

Cosmological perturbations in a big-crunch–big-bang space-time

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A prescription is developed for matching general relativistic perturbations across singularities of the type encountered in the ekpyrotic and cyclic scenarios, i.e., a collision between orbifold planes. We show that there exists a gauge in which the evolution of perturbations is locally identical to that in a model space-time (compactified Milne mod Z_2) where the matching of modes across the singularity can be treated using a prescription previously introduced by two of us. Using this approach, we show that long wavelength, scale-invariant, growing-mode perturbations in the incoming state pass through the collision and become scale-invariant growing-mode perturbations in the expanding hot big-bang phase.

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I. INTRODUCTION

The big-bang singularity is one of the most vexing puzzles in modern cosmology. Tracing time backwards, the field equations of general relativity break down in an apparently irretrievable manner some 1.4×10^{10} years ago when the density of matter and the curvature of space-time diverge. Cosmic inflation does not ameliorate this disaster, but rather tempts us to ignore it by just assuming that the Universe somehow emerged from the singularity in an inflationary state, and that subsequent inflation washed out all of the details of the big bang and how inflation began.

A more fundamental point of view is that the singularity is a manifestation of the breakdown of general relativity at short distances, which needs to be properly dealt with in a more consistent cosmology. String theory and M theory are important suggestions as to what a more fundamental theory might look like, improving on general relativity, for example, by providing consistent perturbative S matrices that include graviton processes. If string theory is a consistent, unitary S -matrix theory, as it is believed to be, then it is reasonable to expect that the cosmic singularity should be resolved within string theory, or a future development of it, in a satisfactory way. In particular, for every “out” state there should be at least one “in” state. The question arises: What could the “in” state have been which produced the hot big bang?

In recent papers, we have explored a concrete, detailed proposal for answering this deep question. In the ekpyrotic [1] and cyclic [2] Universe models, the origin of scale invariant density perturbations and the flatness, homogeneity and horizon puzzles of the standard cosmology are all explained without recourse to a burst of high energy primordial inflation [1,2]. Instead, these puzzles are solved by physical processes occurring prior to the hot big bang [1–4], in a highly economical way employing today’s observed cosmological constant in an integral manner. However, key to the success of these new scenarios is a consistent passage through the big-bang singularity.

At first sight, passing safely through a big crunch/big bang transition seems impossible because many physical

quantities (density, curvature) diverge there. However, in the situation encountered in the ekpyrotic and cyclic brane world models, for the background solution the situation is far less severe [1,2]. When two boundary branes collide, even though this *is* the big-bang singularity in the conventional (Einstein frame) description, in the background solution the density of matter and the space-time curvature of the branes remain finite. Conservation of total energy and momentum across the collision may be consistently imposed [2] and, once the densities of radiation and matter generated on the branes at the collision are fixed (by microscopic physics), the outgoing state is uniquely determined.

However, while the background geometry describing a boundary brane collision seems to be well behaved, it is still mathematically singular in the sense that one dimension disappears at one instant of time. The space-time ceases to be Hausdorff [5], and since the dimensionality of the spatial slice is only three at this moment, it is not a good Cauchy surface. More worryingly, perturbations generally diverge as one approaches the singularity, as the result of the cosmological blueshift associated with the collapse of the extra dimension. Nevertheless, the situation is more manageable than it appears to be at first sight. In certain gauges, the metric perturbations only diverge logarithmically in time [6], and the canonical momenta associated with the perturbations and certain other perturbation variables actually remain finite at the singularity.

Around the brane collision, the space-time geometry may be modeled by a simpler space-time which we shall refer to as “compactified Milne mod Z_2 .” This is locally flat away from the singularity, and may be embedded within Minkowski space-time as shown in Fig. 1. The model space-time may be thought of as describing the collision of two tensionless Z_2 branes separated by a flat bulk. In a study of free fields on this space-time, two of us showed [7] that the construction of a unitary map between incoming and outgoing states is not only possible but essentially unique. As we review in Sec. II, the basic idea is to employ normal propagation of free fields on the Minkowski covering space-time. This rule was shown [7] to satisfy many desirable properties.

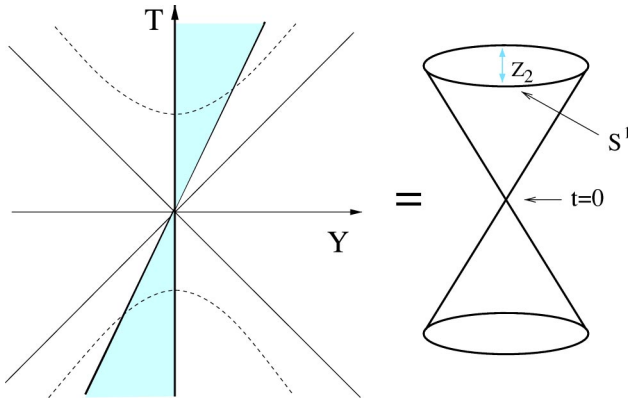


FIG. 1. Locally, the collision of two branes may be embedded in Minkowski space-time. The usual Minkowski space-time coordinates T and Y are expressed as $T=t \cosh y$ and $Y=t \sinh y$, where the Lorentz-invariant coordinate t is constant on the dashed lines. The colliding brane space-time is constructed in two steps. First the y coordinate is compactified by identifying y with $y+2y_0$, to produce the double-conical space-time shown at the right. Second, the circular sections of these cones are orbifolded by the Z_2 symmetry $y \rightarrow 2y_0 - y$. The two fixed points of the Z_2 symmetry are two tensionless branes moving at a relative speed of $\tanh y_0$, which collide and pass through one another at $t=0$.

For example, it defines a vacuum two-point function of the Hadamard form which is also time reversal invariant. And in this idealized situation with no interactions, there turns out to be no particle production associated with passage through the singularity. Some first steps were taken towards studying interactions and these were shown to lead to finite answers provided the coupling constant vanishes sufficiently rapidly near the collision event.

The purpose of the present paper is to extend these ideas to a study of full general relativistic perturbations in space-times possessing singularities of the type shown in Fig. 1. The usual definition of a space-time manifold is that it is a metric space which appears locally flat. This means that in the neighborhood of any point P it should always be possible to choose a coordinate system in which (a) the metric at P is the Minkowski metric, and (b) the first derivatives of the metric with respect to each coordinate vanish at P . The inclusion of singular points of the type shown in Fig. 1 requires an extension of these rules. In particular, the usual notion of general coordinate invariance becomes more subtle. A description of the incoming and outgoing space-times, away from the singularity, should be completely independent of coordinates since only the intrinsic geometry matters. However, connecting the two halves of the space-time across the singularity requires a correspondence between the “incoming” and “outgoing” coordinate systems. What this means in practice is that after solving for the metric and brane perturbations using general relativity in the upper and lower halves (which may be done in any gauge), one needs to choose a set of coordinates, or gauge, common to both halves within which the matching is to be performed.

Our proposal for extending general relativity to this type of singularity is illustrated in Fig. 2. The idea is to insist that the upper and lower halves be connected via a smooth set of

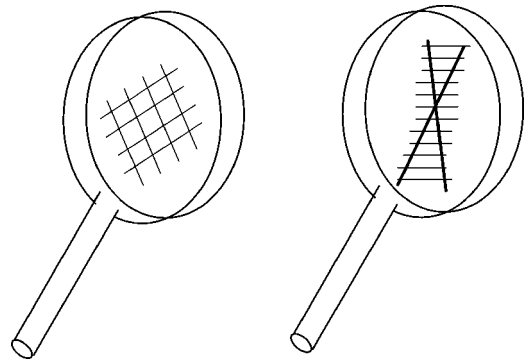


FIG. 2. The definition of a space-time manifold is that when viewed “up close” (left figure), it should appear to be locally flat. We define singular space-times of the type we are interested in here as space-times for which there exists a single coordinate system covering the neighborhood of the singularity in both the incoming and outgoing space-times, within which the collision event appears locally identical to the idealized situation of tensionless Z_2 branes colliding in Minkowski space-time (right figure).

embedding coordinates in which the geometry appears locally identical to that describing the model space-time consisting of the collision of two tensionless branes, i.e. compactified Milne mod Z_2 . This set of embedding coordinates, locally unique up to Lorentz transformations, connects the contracting and expanding phases on either side of the bounce. The fact that fields may propagate across the singularity in the model space-time shown in Fig. 1 and, at the same time, unitarity and all the other desirable physical properties of massless fields propagating in ordinary Minkowski space-time can be maintained [7] makes this minimal extension of general relativity that we propose both reasonable and physically sensible.

In close analogy with the definition of a space-time manifold, we shall define “locally” by insisting that the first two terms in a series expansion of the metric perturbations (specifically the constant and logarithmic terms) behave precisely as free gravitational waves would in a compactified Milne mod Z_2 space-time. The main work of the paper will be to demonstrate that this condition may be precisely formulated, at least for the lowest energy modes, and that it completely fixes the power series expansion in the Lorentz-invariant distance $t = \sqrt{T^2 - Y^2}$ from the singularity. Within the coordinate systems so constructed for the incoming and outgoing space-times, we find a unique rule for matching gravitational perturbations, in a manner entirely analogous to the matching of free scalar fields in the model space-time, as discussed in Ref. [7].

The matching procedure we propose is more subtle than that usually adopted in general relativity. Even in situations where the matter stresses change suddenly on some physically prescribed space-like surface (for example in a phase transition), it is normally only necessary to match the spatial three-metric and its normal time derivative, without worrying about the detailed behavior of the solutions of the field equations. In our case, the metric perturbations diverge at the singularity. One might attempt to cut the divergence off by pasting the incoming and outgoing space-times together on

some arbitrary surface slightly away from the singularity, but it is not known how to do this in a coordinate-invariant manner inevitably leading to ambiguous, and usually cutoff-dependent answers. In contrast, our procedure for massless fields including gravitational waves on compactified Milne mod Z_2 can be formulated in terms of analytic continuation, which is automatically coordinate invariant, or in terms of a real continuation in an embedding Minkowski space with asymptotically flat boundary conditions, also a coordinate-invariant prescription. Both methods produce the same cutoff independent result. Notice also that both involve global aspects of the space-time, and cannot be stated as a purely local matching rule. This seems to be the inevitable price one has to pay for evolving through a singularity where a Cauchy surface does not exist.

We have in mind of course, an application of this proposal to the types of cosmological singularities encountered in ekpyrotic and cyclic models in which two boundary branes collide as shown in Fig. 1. In particular we wish to track scale-invariant perturbations developed via the ekpyrotic mechanism [1,6] in the incoming state across the singularity and into the outgoing hot big-bang phase. The conclusion of our work is that with the prescription adopted here, scale-invariant, growing mode perturbations produced during the pre-big-bang phase [1,2,4] pass through the bounce and become scale-invariant growing mode perturbations in the late Universe.

Let us briefly comment on the relation of this paper to previous studies by ourselves and others. Our first attempt [6] at matching perturbations across the transition was based entirely on the study of the four-dimensional effective theory. As we shall see, this is not sufficient to describe the bounce, which is really five dimensional. Nevertheless, in that work we observed that certain perturbation variables, such as the comoving energy density perturbation ϵ_m were finite at the singularity and could be matched across it. The present (and far more sophisticated) approach confirms this element of the procedure. The problem is that *two* matching conditions are needed in the four-dimensional effective theory and the first time derivative of ϵ_m turns out not to be independent of ϵ_m itself because the differential equation is singular at $t=0$. This leads to an ambiguity in the second matching condition. Based on simplicity, we proposed matching the second derivative and obtained an outgoing scale-invariant spectrum. However, we did not have any real physical justification for this choice.

There were criticisms and alternative proposals for matching conditions [8], including the idea that one should match the curvature perturbation on comoving (or constant density) slices [9–11], a procedure which is often useful in the context of nonsingular, expanding four-dimensional cosmology. In our setting, the comoving curvature perturbation is logarithmically divergent at the singularity [6,12], but if one disregards this and proceeds to match its long wavelength, constant component, this proposal results in the growing, scale-invariant perturbations present in the pre-big-bang phase being matched to a pure decaying mode in the outgoing state [10,11]. The result is a complete absence of long wavelength density perturbations in the big-bang phase. It was subse-

quently pointed out, however, that this null result is atypical in the sense that, for most choices of matching surfaces, scale invariant growing perturbations coming in would match to scale invariant growing perturbations coming out [13].

Some of the alternative proposals are designed specifically for four-dimensional theories in which the bounce from contraction to expansion occurs at a non-zero value of the scale factor [14,15] (see also Ref. [16]). This is accomplished by arranging for the equation of state w to violate the null energy condition near the bounce, i.e., $w < -1$. We emphasize that the ekpyrotic and cyclic scenarios and the considerations here do not fall into this category. The bounce in Fig. 1 corresponds to the zero scale factor in the four-dimensional effective theory and the four-dimensional effective equation of state parameter is strictly positive before the collision.

Our approach is to choose a class of gauges in which the geometry around the collision event appears locally identical to that describing linearized perturbations around the model space-time, compactified Milne mod Z_2 . Then we match the perturbations according to the procedure of Ref. [7] for that space-time. An important feature of our choice of coordinates is that the collision event is *simultaneous* in Milne time and occurs at the background value $t=0$ both for the incoming and outgoing state. That is, the limits $t \rightarrow 0^-$ in the incoming state and $t \rightarrow 0^+$ in the outgoing state correspond to the same physical space-time surface.

In the course of our analysis we shall uncover the problem with matching the curvature perturbation on comoving (or constant energy density) slices in the four-dimensional effective theory, ζ_4 , across the bounce. We shall show that ζ_4 is indeed conserved on long wavelengths both before and after the bounce and, furthermore, that on long wavelengths it is equal to the comoving curvature perturbations on the branes ζ_{\pm} . Why then are these variables not conserved across the bounce? The reason, detailed in Sec. VD, is that the brane collision event is *not simultaneous* in the comoving or constant energy density time slicing. This is a disaster in terms of matching. In these coordinate systems, the $t \rightarrow 0^+$ and $t \rightarrow 0^-$ space-like surfaces do not physically coincide and therefore perturbations should certainly *not* match across them. We find that the collision event is displaced from the $t=0^+$ and $t=0^-$ surfaces in these slicings by a scale-invariant time delay, within which all the information regarding the growing mode perturbation is contained. A determination of the collision-synchronous time slices is only possible within the full five-dimensional theory, and our final result for the spectrum of growing mode perturbations involves five-dimensional parameters which cannot be re-expressed in purely four-dimensional terms.

