Hidden twelve-dimensional super Poincaré symmetry in eleven dimensions

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First, we review a result in our previous paper, of how a ten-dimensional superparticle, taken off-shell, has a hidden eleven-dimensional super Poincaré symmetry. Then, we show that the physical sector is defined by three first-class constraints which preserve the full eleven-dimensional symmetry. Applying the same concepts to the eleven-dimensional superparticle, taken off-shell, we discover a hidden twelve-dimensional super Poincaré symmetry that governs the theory.

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I. FROM TEN TO ELEVEN DIMENSIONS

In Ref. [1], we showed that the quantum algebra of a ten-dimensional superparticle, taken off shell, contains a nonlinear realization of the eleven-dimensional super Poincaré algebra, with some additional constraints. In this section, we review the procedure outlined in Ref. [1] and we write the constraints in a fully covariant fashion.

As a starting point, one can take for instance the Brink-Schwarz action [3] for a ten-dimensional massless superparticle. The phase space of such a particle is spanned by the canonical variables x^μ, θ^α, and their respective momenta p_μ, π_α, where x, p are vectors in ten dimensions and θ, π belong each to a Majorana-Weyl representation of the ten-dimensional Clifford algebra.

The straightforward quantization of this phase space is impeded by the presence of constraints, namely

$$p^2=0, \quad d_\alpha \equiv \pi_\alpha - (\not{p}\theta)_\alpha = 0. \tag{1.1}$$

As in Ref. [1], we make the choice of ignoring the first constraint, because we want to describe the quantum mechanics of a particle off-shell. We retain instead the Fermionic constraints. Off-shell, they are second class and can be treated by an extension of the Dirac quantization method[1] [2,3]: first we reduce by half the number of Fermionic degrees of freedom using the constraints $[\pi_\alpha=(\not{p}\theta)_\alpha]$, then we compute the Dirac brackets for the remaining variables and finally we quantize the Dirac brackets.

It is interesting to express the remaining Fermionic generators in terms of the supersymmetry generators $Q_\alpha \equiv \pi_\alpha + (\not{p}\theta)_\alpha = 2\pi_\alpha$. If we do that, the quantum algebra that we obtain is the following:

$$\{Q_\alpha, Q_\beta\} = 2(\not{p})_{\alpha\beta}, \quad [Q_\alpha, p_\mu]=0, \quad [p_\mu, p_\nu]=0, \tag{1.2}$$

$$[x^\mu, p_\nu] = i\delta^\mu_\nu, \quad [Q_\alpha, x^\mu] = -\frac{i}{2}(\gamma^\mu \not{p}^{-1} Q)_\alpha, \tag{1.3}$$

$$[x^\mu, x^\nu] = -\frac{1}{16p^4} Q\{\gamma^{\mu\nu}, \not{p}\} Q. \tag{1.4}$$

This quantum algebra in ten dimensions is free of constraints. It contains the ten-dimensional supertranslations and has an interesting noncommutativity in the spacetime coordinates. Its consistency can be verified [1] by checking that all Jacobi identities are verified.

Next, we consider the following elements of the algebra:

$$J^\mu \equiv (-p^2)^{1/4} x^\mu (-p^2)^{1/4}, \tag{1.5}$$

$$J^{\mu\nu} \equiv (x^\mu p^\nu - x^\nu p^\mu) + S^{\mu\nu}, \tag{1.6}$$

$$\tilde{Q}_{\dot\alpha} \equiv (-p^2)^{-1/2} (\not{p} Q)_{\dot\alpha}, \tag{1.7}$$

where

$$S^{\mu\nu} \equiv \frac{-i}{16p^2} Q\{\gamma^{\mu\nu}, \not{p}\} Q. \tag{1.8}$$

From them, we can construct the generators of an eleven-dimensional super Poincaré algebra[2] as follows [1]:

$$P^M = (p^\mu, P^{10} \equiv \sqrt{-p^2}), \tag{1.9}$$

[1] There are some subtleties in extending Dirac's method to superspace, but they can be overcome for a class of algebras, of which ours is one.

[2] Whenever square roots appear it is understood that both signs may occur in front of them, so that in particular $P^{10}=\pm\sqrt{-p^2}$ spans the whole range of momentum in the extra dimension. To avoid cluttering our notation we omit the extra $\pm$.

$$J^{MN}=(J^{\mu\nu},J^{\mu,10}\equiv J^{\mu}), \tag{1.10}$$

$$Q_A=\left(Q_\alpha,\tilde{Q}_{\dot\alpha}\equiv\frac{(\not{p}Q)_{\dot\alpha}}{\sqrt{-p^2}}\right), \tag{1.11}$$

where the indices M and N now range from 0 to 10, and the index A is the index of a Majorana representation of the Clifford algebra in eleven dimensions.[3] It is a matter of straightforward computation to show that P, J and Q indeed satisfy the super Poincaré algebra in one more dimension than we started with. We write its commutation and anticommutation relations at the end of the next section.

