In the first two scenarios, there is apparent conflict with the dS/conformal field theory (CFT) proposal. The isometry group of de Sitter involves non-compact boost generators, yielding unitary irreducible representations that are infinite dimensional. This conflicts with finiteness of the entropy. A related argument given in [28] studies a matrix element of a boost $e^{iL(t)}$ assuming finiteness of the entropy and Hermitic-

ity of the Hamiltonian of a comoving observer, and shows this must be quasi-periodic as time t is taken to infinity. The symmetry group instead implies that the matrix element has a finite constant limit. One must then give up either Hermiticity and/or the classical symmetries to obtain a finite entropy. In this paper we take the view that the unitarity of the description is paramount and ultimately led us to construct a formalism where the classical symmetries are relinquished.

Our analysis begins at the classical level with a review in Sec. II of the original dS/CFT correspondence. In Sec. III we then emphasize the point made in [15] that the quantization of a scalar field on de Sitter space with mass larger than the Hubble scale yields a unitary principal series representation of the classical isometry group, as opposed to the standard non-unitary highest-weight representations considered in the original dS/CFT conjecture. It is then natural to propose a new version of the correspondence using the principal series representations of the conformal group.

However, principal series representations are infinite dimensional, conflicting with finite de Sitter entropy. We therefore propose to replace the classical isometry group by a quantum symmetry group. (For an introduction to quantum groups, see [32,33].) The quantum group involves a deformation parameter q , which we take to be a root of unity. The classical symmetry group is recovered in the limit $q \rightarrow 1$. Quantum group deformations have appeared in the context of AdS/CFT in [34–36].

For simplicity we will restrict our attention to the case of two-dimensional de Sitter space; however, we expect the main results to generalize to other dimensions. In Sec. IV we identify finite-dimensional irreducible representations of the quantum group that become the unitary principal series representations of the conformal group in the classical limit. We propose a duality between a q -deformed conformal field theory and a bulk gravitational theory in a q -deformed de Sitter background. Bulk fields corresponding to these representations are then to be identified with corresponding operators in the q -CFT. This new formulation has many of the

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ingredients needed to resolve the entropy finiteness issue mentioned above, and we discuss the analogue of the quasi-periodic correlator calculation of [28] in this new framework. In Sec. V, we end with some discussion of topics such as the q -deformed CFT at the interacting level, geometry of q -deformed de Sitter space, the formulation of gravity in this background, entropy calculations, and the higher dimensional case. Finally, we warmly recommend [34,37] where many related ideas appear.

II. REVIEW OF THE CLASSICAL dS/CFT CORRESPONDENCE

The classical dS/CFT correspondence was proposed by Strominger [1] and studied further in [4], whose notation we follow. Thus far the correspondence does not go as far as specifying the CFT at the interacting level. In the current state of development the correspondence specifies a mapping from free bulk scalar field correlation functions to boundary correlation functions of conformal primary operators.

In terms of global coordinates, the metric on d -dimensional de Sitter space is

$$\frac{ds^2}{L^2} = -dt^2 + \cosh^2 t d\Omega_{d-1}^2,$$

with L the Hubble length, which will be henceforth set equal to unity.

The correspondence of [4] was formulated for a scalar field of mass m in a general α -vacuum. However, as described in [17], only the Euclidean vacuum state ($\alpha = -\infty$) can be consistently coupled to gravity, so we will restrict our attention to that case. On \mathcal{I}^- and \mathcal{I}^+ the field behaves as

$$\begin{aligned} \lim_{\tau \rightarrow -\infty} \phi(t, \Omega) &= \phi_+^{in}(\Omega) e^{h_+ t} + \phi_-^{in}(\Omega) e^{h_- t} \\ \lim_{\tau \rightarrow \infty} \phi(t, \Omega_A) &= \phi_+^{out}(\Omega) e^{-h_+ t} + \phi_-^{out}(\Omega) e^{-h_- t} \end{aligned} \quad (1)$$

where

$$h_{\pm} = \frac{d-1}{2} \pm i\mu, \quad \mu = \sqrt{m^2 - \frac{(d-1)^2}{4}} \quad (2)$$

and Ω_A denotes the point antipodal to Ω on the $d-1$ sphere.