Distinct but closely related are problems raised in recent attempts to directly study string theory on compactified Milne space-times analogous to that shown in Fig. 1 [17,18]. Since these types of backgrounds are locally flat, one can solve [19] the tree level field equations of string theory to all orders in α' , away from the singularity. It is then tempting to calculate string scattering processes using a Lorentzian generalization of standard orbifold techniques to this time-dependent case. Calculations have been performed in analo-

gous backgrounds, for example, the null orbifold and “null-brane” backgrounds [17,20] possessing some remaining supersymmetry. The result is that tree level scattering amplitudes develop infrared divergences which have been attributed to the backreaction of the geometry near the singularity.

It is unclear what the physical significance of these results are yet. The breakdown of string perturbation theory seems to indicate that nonlinear effects must be taken into account. But such nonlinear effects are not necessarily disastrous for cosmology. For example, since the collision takes place on a very short time scale, one plausible possibility is that nonlinearities result in the production of microscopic black holes at the collision. This would be consistent with the conclusion that perturbative string theory breaks down, but it would be unimportant for cosmology. The black holes would radiate and decay rapidly after the bounce without having a significant effect on the long wavelength perturbations that are relevant cosmologically.

The classical theory may provide some insight. For example, consider classical general relativity with a scalar field. As the Universe contracts towards a big crunch singularity, the gradients of the energy density diverge and one might be tempted to argue that gradient terms would dominate the dynamics. However, this conclusion is believed to be wrong. Instead, the behavior of the metric and fields becomes ultralocal [21]. Spatial derivatives become less important as the Universe contracts and, at each point in space, the geometry follows a homogeneous (but in general anisotropic) evolution. This occurs because, although the gradient terms grow, the homogeneous terms grow faster. A description of this subtle situation may well be difficult using string perturbation theory, which relies, for example, upon the existence of a globally good gauge. However, as we shall explain in the conclusions, there is a simple classical picture of where the nonlinearities lead to. And within this picture, we see that the nonlinear corrections would hardly alter our final matching result.

We should also note that the string theoretic calculations have only so far been possible in certain special models for which the technical tools needed are available. In particular, they have all been done in the context of ten-dimensional string theory at fixed coupling, using Lorentzian orbifolding, with one of the *nine* string theory spatial dimensions shrinking away and reappearing. However, this setup is quite different from the case proposed for the ekpyrotic model, where the *tenth* spatial dimension (of 11-dimensional supergravity), separating the two boundary branes, was supposed to collapse and reappear. The 11-dimensional theory reduces, at fixed, small brane separation, to string theory [22] at weak coupling. But in the time-dependent situation we are interested in, the coupling would actually vanish as the branes meet. This situation is qualitatively different from the examples which have been studied so far. In particular, the infinities encountered in Refs. [17] are proportional to the string coupling. But in the ekpyrotic model the coupling vanishes at the singularity.

Progress in the investigation of such singularities within string theory [23] continues to be an active field [24–28]. Analytic continuation methods related to those we employed

for field theory [7] have been applied to constructing string theory on similar backgrounds [29] with less pessimistic conclusions than the above cited works [30]. Other approaches and methods have also been developed [31]. We have continued to develop a simpler field-theoretic approach because it is considerably more manageable and may yield helpful physical insight. We hope that further developments of string theory can be used to check and develop the approach presented here.

The remainder of the paper builds in stages towards a full calculation of the propagation of cosmological perturbations through a bounce of the ekpyrotic/cyclic type:

We first consider the propagation of scalar fields in a fixed background corresponding to two tensionless Z_2 branes colliding in a flat bulk as discussed in Ref. [7] (Sec. II).

We next consider linearized gravitational perturbations of the same model space-time (Sec. IV).

Finally, we consider the full-blown calculation of cosmological perturbations for two colliding branes with tension and a warped bulk. This calculation leads to our central result for the amplitude of the scale-invariant perturbations propagating across the singularity into the hot big-bang phase (Sec. V).

Various tools are developed along the way. Section III develops the moduli space approximation for two colliding branes in a negative cosmological constant bulk [32] which we shall study as our canonical example. We extend this formalism, showing for example that it is exact for empty branes at arbitrary speed and curvature. In Appendix A we show that the four-dimensional effective theory consistently predicts the projected Weyl tensor contribution to the effective Einstein equations on the branes, and is in agreement with the recently developed “covariant curvature” approach [33] as well as earlier metric-based approaches [34,35]. We also match the parameters of four-dimensional effective theory for the homogeneous flat background solution to the parameters of the five-dimensional theory. Appendix B discusses the gauge invariant variables for the five-dimensional theory and how the position of the branes depends on the choice of gauge. Appendix C works out the detailed background geometry near the bounce in a coordinate system convenient for the perturbation calculations. Appendix D concerns the choice of gauge required to have the brane collision simultaneous at all values of the noncompact coordinates \vec{x} .

II. PROPAGATION OF SCALAR FIELDS IN A COLLISION OF TENSIONLESS BRANES

The idealized space-time we shall use as a model for the singularity is just Minkowski space-time subject to two identifications [36]. Expressing the usual Minkowski coordinates as $T = t \cosh y$ and $Y = t \sinh y$, the line element is

$$ds^2 = -dT^2 + dY^2 + d\vec{x}^2 = -dt^2 + t^2 dy^2 + d\vec{x}^2. \quad (1)$$

The incoming and outgoing regions, $t < 0$ and $t > 0$, respectively, are the two halves of Milne space-time $\mathcal{M} \times R^3$. We now compactify the y coordinate by identifying under boosts,

which correspond to translations in y , $y \rightarrow y + 2y_0$. We refer to the resulting space as compactified Milne space-time, or $\mathcal{M}^C \times R^3$. Finally we introduce two tensionless Z_2 branes by identifying fields under reflection across the circle, $y \rightarrow 2y_0 - y$ giving the orbifolded space $\mathcal{M}^C/Z_2 \times R^3$, or compactified Milne mod Z_2 . The branes are separated by a coordinate distance $\Delta y = y_0$ which is the rapidity associated with their relative speed. Later in the paper it will be convenient to choose a Lorentz frame in which the branes are located at equal and opposite values of $y = \pm y_0/2$. Note that any field which is even under the Z_2 must obey Neumann boundary conditions $\partial_y \varphi = 0$ on the two branes.

The problem of propagating a free quantum field through a big-crunch–big-bang singularity of the type shown in Fig. 1 was considered in Ref. [7]. The equation of motion for a scalar field on the background (1) is

$$\ddot{\varphi} + \frac{1}{t} \dot{\varphi} + \frac{k_y^2}{t^2} \varphi + \vec{k}^2 \varphi = 0, \quad (2)$$

where k_y is the momentum in the y direction and \vec{k} that in the uncompactified \vec{x} directions.

In this paper our main interest is in the lowest excitations corresponding to the modes of the four-dimensional effective theory. In this compactified Milne setup these modes are the y -independent fields, trivially satisfying Neumann boundary conditions on the branes and periodicity in y . For these modes, Eq. (2) is just Bessel's equation with index $\nu = 0$. The two linearly independent solutions are $J_0(kt)$ and $N_0(kt)$, behaving for small positive t as

$$J_0(kt) \sim 1 + \dots, \quad N_0(kt) \sim \frac{2}{\pi} [\ln(kt) + \gamma - \ln 2] + \dots, \quad (3)$$

where γ is Euler's constant $0.577\dots$. The positive (negative) frequency outgoing modes $\psi^{(+)}$ ($\psi^{(-)}$) are those which tend to the adiabatic positive (negative) frequency solutions as $t \rightarrow \infty$. They are proportional to the Hankel function $H_0^{(2)} = J_0 - iN_0$ ($H_0^{(1)} = J_0 + iN_0$), and converge rapidly to zero in the lower (upper) half complex t plane. If we split the quantum field $\varphi(t, \vec{x})$ into its positive and negative frequency parts, they are well defined, respectively, in the lower and upper half complex t plane. The unique analytic continuation from negative to positive values of t is then to continue the positive frequency part below and the negative frequency part above the singularity at $t = 0$. Continuing the expressions (3) around a small semicircle below $t = 0$ one infers the relation $H_0^{(2)}(kt) = -H_0^{(1)}(-kt)$ giving the positive frequency mode function at negative values of t . We can translate this into a matching rule for the field φ by writing $\varphi = \sum a \psi^{(+)} + \text{H.c.}$, with a arbitrary and complex. The asymptotic behavior of the field φ is then found to be

$$\begin{aligned} \varphi &\sim Q_{in} + P_{in} \ln k|t|, & t \rightarrow 0^-, \\ \varphi &\sim Q_{out} + P_{out} \ln k|t|, & t \rightarrow 0^+, \end{aligned} \quad (4)$$

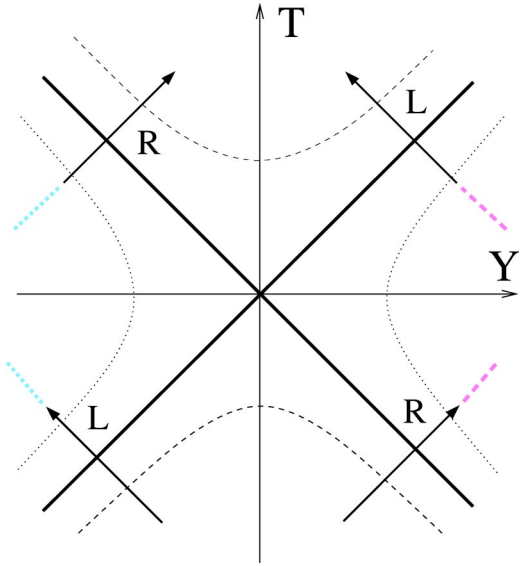


FIG. 3. Continuation of left and right moving modes. A free field propagating in the lower quadrant may be decomposed into left and right movers as it approaches the past light cone of the origin $T=Y=0$. The left movers are regular across $Y=T<0$ and may be continued into the left quadrant $Y<0, |T|<|Y|$. The right movers are regular across the right segment $Y=-T>0$ and may be continued into the right quadrant $Y>0, |T|<|Y|$. If we impose vanishing boundary conditions at large Lorentz-invariant separation from the origin in the left and right quadrants, then once we know the left mover in the left quadrant, the right mover on the null segment $Y=-T<0$ is uniquely determined, and similarly the left mover on $Y=T>0$. One thereby obtains a unique matching rule from the incoming, lower quadrant to the outgoing, upper one.

and the above continuation implies that

$$Q_{out} = -Q_{in} + 2(\gamma - \ln 2)P_{in}, \quad P_{out} = P_{in}. \quad (5)$$

The canonical momentum of the field $|t|\dot{\varphi}$ is actually proportional to $\text{sign}(t)P$. Hence, the field momentum reverses at $t=0$ with this matching rule. Note, however, that the constant term Q is not preserved across $t=0$. Hence this matching rule is not simply time reversal at $t=0$, and there is an arrow of time across $t=0$.

There is another way of looking at this rule which is illustrated in Fig. 3. Take a field configuration on one copy of the incoming wedge and repeatedly reflect it through the boundary branes to fill out the lower quadrant. The resulting configuration obeys the field equation (even with nonlinear interactions), as long as the equation is Z_2 invariant. The solutions to the field equation then naturally split into left and right movers as one approaches the light cone. The left movers are regular on $Y=-T$ and the right movers on $Y=T$. Each can therefore be uniquely matched across the appropriate segments of the past and future light cone of the singularity (Fig. 3).

In this way, incoming data in the lower quadrant uniquely determines the left moving modes entering the left quadrant and the right moving modes entering the right quadrant. The solutions in the left and right quadrants may be fully specified by choosing boundary conditions. It is natural to demand that the fields vanish at space-like infinity. Once the solution

in the left and right quadrants is determined then the left movers from the right quadrant and the right movers from the left quadrant may be uniquely matched to the left and right movers in the upper quadrant, completely determining the solution in the outgoing state. Again, in the context of our model space-time compactified Milne mod Z_2 , this prescription yields exactly the same matching rule (5). The advantage of this derivation is that it gives the clearest explanation for the sign change in the constant contribution Q , between the “in” and “out” states. This is just due to our having imposed a “reflecting” boundary condition at space-like infinity. Since in passing from the lower to the upper quadrant one such reflection is involved, a relative minus sign is acquired. And as we shall explain in Sec. IV, precisely the same matching rule may be applied for gravitational perturbations on compactified Milne mod Z_2 . In this case one can see that the condition of asymptotic flatness imposed in the two unphysical quadrants is actually coordinate invariant.

In the case of cosmological interest where the branes have tension and the bulk is warped, the sign change of Q in Eq. (5) is still guaranteed provided two reasonable conditions are fulfilled. Assume that the low energy modes in the space-like regions (which are just the analytic continuation of the corresponding modes in the lower quadrant, obtained by setting $t = is$ and $y = \rho - i\pi/2$, where $T = s \sinh \rho$ and $Y = s \cosh \rho$) depend only on s as $s \rightarrow 0$ (i.e. behave as the Kaluza-Klein zero modes). Second, assume that the mode selected by the imposed boundary condition at space-like infinity behaves, near $s=0$, as $D + \ln(k|s|)$ with D a model-dependent constant. This is the generic behavior—for compactified Milne mod Z_2 we have $D = \gamma - \ln 2$. Then it is straightforward to show by explicit calculation that matching the left/right movers across the light cone from the lower quadrant into the left/right quadrants and then into the upper quadrant, one obtains $P_{out} = P_{in}$ and $Q_{out} = -Q_{in} + 2DP_{in}$. Hence we see the sign change of Q is universal but the coefficient D is not.

It is important to emphasize that all of these arguments for the matching rule (5) involve the detailed *global* structure of the embedding space-time. In particular the $\gamma - \ln 2$ term in Eq. (5) is peculiar to the Minkowski embedding space-time appropriate for compactified Milne mod Z_2 . If the embedding space-time is warped, the corresponding constant would be altered to some constant D as explained above. Fortunately it shall turn out that for the case we are interested in, $P_{in} \ll Q_{in}$ at long wavelengths and hence we are insensitive to the value of D . The correspondence $Q_{out} \approx -Q_{in}$ is, however, universal as argued above and therefore reliable even in the warped case. It turns out that this sign change is crucial in allowing scale invariant growing perturbations to propagate across the singularity, in the absence of radiation. Furthermore, the sign change is interesting and important in the nonlinear theory, as we explain in the conclusions.