Note that this eleven-dimensional algebra is realized nonlinearly in the original ten-dimensional algebra, but it also contains it, because among its generators are in particular $J^{\mu}\equiv J^{\mu,10}$, p_μ and Q_α which in turn generate the original algebra.[4] Hence, the two algebras are actually the same. This apparent paradox is resolved once we realize that the new algebra is not free. In the next section we describe the constraints it is subject to.

Note also that so far we have assumed a timelike momentum for the off-shell particle, so that $\sqrt{-p^2}$ is real. More generally we should allow also an off-shell spacelike momentum. In that case the extra dimension is timelike because the momentum in the additional dimension is purely imaginary and given by $i\sqrt{p^2}$. In the latter case, all our constructions can be extended and the corresponding formulas can be obtained by analytic continuation $\sqrt{-p^2}\to i\sqrt{p^2}$ from the ones given below. In the following, we will let it be understood that when $\sqrt{-p^2}$ is real the extra dimension is spacelike, and when it is imaginary the extra dimension is timelike.

II. CONSTRAINTS

To begin with, the eleven-dimensional algebra satisfies the constraints

$$P^M P^N \eta_{MN}=0, \tag{2.1}$$

$$P^M(\Gamma_M)_A{}^B Q_B=0, \tag{2.2}$$

where η_{MN} is the Minkowski metric in eleven dimensions, the last dimension taken to be spacelike, and the Γ^M form a representation of the Clifford algebra in eleven dimensions, whose expression in terms of $\gamma^\mu_{\alpha\beta}$ matrices (and their antichiral counterpart $\gamma^\mu_{\dot\alpha\dot\beta}$) is

$$\Gamma^\mu=\begin{pmatrix}0 & \gamma^\mu_{\alpha\beta}\\ \gamma^\mu_{\dot\alpha\dot\beta} & 0\end{pmatrix},\quad \Gamma^{10}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2.3}$$

Specifically, Eq. (2.1) encodes in the new algebra the definition (1.9) of P in terms of quantities in the old algebra, and similarly Eq. (2.2) encodes Eq. (1.11).

The algebra has also a constraint that encodes the definition (1.10). It is easy to write it in a ten-dimensional covariant way by combining Eqs. (1.5), (1.6), and (1.8),

$$J^{\mu\nu}-(-p^2)^{-1/4}(J^\mu p^\nu-J^\nu p^\mu)(-p^2)^{-1/4}$$
$$=-\frac{i}{16p^2}Q\{\gamma^{\mu\nu},\not{p}\}Q. \tag{2.4}$$

This constraint contains a nontrivial relation between the Bosonic and the Fermionic parts of the eleven-dimensional algebra. It also allows us to express $J^{\mu\nu}$ in terms of J^μ, p^μ and Q_α, making explicit the fact that the constrained eleven-dimensional algebra has the same number of independent generators as the algebra in ten dimensions.

It is desirable to express Eq. (2.4) in an eleven-dimensionally covariant form. For that purpose, first we rewrite every quantity in the constraint explicitly as a generator of the eleven-dimensional algebra, then we perform a few algebraic steps and we obtain

$$J^{\mu\nu}P^{10}+J^{\nu 10}P^\mu+J^{10\mu}P^\nu=\frac{i}{16}(Q\gamma^{\mu\nu}\tilde{Q}+\tilde{Q}\gamma^{\mu\nu}Q). \tag{2.5}$$

The left-hand side of Eq. (2.5) can be written as $3\times W^{10\mu\nu}$, with

$$W^{LMN}\equiv J^{<LM}P^{N>}\equiv\frac{1}{3!}(J^{LM}P^N\pm\text{permutations}).^5 \tag{2.6}$$

Here and in the following, the angular brackets indicate complete antisymmetrization. To rewrite covariantly the right-hand side, we need to find spinor bilinears with tensorial transformation properties. Let us define

$$\overline{Q}\equiv Q^T\mathcal{C},\quad \mathcal{C}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

The matrix $\mathcal{C}$ is chosen so that it satisfies $\mathcal{C}\Gamma^M\mathcal{C}^\dagger=-(\Gamma^M)^T$. Then $\overline{Q}\Gamma^{M_1\cdots M_p}Q$ transforms[6] as an antisymmetric p tensor under the Lorentz group in eleven dimensions. An explicit computation shows that the right-hand side of Eq. (2.5) is $3\times S^{10\mu\nu}$, with

$$S^{LMN}\equiv-\frac{i}{3\times 16}\overline{Q}\Gamma^{LMN}Q, \tag{2.7}$$

[3]We have indicated both the ten-dimensional operators Q_α and the eleven-dimensional operator Q_A with the same letter. However, it should be clear from the context which is which. In particular in the following two sections all the instances of the letter Q refer to the eleven-dimensional Majorana spinor.