The correspondence proceeds by examining the bulk Wightman function in the limit that points are taken to infinity:

$$\begin{aligned} \lim_{\tau, \tau' \rightarrow -\infty} G_E(x, x') &= e^{h_+(t+t')} \Delta_+(\Omega, \Omega') \\ &+ e^{h_-(t+t')} \Delta_-(\Omega, \Omega'), \end{aligned} \quad (3)$$

where Δ_{\pm} is proportional to the two-point function on the sphere of a conformal primary of weight h_{\pm} . The expressions (3) determine two-point functions of operators in the dual CFT,

$$\langle 0 | \mathcal{O}_{\pm}(\Omega) \mathcal{O}_{\pm}(\Omega') | 0 \rangle = -\frac{\mu^2}{4} \Delta_{\pm}(\Omega, \Omega'), \quad (4)$$

together with the contact terms

$$\begin{aligned} \langle 0 | \mathcal{O}_-(\Omega) \mathcal{O}_+(\Omega') | 0 \rangle &= \frac{\mu}{4} \delta^{(2)}(\Omega, \Omega') \\ \langle 0 | \mathcal{O}_+(\Omega) \mathcal{O}_-(\Omega') | 0 \rangle &= -\frac{\mu}{4} \delta^{(2)}(\Omega, \Omega'). \end{aligned} \quad (5)$$

We note that for bulk fields with masses $m > (d-1)/2$ (in units of the Hubble scale), which we will concentrate on here, the quantity μ is real and so the corresponding conformal primaries have complex conformal weights.

For the case of dS_3 , the two-point functions in the CFT (4) can be used to define the Zamolodchikov norm in the standard way:

$$\langle \mathcal{O}_{\pm}(\Omega) | \mathcal{O}_{\pm}(\Omega) \rangle = \Delta_{\pm}(\Omega, -\Omega_A). \quad (6)$$

This norm corresponds to an inner product defined on fields in the bulk that involves an additional action of CPT as compared to the standard Klein-Gordon inner product [4,38].

For future reference let us define precisely what we mean by unitarity in this context [39]. A Hermitian inner product satisfies $\langle c\psi | \chi \rangle = \bar{c} \langle \psi | \chi \rangle = \langle \psi | \bar{c}\chi \rangle$ for $c \in \mathbb{C}$, together with the condition $\langle \psi | \chi \rangle = \langle \chi | \psi \rangle$. This inner product is said to be invariant if $\langle \psi | g\chi \rangle = \langle g^* \psi | \chi \rangle$ where g is an element of the algebra, and g^* is the adjoint of g . Unitarity adds the condition that the Hermitian inner product be positive definite.

Representations with complex conformal weights are therefore non-unitary [40] with respect to the inner product (6), for which $L_n^* = L_{-n}$. For example,

$$\langle L_0 h_+ | L_0 h_+ \rangle = \langle h_+ | L_0^2 | h_+ \rangle = h_+^2 \langle h_+ | h_+ \rangle$$

which is not positive definite for nontrivial $|h_+\rangle$.

III. NEW CLASSICAL dS/CFT PROPOSAL

As we have seen, the conformal field theory as defined above is not unitary. This issue arose because one insisted on establishing a correspondence with the standard highest-weight representations considered in conformal field theory. We propose instead to formulate the correspondence using the unitary principal series representations of the de Sitter isometry group and conformal group $SO(d,1)$. Closely related proposals appear in [15], though we differ on some of the details and interpretation.

To keep the discussion as explicit as possible, we will now restrict our attention to two-dimensional de Sitter space. The classical de Sitter isometry group is $SO(2,1)$, which at the level of the algebra is isomorphic to $sl(2, \mathbb{R}) \approx su(1,1)$. Let us choose global coordinates (t, θ) for dS_2 :

$$ds^2 = -dt^2 + \cosh^2 t d\theta^2.$$

When acting on scalar field modes, the $sl(2, \mathbb{R})$ generators take the form

$$\begin{aligned}
 L_0 &= \cos \theta \frac{\partial}{\partial t} - \sin \theta \tanh t \frac{\partial}{\partial \theta} \\
 L_{-1} - L_1 &= 2 \sin \theta \frac{\partial}{\partial t} + 2 \cos \theta \tanh t \frac{\partial}{\partial \theta} \\
 L_1 + L_{-1} &= 2 \frac{\partial}{\partial \theta}
 \end{aligned} \tag{7}$$

and satisfy the Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n}. \tag{8}$$

We will now show that the representation of $su(1,1) \approx sl(2, \mathbb{R})$ one gets when quantizing a scalar field in de Sitter, in the range of masses under consideration ($m > 1/2$), is in fact a principal series representation. These representations are part of a larger family of representations labeled by a continuous complex parameter τ and an index ε that can take the discrete values 0 and 1/2; we will be interested only in the case $\varepsilon = 0$. They can be realized [41] in terms of generators H, X_+, X_- that act on periodic functions on the circle, $f(\theta)$ with $\theta \in [0, 2\pi)$. Choosing $e^{-ik\theta}$, $k \in \mathbb{Z}$ as a basis on this space of functions, the action of the generators is given by

$$\begin{aligned}
 H e^{-ik\theta} &= 2k e^{-ik\theta}, \\
 X_+ e^{-ik\theta} &= (k - \tau) e^{-i(k+1)\theta}, \\
 X_- e^{-ik\theta} &= -(k + \tau) e^{-i(k-1)\theta}.
 \end{aligned} \tag{9}$$