III. THE 4D EFFECTIVE THEORY

In subsequent sections we shall extend the matching rule just discussed for free scalar fields to full general relativistic perturbations. There are two major complications. The first is the gauge invariance of general relativity which, as explained

above, is unusually subtle for singular space-times such as we are dealing with. The second is that the bulk space-time is not globally Minkowski space-time but is warped and has non-negligible y dependence. Of course, this is related via Israel matching (see, e.g. Ref. [37]) to the fact that the brane tensions are nonzero.

We want to solve the linearized Einstein field equations for five-dimensional gravity coupled to a pair of colliding orbifold (Z_2) branes. For the cosmological applications, we need to follow the system from times well before the brane collision, when the scale-invariant perturbations were generated, through the collision and into the far future. In general this would involve solving a system of coupled partial differential equations in y and t for the bulk gravitational fields with mixed boundary conditions following from the Israel matching conditions on the branes, and would be well beyond an analytic treatment.

However, there is a powerful tool we can call upon which makes the task surprisingly tractable: the moduli space approximation.

A. The moduli space approximation

On general grounds one expects the long wavelength, low energy modes of the system to be described by a four-dimensional effective theory, and we are only interested in low energy incoming states which are well described by this theory. We shall show that the four-dimensional effective theory may be consistently used to predict the brane geometries all the way to collision, thereby providing boundary data for the bulk five-dimensional equations which we solve as an expansion in t about the collision event. After the collision, the four-dimensional effective theory plays an equally important role, enabling us to track the behavior of perturbations into the far future of the collision event (Fig. 4). The technique we describe forms the basis for our analysis of the singularity described in later sections, but it is also of considerable generality and use in its own right, since almost all of the late Universe phenomenology of brane worlds can be most efficiently described using the effective theory alone.

In this paper we concentrate on the simplest two-brane world model consisting of one positive and one negative tension brane bounding a bulk with a negative cosmological constant $\Lambda = -6M_5^3/L^2$ where L is the AdS radius and M_5 the five-dimensional Planck mass. If the brane tensions σ_{\pm} are fine tuned to the special values $\pm 6M_5^3/L$, the system allows a two-parameter family of static solutions in which the scale factor on each brane is a free parameter, or modulus. The idea of the moduli space approach is that such parameters are promoted to space-time-dependent fields within the four-dimensional effective theory. In passing, we note that many of the methods we use in this paper should in principle extend to more complicated theories, such as the Horava-Witten theory, in which the family of static solutions exists without the need for a fine tuning of the brane tensions.

In Khoury *et al.* [1], the effective action for the moduli in this system was computed in the low velocity approximation, and shown to be equivalent to Einstein gravity plus a scalar

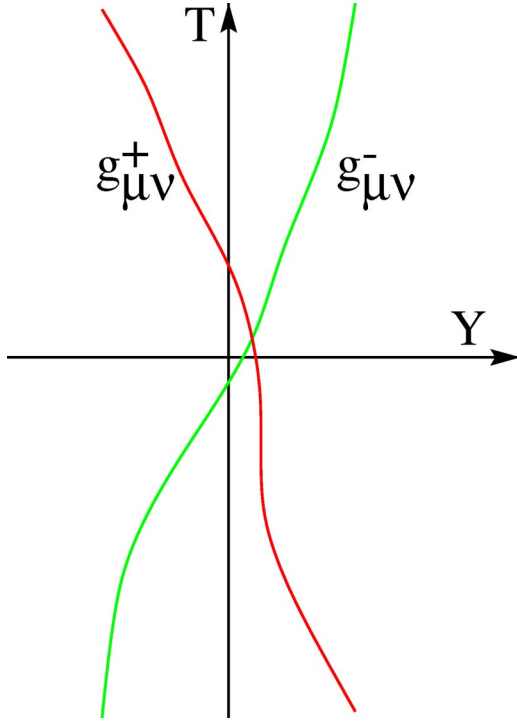


FIG. 4. The world lines of the positive and negative tension branes are plotted for some fixed value of the uncompactified coordinates \vec{x} . The four-dimensional effective theory is used to predict the intrinsic geometries of the positive and negative tension branes, i.e. their space-time metrics $g_{\mu\nu}^+$ and $g_{\mu\nu}^-$, according to Eq. (10). The four-dimensional effective theory is used to describe the incoming and outgoing perturbed branes far to the past or future of the collision event. The brane metrics also provide boundary data for the five-dimensional bulk metric which we solve for as a power series expansion in time about the collision event.

field which couples non-minimally to the matter on each brane (see also Ref. [38]). The derivation given here, while more specific to the simplest brane models, is both simpler and more powerful. It shows that the same effective action actually has a broader range of validity than originally anticipated, turning out to be exact for empty brane configurations with cosmological symmetry, for arbitrary spatial curvature and velocity (or expansion rate). When matter is present, the effective theory is a good approximation as long as the density of matter is small compared to the brane tension. The fact that the four-dimensional effective theory is so accurate is likely to be a special feature associated with the lack of bulk degrees of freedom in the simplest brane world model we are focusing on: for configurations with cosmological symmetry, a generalized Birkhoff theorem [39,40] holds which guarantees that no radiation is emitted into the bulk.

Consider a positive or negative tension brane with cosmological symmetry but which moves through the five-dimensional bulk. The motion through the warped bulk induces expansion or contraction of the scale factor on the brane. As shown in Ref. [36], the scale factor on the brane obeys a “modified Friedmann” equation,

$$H_{\pm}^2 = \pm \frac{1}{3M_5^3 L} \rho_{\pm} + \frac{\rho_{\pm}^2}{36M_5^6} - \frac{K}{b_{\pm}^2} + \frac{C}{b_{\pm}^4}, \quad (6)$$

where ρ_{\pm} is the density (not including the tension) of matter or radiation confined to the brane, b_{\pm} is the brane scale factor, and H_{\pm} is the induced Hubble constant on the positive (negative) tension brane. We work in units such that the coefficient of the Ricci scalar in the five-dimensional Einstein action is $M_5^3/2$. The last term is the “dark radiation” term, where the constant C is related to the mass of the black hole in the Schwarzschild-AdS solution discussed in Appendix C.

We shall show that the solutions to these equations are precisely reproduced by a four-dimensional effective theory, with the only approximation necessary being that the density of matter or radiation confined to the branes, ρ_{\pm} be much smaller than the magnitudes of the brane tensions, so that the ρ_{\pm}^2 terms in Eq. (6) are negligible. For the particular concerns in this paper, namely the accurate calculation of the long wavelength curvature perturbation on the branes, it is reassuring that the four-dimensional effective theory description is such a well-controlled approximation, even at large brane velocities, in the long wavelength limit.

Choosing conformal time on each brane, and neglecting the ρ^2 terms Eqs. (6) become

$$b_+^{\prime 2} = + \frac{1}{3M_5^3 L} \rho_+ b_+^4 - K b_+^2 + C, \\ b_-^{\prime 2} = - \frac{1}{3M_5^3 L} \rho_- b_-^4 - K b_-^2 + C, \quad (7)$$

where the prime denotes conformal time derivative. The corresponding acceleration equations for b_+'' and b_-'' , from which C disappears, are derived by differentiating equations (7) and using $d(\rho b^4) = b^3(\rho - 3P)db$, with P being the pressure of matter or radiation on the branes. We now show that these two equations can be derived from a single action provided we equate the conformal times on each brane. Consider the action

$$\mathcal{S} = \int dt N d^3x [-3M_5^3 L (N^{-2} b_+^{\prime 2} - K b_+^2) - \rho_+ b_+^4 \\ + 3L (N^{-2} b_-^{\prime 2} - K b_-^2) - \rho_- b_-^4], \quad (8)$$

where N is a lapse function introduced to make the action time reparametrization invariant. Varying with respect to b_{\pm} and then setting $N=1$ gives the correct acceleration equations for b_+'' and b_-'' following from Eq. (7). These equations are equivalent to Eq. (7) up to two integration constants. The constraint equation, following from varying with respect to N and then setting $N=1$, is just the difference of the two equations (7) and ensures that one combination of the integration constants is correct. The constant C is then seen to be just the remaining constant of integration of the resulting system of

equations and can in effect be determined by the solutions of equations of motion following from the action (8) [41].

Having shown that the modified Friedmann equations (with the neglect of ρ^2 terms) follow from an action in which \mathcal{C} does not appear, we are now able to change variables to those in which the system appears as conventional Einstein gravity coupled to a scalar field plus matter. We rewrite the action (8) in terms of a four-dimensional effective scale factor a and a scalar field ϕ , defined by $b_+ = a \cosh(\phi/\sqrt{6})$, $b_- = -a \sinh(\phi/\sqrt{6})$. Clearly, a and ϕ transform as a scale factor and as a scalar field under rescalings of the spatial coordinates \vec{x} . To interpret ϕ more physically, note that for static branes the bulk space-time is perfect anti-de Sitter space with line element $dY^2 + e^{2Y/L}(-dt^2 + d\vec{x}^2)$. The separation between the branes is given by $d = L \ln(a_+/a_-) = L \ln[-\coth(\phi/\sqrt{6})]$, so d tends from zero to infinity as ϕ tends from minus infinity to zero.

In terms of a and ϕ , the action (8) becomes

$$\mathcal{S} = \int dt d^3x \left[-3M_5^3 L (\dot{a}^2 - K a^2) + \frac{1}{2} a^2 \dot{\phi}^2 \right] + \mathcal{S}_m, \quad (9)$$

which is recognized as the action for Einstein gravity with line element $a^2(t)(-dt^2 + \gamma_{ij} dx^i dx^j)$, γ_{ij} being the canonical metric on H^3 , S^3 or E^3 with curvature K , and a minimally coupled scalar field ϕ . The matter action \mathcal{S}_m is conventional, except that the scale factor appearing is not the Einstein-frame scale factor but instead $b_+ = a \cosh(\phi/\sqrt{6})$ and $b_- = -a \sinh(\phi/\sqrt{6})$ on the positive and negative tension branes, respectively.

Now we wish to make use of two very powerful principles. The first is the assertion that even in the absence of symmetry, the low energy modes of the five-dimensional theory should be describable with a four-dimensional effective action. The second is that since the original theory was coordinate invariant, the four-dimensional effective action must be coordinate invariant too. Since the five-dimensional theory is local and causal, it is reasonable to expect these properties in the four-dimensional theory. Furthermore, if the relation between the four-dimensional induced metrics on the branes and the four-dimensional fields (i.e. the four-dimensional effective metric and the scalar field ϕ) is local (as one expects for the long wavelength, low energy modes we are interested in), then covariance plus agreement with the above results forces the relation to be

$$g_{\mu\nu}^+ = [\cosh(\phi/\sqrt{6})]^2 g_{\mu\nu}^{4d}, \quad g_{\mu\nu}^- = [-\sinh(\phi/\sqrt{6})]^2 g_{\mu\nu}^{4d}. \quad (10)$$

When we couple matter to the brane metrics, these expressions should enter the action for matter confined to the positive and negative tension branes respectively. Likewise we can from (9) and covariance immediately infer the effective action for the four-dimensional theory:

$$\mathcal{S} = \int d^4x \sqrt{-g} \left(\frac{M_4^2}{2} R - \frac{1}{2} (\partial_\mu \phi)^2 \right) + \mathcal{S}_m^- [g^-] + \mathcal{S}_m^+ [g^+], \quad (11)$$

where we have defined the effective four-dimensional Planck mass $M_4^2 = (8\pi G_4)^{-1} = M_5^3 L$.

B. Branes with non-zero matter density

For most of this paper we shall only study the especially simple case of radiation on the branes (which are 3+1 dimensional). The matter action is then independent of ϕ as a result of the conformal invariance of radiation in 3+1 dimensions, and this will greatly simplify our analysis. But as an aside let us for a moment consider nonrelativistic matter on the branes. Then there is a non-minimal coupling with ϕ , leading to a source term in the scalar field equation:

$$\partial^2 \phi = -\frac{1}{4} \left[\cosh\left(\frac{\phi}{\sqrt{6}}\right) \right]_{,\phi}^4 T^+ - \frac{1}{4} \left[\sinh\left(\frac{\phi}{\sqrt{6}}\right) \right]_{,\phi}^4 T^-, \quad (12)$$

where primes denote ϕ derivatives and the $T^{(\pm)}$ are the traces of the stress tensors for matter on the two branes contracted with respect to the relevant brane metric. It is interesting to see how these results compare with what is known about brane world gravity from prior studies [42]. For perfect fluids, the effective matter Lagrangian [43] reads $-\int d^4x \sqrt{-g^\pm} \rho_\pm$. Hence matter on the branes couples to the four-dimensional (Einstein frame) effective theory in the combination $\rho_4 = \cosh(\phi/\sqrt{6})^4 \rho_+ + \sinh(\phi/\sqrt{6})^4 \rho_-$. As the inter-brane distance grows, the field ϕ tends to zero. Since the cosh tends to unity, we see that a matter source on the positive tension brane with physical density ρ_+ contributes the same amount to the density seen by Einstein gravity in the four-dimensional effective theory. Furthermore, from Eq. (12), the coupling of such matter to the dilaton vanishes as ϕ . Hence the dilaton decouples and ordinary Einstein gravity is reproduced in this limit. Matter on the negative tension brane behaves very differently. If its density as seen by Einstein gravity in the four-dimensional effective theory is ρ_4 , then its physical density on the brane is much larger, $\rho_- \sim \phi^{-4} \rho_4$, and from Eq. (12) it sources the dilaton field as $\phi^{-1} \rho_4$. Hence at small ϕ the source for the dilaton diverges and Einstein gravity is never reproduced.

The derivation we have just given of the four-dimensional effective action starting from the modified Friedmann equations is in the present context both simpler and more powerful than previous derivations. It shows that the induced geometries on the branes are correctly predicted for branes with cosmological symmetry, for arbitrary curvature and speed of the branes provided only that the ρ^2 matter terms are negligible. For these cosmological backgrounds, the four-dimensional effective theory accurately describes the brane collision even though from the Einstein frame point of view such a collision is highly singular in the sense that the 4D effective scale factor a tends to zero and the Riemann ϕ tends to minus infinity in finite time. Nevertheless, the brane geometries and densities described by $g_{\mu\nu}^\pm$ and ρ_\pm are finite and well behaved at all times.