[4]x^μ can be expressed in terms of J^μ by inverting Eq. (1.5).

[5]In four dimensions, Eq. (2.6) is the dual of the Pauli-Lubański vector, so that W^{LMN} should be thought of as its generalization to higher dimensions.

[6]$\Gamma^{M_1\cdots M_p}$ indicates the antisymmetric combination of p gamma matrices, $\Gamma^{<M_1}\cdots\Gamma^{M_p>}$.

so that the third constraint (2.4) reads simply $W^{10\mu\nu} = S^{10\mu\nu}$. Furthermore, the equality holds also for the other components of W and S. This can be checked by explicit computation in ten-dimensional language and, again, it is a consequence solely of Eq. (2.4). In conclusion, the third constraint (2.4) can be written as

$$\Delta^{LMN} \equiv W^{LMN} - S^{LMN} = 0. \tag{2.8}$$

It should be clear from the previous line of reasoning that not all components of Eq. (2.8) are independent of one another. Indeed the number of independent components has to be that of Eq. (2.4) from which we started. That number is

$$\binom{10}{2}.$$

There is a more elegant and fully covariant way to see that the number of independent components in Eq. (2.8) is indeed

$$\binom{10}{2},$$

and we show it in the next subsection.

Thus, the conclusion of our analysis is that the Hilbert space of a ten-dimensional superparticle taken off-shell is also the Hilbert space of the eleven-dimensional super Poincaré algebra

$$[P^M, P^N] = 0, \ [P^M, Q_A] = 0,$$

$$[P^M, J^{NQ}] = i\eta^{MQ}P^N - i\eta^{MN}P^Q \tag{2.9}$$

$$\{Q_A, \bar{Q}_B\} = 2\not{P}_{AB}, \quad [J^{MN}, Q_A] = \frac{i}{2}(\Gamma^{MN}Q)_A \tag{2.10}$$

$$[J^{MN}, J^{PQ}] = i\eta^{MP}J^{NQ} - i\eta^{MQ}J^{NP} + i\eta^{NQ}J^{MP} - i\eta^{NP}J^{MQ} \tag{2.11}$$

constrained as follows:

$$P^2 = 0, \quad \not{P}Q = 0, \quad \Delta^{LMN} = 0. \tag{2.12}$$

The first two constraints are well known in the context of the massless superparticle in eleven dimensions. The last constraint $\Delta^{LMN} = 0$, and some additional ones to be discussed in the next section, are newly realized. As is well known, the spectrum of quantum states that satisfy the first two constraints is precisely the supergravity multiplet in eleven dimensions, consisting of the metric g_{MN}, 3-index antisymmetric tensor A_{LMN}, and the gravitino ψ_A^M. The last constraint, and the additional ones discussed in the next section, are also satisfied covariantly by this supermultiplet.

In fact, other than the 11D supergravity multiplet, there are no other supermultiplets that satisfy these constraints. This can be seen by solving the constraints explicitly in terms of the 10D unconstrained degrees of freedom, which correspond to the off-shell 10D superparticle whose quantum states correspond to the 11D supergravity multiplet but in a 10D notation. This point can also be understood by solving the constraints in the lightcone gauge of the eleven-dimensional superparticle, which also indicates the same set of quantum states in a fixed gauge; namely 128 bosons and 128 fermions consisting of only the SO(9) covariant transverse degrees of freedom of the supergravity multiplet, $g_{ij}, A_{ijk}, \psi_a^i$. Incidentally, in the context of the 11D superparticle, we should emphasize that our approach provides a ghost-free SO(9,1) covariant quantization of the 11D superparticle. This displays more symmetry as compared to the ghost-free light-cone quantization.

A. Counting constraints covariantly

To count correctly the number of independent constraints in Eq. (2.8), we need to take into account that not all components of that equation are independent. One way to show that this is the case is to show that there is a constraint on the constraint. Indeed it can be checked that

$$\Delta^{LMNO} \equiv \Delta^{<LMN}P^{O>} = 0, \tag{2.13}$$

by virtue of the definition of Δ^{LMN} alone, without using the condition $\Delta^{LMN} = 0$. It holds trivially for the W part of Δ, while a brief computation is required to show that it holds also for the S part.[7] Therefore Eq. (2.13) is an honest constraint on the constraint. Again, not all components of Eq. (2.13) are independent of one another and indeed they are subject to a constraint themselves, namely

$$\Delta^{LMNOP} \equiv \Delta^{<LMNO}P^{P>} = 0, \tag{2.14}$$

and so on.