In terms of differential operators, the generators are

$$\begin{aligned}
 H &= 2i \frac{\partial}{\partial \theta}, \\
 X_+ &= i e^{-i\theta} \frac{\partial}{\partial \theta} - \tau e^{-i\theta}, \\
 X_- &= -i e^{i\theta} \frac{\partial}{\partial \theta} - \tau e^{i\theta}.
 \end{aligned} \tag{10}$$

The principal series representations correspond to the case $\tau = -1/2 + i\rho$, with ρ real [41]. They are unitary with respect to the canonical inner product

$$(f_1 | f_2) = \frac{1}{2\pi} \int_0^{2\pi} d\theta f_1(e^{i\theta}) \overline{f_2(e^{i\theta})}. \tag{11}$$

The associated notion of conjugation is

$$H^* = H, \quad X_+^* = -X_-, \quad X_-^* = -X_+. \tag{12}$$

Now, in the dS/CFT context, one can define a boundary-to-bulk map by promoting the modes on the circle to modes on de Sitter,

$$e^{-ik\theta} \rightarrow y_k(t) e^{-ik\theta}, \tag{13}$$

where the $y_k(t)$ are the normalized Euclidean vacuum modes, constructed explicitly for arbitrary-dimensional de Sitter in (3.37) of [4]. Using Eq. (11) the boundary propagator is simply $\delta_{k,k'}$, or equivalently $\sum_{k=-\infty}^{\infty} e^{-ik(\theta' - \theta)} = \delta(\theta' - \theta)$ in coordinate space. The bulk Wightman propagator in the Euclidean vacuum is then recovered from the boundary propagator $\delta_{k,k'}$ as

$$\begin{aligned}
 &\langle E | \phi(t', \theta') \phi(t, \theta) | E \rangle \\
 &= \sum_{k, k' = -\infty}^{\infty} y_{k'}(t') y_k^*(t) e^{-ik'\theta' + ik\theta} \delta_{k, k'} \\
 &= G_E(t', \theta'; t, \theta)
 \end{aligned} \tag{14}$$

where G_E is the Euclidean vacuum propagator as defined, for example, in [4].

We can act on the set of modes (13) with the generators (7) to determine the relation with Eq. (10). This requires the use of a non-trivial hypergeometric function identity, noted in formula (161) of [15], with the result

$$\begin{aligned}
 L_0 &= h_+ \cos \theta + \sin \theta \frac{\partial}{\partial \theta}, \\
 L_{-1} - L_1 &= 2h_+ \sin \theta - 2 \cos \theta \frac{\partial}{\partial \theta}, \\
 L_1 + L_{-1} &= 2 \frac{\partial}{\partial \theta}.
 \end{aligned}$$

Comparing with Eq. (10), we see that we indeed have a principal series representation with $\tau = -h_+ = -1/2 - i\mu$, and

$$\begin{aligned}
 H &= i(L_1 + L_{-1}), \\
 X_+ &= \frac{i}{2}(L_1 - L_{-1}) + L_0, \\
 X_- &= -\frac{i}{2}(L_1 - L_{-1}) + L_0.
 \end{aligned} \tag{15}$$

For completeness, let us observe that when $m < 1/2$ the quantity μ is pure imaginary and so the parameter τ is real. As noted in [13], in the range $0 < m < 1/2$ the representation one finds belongs to the supplementary series ($-1 < \tau < 0$), whereas for special values in the range $m < 0$ one makes contact with the discrete series representations. These two additional types of representations are unitary with respect to a different boundary inner product [32].

Notice that our boundary basis functions $e^{-ik\theta}$ [or, equivalently, our bulk basis functions $y_k(t) e^{-ik\theta}$] are eigenfunctions not of L_0 , but of $L_1 + L_{-1}$. Eigenfunctions of L_0 with eigenvalue $i\omega$ are of the form

$$F_\omega(\theta) = \tan^{i\omega} \frac{\theta}{2} \sin^\tau \theta, \tag{16}$$

with ω real. These functions are periodic and so admit a Fourier series expansion; they are however singular at $\theta = 0, \pi$. They satisfy the orthogonality relation [15]

$$\int_0^{2\pi} d\theta \overline{F_\omega(\theta)} F_{\omega'}(\theta) \propto \delta(\omega - \omega') \quad (17)$$

provided the singularities at $\theta = 0, \pi$ are regulated in a suitable way.