One surprising point about the map from five dimensions to four is that the effective theory with a scalar field sourced

by the combined energy density $\rho_4 = \cosh(\phi/\sqrt{6})^4 \rho_+ + \sinh(\phi/\sqrt{6})^4 \rho_-$ manages to correctly predict the solutions to the Friedmann equations on each brane even though these are separately sourced by ρ_+ and ρ_- . This is possible because of the integration constants. In the four-dimensional effective theory the basic equations can be taken to be the Friedmann equation ($a'^2 = \dots$) which has one integration constant and the scalar field equation $[(a^2 \phi)'] = \dots$ which has two. So there is a three-parameter set of solutions, although one of these is not physical as it is just a rescaling of a . On the other hand, the two-brane Friedmann equations have two integration constants along with the additional constant \mathcal{C} which is the dark radiation term. Consequently we have a precise match between the integration constants showing that there is a one-to-one map between the solutions of the two sets of equations. In performing an explicit check we find that the missing information on how much matter is contained on each brane is contained in the integration constants for the dilaton equation.

C. Relation between 4D effective theory and 5D brane parameters

The five-dimensional background we seek to describe consists of two parallel, flat Z_2 -symmetric three-branes bounding a bulk with a negative cosmological constant. In the incoming state, as they head towards a collision, the branes are assumed to be empty. In the ekpyrotic scenario, it is assumed that the brane collision event fills them with radiation. In this section we shall see how to describe this background setup in terms of the four-dimensional effective theory, and in particular we shall determine precise relations between the parameters of the four- and five-dimensional theories. The two brane geometries are determined according to formulas (10), and the background solution relevant post-collision is assumed to consist of two flat, parallel branes with radiation densities ρ_{\pm} . The corresponding four-dimensional effective theory has radiation density ρ_r , and a massless scalar field with kinetic energy density ρ_{ϕ} . It is convenient to work in units where the four-dimensional reduced Planck mass $M_4 = (8\pi G)^{-1/2}$ is unity. The four-dimensional Friedmann equation in conformal time then reads

$$a'^2 = \frac{1}{3}(\rho_r a^4 + \rho_{\phi} a^4) \equiv 4A_4 \left(r_4 + \frac{A_4}{a^2} \right), \quad (13)$$

where we have defined the constants A_4 and r_4 , and used the fact that the massless scalar kinetic energy $\rho_{\phi} \propto a^{-6}$. The reason for this choice of constants will become clear momentarily.

The solution to Eq. (13) and the massless scalar field equation $(a^2 \phi)' = 0$ is

$$a^2 = 4A_4 \tau (1 + r_4 \tau), \quad \phi = \sqrt{\frac{3}{2}} \ln \left(\frac{A_4 \tau}{(1 + r_4 \tau)} \right). \quad (14)$$

From these solutions, we reconstruct the scale factors on the branes according to Eqs. (10), obtaining

$$b_{\pm} = 1 \pm A_4 \tau + r_4 \tau, \quad (15)$$

so we see that with the choice of normalization for the scale factor a made in Eq. (13), the brane scale factors are unity at collision. For comparison, in Ref. [2] we parametrized the radiation density appearing in the four-dimensional effective theory using the Hubble constant H_r at equal density of the radiation and scalar kinetic energy, $H_r = (2r_4)^{3/2}/A_4^{1/2}$. Also, the parameter H_5 used there to describe the contraction rate of the fifth dimension may be expressed, for low radiation densities at collision, $\rho_{\pm} L^2 \ll 1$, and slow velocities as $2A_4$.

We may now directly compare the predictions (15) with the exact five-dimensional solution given in Eqs. (C23) of Appendix C, equating the terms linear in τ to obtain

$$A_4 = (1/L) \left(1 + \frac{L^2(r_+ - r_-)}{12} \right) \tanh(y_0/2),$$

$$r_4 = \frac{L(r_+ + r_-)}{12 \tanh(y_0/2)}, \quad (16)$$

where y_0 is the rapidity associated with the relative velocity of the branes at collision $V = \tanh(y_0)$ and r_{\pm} is the value of the radiation density ρ_{\pm} on each brane at collision. These formulas are the exact expressions for the four-dimensional parameters in terms of the five-dimensional parameters neglecting contributions of order ρ^2 . In fact, at leading order in τ they are better than this since to this order the four-dimensional prediction is exact.

For later purposes it will also be useful to define the fractional density mismatch on the two branes as

$$f = \frac{r_+ - r_-}{r_+ + r_-}, \quad (17)$$

so that we have

$$r_+ - r_- = \frac{12f r_4}{L} \tanh(y_0/2). \quad (18)$$

D. Four-dimensional perturbation equations

In this section we describe the perturbations of the brane-world system in terms of the four-dimensional effective theory. The only cases we consider in detail are where the branes are empty or carry radiation. The conformal invariance of radiation in four dimensions greatly simplifies matters since the scalar field then has no direct coupling to the radiation and hence the latter evolves as a free fluid in the four-dimensional effective theory. We elaborate on the significance of this conformal invariance in Sec. VI C.

We shall now describe the scalar perturbations, in longitudinal (conformal Newtonian) gauge with a spatially flat background where the scale factor and the scalar field are given by (14). The perturbed line element is

$$ds^2 = a^2(\tau) [-(1 + 2\Phi) d\tau^2 + (1 - 2\Psi) d\vec{x}^2]. \quad (19)$$

Since there are no anisotropic stresses in the linearized theory, we have $\Phi = \Psi$ (see, e.g. Ref. [43]).

A complete set of perturbation equations consists of the radiation fluid equations, the scalar field equation of motion and the Einstein momentum constraint:

$$\begin{aligned}\delta_r' &= -\frac{4}{3}(k^2 v_r - 3\Phi') \\ v_r' &= \frac{1}{4}\delta_r + \Phi \\ (\delta\phi)'' + 2\mathcal{H}(\delta\phi)' &= -k^2(\delta\phi) + 4\phi'\Phi' \\ \Phi' + \mathcal{H}\Phi &= \frac{2}{3}a^2\rho_r v_r + \frac{1}{2}\phi'(\delta\phi),\end{aligned}\quad (20)$$

where primes denote τ derivatives, δ_r is the fractional perturbation in the radiation density, v_r is the scalar potential for its velocity, i.e. $\vec{v}_r = \vec{\nabla}v_r$, $\delta\phi$ is the perturbation in the scalar field, and from Eq. (14) we have the background quantities $\mathcal{H} \equiv a'/a = (1 + 2r_4\tau)/[2\tau(1 + r_4\tau)]$, and $\sqrt{= 1/[\tau(1 + r_4\tau)]}$.

We are interested in solving these equations in the long wavelength limit, $|k\tau| \ll 1$. There are only two independent solutions to Eqs. (20), namely a growing and a decaying mode, provided that we specify that the perturbations are *adiabatic*. Recall that the idea of adiabaticity in the cosmological context is that for long wavelength perturbations, there should be nothing in the state of the matter to locally distinguish one region of the Universe from another. At each spatial location the evolution of the densities of all the different fluids (radiation, baryons, dark matter) share a single history in which each fluid evolves with the scale factor a according to $d\rho_i = -3(\rho_i + P_i)d\ln a = -3\rho_i(1 + w_i)d\ln a$ where ρ_i is its density, P_i is its pressure and w_i parametrizes the equation of state. Likewise the total density evolves as $d\rho = -3(\rho + P)d\ln a = -3\rho(1 + w)da$. Since the history is parametrized uniquely by the scale factor a , an adiabatic perturbation can be thought of as arising from a fluctuation $\delta\ln a$. Hence solving all the above equations for $\delta\ln a$, one finds

$$\frac{\delta_i}{(1 + w_i)} \approx \frac{\delta}{(1 + w)}, \quad i = 1, \dots, N, \quad (21)$$

for adiabatic perturbations.

For the case at hand, the components of the background energy density in the four-dimensional effective theory are scalar kinetic energy, with $w_\phi = 1$, and radiation, with $w_r = \frac{1}{3}$. It follows that for adiabatic perturbations, at long wavelengths we must have

$$\delta_\phi \approx \frac{3}{2}\delta_r. \quad (22)$$

In longitudinal gauge, the fractional energy density perturbation and the velocity potential perturbation in the scalar field (considered as a fluid with $w = 1$) are given by

$$\delta_\phi = 2\left(\frac{(\delta\phi)'}{\phi'} - \Phi\right), \quad v_\phi = \frac{\delta\phi}{\phi'}. \quad (23)$$

From Eqs. (20) above (and using $\phi' \propto a^{-2}$) it follows that

$$\left(\delta_\phi - \frac{3}{2}\delta_r\right)' = 2k^2\left(v_r - \frac{\delta\phi}{\phi'}\right). \quad (24)$$

Maintaining the adiabaticity condition (22) up to order $(k\tau)^2$ then requires that the fractional velocity perturbations for the scalar field and the radiation should be equal: $v_r \approx \delta\phi/\phi'$. Expressing the radiation velocity in terms of $\delta\phi$, the momentum constraint [the last equation in (20)] then yields

$$\delta\phi \approx \left(1 + \frac{2}{3}\frac{\rho_r}{\rho_\phi}\right)^{-1}\left(\frac{2(\Phi' + \mathcal{H}\Phi)}{\phi'}\right), \quad (25)$$

where $\rho_\phi = \frac{1}{2}\phi'^2 a^{-2}$.

The above equations may be used to determine the leading terms in an expansion in $|k\tau|$ of all the quantities of interest about the singularity. In order to compare with Ref. [6], we shall choose to parametrize the expansions in terms of the parameters describing the comoving energy density perturbation, $\epsilon_m = -\frac{2}{3}\mathcal{H}^{-2}k^2\Phi$, which has the following series expansion about $\tau = 0$:

$$\epsilon_m = \epsilon_0 D(\tau) + \epsilon_2 E(\tau), \quad (26)$$

where ϵ_0 and ϵ_2 are arbitrary constants, and

$$\begin{aligned}D(\tau) &= 1 - 2r_4\tau - \frac{1}{2}k^2\tau^2\ln|k\tau| + \dots, \\ E(\tau) &= \tau^2 + \dots\end{aligned}\quad (27)$$

For adiabatic perturbations, we obtain

$$\begin{aligned}\delta_\phi &= \epsilon_0\left(-\frac{9}{4k^2\tau^2} - \frac{3}{8}\ln|k\tau| + \frac{1}{4} - \frac{3}{4}\frac{r_4^2}{k^2}\right) + \epsilon_2\frac{3}{4k^2} \\ &\quad + O(\tau, \tau\ln|k\tau|)\end{aligned}$$

$$v_\phi = \epsilon_0\left(\frac{3}{4k^2\tau}(1 - r_4\tau)\right) + O(\tau, \tau\ln|k\tau|),$$

$$\delta_r = \frac{2}{3}\delta_\phi + O(\tau^2, \tau^2\ln|k\tau|),$$

$$v_r = v_\phi + O(\tau, \tau\ln|k\tau|),$$

$$\begin{aligned}\Phi &= \epsilon_0\left(-\frac{3}{8k^2\tau^2} + \frac{3}{16}\ln|k\tau| + \frac{15}{8}\frac{r_4^2}{k^2}\right) - \epsilon_2\frac{3}{8k^2} \\ &\quad + O(\tau, \tau\ln|k\tau|),\end{aligned}$$

$$\begin{aligned}
 \frac{(\delta\phi)}{\sqrt{6}} &= \epsilon_0 \left(\frac{3}{8k^2\tau^2} (1 - 2r_4\tau) + \frac{1}{16} \ln|k\tau| + \frac{1}{8} + \frac{13}{8} \frac{r_4^2}{k^2} \right) \\
 &\quad - \epsilon_2 \frac{1}{8k^2} + O(\tau, \tau \ln|k\tau|), \\
 \zeta_{4,M} &= -\frac{1}{2k^2} \epsilon_2 + \epsilon_0 \left(\frac{1}{8k^2} (k^2 + 16r_4^2) + \frac{1}{4} \ln|k\tau| \right) \\
 &\quad + O(\tau, \tau \ln|k\tau|), \tag{28}
 \end{aligned}$$

where $\zeta_{4,M}$ is the spatial curvature perturbation on comoving slices [43].

In an expanding Universe the adiabatic growing mode corresponds to a curvature perturbation, conveniently parametrized by $\zeta_{4,M}$. The decaying mode perturbation is really a local time delay since the big bang, to which $\zeta_{4,M}$ is insensitive but Φ is not. As detailed in Ref. [6], in a contracting Universe these modes switch roles so that the time delay mode is the growing perturbation and the curvature perturbation is the decaying perturbation as one approaches the big crunch.

The perturbations generated in the ekpyrotic/cyclic scenarios consist of growing mode scale-invariant perturbations in the incoming state with no decaying mode component. These perturbations are parametrized by ϵ_0/k^2 having a scale invariant spectrum, and since there is no decaying mode, $\zeta_{4,M}$ is zero on long wavelengths. After the collision, from the four-dimensional effective theory view the Universe is expanding. Now, the growing mode perturbation is proportional to the long wavelength part of $\zeta_{4,M}$. The key question is whether with our five-dimensional prescription matches the growing mode in the incoming state onto the growing mode in the outgoing state, parametrized by $\zeta_{4,M}$, with non-zero amplitude. For this to occur, the long wavelength piece of $\zeta_{4,M}$ must jump across the bounce. We shall see below that this indeed occurs.

IV. PROPAGATION OF GRAVITATIONAL PERTURBATIONS IN A COLLISION OF TENSIONLESS BRANES

In this section we consider the propagation of metric perturbations through a collision of tensionless branes where the background space-time is precisely $\mathcal{M}^C/Z_2 \times R^3$. The analysis follows closely Sec. II, which considered the propagation of generic scalar fields in this same background. The results here are essential to our analysis for the physically relevant case of colliding branes with tension (Sec. V) since our approach is based on finding a gauge where the propagation of metric perturbations through the bounce is as close as possible to the case for fixed tensionless branes.