There is an end to this chain of constraints, because each of the constraints is completely antisymmetric in its indices and so it can have at most 11 indices. Incidentally, the operation of adding a power of P and antisymmetrizing can be thought of as a cohomological operation, akin to taking the exterior derivative in De Rham cohomology. To count the correct number of degrees of freedom then, we ought to start from the end. The last constraint is $\Delta^{L_1\cdots L_{11}} = 0$. Because of antisymmetry, this has only

$$\binom{11}{11} = 1$$

[7]For instance, one can start with the equations $\bar{Q}\{\not{P}, \Gamma^{M_1\cdots M_p}\}Q = \bar{Q}[\not{P}, \Gamma^{M_1\cdots M_p}]Q = 0$, which follow from the constraint $\not{P}Q = 0$, and evaluate the anticommutator or commutator using the Clifford algebra. From this one can show that $\bar{Q}\Gamma^{<M_1\cdots M_{p-1}}QP^{M_p>} = 0$ and $\bar{Q}\Gamma^{M_1\cdots M_{p+1}}QP_{M_{p+1}} = 0$ for an arbitrary number of indices. Applying this to $p = 1$ we derive $\bar{Q}Q = 0$. In addition, it is possible to show that $\bar{Q}\Gamma^{M_1\cdots M_p}Q$ vanishes by itself for $p = 2,5,6,9$, which corresponds to the vanishing of the D-brane charges in the 11D superalgebra. The $p = 1,10$ cases are simple $\bar{Q}\Gamma^M Q = 32P^M$, $\bar{Q}\Gamma^{M_1\cdots M_{10}}Q = 32\epsilon^{M_1\cdots M_{11}}P_{M_{11}}$, while the remaining cases $p = 3,4,7,8$ satisfy the above constraints nontrivially.

independent components and it constrains the previous equation in the chain $\Delta^{L_1\cdots L_{10}}=0$, so that the latter has only

$$\binom{11}{10}-\binom{11}{11}=10-1$$

independent components. These are the number of components that should be subtracted from the number of components in the previous equation yet, and so on backwards along the chain. Consequently, the number of independent components in the third constraint (2.8) must be

$$\binom{11}{3}-\binom{11}{4}+\binom{11}{5}-\cdots-\binom{11}{10}+\binom{11}{11}=\binom{10}{2},$$

as expected.

III. ROLE OF SUPERSYMMETRY

Of the three constraints that the eleven-dimensional algebra is subject to, the third appears somewhat peculiar, especially on account of the coefficient entering the definition of S^{LMN}. We found that some light can be shed by examining the transformation properties of the constraints under supersymmetry.

Before we do that, let us begin with a premise. In the previous section, we expressed the constraints on the eleven-dimensional algebra as the vanishing of tensorial and spinorial quantities in eleven dimensions. As such, the constraints are automatically consistent with the Lorentz part of the algebra, in the sense that their variation under Lorentz transformations[8] vanishes once we impose the constraints themselves. More specifically,

$$\delta_J P^2=0, \quad \delta_J(\not{P}Q)=\frac{i}{2}\Gamma^{MN}\not{P}Q,$$

$$\delta_J\Delta^{LMN}=i\,\eta^{RL}\Delta^{SMN}+\cdots, \tag{3.1}$$

and these variations are zero modulo $\not{P}Q, \Delta^{LMN}$.

A different way to put it, is that if we represent the algebra on a Hilbert space of states, the states that satisfy the constraints are invariant under the Lorentz subalgebra. The same holds for the translations, because all constraints commute with P,

$$\delta_P P^2=0, \quad \delta_P(\not{P}Q)=0, \quad \delta_P\Delta^{LMN}=0. \tag{3.2}$$

The next natural step is to check what happens with the supersymmetry transformations. We find the following:

$$\delta_Q(P^2)=0, \quad \delta_Q(\not{P}Q)_B=2P^2\mathcal{C}_{AB},$$

$$\delta_Q(\Delta^{LMN})=-\frac{i}{12}(\Gamma^{LMN}\not{P}Q)_A. \tag{3.3}$$

So again we find that the supersymmetry transformations vanish once we assume the constraints to hold, and therefore that the states that satisfy the constraints are invariant under the supersymmetry transformations as well. Hence, the constraints are such that they preserve the full symmetry of the super Poincaré algebra. In addition, the commutators (or anticommutators) of the constraints with one another can be computed using Eqs. (3.1), (3.2) and (3.3). One finds that they vanish modulo the constraints themselves (but with coefficients that depend on the dynamical variables). In other words, the constraints are first class. They generate some supergroup of transformations.