A very interesting point is that we see a single irreducible representation of the conformal group appear, rather than the two distinct highest-weight representations that appear in the original version of the correspondence described above. If we were to try to represent a non-trivial α -vacuum other than the Euclidean vacuum (which as explained in [17] is unlikely to make sense in the interacting theory coupled to gravity), we would need to use a linear combination of $y_k(t)$ and $\overline{y_k(t)}$ in Eq. (13), and two distinct representations with $\tau = -h_\pm$ would appear. These representations are actually equivalent, but the transformation involves a non-trivial change of basis [41], which takes the form

$$\overline{|k\rangle} = \frac{\Gamma(h_- - k)}{\Gamma(h_+ - k)} |k\rangle$$

for $|k\rangle$ a representation satisfying Eq. (9) with $\tau = -h_+$, and $\overline{|k\rangle}$ a representation satisfying Eq. (9) with $\tau = -h_-$. In coordinate space, this is rewritten [15]

$$\overline{f_k(\theta)} \propto \int_0^{2\pi} d\theta' [1 - \cos(\theta - \theta')]^{-h_-} e^{-ik\theta'};$$

i.e., for the basis $e^{-ik\theta}$ satisfying Eq. (9) with $\tau = -h_+$, the basis $\overline{f_k}$ will satisfy the same relations with $h_+ \rightarrow h_-$. Note the appearance of the propagator of a conformal primary of weight h_- in this change of basis.

It is easy to check that the Klein-Gordon inner product for a free quantum scalar field reduces to the inner product (11), up to a positive normalization constant (the simplest way is to look at a time slice that approaches \mathcal{I}^-). The adjoint of the generators with respect to this inner product is therefore given by Eq. (12), which implies

$$L_n^* = -L_n.$$

This notion of conjugation is also emphasized in [15], and discussed in [13]; it amounts to the statement that the $SL(2, \mathbb{R})$ group elements, $\exp(c_n L_n)$ with $c_n \in \mathbb{R}$, are unitary. Notice that it differs from the definition of adjoint considered in [4,38], $L_n^* = L_{-n}$. The difference between these two definitions becomes even more significant in the higher-dimensional case: quantization of a scalar field in d dimensions yields a unitary principal series representation of the conformal group in $d-1$ Euclidean dimensions, $SO(d,1)$, whereas the notion of conjugation employed in [4,38] makes contact instead with the Lorentzian conformal group $SO(d-1,2)$ [15].

To summarize, we have seen that scalar fields in de Sitter yield principal series representations of $SU(1,1) \approx SL(2, \mathbb{R})$. All these representations are irreducible for generic m , infinite dimensional and without highest or lowest weights. We have also constructed a bulk-to-boundary map at the level of the field modes: field configurations on the boundary are mapped into linear combinations of positive-frequency Euclidean vacuum modes.

It is natural then to propose a new version of the dS_2/CFT_1 correspondence, where bulk correlation functions of a scalar field are determined by matrix elements in a CFT that lives on a circle, built out of the unitary principal series representation with $\tau = -h_+$. It is most natural to think of this correspondence in terms of duality of the bulk theory with a CFT living on a single spatial boundary in the infinite past \mathcal{I}^- , where the bulk to boundary map (13) matches the definition of the Euclidean vacuum (14). Here we differ with the interpretation proposed in [15] in terms of two entangled CFTs living on both past and future infinities of de Sitter. Formulating the correspondence using a single boundary also allows for the possibility that the CFT may describe states that are asymptotically de Sitter in the past, but do not evolve to an asymptotically de Sitter final state (or vice versa if one takes the boundary at \mathcal{I}^+). Aside from some minor differences in the formulas, all we have said generalizes to the general case dS_d/CFT_{d-1} .

IV. QUANTUM dS/CFT PROPOSAL

The representations of the $SO(d,1)$ group discussed above are infinite dimensional, and so do not naturally explain the finiteness of the horizon entropy in the dS/CFT framework. To ameliorate this problem, we will show for the case of dS_2 that a quantum deformation of the symmetry group yields finite-dimensional unitary representations that go over to the unitary principal series representations in the classical limit.

A. Quantum groups

Our basic building block will be the quantum group $SL_q(2, \mathbb{C})$ or, more precisely, the universal enveloping algebra $U_q(\mathfrak{sl}(2, \mathbb{C}))$ with complex coefficients built out of generators K, X_+, X_- satisfying [32]

$$KK^{-1} = K^{-1}K = 1, \quad KX_\pm K^{-1} = q^{\pm 2}X_\pm,$$

$$[X_+, X_-] = \frac{K - K^{-1}}{q - q^{-1}}. \quad (18)$$

The universal enveloping algebra consists of the space spanned by the monomials

$$(X_+)^n K^m (X_-)^l \quad (19)$$

with $m \in \mathbb{Z}$ and n, l non-negative integers. It is a Hopf algebra with comultiplication Δ defined as

$$\Delta(K) = K \otimes K$$

$$\Delta(X_+) = X_+ \otimes K + 1 \otimes X_+$$

$$\Delta(X_-) = X_- \otimes 1 + K^{-1} \otimes X_-$$

and with antipode S and counit ε defined as

$$S(K) = K^{-1}, \quad S(X_+) = -X_+ K^{-1}, \quad S(X_-) = -K X_-,$$

$$\varepsilon(K) = 1, \quad \varepsilon(X_{\pm}) = 0.$$

It is also useful to define a related comultiplication

$$\Delta' \equiv \sigma \circ \Delta, \quad \sigma(a \otimes b) \equiv b \otimes a.$$