In this problem it is simplest to choose coordinates in which the branes remain at fixed locations and all the fluctuations in the geometry are accounted for by the bulk metric perturbations. Recall that, ignoring gravity, the background metric is

$$ds^2 = -dt^2 + t^2 dy^2 + d\vec{x}^2, \tag{29}$$

but with y identified under translations $y \rightarrow y + 2y_0$, and the reflection $y \rightarrow 2y_0 - y$. The orbifold fixed points located at $y = \pm y_0/2$ are the trajectories of two tensionless orbifold branes. In Sec. II we considered matching a scalar field across the singularity in this space-time [7] and now we generalize the methods considered there to the case of gravitational waves.

A gravitational wave in five dimensions has five independent propagating components. If the y dependence may be ignored these five components split up in synchronous gauge into tensor (δg_{ij}), vector (δg_{iy}) and scalar (δg_{yy}) components, possessing two, two and one propagating degree of freedom, respectively. As usual in four-dimensional cosmological perturbation theory the most interesting piece is the scalar as this transforms nontrivially under coordinate transformations and couples to the matter density perturbations. The tensor pieces are especially simple since they are trivially gauge invariant and decouple from the matter. Finally, the vector pieces only couple to the curl component of the matter velocities and not to the matter density perturbation. They require a separate analysis which will not be given here. Furthermore, in our setup the vector modes are naturally projected out because δg_{iy} must be odd under the Z_2 . Hence the vector modes must vanish on the branes, and this is why there are no vector degrees of freedom in the four-dimensional effective theory.

We shall, therefore, need only to consider the scalar sector in what follows. The form we take for the five-dimensional cosmological background metric is

$$ds^2 = n^2(t, y) (-dt^2 + t^2 dy^2) + b^2(t, y) \delta_{ij} dx^i dx^j, \tag{30}$$

and we write the most general scalar metric perturbation about this as

$$\begin{aligned}
 ds^2 &= n^2(t, y) [- (1 + 2\Phi) dt^2 - 2W dt dy + t^2 (1 - 2\Gamma) dy^2 \\
 &\quad - 2\nabla_i \alpha dx^i dt + 2t^2 \nabla_i \beta dy dx^i] \\
 &\quad + b^2(t, y) [(1 - 2\Psi) \delta_{ij} - 2\nabla_i \nabla_j \chi] dx^i dx^j. \tag{31}
 \end{aligned}$$

For perturbations on $\mathcal{M}^C \times R^3$ it is straightforward to find a gauge in which the metric takes the form

$$\begin{aligned}
 ds^2 &= \left(1 + \frac{4}{3} k^2 \chi \right) (-dt^2 + t^2 dy^2) \\
 &\quad + \left[\left(1 - \frac{2}{3} k^2 \chi \right) \delta_{ij} + 2k_i k_j \chi \right] dx^i dx^j, \tag{32}
 \end{aligned}$$

and χ satisfies a massless scalar equation of motion on $\mathcal{M}^C \times R^3$. To be precise, the gauge is

$$\alpha = \beta = 0, \quad \Gamma = \Phi - \Psi - k^2 \chi,$$

$$\Phi = \frac{2}{3} k^2 \chi, \quad \Psi = \frac{1}{3} k^2 \chi,$$

$$W = 0. \tag{33}$$

Notice that the non-zero variables can all be related to χ according to

$$(\Gamma, \Phi, \Psi) = \left(-\frac{2}{3}, +\frac{2}{3}, +\frac{1}{3} \right) k^2 \chi. \quad (34)$$

We shall, henceforth, refer to these as the ‘‘Milne ratio conditions.’’ Furthermore, imposing the Z_2 symmetry, we obtain Neumann boundary conditions on χ ,

$$\chi'(y_{\pm}) = 0, \quad (35)$$

where $y_{\pm} = \pm y_0/2$ are the locations of the two Z_2 fixed points.

In the model space-time, the lowest energy mode for χ is y independent and has the asymptotic form

$$\chi(t, y) = Q + P \ln|kt|, \quad (36)$$

with Q and P being arbitrary constants, just like the case of scalar fields in Sec. II. Our matching proposal for all the perturbation modes is then simply the analogue of the scalar field rule given in Sec. II, namely,

$$Q_{out} = -Q_{in} + 2(\gamma - \ln 2)P_{in}, \quad P_{out} = P_{in}. \quad (37)$$

These relations are sufficient to determine the metric fluctuations after the bounce.

In later applications, we are only interested in the long-wavelength part of the spectrum, and, for the cases of interest, P is suppressed by k^2 compared to Q . As a result, we obtain the approximate matching rule

$$Q_{out} = -Q_{in}, \quad P_{out} = P_{in}. \quad (38)$$

The key conditions (33) through (35) are satisfied precisely for all time in a compactified Milne mod Z_2 background. When tension is added to the brane and the bulk is warped, our approach is to find a gauge which takes us as close as possible to these conditions in the limit as t tends to zero, where the same matching rule may then be applied.

V. 5D COSMOLOGICAL PERTURBATIONS FOR BRANES WITH TENSION IN A WARPED BACKGROUND

Our strategy for computing propagation of perturbations when the branes are dynamical and have tension (so the bulk is warped) is conceptually simple:

(1) We use the four-dimensional effective (moduli) theory described in Sec. III to provide boundary data for the five-dimensional bulk fields. In particular, we will be interested in the case where a nearly scale-invariant perturbations has been generated well before the bounce when the four-dimensional effective theory is an excellent approximation, as occurs in ekpyrotic and cyclic models.

(2) In the five-dimensional theory, we find a gauge which approaches the Milne conditions (33) through (35) as $t \rightarrow 0$. In the gauge, the perturbation variables satisfy the massless scalar field equations of motion.

(3) We use the conditions in (38) to propagate all perturbation variables through the collision.

(4) We match onto the four-dimensional (moduli) theory to determine the cosmological results for long wavelength perturbations.

One might worry that the four-dimensional effective theory we use to predict the boundary data for five-dimensional general relativity breaks down close to the bounce. However, there are reasons to expect the effective theory remains accurate as an approximation to general relativity even at small times. First, in Kaluza-Klein theory, the effective four-dimensional theory is a consistent truncation and hence provides exact solutions of the five-dimensional theory even in situations of strong curvature and anisotropy. In our case, as the branes come close, the warp factor should become irrelevant so that the Kaluza-Klein picture should become more and more valid. Second, in the approach to the singularity in general relativity [44] (based on the classic BKL work [21]), the decomposition of fields according to dimensional reduction does correctly predict the asymptotics of the solutions in the limit as $t \rightarrow 0$. This suggests that the effective field theory indeed captures the correct behavior of full five-dimensional gravity near the singularity. In our detailed study of the linearized theory, we shall find a remarkable consistency between the predictions of the four-dimensional effective theory near $t=0$ and the full five-dimensional cosmological perturbation equations, and these consistency checks are the main justification for our use of the effective theory all the way to the brane collision. Of course, the use of five-dimensional general relativity near the singularity may itself be doubted since stringy corrections may be large there. But this objection can only be addressed in a detailed calculation within a string or M-theory context, which is beyond the scope of the present paper.

We first infer the boundary geometry in longitudinal gauge (Sec. V A) for which there is a simple and precise correspondence between the four- and five-dimensional perturbations and both are completely gauge fixed (see also Appendix B). However, in this gauge the metric perturbations diverge much more rapidly (as $1/t^2$) than a massless scalar near $t=0$. We shall need to transform to a gauge where (a) all the components of the metric are only logarithmically divergent and (b) in which the components of the metric are in the same ratios and obey the same boundary conditions asymptotically as $t \rightarrow 0$, as for the perturbed model space-time with two tensionless branes in $\mathcal{M}^C/Z_2 \times R^3$ (Sec. V C). In this gauge we can treat the components as massless fields and match across the singularity as in Sec. II (Sec. V D).

We wish to emphasize that the choice of gauge we are making is fully five-dimensional and is quite unlike that usually made in four-dimensional cosmology for several reasons. In four-dimensional cosmology, the matter present is often used to define a gauge—for example, one may choose gauges in which the total density or velocity perturbation is zero. However, in the five-dimensional bulk there is never any matter present, just the cosmological term which is constant and, therefore, does not define any preferred time slicing. One might choose surfaces of constant extrinsic curva-

ture, but these are not in any way preferred by the physics involved. Instead, our approach focuses on the asymptotic geometry near $t=0$, and identifying it with the model space-time $\mathcal{M}/Z_2 \times R^3$. In addition to approximating the model space-time, it is essential that, for the same gauge choice, the brane collision be *simultaneous* at all \vec{x} , so that the $t=0^-$ and $t=0^+$ surfaces physically coincide. We shall show that our gauge choice satisfies this latter criterion, but the standard four-dimensional gauge choices, e.g., constant density or velocity gauges, do not.

A. Longitudinal gauge moduli predictions

In this section we wish to use the four-dimensional effective (moduli) theory discussed in Sec. III to infer the boundary data for the five-dimensional bulk perturbations. In any four-dimensional gauge, the four-dimensional metric perturbation $h_{\mu\nu}$ and scalar field perturbation $\delta\phi$ determine the induced metric perturbations on the branes (in a related but not equivalent gauge) via the formulas (10):

$$h_{\mu\nu}^{\pm} = h_{\mu\nu} + 2(\ln \Omega_{\pm})_{,\phi} \delta\phi g_{\mu\nu}, \quad (39)$$

where $\Omega_+ = \cosh(\phi/\sqrt{6})$ and $\Omega_- = -\sinh(\phi/\sqrt{6})$ and the metric perturbations are fractional i.e. $\delta g_{\mu\nu} = a^2 h_{\mu\nu}$, $\delta g_{\mu\nu}^{\pm} = b_{\pm}^2 h_{\mu\nu}^{\pm}$.

This formula is particularly easy to use in five-dimensional longitudinal gauge. (Our definition follows that of Ref. [45], where many useful formulas are given.) This gauge may always be chosen, and it is completely gauge fixed as we explain in Appendix B. In this gauge the five-dimensional metric takes the form

$$ds^2 = n^2(t,y) [-(1+2\Phi_L)dt^2 - 2W_L dt dy + t^2(1-2\Gamma_L)dy^2] + b^2(t,y) [(1-2\Psi_L)\delta_{ij}] dx^i dx^j. \quad (40)$$

Furthermore, as explained in Appendix B, in the absence of anisotropic stresses the brane trajectories are unperturbed in this gauge. An immediate consequence is that the four-dimensional longitudinal gauge scalar perturbation variables Φ_{\pm} and Ψ_{\pm} , describing perturbations of the *induced* geometry on each brane

$$ds_{\pm}^2 = b_{\pm}^2(\tau_{\pm}) [-(1+2\Phi_{\pm})d\tau_{\pm}^2 + (1-2\Psi_{\pm})d\vec{x}^2], \quad (41)$$

are precisely the boundary values of the five-dimensional longitudinal gauge perturbations $\Phi_{\pm} \equiv \Phi_L(y_{\pm})$ and $\Psi_{\pm} \equiv \Psi_L(y_{\pm})$. Using Eqs. (39) and (41) we find for the induced perturbations

$$\Phi_+ = \Phi_4 + \frac{1}{\sqrt{6}} \tanh(\phi/\sqrt{6}) \delta\phi,$$

$$\Psi_+ = \Phi_4 - \frac{1}{\sqrt{6}} \tanh(\phi/\sqrt{6}) \delta\phi,$$

$$\Phi_- = \Phi_4 + \frac{1}{\sqrt{6}} \coth(\phi/\sqrt{6}) \delta\phi,$$

$$\Psi_- = \Phi_4 - \frac{1}{\sqrt{6}} \coth(\phi/\sqrt{6}) \delta\phi. \quad (42)$$

One subtlety in utilizing these formulas is that if Φ_4 and $\delta\phi$ are expressed as functions of four-dimensional conformal time, then they give the correct predictions for Φ_{\pm} and Ψ_{\pm} on the branes in terms of the conformal time τ_{\pm} on each brane. However, when we use them as boundary values of the five-dimensional metric it will be necessary to consider all the perturbation variables as functions of the five-dimensional time t entering in the background metric (30). The brane conformal times may be expressed in terms of t by integrating

$$\tau_{\pm} = \int_0^t \frac{dt}{q(t, y_{\pm})}, \quad (43)$$

where $q \equiv b/n$. So, for example, the boundary value of the bulk metric perturbation Φ_L on the positive tension brane is given explicitly by

$$\begin{aligned} \Phi_L(t, y_+) &= \Phi_4 \left(\int q(t, y_+)^{-1} dt \right) \\ &+ \frac{1}{\sqrt{6}} \tanh \left[\phi \left(\int q(t, y_+)^{-1} dt \right) / \sqrt{6} \right] \\ &\times \delta\phi \left(\int q(t, y_+)^{-1} dt \right), \end{aligned} \quad (44)$$

where y_+ is the location of the positive tension brane. As noted, in this gauge even when we include perturbations the branes are static and the Israel matching conditions are easily found to be

$$\begin{aligned} \frac{b'}{b}(y_{\pm}) &= \pm \frac{L}{6} n t \rho_{\pm}, \\ \frac{q'}{q}(y_{\pm}) &= \pm \frac{L}{2} n t (p_{\pm} + \rho_{\pm}) \end{aligned} \quad (45)$$

for the background solution and

$$\begin{aligned} \Psi'_L(y_{\pm}) &= \frac{\dot{b}}{b} W_L \mp \frac{L}{6} n t (\delta\rho_L^{\pm} - \Gamma_L \rho_{\pm}), \\ \Phi'_L(y_{\pm}) &= - \left(\frac{\dot{n}}{n} + \frac{\partial}{\partial t} \right) W_L \mp \frac{L}{3} n t (\delta\rho_L^{\pm} - \Gamma_L \rho_{\pm}) \\ &\quad \mp \frac{L}{2} n t (\delta p_L^{\pm} - \Gamma_L p_{\pm}), \\ W_L(y_{\pm}) &= \pm \frac{b^2 L t}{n} (p_{\pm} + \rho_{\pm}) v_L^{\pm} \end{aligned} \quad (46)$$

for the perturbations, where the right-hand sides are all evaluated at $y=y_{\pm}$, the locations of the positive and negative tension branes. (From now on a prime shall denote $\partial/\partial y$ and a dot shall denote $\partial/\partial t$.) We can re-express W_L on the branes as

$$W_L(y_{\pm}) = (q^2)' v_L^{\pm}, \quad (47)$$

where v_L^{\pm} is the longitudinal gauge velocity perturbation of the matter on each brane. From this one sees, for example, that for empty branes, W_L vanishes on the branes.