We should point out that the value of $\delta_Q(\Delta^{LMN})$ depends critically on the choice of coefficient for S^{LMN} in Eq. (2.7). With a different coefficient, there would be residual pieces which are not proportional to any of the constraints. However, when the coefficient is chosen to be precisely as in Eq. (2.7), a cancellation occurs between $\delta_Q(W^{LMN})$ and some terms in $\delta_Q S^{LMN}$ and the only term left is the one given in Eq. (3.3).

More importantly, we should note that the interplay of the constraints is more interesting for supersymmetry transformations, because now the variation of the third constraint $\delta_Q(\Delta^{LMN})$ vanishes only modulo the second constraint $\not{P}Q$, and similarly $\delta_Q(\not{P}Q)$ vanishes only modulo P^2. In other words, $\Delta^{LMN}=0$ is consistent with supersymmetry only if we also require $\not{P}Q=0$. Similarly for $\not{P}Q$ and P^2.

Let us also mention that in four dimensions a generalization of the Pauli-Lubański vector was discussed in Refs. [8,9]. It was given as $C_a=W_a-(i/8)\bar{Q}\gamma_a\gamma_5 Q$, where $W_a=\frac{1}{2}\varepsilon_{abcd}p^a J^{cd}$, and the Latin indices are four-dimensional space-time indices. If we specialize our off-shell superparticle approach to four dimensions (with hidden five dimensional symmetry), our five dimensional Δ^{LMN} has a four dimensional component $\Delta^{lmn}\sim\varepsilon^{lmna}\Delta_a$ which we can attempt to compare to C_a. We find that Δ_a is different from C_a by an additional crucial term, $\Delta_a=C_a+(i/8p^2)p_a\bar{Q}\not{p}\gamma_5 Q$. The supersymmetry variation of C_a is $\delta_Q C_a=-(i/2)\gamma_5 Q p_a$. However, the supersymmetry variation of Δ_a is $\delta_Q\Delta_a=0$. For this reason the constraint $\Delta_a=0$, or more generally the five dimensional $\Delta^{LMN}=0$ can be imposed without breaking supersymmetry in five dimensions. In more general representations where $\Delta_a\neq 0$, we note that Δ_a commutes with p_b and Δ^2 commutes also with the Lorentz generators. So Δ^2 is a Casimir invariant of the full super Poincaré algebra in four dimensions. For comparison to Refs. [8,9] one may also construct from the Δ_a the tensor $C_{ab}\equiv p_a\Delta_b-p_b\Delta_a$ which coincides with $p_aC_b-p_bC_a$. Then $C_{ab}C^{ab}$ is also a Casimir for the full algebra related to Δ^2. The eigenvalue of $C_{ab}C^{ab}$ is proportional to $Y(Y+1)$ where Y is integer or half integer. Y was called ''superspin'' in Refs. [8,9].

IV. FROM ELEVEN TO TWELVE DIMENSIONS

We repeat the reasoning of the previous sections by taking as the starting point the off-shell eleven-dimensional superparticle. The dynamical quantum operators of interest are the

[8]We define the Lorentz transformations as $\delta_J(\cdot)\equiv[J^{RS},\cdot]$. Similarly $\delta_P(\cdot)\equiv[P^R,\cdot]$ and $\delta_Q(\cdot)\equiv[Q_A,\cdot]$ for bosons, $\equiv\{Q_A,\cdot\}$ for fermions.

11-component vectors (X^M,P_M) and the supercharge Q_A which is a 32-component spinor in eleven-dimensional spacetime. By following the same procedure, the nonlinear quantum algebra that we obtain has a similar form to the ten-dimensional one,

$$\{Q_A,Q_B\}=-2(\not{P}\mathcal{C})_{AB},\quad [Q_A,P_M]=0,\quad [P_M,P_N]=0, \tag{4.1}$$

$$[X^M,P_N]=i\delta^M_N,\quad [Q_A,X^M]=-\frac{i}{2}(\Gamma^M\not{P}^{-1}Q)_A, \tag{4.2}$$