Roughly speaking, we can think of the element K as q^H , where H becomes the usual Cartan generator of $SL(2)$ when we take the classical $q \rightarrow 1$ limit. Written in terms of H , the algebra becomes

$$[H, X_{\pm}] = \pm 2X_{\pm}, \quad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}.$$

However, it will be important later that the quantum group involves the restriction to products of generators of the form (19). We refer the reader to [32,33] for an introduction to quantum groups.

We will be interested in defining a notion of conjugation on these generators, in order to specify inner products. This pairing is known as a $*$ -structure. In the mathematics literature, this is usually defined to preserve the form of the comultiplication, in the sense that $\Delta(a^*) = (* \otimes *) \Delta(a)$. We will be interested however in a generalization of this notion of $*$ -structure where $\Delta(a^*) = (* \otimes *) \Delta'(a)$, which is discussed extensively in the introductory sections of [42]. There exist a number of different choices of this $*$ -structure for $U_q(sl(2, \mathbb{C}))$, which are discussed in [36,39,42]. We will be interested in the specific choice

$$X_{\pm}^* = -X_{\mp}, \quad K^* = K^{-1}, \quad (20)$$

defined for q a root of unity. This is the quantum counterpart of Eq. (12), which as we saw in Sec. III is the notion of conjugation relevant to a scalar field on dS_2 . The map (20) is anti-linear (acts by complex conjugation on c numbers), involutive and is an anti-morphism (reverses the order of generators). The Hopf algebra $U_q(sl(2, \mathbb{C}))$ combined with this $*$ -structure is known as $U_q(su(1,1))$, which again is to be thought of as an enveloping algebra with complex coefficients. For the special case that q is a root of unity, we can extract a real subalgebra of this Hopf algebra, which we denote $U_q(su(1,1))_{\mathbb{R}}$. This is done by defining a map

$$\theta(X^{\pm}) = -X^{\pm}, \quad \theta(K) = K^{-1}, \quad (21)$$

and showing that the restriction $a^* = \theta(a)$ where $a \in U_q(su(1,1))$ is compatible with the algebra and comultiplication structure.

In more detail: one may define the basis of generators

$$J_3 = iH, \quad J_1 = X^+ + X^-, \quad \text{and} \quad J_2 = i(X^+ - X^-),$$

which become a canonical basis for $su(1,1)$ in the classical limit $q \rightarrow 1$. The $*$ -structure (20) defines an involution of the algebra $U_q(sl(2, \mathbb{C}))$. The map θ (21) maps the Hopf algebra into itself, provided we permute factors in the comultiplication (i.e., θ is an anti-comorphism). It is then easy to check that

$$J_i^* = -J_i$$

for $i=1,2,3$ and that this restriction is compatible with the Hopf algebra structure. The algebra generated by the J_1, J_2 and $K = q^{-iJ_3}$ with real coefficients is then a real Hopf algebra, which we denote $U_q(su(1,1))_{\mathbb{R}}$.

For q a root of unity (for simplicity we will mostly consider the case $q = e^{2\pi i/N}$ with N odd¹), the center of the algebra involves not just the usual quadratic Casimir

$$C = X_- X_+ + \frac{Kq + K^{-1}q^{-1}}{(q - q^{-1})^2}$$

but also the elements

$$X_+^N, X_-^N, K^N, \quad \text{and} \quad K^{-N}. \quad (22)$$

This implies that any irreducible representation of the algebra is finite dimensional [32].

B. Classical limit of the quantum group representation

For q a root of unity (we will take $q = e^{2\pi i/N}$ with N odd) there exists a class of finite-dimensional irreducible representations of the quantum group that can be realized on the N -dimensional basis $|m\rangle$ with $m=0, \dots, N-1$ [32] and parametrized by the complex numbers a, b, λ :

$$K|m\rangle = q^{-2m\lambda}|m\rangle$$

$$X_+|m\rangle = \left(ab + \frac{q^m - q^{-m}}{q - q^{-1}} \frac{\lambda q^{1-m} - \lambda^{-1} q^{m-1}}{q - q^{-1}} \right) |m-1\rangle$$

$$X_-|m\rangle = |m+1\rangle$$

supplemented by the additional cyclic operations

$$X_+|0\rangle = a|N-1\rangle, \quad X_-|N-1\rangle = b|0\rangle. \quad (23)$$

For $a, b \neq 0$ there are no highest- or lowest-weight states and the representation is called cyclic.