As long as the bulk matter is isotropic, as it is in our case, the Einstein equations lead to a constraint which may be written

$$G^1_1 - G^2_2 = 0. \quad (48)$$

In longitudinal gauge this reads [45]

$$\Gamma_L = \Phi_L - \Psi_L \quad (49)$$

everywhere in the bulk. This is the five-dimensional analogue of the well known four-dimensional no-shear condition $\Phi = \Psi$ in longitudinal gauge. Equation (49) serves to define Γ_L on the branes in longitudinal gauge. Consequently we have sufficient boundary data for all the components of the five-dimensional metric in this gauge. We can then perform an arbitrary five-dimensional diffeomorphism to infer the boundary data in any gauge we choose. Equivalently, Eq. (49) may be interpreted as a condition in any gauge by using the gauge invariant variables defined in Appendix B.

B. Stress energy conservation

In this paper we consider perturbations in the ‘‘in’’ state which may be described as local fluctuations in a single scalar field ϕ representing the inter-brane separation. We are interested in long wavelength modes which are completely frozen-in during the collision event. Hence the local processes describing the production of radiation at the bounce should be identical at each \vec{x} , and in the usual sense employed in cosmology, described in Sec. III D, the perturbations should be ‘‘adiabatic.’’

As is well known, the conservation of stress energy leads to powerful constraints on adiabatic density perturbations, in particular implying that the amplitude of the growing mode perturbation cannot be altered on super-horizon scales. In this section we discuss this constraint and show how it implies the spatial curvature of comoving (or constant energy density) slices is conserved on large scales both for the brane geometries and for the four-dimensional effective theory. We shall restrict ourselves to considering only radiation on each brane. This considerably simplifies the analysis because when the matter on each brane is conformally invariant, as explained above, in the four-dimensional effective theory the scalar field decouples from the matter and can be treated as an independent fluid.

First we need to generalize the usual notion of adiabaticity to deal with perturbations in the radiation densities on each brane. As mentioned above, radiation couples to the

scale factor $\Omega_+(\phi)a$ on the positive tension brane and $\Omega_-(\phi)a$ on the negative tension brane with notation as in the previous section. Conservation of the radiation density on each brane reduces at long wavelengths to $d\rho_{\pm} = -4\rho_{\pm}d \ln a - 4\rho_{\pm}d \ln \Omega_{\pm}$. Likewise we have for the radiation density in the four-dimensional effective theory $d\rho_4 = -4\rho_4d \ln a$. Hence solving for $\delta \ln a$ as in Sec. III D, we infer the adiabaticity condition for radiation on the branes to be

$$\delta_{\pm} = \delta_4 - 4(\ln \Omega_{\pm})_{,\phi} \delta\phi. \quad (50)$$

The equation for conservation of energy in four dimensions can be written in the form [45]

$$\dot{\zeta}_B = \frac{1}{3}k^2 v_L, \quad (51)$$

where

$$\zeta_B = \Psi - \frac{1}{3(1+w)} \delta \quad (52)$$

is the gauge-invariant variable measuring the spatial curvature perturbation on constant density hypersurfaces, as originally defined by Bardeen [46]. The quantity v_L is the gauge invariant scalar velocity potential, equal to the velocity potential in longitudinal gauge (so that $\nabla_i v_L$ is the scalar part of the velocity perturbation).

At long wavelengths $k \rightarrow 0$, Eq. (51) implies that ζ_B is conserved, provided the velocity perturbation does not grow with scale. This property is very powerful since it means that under most circumstances, as long as modes remain outside the horizon ζ_B can be trivially extrapolated from the early to the late Universe, where it gives the amplitude of the growing mode adiabatic density perturbation, the main quantity of observational interest today.

The above definition (52) applies equally on each brane and in the four-dimensional effective theory, provided the terms on the right hand side are appropriately interpreted. On the branes, we have

$$\zeta_{B,\pm} = \Psi_{\pm} - \frac{1}{4} \delta_{\pm}, \quad (53)$$

where δ_{\pm} are the fractional perturbations in the radiation densities on each brane, and Ψ_{\pm} is the perturbation in the brane spatial metric. Using (42), written as

$$\Psi_{\pm} = \Psi_4 - (\ln \Omega_{\pm})_{,\phi} \delta\phi, \quad (54)$$

and the adiabaticity condition (50) we see that the four-dimensional effective value of Bardeen’s variable, $\zeta_{B,4} \equiv \Phi_4 - \frac{1}{4} \delta_4$ is in fact identical to $\zeta_{B,\pm}$ on long wavelengths.

Our final result will in fact more naturally emerge in terms of another gauge invariant variable, the curvature perturbation on comoving slices, emphasized by Mukhanov and others [43]. This is defined as

$$\zeta_M = \Psi + \mathcal{H}v, \quad (55)$$

with v the velocity potential and $\mathcal{H} \equiv d \ln a(\tau)/d\tau$ the conformal Hubble constant. Again this may be interpreted on either brane or in the four-dimensional effective theory. But adiabaticity requires that the fluid velocities be identical on long wavelengths for each fluid component. Therefore we must have $v_{\pm} = v_4 = \delta\phi/\phi'$ [from (23)]. This is also seen to be consistent with (51) and the equality of the Bardeen variables $\zeta_{B,\pm} = \zeta_{B,4}$ which we have just shown.

The scale factors on each brane are related to the four-dimensional effective scale factor via $b_{\pm} = \Omega_{\pm} a$. Recalling that the conformal times on the branes are the same as that in the effective theory, we have $\mathcal{H}_4 = \mathcal{H}_{\pm} - (\ln \Omega_{\pm})_{,\phi} \phi_{,\tau}$. Using $v_4 = v_{\pm} = v_{\phi} = \delta\phi/(\phi_{,\tau})$ we find

$$\begin{aligned} \zeta_{M,4} &\equiv \Psi_4 + \mathcal{H}_4 v_4 = \Psi_4 + \mathcal{H}_{\pm} v_{\pm} - (\ln \Omega_{\pm})_{,\phi} \delta\phi \\ &= \Psi_{\pm} + \mathcal{H}_{\pm} v_{\phi}, \end{aligned} \quad (56)$$

which is just $\zeta_{M,\pm}$. So for adiabatic perturbations and at long wavelengths, the comoving curvature perturbations on the branes are both equal to that in the four-dimensional effective theory. As is well known, the latter is conserved for adiabatic perturbations at long wavelengths. It follows that away from the bounce, $\zeta_{M,\pm}$ are both conserved as well. As we discussed in the Introduction, and will detail below, this does *not* imply they are conserved across the bounce.

We will use (56) below, but we should point out one minor subtlety. We shall be performing all our calculations in five-dimensional time t , not four-dimensional conformal time. The velocity v_{ϕ} is not a scalar under coordinate transformations, and we shall need to multiply v_{ϕ} by a factor of q when we re-interpret Eq. (56) in terms of the five-dimensional time t .

C. Transformation to Milne gauge

Our philosophy is to evolve cosmological perturbations through the bounce in a ‘‘Milne gauge’’ where they behave as closely as possible to gravitational waves on $\mathcal{M}^C/Z_2 \times R^3$, as described in Sec. IV. Then, we can use the same matching conditions (38) to determine the perturbation spectrum after the bounce.

The Milne gauge we use is chosen to match the gauge choice (33) in Sec. IV up to corrections of order t and $t \ln|kt|$ due to the finite brane tension, radiation densities and the warp factor. We still have enough coordinate freedom to set three linear combinations of the metric perturbations equal to zero for all t , and we choose

$$\alpha = \beta = 0, \quad \Gamma = \Phi - \Psi - k^2 \chi. \quad (57)$$

A remarkable feature of this choice is that the constraint equation (49) implies that χ obeys the equation for a massless scalar field on the unperturbed background for all times. From Eqs. (57) and (B5) in Appendix B, we find

$$\begin{aligned} \nabla^2 \chi &= -\frac{1}{t} \frac{\partial}{\partial t} \left(t \frac{\partial \chi}{\partial t} \right) + \frac{1}{t^2} \frac{\partial^2 \chi}{\partial y^2} - 3 \frac{\dot{b}}{b} \frac{\partial \chi}{\partial t} \\ &+ \frac{3}{t^2} \frac{b'}{b} \frac{\partial \chi}{\partial y} - \frac{k^2 b^2}{n^2} \chi = 0. \end{aligned} \quad (58)$$

This result is remarkable in that it is independent of the precise details of the background bulk geometry and the form of the stress energy in the bulk, assuming only that no anisotropic stresses are present.

The remaining gauge freedom is of the form $x^{\mu} \rightarrow x^{\mu} + \xi^{\mu}$ where

$$\xi^t = \frac{b^2}{n^2} \xi^s, \quad \xi^y = -\frac{b^2}{n^2 t^2} \xi'^s, \quad (59)$$

provided that ξ^s also satisfies a massless scalar field equation

$$\nabla^2 \xi^s = 0. \quad (60)$$

Since χ transforms as $\chi \rightarrow \chi + \xi^s$, and χ is zero in longitudinal gauge, it follows that χ in the gauge we use is, in fact, precisely the value of the spatial coordinate transformation ξ^s needed to get to a gauge satisfying (57) from five-dimensional longitudinal gauge. Furthermore, ξ^t and ξ^y may be inferred from $\chi = \xi^s$ via Eqs. (59).

To completely fix the gauge within the family specified by (57), we need to specify boundary conditions for the field χ on the two branes, and initial conditions on some space-like surface. As a first guess, one might consider choosing to fix the gauge by specifying Neumann boundary conditions on the branes [i.e. $\chi'(t, y_{\pm}) = 0$] for all time, as in Sec. IV. One can easily prove that in this gauge, as in the longitudinal gauge, the brane trajectories are unperturbed. This follows from the formula (59) upon setting $\xi^s = \chi$ as noted above. This is very important: it follows that in this Neumann gauge the brane collision is simultaneous and occurs at precisely $t = 0$ for all \vec{x} . Furthermore, the Neumann gauge $\chi'(t, y_{\pm}) = 0$ for all t is a good gauge in the sense that none of the metric components diverge worse than logarithmically.

However, it turns out that setting $\chi'(t, y_{\pm}) = 0$ for all time is too strong a condition. One cannot choose Neumann gauge for all time and also have

$$W = 0 + \mathcal{O}(t, t \ln|kt|) \quad (61)$$

$$\Phi = \frac{2}{3} k^2 \chi + \mathcal{O}(t, t \ln|kt|) \quad (62)$$

$$\Psi = \frac{1}{3} k^2 \chi + \mathcal{O}(t, t \ln|kt|), \quad (63)$$

consistent with the behavior in the model space-time (33) at leading order in t and $t \ln|kt|$. The resolution is simple: we need to perform a small gauge transformation away from Neumann gauge in which we maintain only the asymptotic vanishing of the proper normal derivative of χ as t tends to zero, i.e. we impose that

$$n^{-1}t^{-1}\chi'(y_{\pm})=0+\mathcal{O}(t,t\ln|kt|) \quad (64)$$

on the two branes. With this choice we are able to impose all of the conditions in (63) as well as (57). This small gauge transformation away from Neumann gauge shifts the locations of the branes, y_{\pm} , but only by a finite amount. As discussed in Appendix D, this means that the rapidities of the branes are perturbed in our chosen gauge, but the collision event is still simultaneous.

Our reason for expecting that we can choose a gauge specified by Eqs. (63) and (64) is that when the branes approach the warp factor should become increasingly irrelevant and the real background space-time should asymptotically approach the model space-time $\mathcal{M}/Z_2 \times R^3$. We expect the low energy modes we are interested in to behave as the lowest Kaluza-Klein modes in this limit, i.e. becoming independent of y . Within the class of gauges specified by Eq. (57) we shall indeed see that there are solutions for the perturbations in which all the perturbation components behave like $Q + P \ln|kt|$ as t tends to zero. The Milne ratio condition (34) turns out to be automatically satisfied by the coefficients of the logarithms. Fixing the constant terms to be in the Milne ratios further fixes the gauge up to a residual two-parameter family and imposing asymptotically Neumann boundary conditions (64) on both branes then completely fixes the gauge.

Imposing asymptotically Neumann boundary conditions turns out to have various other natural consequences. For example, in this gauge all the metric perturbation components possess identical asymptotic behavior (i.e. constant and logarithmic terms) on the two branes, as t tends to zero, consistent with their behavior as a lowest Kaluza-Klein mode. Furthermore, there is a simple geometrical consequence of this choice which we explain in Appendix D, namely that in this gauge the perturbations to the embedding (T, Y) coordinates of the brane collision event actually vanish so the branes collide at precisely the background values of T and Y .

The non-zero perturbations in our chosen class of gauges are Φ , Ψ , W and χ along with Γ which is fixed by the gauge choice (57). All the gauge freedom is contained in the solution for χ . To see this we note that if we know the solutions for χ we can immediately infer Φ , Ψ and W from the values in longitudinal gauge via the formulas from Appendix B, Eqs. (B5), which with Eq. (57) imply

$$\begin{aligned} \Phi &= \Phi_L - (q^2 \dot{\chi}) - \frac{\dot{n}}{n} q^2 \dot{\chi} + \frac{n'}{n} \left(\frac{q^2 \chi'}{t^2} \right), \\ W &= W_L - (q^2 \dot{\chi})' - t^2 \left(\frac{q^2 \chi'}{t^2} \right), \\ \Psi &= \Psi_L + \frac{\dot{b}}{b} q^2 \dot{\chi} - \frac{b'}{b} \frac{q^2 \chi'}{t^2}. \end{aligned} \quad (65)$$

Here as above, $q \equiv b/n$.

Our goal then is simply to determine χ to sufficient order in t to be able to compute all the other components from Eqs.

(65). As we have already explained, in our chosen class of gauges χ satisfies the massless scalar equation (58) at all times. To specify a complete solution we need to specify both Cauchy data on some constant t hypersurface between the two-brane world sheets, plus boundary conditions on the two branes. The boundary data will be obtained from the four-dimensional effective theory, and we make the conjecture that the bulk solution which is consistent with these data will behave near $t=0$ like a Kaluza-Klein zero mode on $\mathcal{M}^C/Z_2 \times R^3$, which is to say that the perturbations should be independent of y as t tends to zero. In practice this means we will look for a solution which is asymptotically of the form $\chi = Q + P \ln|kt|$, independent of y . This assumption formally provides the Cauchy data once we determine Q and P (see below).