$$[X^M,X^N]=-\frac{1}{16P^4}\overline{Q}\{\Gamma^{MN},\not{P}\}Q. \tag{4.3}$$

Note that we are now using the 32×32 gamma matrices Γ^M given above. We inserted explicitly the charge conjugation matrix $\mathcal{C}$ which satisfies $\mathcal{C}^{-1}\Gamma^M C=-(\Gamma^M)^T$, and have defined $\overline{Q}\equiv Q^T\mathcal{C}$. The matrices $(\Gamma^M\mathcal{C})_{\alpha\beta}$, $(\Gamma^{MN}\mathcal{C})_{\alpha\beta}$, $(\Gamma^{M_1\cdots M_5}\mathcal{C})_{\alpha\beta}$ are 32×32 symmetric, and $(\Gamma^{MNL}\mathcal{C})_{\alpha\beta}$, $(\Gamma^{MNLK}\mathcal{C})_{\alpha\beta}$ are 32×32 antisymmetric. This algebra has no constraints. The first line is the standard eleven-dimensional super Poincaré algebra, and the rest is a new nonlinear extension for the case of the off-shell superparticle.[9] The consistency of this algebra can be verified as in Ref. [1] by checking that all Jacobi identities hold.

Next, as before, we consider the following elements of the algebra:

$$J^M\equiv(-P^2)^{1/4}X^M(-P^2)^{1/4}, \tag{4.4}$$

$$J^{MN}\equiv(X^MP^N-X^NP^M)+S^{MN}, \tag{4.5}$$

$$\tilde{Q}_A\equiv(-P^2)^{-1/2}(\not{P}Q)_A, \tag{4.6}$$

where

$$S^{MN}\equiv\frac{-i}{16P^2}\overline{Q}\{\Gamma^{MN},\not{P}\}Q. \tag{4.7}$$

Note that $\tilde{Q}_A$ and Q_A are both in the 32-component spinor representation, unlike the ten-dimensional case where $(Q_\alpha,\tilde{Q}_{\dot{\alpha}})$ were in different representations, namely (16,16*). Therefore, we will use an additional index $i=1,2$ to identify $Q^i_A=(Q_A,\tilde{Q}_A)$ as two supercharges that belong to a $N=2$ supersymmetry in 11 dimensions. These two supercharges satisfy the SO(10,1)$\times$SO(2) covariant constraint

$$(\not{P}Q^i)_A-(-P^2)^{1/2}\varepsilon^{ij}Q^j_A=0, \tag{4.8}$$

and S^{MN} takes the SO(2) invariant form

$$S^{MN}=-\frac{i}{16\sqrt{-P^2}}\overline{Q}^i\Gamma^{MN}Q^j\varepsilon_{ij}, \tag{4.9}$$

where ε^{ij} is antisymmetric and $\varepsilon^{12}=-\varepsilon_{12}=+1$. The nonlinear algebra above may now be rewritten as a nonlinear extension of the $N=2$ eleven-dimensional super Poincaré algebra consistent with SO(2),

$$\{Q^i_A,Q^j_B\}=-2\delta^{ij}(\not{P}\mathcal{C})_{AB}-2\varepsilon^{ij}(-P^2)^{1/2}\mathcal{C}_{AB},$$

$$[Q^i_A,P_M]=0,\ [P_M,P_N]=0, \tag{4.10}$$

$$[J^M,P_N]=i(-P^2)^{1/2}\delta^M_N,$$

$$[J^M,Q^i_A]=-\frac{i}{2}\varepsilon^{ij}(\Gamma^MQ^j)_A,$$

$$[J^M,J^N]=iJ^{MN}. \tag{4.11}$$

The J^{MN} which was given above in terms of X^M, is rewritten in terms of J^M,P^M, and Q^i_A in the SO(10,1)$\times$SO(2) covariant notation as

$$J^{MN}=(-P^2)^{-1/4}(J^MP^N-J^NP^M)(-P^2)^{-1/4}+\frac{i}{16\sqrt{-P^2}}\overline{Q}^i\Gamma^{MN}Q^j\varepsilon_{ij}. \tag{4.12}$$

By using the nonlinear algebra above, it is straightforward to show that J^{MN} satisfies the standard Lorentz algebra in 11 dimensions and is the generator of 11D Lorentz transformations for all the 11D vectors and spinors that have appeared so far above.

It is also possible to construct the generators of a twelve-dimensional superalgebra from the unconstrained operators P^M,J^M,Q^i_A, as follows. We construct the twelve-dimensional operators as

$$P^m=(P^M,P^{11}\equiv\sqrt{-P^2}), \tag{4.13}$$

$$J^{mn}=(J^{MN},J^{M,11}\equiv J^M), \tag{4.14}$$

$$q_a=\frac{1}{\sqrt{2}}(Q^1_A+iQ^2_A)=\frac{1}{\sqrt{2}}\{[1+i(-P^2)^{-1/2}\not{P}]Q\}_A, \tag{4.15}$$

$$\overline{q}_{\dot{a}}=\frac{1}{\sqrt{2}}(\overline{Q}^1_A-i\overline{Q}^2_A)=\frac{1}{\sqrt{2}}\{\overline{Q}[1+i(-P^2)^{-1/2}\not{P}]\}_A. \tag{4.16}$$