To try to establish a connection with the principal series representations, let us take the $q \rightarrow 1$ classical limit of the above expressions, with $\lambda = q^{2l}$, $K = q^H$ and allowing for a change in normalization of the basis elements, $|m\rangle \rightarrow B(m)|m\rangle$:

¹The main difference for even N is that the exponents in Eq. (22) and, consequently, the dimensions of the irreducible representations discussed in the next subsection become $N/2$ [32].

$$\begin{aligned}
H|m\rangle &= 2(l-m)|m\rangle \\
X_+|m\rangle &= (ab+m(2l+1-m))\frac{B(m-1)}{B(m)}|m-1\rangle \\
X_-|m\rangle &= \frac{B(m+1)}{B(m)}|m+1\rangle.
\end{aligned} \tag{24}$$

Setting $B(m+1)/B(m) = m - \tau - l$,

$$ab = \tau^2 + \tau - l^2 - l, \tag{25}$$

and defining a new basis $|k\rangle' = |l-m\rangle$, Eq. (24) becomes

$$\begin{aligned}
H|k\rangle' &= 2k|k\rangle' \\
X_+|k\rangle' &= (k-\tau)|k+1\rangle' \\
X_-|k\rangle' &= -(\tau+k)|k-1\rangle'
\end{aligned}$$

as for the principal series representation (9). This makes sense provided we identify l with an integer. Since τ is complex, ab is in general complex. Furthermore, since $m = 0, \dots, N-1$ we get $k = l - N + 1, \dots, k$. Therefore, if we take $l = (N-1)/2$ (odd N), we get $k = -(N-1)/2, \dots, (N-1)/2$ which gives us the principal series basis as $N \rightarrow \infty$. Notice that up to these conditions on l and Eq. (25), the individual values of a and b are undetermined. As we will see in the next subsection, invariance of the inner product fixes a and b up to a phase, and this phase drops out of the product ab which appears in the action of the generators (24).

In the basis where $L_1 + L_{-1} = -iH$ is diagonal, we approach the classical principal series representation in a smooth way. There is a subtlety, however, if we attempt to change basis to diagonalize the operator $L_0 = (X_+ + X_-)/2$ and then take the $q \rightarrow 1$ limit. Because $X_{\pm}^N \propto 1$, the spectrum of X_{\pm} is N evenly spaced points around a circle centered at the origin. Therefore the spectrum of L_0 will also be made up of N discrete points, and it turns out their spacing remains constant as $N \rightarrow \infty$, so one does not reproduce the continuous spectrum of L_0 expected in the classical limit (16). This implies the $q \rightarrow 1$, $N \rightarrow \infty$ limit does not commute with the regularization needed to make sense of the completeness relation (17).

C. Unitarity

We wish to investigate unitarity of the cyclic quantum group representations under the conditions (25) and $l = (N-1)/2$. For the unitary principal series representations in the classical limit we have $\tau = -1/2 + i\rho$, which implies ab is always a negative real number. The eigenvalues of H are real, since l is an integer. It remains to examine

$$\begin{aligned}
\langle X_+m|X_+m\rangle &= -\langle m|X_-X_+|m\rangle \\
&= -\left(ab + \frac{q^m - q^{-m}}{q - q^{-1}} \frac{\lambda q^{1-m} - \lambda^{-1} q^{m-1}}{q - q^{-1}}\right) \\
&\quad \times \langle m|m\rangle
\end{aligned} \tag{26}$$

where we have used the notion of conjugation defined by the $*$ -structure (20). Substituting in for ab and using $\bar{\tau} = -1 - \tau$, we need to check whether

$$v = l^2 + l + |\tau|^2 - \left(\frac{q^m - q^{-m}}{q - q^{-1}} \frac{\lambda q^{1-m} - \lambda^{-1} q^{m-1}}{q - q^{-1}}\right) > 0.$$

This can be expressed as

$$\begin{aligned}
v &= l^2 + l + |\tau|^2 - \frac{\sin\left(\frac{2\pi(l-k)}{2l+1}\right) \sin\left(\frac{2\pi(l+1+k)}{2l+1}\right)}{\sin^2\left(\frac{2\pi}{2l+1}\right)} \\
&= l^2 + l + |\tau|^2 + \frac{\sin^2\left(\frac{2\pi(l-k)}{2l+1}\right)}{\sin^2\left(\frac{2\pi}{2l+1}\right)}
\end{aligned}$$

which is manifestly positive. The same is true for the special cases on the edges ($m=0, N-1$). Similar results are obtained for $\langle X_-m|X_-m\rangle$. There is one additional relation that comes from demanding invariance of the inner product under X_+^N ,

$$\langle X_-^N 0|0\rangle - \langle 0|(X_-^N)^*0\rangle = 0,$$

which leads to the condition

$$|b|^2 = -\prod_{j=0}^{N-1} s(j)$$

where

$$s(j) \equiv ab + \frac{q^m - q^{-m}}{q - q^{-1}} \frac{\lambda q^{1-m} - \lambda^{-1} q^{m-1}}{q - q^{-1}}.$$

This fixes a and b up to an overall phase. Under these conditions, the cyclic irreducible representations are unitary for arbitrary N .