At higher orders in t , we shall allow for arbitrary Neumann boundary conditions, which we shall parameterize as

$$\chi'(y_{\pm}) = \frac{1}{2} a_2^{\pm} t^2 + \mathcal{O}(t^3, t^3 \ln|kt|). \quad (66)$$

As explained above, we shall adjust the coefficients a_2^{\pm} to obtain the correct Milne ratios. Note that there can be no $O(1)$ term since we are assuming that χ is asymptotically of the form $\chi = Q + P \ln t$, independent of y , and the $O(t)$ term is prohibited by our condition (64). In principle we could also include $t \ln t$ and $t^2 \ln t$ terms but we shall find that the Milne ratio conditions (34) are sufficient to rule these terms out.

The form of the series expansion for χ , implied by its equation of motion (58), is

$$\begin{aligned} \chi(t, y) &= \{Q + [f_1(y) + c_1 \cosh y + c_2 \sinh y]t \\ &\quad + f_2(y)t^2/2 + \mathcal{O}(t^3)\} \\ &\quad + P \ln|kt| \left(1 - \frac{1}{4} k^2 t^2 + \mathcal{O}(t^3) \right), \end{aligned} \quad (67)$$

where $f_1(y)$ and $f_2(y)$ are two functions of y that are obtained as solutions of second order differential equations in y with boundary conditions derived from (66). We choose to define f_1 so that $f_1'(y_{\pm}) = 0$. Therefore if χ satisfies the asymptotically Neumann condition (64) on both branes, we must have $c_1 = c_2 = 0$. A geometrical interpretation of this condition is explained in Appendix D.

Using the expressions for $b(t, y)$ given in Appendix C, Eq. (C23), in the equation of motion (58), for χ , we find at order t^{-1} the following differential equation must be satisfied by $f_1(y)$:

$$\begin{aligned} f_1'' - f_1 - \frac{P}{2L \sinh y_0} \left[(6 + r + L^2) \cosh \left(y + \frac{y_0}{2} \right) \right. \\ \left. - (6 - r - L^2) \cosh \left(y - \frac{y_0}{2} \right) \right] = 0. \end{aligned} \quad (68)$$

A similar equation for f_2 is found at order t^0 . The solutions are messy in general but simpler when no radiation is present; for example, in this case we have

$$f_1(y) = \frac{3P}{2L \cosh(y_0/2)} \left[y \cosh y - \left(1 + \frac{y_0}{2} \tanh \frac{y_0}{2} \right) \sinh y \right]. \quad (69)$$

By substituting (67) into (58) and imposing the boundary conditions (66) at each order, the solution for χ up to t^3 corrections is completely determined in terms of the four constants in total: Q , P , and a_2^\pm . From this solution for χ , Eq. (65) then determines all the other components of the metric perturbations at leading order in t , on each brane.

Let us start by determining the spatial curvature perturbation Ψ on each brane. From Eqs. (56) and (65) we find

$$\Psi = \zeta_{4,M} + q \frac{\dot{b}}{b} (q\dot{\chi} - v_\phi) - \frac{b'}{b} \frac{q^2 \chi'}{t^2}. \quad (70)$$

We require that Ψ be only logarithmically divergent. Since from Eqs. (28) we have that $v_\phi = 3\epsilon_0/(4k^2\tau) + O(1)$, diverging as t^{-1} as $t \rightarrow 0$, we see from Eq. (67) and the expressions for the background metric functions in Appendix C that only $\dot{\chi}$ can cancel that divergence, which requires that

$$P = \frac{3\epsilon_0}{4k^2}. \quad (71)$$

This condition ensures that the curvature Ψ in our gauge and the comoving curvature $\zeta_{4,M}$ in the four-dimensional effective theory only differ by a constant at leading order in t . However, it shall be very important that the constant is non-zero. As we shall see, the constant represents the time delay between the two time slicings, and it is the key to why $\zeta_{4,M}$ jumps across the singularity.

We shall now show that it is possible to choose the three remaining gauge constants Q and a_2^\pm so that the metric takes the canonical Milne gauge form asymptotically as t tends to zero. First, in this gauge all the metric perturbations behave as $Q + P \ln|kt|$, as t tends to zero, but with different constants Q and P for each component. Substituting Eqs. (67) and (71) into Eq. (65), with Ψ_L given from Eq. (56), Φ_L given from Eq. (44) and W_L given from Eq. (47), one finds that the logarithmic terms are actually all in the correct Milne ratios Eq. (34), and also that W vanishes to leading order, independently of the undetermined constants. Furthermore, the logarithmic terms obey $\Phi(y_+) - \Phi(y_-) = 0$ and $\Psi(y_+) - \Psi(y_-) = 0$, consistent with our assumption that the Kaluza-Klein zero mode dominates.

The gauge constants Q , a_1^\pm and a_2^\pm do, however, affect the t -independent constant terms in each metric perturbation component. Two of the constants are fixed once one sets the constant terms in Φ and Ψ to their Milne ratio values $(2/3)k^2\chi$ and $(1/3)k^2\chi$. We also want to ensure that all components of the metric perturbations behave asymptotically like a Kaluza-Klein zero mode, becoming independent of y as t tends to zero. We check this by comparing the values of Φ , Ψ and W on the two branes. The difference of Ψ on the two branes turns out to be independent of the choice of the gauge constants as t tends to zero,

$$\Psi(y_+) - \Psi(y_-) = O(\rho_\pm^2 L^2). \quad (72)$$

Since the moduli space approximation was derived neglecting $\rho_\pm^2 L^4$ corrections, to the order we can trust the calculation, Ψ is equal at the two brane locations. The difference of Φ on the two branes is not automatically zero at leading order, however, and setting it to zero provides the additional equation needed to determine the third constant.

The result of these calculations is that the solution for χ up to $O(t^3)$ and the leading order behavior of the other components of the metric is completely determined. Explicitly we find [47]

$$\begin{aligned} \Psi(t) = & \zeta_{4,M}(t) + \frac{\epsilon_0 \tanh(y_0/2)}{32k^2 L^2 \cosh^2(y_0/2)} \{ 18[y_0 - \sinh(y_0)] \\ & - L^2(r_+ - r_-)[-3y_0 + \sinh(y_0)] \} \\ & + O(\rho_\pm^2 L^2, t, t \ln|kt|). \end{aligned} \quad (73)$$

Since Ψ is one of the variables which we match in our chosen gauge, it follows that our prescription is quite different to matching the comoving curvature perturbation $\zeta_{4,M}$ four-dimensional effective theory. As we shall explain, the additional terms in (73) allow the propagation of growing mode perturbations across the singularity.

D. Matching proposal

The requirement that around the collision event the geometry looks locally like $\mathcal{M}^C/Z_2 \times R^3$ has completely fixed the gauge in the incoming and outgoing states. As elaborated in Appendix D, the asymptotically Neumann boundary condition (64) further ensures that the collision event is simultaneous in our gauge, an essential property for matching perturbations since the space-like surfaces defined by $t \rightarrow 0^+$ and $t \rightarrow 0^-$ then physically coincide.

Furthermore, as we have discussed this gauge is special in that the induced geometry on each brane is asymptotically the same at collision. In general if a brane is moving, the values of the bulk perturbations Φ and Ψ evaluated on the branes differ from the induced values Φ_\pm and Ψ_\pm . The differences are given by

$$\begin{aligned} \Phi_\pm - \Phi(y_\pm) &= - \frac{b^2}{n^2 t^2} \frac{n'}{n} \chi' \\ \Psi_\pm - \Psi(y_\pm) &= + \frac{b^2}{n^2 t^2} \frac{b'}{b} \chi'. \end{aligned}$$

Since $n'/n \propto t$ and $b'/b \propto t$ as $t \rightarrow 0$, in the presence of matter on the branes, if we make the requirement that the metrics on each brane are asymptotically identical this fixes $\chi' = 0 + O(t^2)$, which is what we have required. Physically this seems a natural choice of gauge because when two ordinary branes collide, the induced geometries are identical at the collision moment. This interpretation is also consistent with the predictions from the four-dimensional effective theory where the brane metrics are given by $g_{\mu\nu}^+$

$=[\cosh(\phi/\sqrt{6})]^2 g_{\mu\nu}$ and $g_{\mu\nu}^- = [-\sinh(\phi/\sqrt{6})]^2 g_{\mu\nu}$. Since the collision corresponds to $\phi \rightarrow -\infty$, in the limit there is no difference between the two conformal factors, and the brane geometries appear identical.

So in the gauge we have fixed by the requirement that the perturbations behave asymptotically like those on the model space-time $\mathcal{M}/Z_2 \times R^3$, the collision event is synchronous, the Milne ratio conditions are satisfied, the boundary conditions are asymptotically Neumann, and the geometries on each brane are asymptotically the same both before and after the collision. Our matching proposal amounts to relating the geometry on these chosen time slices across the collision. We believe that these are sufficiently desirable properties to justify this as the natural gauge in which to perform the matching, and from now on we shall take this to be our complete gauge fixed matching gauge.

Let us now return to our final formula (73) to infer its meaning in the context of ekpyrotic and cyclic models. In those scenarios [6] the quantity ϵ_0/k^2 has an approximately scale-invariant long wavelength spectrum in the incoming state. The first point to make is that even in an in-state with no radiation present the dimensionless curvature perturbation on spatial slices Ψ in our gauge has a scale invariant spectrum, since

$$\Psi = \zeta_{4,M} + \frac{9\epsilon_0 \tanh(y_0/2)}{16k^2 L^2 \cosh^2(y_0/2)} [y_0 - \sinh(y_0)]. \quad (74)$$

Recall that y_0 is the relative rapidity and $V_{in} \equiv \tanh(y_0)$ is the incoming relative velocity between the two branes. Then, at small velocities this gives

$$\Psi = \zeta_{4,M} - \frac{3}{64} \frac{\epsilon_0}{k^2 L^2} V_{in}^4. \quad (75)$$

We may interpret this geometrically as follows. In the absence of radiation there is no real meaning to the curvature perturbation on the branes but if we imagine that there is a small density of radiation coming in, and the perturbations are adiabatic, we can infer the comoving curvature perturbation on the brane, $\zeta_{\pm,M}$, so Eq. (75) becomes for long wavelengths

$$\Psi_{\pm} = \zeta_{\pm,M} - \frac{3}{64} \frac{\epsilon_0}{k^2 L^2} V_{in}^4. \quad (76)$$

Since $\zeta_{\pm,M}$ and Ψ are the spatial curvature perturbations of the branes as respectively measured in the comoving time slicing and in our chosen time slicing (in which the collision is at $t=0$), it must be that the additional piece arises from a time translation between the two gauges. That this is so is verified when one traces back the origin of this term to the second term in Eq. (70). As explained in the Introduction, comoving gauge (or equivalent constant energy density gauge) are bad gauges to match in because the brane collision is not simultaneous in those gauges. Since our prescription is to propagate Ψ across the collision, the jump in ζ_{\pm} is due to the time delay occurring between the collision-

synchronous surfaces in our gauge, and those of the comoving/constant density surfaces. The key to our result is that in the comoving or constant density gauge the time delay between $t=0$ and the actual brane collision event has a scale invariant spectrum.

In fact, using Eqs. (28) and (73) to find Q and P before and after the bounce for all components of the metric perturbations and matching according to the rule given in Eq. (38) results in $\zeta_{4,M}$ inheriting *two* separate scale-invariant long wavelength contributions in the post-singularity state. The first occurs as a direct consequence of the sign change in Eq. (38), and is independent of the amount of radiation generated at the singularity. The second is proportional to the difference in the densities of the radiation on the two branes. At leading order in velocities we have

$$\Delta \zeta_{4,M} = \frac{3}{64} \frac{\epsilon_0}{k^2 L^2} (V_{in}^4 + V_{out}^4) - \frac{(r_+ - r_-) \epsilon_0 V_{out}^2}{32k^2} + O(r_{\pm} V^3, V^5 L^{-2}, \rho_{\pm}^2 L^2), \quad (77)$$

where V_{in} and V_{out} are the relative velocities of the branes before and after collision. Note that since $P \propto \epsilon_0$, matching P is in fact equivalent to matching ϵ_0 across the collision as proposed in Ref. [6]. In terms of four-dimensional parameters defined in Sec. III C, including r_4 given in Eq. (18) defining the abundance of the radiation and the fractional density mismatch f defined in Eq. (17), we find again at leading order in velocities

$$\Delta \zeta_{4,M} = \frac{3}{64} \frac{\epsilon_0}{k^2 L^2} (V_{in}^4 + V_{out}^4) - \frac{3\epsilon_0}{16k^2} \frac{f r_4 V_{out}^3}{L}. \quad (78)$$

This is our final result, relevant to tracking perturbations across the singularity in the ekpyrotic and cyclic models. We see it consists of two essentially independent terms. The first is proportional to the radiation density mismatch on the two branes after collision. Note that just such a mismatch (with more radiation on the negative tension brane) was required in order to enable the cyclic solution of Ref. [2] to work. The second term exists, however, even in the limit of no radiation generated on the branes. As we have noted above, it is non-zero even if $V_{in} = V_{out}$, and it originates in the sign change of the parameter Q in our matching rule, which yields an arrow of time across the collision as explained in Sec. II. Going back to the original formula (73) in which we have not made the small velocity approximation, we note that both the radiation-dependent and radiation-independent terms possess a well defined limit as the brane collision becomes relativistic (as the rapidity $y_0 \rightarrow \infty$),

$$\Delta \zeta_{4,M} \approx \frac{\epsilon_0}{k^2} \left(\frac{9}{4L^2} + \frac{(r_+ - r_-)}{8} \right). \quad (79)$$

Recall, we need the radiation densities on the branes to be much smaller than their tension, i.e. $r_{\pm} L^2 \ll 1$, in order that the four-dimensional effective theory be valid (Sec. III). Therefore in the high velocity limit, the radiation-

independent term dominates. Conversely, from Eq. (78), in the low velocity limit [with $(r_+ - r_-)L^2$ fixed] the radiation-dependent term dominates.