The indices m and n now range from 0 to 11, and the indices $a,\dot{a}$ denote the complex spinors of SO(11,1) which are **32**

[9]The X^M become commutative on shell, since then $\not{p}Q=0$ for the massless superparticle.

and **32***.[10] It can then be shown that the $N=2$ nonlinear algebra in 11 dimensions now takes the form of the linear 12-dimensional super Poincaré algebra given below:

$$\{q_a,\overline{q_{\dot b}}\}=2(\not{P})_{a\dot b},\quad [q_a,P_m]=0,\quad [P_m,P_n]=0, \tag{4.17}$$

$$[J^{mn},P^l]=i(\Sigma^{mn})^l_k P^k,$$

$$[J^{mn},q_a]=\frac{i}{2}(\Gamma^{mn}q)_a,\quad [J^{mn},J^{kl}]=if^{mn,kl}_{rs}J^{rs}, \tag{4.18}$$

where the first line is the standard supersymmetry algebra in twelve dimensions with $(\Gamma^m)_{a\dot b}=((\Gamma^M)_{AB},-i\delta_{AB})$, while the second line contains the expected commutation properties of the SO(11,1) generator J^{mn}, with

$$(\Sigma^{mn})^l_k=\eta^{ml}\delta^n_k-\eta^{nl}\delta^m_k, \tag{4.19}$$

$$f^{mn,kl}_{rs}=[\eta^{mk}\delta^n_r\delta^l_s-(k\leftrightarrow l)]-[m\leftrightarrow n], \tag{4.20}$$

$$(\Gamma^{mn})_{ab}=((\Gamma^{MN})_{AB},\ (\Gamma^{M,11}=i\Gamma^M)_{AB}). \tag{4.21}$$

The antichiral counterparts of the matrices $(\Gamma^m)_{a\dot b}$ and $(\Gamma^{mn})_{ab}$ above are respectively

$$(\Gamma^m)_{\dot a b}=((\Gamma^M)_{AB},i\delta_{AB}), \tag{4.22}$$

$$(\Gamma^{mn})_{\dot a\dot b}=((\Gamma^{MN})_{AB},\ (\Gamma^{M,11}=-i\Gamma^M)_{AB}). \tag{4.23}$$

This algebra is subject to the three 12D covariant constraints

$$P^mP_m=0,\ P_m(\Gamma^m)_{\dot a b}q_b=0,\ \Delta^{mnl}=0, \tag{4.24}$$

where the last two are a 12D covariant rewriting of the 11D constraints in Eqs. (4.8), (4.12). The tensor Δ^{mnl} is defined by $\Delta^{lmn}=W^{lmn}-S^{lmn}$, where W^{lmn} and S^{lmn} are the generalized Pauli-Lubański and spin tensors in 12D,

$$W^{lmn}\equiv J^{<lm}P^{n>}\equiv\frac{1}{3!}(J^{lm}P^n\pm\text{permutations}),$$

$$S^{lmn}\equiv-\frac{i}{3\times16}\overline{q}\Gamma^{lmn}q. \tag{4.25}$$

We see that the constraint in Eq. (4.12) corresponds to

$$J^{\mu\nu}P^{11}+J^{\nu11}P^\mu+J^{11\mu}P^\nu-\frac{i}{16}\varepsilon^{ij}\overline{Q^i}\gamma^{\mu\nu}Q^j=3\Delta^{11mn}=0. \tag{4.26}$$

The number of independent components in the tensor Δ^{lmn} can be computed covariantly in 12 dimensions as in the previous section,

$$\sum_{k=3}^{12}\binom{12}{k}(-1)^{k+1}=55=(1-1)^{12}-\left[-1+\binom{12}{1}-\binom{12}{2}\right]. \tag{4.27}$$

This is the same number of components as in Eq. (4.12), namely $11\times10/2=55$.

The supermultiplet of the quantum states that provide a representation of the constrained 12D super Poincaré algebra can be easily computed in the SO(10) covariant lightcone gauge. After solving all the constraints explicitly, the degrees of freedom reduce to x^i,p^i,χ^a where the SO(10) vectors x^i,p^i are canonical and the 32-components of χ^a [two SO(10) spinors] satisfy the Clifford algebra $\{\chi^a,\chi^b\}=2\delta^{ab}$. Therefore the quantum states are $|\alpha,p^i\rangle$ where α indicates 2^{15} bosons and 2^{15} fermions corresponding to the two spinor representations of SO(32).