D. qdS/CFT proposal

Underlying our proposal that, at the quantum level, the isometry group of two-dimensional de Sitter space should be q deformed lies of course the idea that dS_2 itself should be similarly deformed, to produce a geometry on which $U_q(su(1,1))_{\mathbb{R}}$ has a natural action. At the classical level, dS_2 can be understood as the quotient of the isometry group $SU(1,1)$ by the non-compact $U(1)$ associated with one of the boost generators, and so, at the quantum level, one is

naturally led to consider $U_q(su(1,1))_{\mathbb{R}}/U(1)$. Similar constructions have been explored in [35,43]. We will make some additional comments about this q -deformed geometry in the concluding section, but a more detailed analysis is left for future work.

Our overall proposal is then that the N -dimensional representations described above can be used to formulate a new correspondence between a gravitational theory in a bulk q -deformed de Sitter geometry and a q -deformed holographic CFT. We emphasize that the representations in question are unitary for arbitrary N , and in the $N \rightarrow \infty$ limit, they make contact with the reformulation of the classical dS/CFT correspondence in terms of unitary principal series representations discussed in Sec. III. We believe this provides a promising microscopic framework to account for the finite entropy of de Sitter space.

E. Relation to “the trouble with de Sitter space”

Now that we have a q -deformed version of the theory with finite dimensional representations, we can reanalyze the arguments of [28,44] arguing for quasi-periodic matrix elements for any description of de Sitter compatible with finite entropy, Hermiticity of the static patch Hamiltonian, and covariance under the classical symmetries. In the static patch with coordinates

$$ds^2 = \frac{1}{\cosh^2 r} (-dt_s^2 + dr^2)$$

the $sl(2, \mathbb{R})$ generators take the form

$$L_0 = -\frac{\partial}{\partial t_s}$$

$$L_{-1} - L_1 = -2 \cosh t_s \sinh r \frac{\partial}{\partial t_s} - 2 \sinh t_s \cosh r \frac{\partial}{\partial r}$$

$$L_1 + L_{-1} = 2 \cosh t_s \sinh r \frac{\partial}{\partial t_s} + 2 \cosh t_s \cosh r \frac{\partial}{\partial r}.$$

Thus the static patch Hamiltonian is to be identified with

$$H_s = -iL_0 = -i(X_+ + X_-)/2. \quad (27)$$

The argument of [28,44] proceeds by analyzing the general matrix element of a Hermitian boost generator $L = iL_{-1} = (X_- - X_+ + H)/2$, which obeys

$$[H_s, L] = iL.$$

The classical argument proceeds by studying

$$\langle \psi | e^{iH_s t} e^{iL} e^{-iH_s t} | \psi \rangle = \langle \psi | e^{iLe^{-t}} | \psi \rangle$$

with $|\psi\rangle$ a general state. This matrix element approaches 1 as $t \rightarrow \infty$. On the other hand, under the assumption that H_s has a discrete spectrum, the authors of [28,44] show the matrix element must be quasi-periodic in time, and so in particular cannot approach a constant.

In our q -deformed framework, however, e^{-iL} is not in the universal enveloping algebra, so the argument does not go through. Instead one is restricted to operators built out of products of K , K^{-1} , X_+ and X_- . Since the spectrum of H_s in our representations is manifestly discrete, correlation functions are guaranteed to display the expected quasi-periodicity. The main virtue of our approach is that this is achieved without giving up the Hermiticity of H_s .

V. DISCUSSION

In this paper we have made two main points. First, we have emphasized that, as observed in [15], the dS/CFT correspondence must be formulated in terms of principal series representations of the isometry-conformal group, as opposed to the standard highest-weight representations usually considered in CFT. Such a reformulation of dS/CFT is natural from the bulk point of view, since quantization of a scalar field on dS_d yields representations of the former, and not the latter, type. In particular, the ordinary Klein-Gordon inner product directly coincides with the scalar product of the principal series representations, and differs from the one considered in [4,38], which is on the other hand associated with the usual CFT notion of adjoint. But the reformulation is in fact also natural from the perspective of the boundary theory, because the putative dual CFT lives on a $(d-1)$ -dimensional space that is Euclidean from the start, and is not as in the usual case obtained by analytic continuation from an originally Lorentzian spacetime. The relevant conformal group is consequently $SO(d,1)$ and not $SO(d-1,2)$ [15]. Most importantly, the principal series representations are unitary, so in the new formulation one avoids the problems associated with the non-unitarity of the highest-weight representations that appear in [1].