These are precisely the quantum states of the first massive level of the 11D supermembrane, as computed in Ref. [4]. They also correspond to the first massive level of the type-IIA closed string, which gives a first signal of the relationship to the 11D M theory as given in Ref. [5].

The SO(10) covariant multiplets of bosons and fermions given in Ref. [4] are massive 11D states but, through the present work, they are now being interpreted as massless in 12 dimensions. These $2^{15}+2^{15}$ states provide a representation of the 12D constrained super Poincaré algebra or of the unconstrained 11D nonlinear superalgebra.

We emphasize that $2^{15}+2^{15}$ are just the transverse SO(10) components of covariant fields in 12 dimensions. By extending the tensor and spinor indices of these states to covariant 12D indices, one should be able to identify the SO(11,1) covariant tensors and spinors that describe the 12D massless supermultiplet and provide a representation of the constrained super Poincaré algebra covariantly. In turn, by reduction from 12 to 11, these can also be understood as 11D covariant states that correspond to the first massive level of the supermembrane.

V. REMARKS

We have shown that there is a sense in which the super Poincaré algebra in twelve dimensions exists: it leads to a nonlinear algebra in eleven dimensions which contains the 11D super Poincaré algebra and which is interpreted as the off-shell superparticle in eleven dimensions, as given in Eqs. (4.1)–(4.3). This algebra necessarily contains a noncommutative spacetime in eleven dimensions, $[X^\mu,X^\nu]\neq0$. The algebra is represented on the quantum states of the first massive level of the 11D supermembrane, or first massive level of the type-IIA closed superstring which has a close relationship to 11D M theory.

The 11D aspect is an indication of M theory, while the 12D aspect hints a possible relationship to F theory [6] or S theory [7].

[10]Note that if the off-shell momentum in eleven dimensions is spacelike, then we would obtain two *real* chiral spinors in twelve dimensions belonging respectively to **32** and **32′** representations of SO(10,2). Indeed the analytic continuations of q_a and $\overline{q_{\dot a}}$ for P^2 positive are both real.

In the enlarged space including the extra dimension the algebra is the standard super Poincaré algebra, but with covariant constraints. We found that some of the constraints were unfamiliar. For example, in 11 dimensions the constraints $P^2=0$ and $\not{P}Q=0$ are standard, but the constraints $\Delta^{M_1M_2\cdots M_p}=0$ for $p=3,4,\ldots,11$ were not noticed before. The story is similar in the twelve-dimensional case, with the new constraints $\Delta^{m_1m_2\cdots m_p}=0$ for $p=3,4,\ldots,12$. These new Δ's commute with the translation generators P and supersymmetry generators Q. In general they would be related to additional quantum numbers that label the representation. But in our case we have a special representation in which the additional quantum numbers all vanish. In this representation all constraints are solved explicitly by writing the algebra as a nonlinear algebra in one lower dimension. The representation space that realizes the algebra is the massless particle in the higher dimension, which is also interpreted as the off-shell particle in one lower dimension.

It is clear from the explicit discussion in ten and eleven dimensions that the same kind of analysis can be applied in any number of dimensions. This could shed light on the meaning of supersymmetry in dimensions higher than 11.[11]

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[11]Without giving the details we state the result for the case of 9 to 10 dimensions, which could be done as an exercise by the reader. The structure is similar. Namely, the nonlinear nine dimensional off-shell $N=1$ superalgebra is related to the type IIA constrained super Poincaré algebra in ten dimensions. The representation space consists of the massless states of type IIA supergravity. Similarly some aspects of the case of 4 to 5 dimensions is briefly discussed at the end of Sec. III.

[1] I. Bars, C. Deliduman, A. Pasqua, and B. Zumino, Phys. Rev. D **68**, 106006 (2003), and references therein.
[2] R. Casalbuoni, Phys. Lett. **62B**, 49 (1976); Nuovo Cimento A **33**, 389 (1976).
[3] L. Brink and J. Schwarz, Phys. Lett. **100B**, 310 (1981).
[4] I. Bars, Nucl. Phys. **B308**, 462 (1988).
[5] I. Bars, Phys. Rev. D **52**, 3567 (1995).
[6] C. Vafa, Nucl. Phys. **B469**, 403 (1996).
[7] I. Bars, Phys. Rev. D **55**, 2373 (1997); in *Proceedings of the Second International A.D. Sakharov Conference on Physics*, Moscow, Russia, 1996, edited by A.M. Semikhatov (World Scientific, Singapore, 1997), pp. 355–363, hep-th/9608061; Phys. Rev. D **54**, 5203 (1996).
[8] E. Sokatchev, Nucl. Phys. **B99**, 96 (1975).
[9] C. Pickup and J.G. Taylor, Nucl. Phys. **B188**, 577 (1981).