Of course, one of the motivations of [4,38] for concentrating on a bulk inner product that differs from the ordinary one was to try to obtain a framework that departs from the standard perturbative quantization of the scalar field on dS space, and has consequently at least some chance of making contact with the finite-dimensional (or at least discrete-energy) Hilbert space that the finite entropy of de Sitter seems to hint at [23,24,27,28,31]. Our second main point in this paper has been that it is possible to achieve this goal without losing contact with the principal series story, as long as we are willing to give up the classical symmetries and trade them for a q -deformed version, with q a root of unity. We gladly pay this price because in exchange we have obtained a finite-dimensional framework that is *manifestly unitary*.

Thus far, a precise description of the q -deformed de Sitter geometry and its spacetime physics is lacking, but let us make a few general comments based on the structure of the algebra and the representations we are considering. We hope to return to this set of issues in more detail in future work. The cyclic relation (22) implies that all irreducible representations are finite dimensional. The index k that labels our basis of states $|k\rangle'$ is interpreted as momentum around the circle in the classical limit, so the q deformation can be thought of as enforcing an ultraviolet cutoff on this momentum. Thus the Euclidean boundary space of the dS/CFT cor-

respondence, dual to this momentum, can be roughly thought of as a discrete set of points.² Likewise, one deduces the spectrum of the operators X^+ and X^- must also be discrete with an ultraviolet cutoff of order N . This implies the same is true of the Hamiltonian (27), so it seems the time direction of the bulk de Sitter spacetime also becomes discretized.

Systems with finite dimensional Hilbert spaces undergo Poincaré recurrence, as has been discussed extensively in the context of de Sitter spacetime in [23,28,31,44,45]. If the dimension of the Hilbert space is e^S where S is the statistical mechanical entropy, then there will be a recurrence time $t_r \propto e^S$. Likewise energy measurements will always be uncertain at order $1/t_r = e^{-S}$. In our setup, if the dimension of Hilbert space is of the same order of that of a single q -deformed representation, we would obtain the same relations with $N = e^S$.

Of course so far we have only considered the properties of a single irreducible representation. One might expect the full Hilbert space to be built out of arbitrary tensor products of these representations, which would enlarge the number of states to infinity. Interactions are built using the fusion rules for these representations, which have been thoroughly studied in [46,47]. However as we have said, space should be thought of as a finite number of points, so we are far from the situation where we have a spacetime with a well-defined asymptotic region where multi-particle states can be built neglecting interactions. Nevertheless, even if we assume that this infinite-dimensional Hilbert space is the correct description, the energy eigenvalues can remain discrete, which can still lead to Poincaré recurrence [45] and, more importantly, can be compatible with the finite entropy of de Sitter space [28]. In the static patch, a cutoff is naturally implemented in the form of a Boltzmann weighting of states at finite temperature, so the entropy can be accounted for by a finite number of states below an energy of order the Hubble scale [28].³

²See [37] for an interesting set of related examples of q -deformed spaces.

³We should also bear in mind that the condition that the spacetime be close to de Sitter space in the past and future in a suitable classical limit could translate into a cutoff on the number of states relevant to the de Sitter entropy calculation. Conversely, the full Hilbert space may then contain states that do not approach de Sitter in the past and/or future as one takes the classical limit.

Let us describe this in a little more detail. To isolate the states defined on the causal diamond of an observer on the north pole of de Sitter, it is necessary to trace over modes on the southern diamond [4]. This can be implemented by viewing the modes on the southern diamond as a thermofield double of modes on the northern diamond [28]. Provided we start in the Euclidean vacuum, integrating out the southern modes then gives rise to a thermal Boltzmann density matrix for the northern diamond modes. To make contact with this paper, eigenfunctions of L_0 with $\omega > 0$ should be viewed as static patch modes on the northern diamond, and likewise $\omega < 0$ modes correspond to the southern diamond. The bulk to boundary map places these modes in the Euclidean vacuum, so for the classical version of the dS/CFT correspondence described in Sec. III, the same story will carry over. With a better understanding of the q -deformed de Sitter geometry, and the bulk to boundary map in particular, we hope a similar story will also hold in the q -dS/CFT case.

Most of the ideas studied in this paper generalize directly to higher-dimensional de Sitter space. In particular, the classical de Sitter isometry groups $SO(d,1)$ have unitary principal series representations. The unitary norm is naturally defined on functions on the $d-1$ sphere, which we identify with the holographic boundary. The adjoint on this norm again differs from that proposed in [4,38], as also emphasized in [15]. We conjecture there will likewise be a sensible q deformation of the classical isometry group and that the associated q -deformed holographic conformal field theory based on a deformation of the unitary principal series representations will be dual to a gravitational theory in the bulk q -deformed de Sitter geometry. We hope to further elaborate on the details of this higher-dimensional correspondence in future work.

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