We should stress once more that the dependence upon parameters in Eq. (78) indicates its thoroughly five-dimensional origin. It cannot be expressed in purely four-dimensional terms. In previous work [6] with Khoury and Ovrut, two of us employed a more naive matching prescription framed entirely in terms of the four-dimensional effective theory. This prescription was based upon using the comoving energy density perturbation ϵ_m , which is finite at the singularity, as the matching variable. Unfortunately, since the differential equation governing ϵ_m is singular at $t=0$, the first time derivative of ϵ_m is not an independent quantity at the collision and hence could not be independently matched. Instead we proposed matching the second time derivative. This has the virtue of at least yielding a dimensionally correct result, but it is ambiguous since there are other choices of finite variables. Now we understand the source of the ambiguity better. There is simply not enough information present in the four-dimensional theory to fix the gauge. For that, the five-dimensional picture is essential as we have seen here.

In summary, we have found that a spectrum of scale invariant, growing, long wavelength perturbations generally propagate across the singularity even in the limit when no radiation is produced. The radiation-independent contribution rests upon the sign change of Q in the matching rule (5). If radiation is produced at the bounce, then, for the long wavelength modes we are interested in, we believe it is reasonable to model the production of radiation as occurring suddenly, taking into account the conservation of energy and momentum as was done in Ref. [2]. In this case, we find an additional contribution to the long wavelength scale-invariant perturbations emerging from the singularity, which is proportional to the difference in the radiation densities on the two branes.

VI. CONCLUSIONS

In this paper we have developed an unambiguous and, we believe, compelling rule for matching perturbations across the types of singularity encountered in the ekpyrotic and cyclic Universe scenarios. In the simplest realization of these scenarios, involving the collision of two Z_2 branes in a bulk with a negative cosmological constant, we have shown that the proposed rule leads unambiguously to a spectrum of scale invariant growing density perturbations in the ensuing hot big-bang phase, even in the limit when only a small amount of radiation is produced at the collision. The result provides support for a key assumption of the ekpyrotic and cyclic models.

We have dealt here only with the linear theory, treating the perturbations as free massless fields which we match across the singularity. This treatment clearly is not fully consistent since the perturbations are divergent at the singularity and nonlinear effects must become important there. However, there are reasons to expect that in the nonlinear theory, a similar matching rule will apply. In the linear theory, we

have seen that the metric components typically behave as $1 + \epsilon \ln|t|$ as $t \rightarrow 0$. This is just the small ϵ expansion of $|t|^\epsilon$, the generic behavior expected in the full nonlinear Kasner solutions of general relativity that describe the generic approach to a space-like singularity. The natural extension of our proposal to the nonlinear theory is, therefore, that we should match the Kasner exponents across the singularity. As in the linear theory, the canonical momenta associated with the three-metric are finite and our proposal amounts to matching them with a sign flip. But our matching proposal in Eq. (5) also reverses the long wavelength component of the constant term in the metric perturbation. Generalizing to the nonlinear case, we may anticipate that when the metric tends to the Kasner form with spacelike components $\sim e^{Q|t|^P}$, with Q and P of order ϵ , these components will match to $e^{-Q|t|^P}$ in the outgoing state. If ϵ is small as expected in the ekpyrotic/cyclic scenarios, nonlinear corrections will be of order ϵ^2 and hence negligible.

Strongly supporting the idea of a local matching rule is the classic conjecture that in general relativity the behavior of the metric and fields becomes ultralocal in the approach to this type of singularity [21]. That is, the spatial derivatives become unimportant and the geometry at each point in space follows a homogeneous Friedmann-Robertson-Walker evolution that just depends on local conditions. One might worry that contraction also leads to chaotic mixmaster behavior in which the Universe moves from one kind of Kasner contraction to another and the Kasner exponents change unpredictably. However, the existence of mixmaster behavior depends on the number and types of fields. We discuss elsewhere [48] how the mixmaster behavior is naturally suppressed in ekpyrotic and cyclic models.

Finally, with a precise matching rule for propagating perturbations through the singularity in place, we believe that the cyclic and ekpyrotic models are now on firmer footing. A detailed study applying the above results to these cosmological scenarios will be given elsewhere [49].

Note added. Since the appearance of this work, Craps and Ovrut [50] have investigated the propagation of perturbations within string theory in a class of backgrounds possessing big-crunch-big-bang singularities qualitatively similar to those studied here. In a certain limit, the space-times they study reduce to compactified Milne space-time. In this limit they obtain results identical to those discussed here and in Ref. [7]. The agreement is gratifying. However, their string theory examples have not yet been studied beyond linear theory, within which there is no significant difference between string theory calculations and those in field theory. They have not yet studied perturbations of fields which are time varying in the background, and which therefore couple to metric perturbations, as we have here. Nevertheless these and other examples are very instructive and we are fully in agreement with them about the importance of performing calculations of the type reported here within string and M theory.

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APPENDIX A: PROJECTED EINSTEIN EQUATION

In Sec. III we derived the four-dimensional effective action for solutions with cosmological symmetry and then used general coordinate invariance to infer the covariant moduli space action. While we have shown that this approach recovers the cosmological solutions perfectly at low densities, we have obtained the low energy effective action describing the general (asymmetrical) case by simply assuming locality and imposing covariance. While this is plausible it is important to check it explicitly. This has in fact been done in Refs. [33–35,51] which further clarify the conditions under which the moduli approximation is valid. We shall compare the results of these works with those of the moduli space approach.

We shall first show that the effective theory we have derived satisfies one non-trivial check. One way of formulating a low energy theory for the brane geometries is the so-called Gauss-Codazzi formalism developed in Ref. [50]. Here we take the five-dimensional Einstein equations and project them onto the brane to infer an equation for the brane geometry. One finds

$$G_{\mu\nu}^{\pm} = \pm \frac{1}{M_5^3 L} T_{\mu\nu}^{\pm} + \frac{1}{M_5^6} S_{\mu\nu}^{\pm} - E_{\mu\nu}^{\pm}, \quad (\text{A1})$$

where $T_{\mu\nu}^{\pm}$ is the stress-energy on the brane, not including the tension. This looks like the four-dimensional Einstein equations except for two additional source terms. One contains stress energy squared terms,

$$S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_{\mu\alpha} T_{\nu}^{\alpha} + \frac{1}{24} g_{\mu\nu} (3 T_{\alpha\beta} T^{\alpha\beta} - T^2), \quad (\text{A2})$$

where $T = T^{\lambda}_{\lambda}$, and the second $E_{\mu\nu}^{\pm}$ is obtained from projecting the ‘‘electric’’ part of the bulk Weyl tensor onto the brane

$$E_{\mu\nu} = \frac{\partial x^A}{\partial x^{\mu}} \frac{\partial x^B}{\partial x^{\nu}} E_{AB}, \quad E_{AB} = C_{ACBD} n^C n^D, \quad (\text{A3})$$

where n^A is the normal to the brane. Note that by definition $E_{\mu\nu}$ is symmetric. Since this term contains information about the second ‘‘y’’ derivatives of the bulk geometry we cannot calculate it in any purely four-dimensional way and so although the above equations strongly resemble Einstein’s equations they are purely formal. However, we can construct one purely four-dimensional equation because $E_{\mu\nu}$ does satisfy the exact condition

$$E_{\pm\mu}^{\mu} = 0. \quad (\text{A4})$$

The moduli space approximation only works in the limit in which the stress energy of the matter on the brane is much smaller than the brane tension. This amounts to neglecting the T^2 terms in the above action leaving

$$G_{\mu\nu}^{\pm} = \pm \frac{1}{M_4^2} T_{\mu\nu}^{\pm} - E_{\mu\nu}^{\pm}. \quad (\text{A5})$$

From now on we shall for convenience use units where $M_4 = (8\pi G)^{-1/2}$ is unity. As a consequence of the Bianchi identities it follows that in this ‘‘low energy’’ approximation the following condition must be true:

$$\nabla_{\mu} E_{\pm\nu}^{\mu} = 0. \quad (\text{A6})$$

Since $E_{\mu\nu}$ is conserved and traceless it means that the influence of the bulk on the brane geometry is identical in form to that of the stress energy of a conformal field theory. If we look for a cosmological solution, the vanishing trace condition tells us that the only non-zero components of E_{ν}^{μ} are $E_0^0 = f(b)$ and $E_j^i = -\frac{1}{3} f(b) \delta_j^i$, where $f(b)$ is an arbitrary function of the scale factor on the brane. In addition the condition that $\nabla_{\mu} E_{\pm\nu}^{\mu} = 0$ tells us that $f(b) = C/b^4$ and so the effect of this term is gravitationally indistinguishable from radiation, and it may be thought of as a dark radiation term. This is the import of Birkhoff’s theorem in the bulk, viewed from the brane.

The moduli space approximation as we have developed it provides a precise prediction for $E_{\mu\nu}^{\pm}$. A non-trivial check on this approximation is that the predicted value of $E_{\mu\nu}^{\pm}$ is traceless. This condition of tracelessness is built in at the start in the other formalisms [33–35], but is a nontrivial check of our approach. We can compute the trace by simply conformally transforming the trace of the Einstein equation in the four-dimensional effective theory. Writing the brane metrics as $g_{\pm\mu\nu} dx^{\mu} dx^{\nu} = \Omega_{\pm}^2 g_{\mu\nu} dx^{\mu} dx^{\nu}$ we find

$$\begin{aligned} E^{\mu}_{\mu} &= -G^{\mu}_{\mu} \pm T_{\pm} = R_{\pm} \pm T_{\pm} \\ &= \Omega_{\pm}^{-2} \left(R - \frac{6}{\Omega_{\pm}} \nabla^2 \Omega_{\pm} \right) \pm T_{\pm} \\ &= \Omega_{\pm}^{-2} \left[-T_4 + (\nabla \phi)^2 \left(1 - 6 \frac{(\Omega_{\pm})_{,\phi} \phi}{\Omega_{\pm}} \right) - \frac{6(\Omega_{\pm})_{,\phi}}{\Omega_{\pm}} \nabla^2 \phi \right] \\ &\quad \pm T_{\pm} \\ &= \Omega_{\pm}^{-2} \left(-T_4 - \frac{6(\Omega_{\pm})_{,\phi}}{\Omega_{\pm}} \nabla^2 \phi \right) \pm T_{\pm}, \end{aligned} \quad (\text{A7})$$

where $T_{\pm} = T^{\mu}_{\mu}$ and in the last step we have used $\Omega_{+} = \cosh(\phi/\sqrt{6})$, $\Omega_{-} = -\sinh(\phi/\sqrt{6})$. Finally, making use of the equation of motion for the scalar field

$$\nabla^2 \phi = -\frac{1}{4} (\Omega_{+}^4)_{,\phi} T^{+} - \frac{1}{4} (\Omega_{-}^4)_{,\phi} T^{-}, \quad (\text{A8})$$

we find that

$$E_{\pm}^{\mu}{}_{\mu} = 0. \quad (\text{A9})$$

It is interesting to note that the intermediate steps in this calculation *require* that the conformal factors on the positive and negative tension branes are of the forms described above involving $\cosh(\phi/\sqrt{6})$ or $\sinh(\phi/\sqrt{6})$.

In order to compute the projected Weyl curvature in general it is helpful to work at the level of the action. We start with the action for the four-dimensional effective theory

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} [R - (\nabla\phi)^2] + \Omega_+^4 \mathcal{L}_+ + \Omega_-^4 \mathcal{L}_- \right). \quad (\text{A10})$$

To get the action for the metric on the positive tension brane we simply perform the conformal transformation, taking us out of Einstein frame

$$S = \int d^4x \sqrt{-g_+} \Omega_+^{-4} \left(\frac{\Omega_+^2}{2} [R_+ - 6\Omega_+ \nabla_+^2 \Omega_+^{-1} - (\nabla_+ \phi)^2] + \Omega_+^4 \mathcal{L}_+ + \Omega_-^4 \mathcal{L}_- \right), \quad (\text{A11})$$

then defining $\Psi = \Omega_+^{-2}$ and performing an integration by parts we obtain the following action for the metric on the positive tension brane:

$$S_+ = \int d^4x \sqrt{-g_+} \left[\frac{1}{2} \left(\Psi R_+ - \frac{3}{2(1-\Psi)} (\nabla_+ \Psi)^2 \right) + \mathcal{L}_+ + (1-\Psi)^2 \mathcal{L}_- \right], \quad (\text{A12})$$

and a similar calculation on the negative tension brane defining $\Phi = \Omega_-^{-2}$ gives

$$S_- = \int d^4x \sqrt{-g_-} \left[\frac{1}{2} \left(\Phi R_- + \frac{3}{2(1+\Phi)} (\nabla_- \Phi)^2 \right) + \mathcal{L}_- + (1+\Phi)^2 \mathcal{L}_+ \right]. \quad (\text{A13})$$

These results are in perfect agreement with the low energy approximation developed in Refs. [34,35] using a metric-based approach and in Ref. [33] using the covariant curvature formalism. After deriving the equations of motion by varying these actions we can simply read off the predictions for the projected Weyl tensor on the positive tension brane as

$$E_{+\nu}^{\mu} = T_{+\nu}^{\mu} \left(1 - \frac{1}{\Psi} \right) - \frac{(1-\Psi)^2}{\Psi} T_{-\nu}^{\mu} - \frac{1}{\Psi} (\nabla_+^{\mu} \nabla_+^{\nu} \Psi - \delta_{\nu}^{\mu} \nabla_+^2 \Psi) - \frac{3}{2} \frac{1}{\Psi(1-\Psi)} \left(\nabla_+^{\mu} \Psi \nabla_+^{\nu} \Psi - \frac{1}{2} \delta_{\nu}^{\mu} (\nabla_+ \Psi)^2 \right), \quad (\text{A14})$$

and on the negative tension brane

$$E_{-\nu}^{\mu} = -T_{-\nu}^{\mu} \left(1 + \frac{1}{\Phi} \right) - \frac{(1+\Phi)^2}{\Phi} T_{+\nu}^{\mu} - \frac{1}{\Phi} (\nabla_-^{\mu} \nabla_-^{\nu} \Phi - \delta_{\nu}^{\mu} \nabla_-^2 \Phi) + \frac{3}{2} \frac{1}{\Phi(1+\Phi)} \left(\nabla_-^{\mu} \Phi \nabla_-^{\nu} \Phi - \frac{1}{2} \delta_{\nu}^{\mu} (\nabla_- \Phi)^2 \right). \quad (\text{A15})$$

An especially interesting limit of these equations is obtained by $\phi \rightarrow 0$ implying $\Psi \rightarrow 1$ and $\Phi \rightarrow \infty$ which corresponds to the distance between the branes becoming infinite. Providing we can neglect the derivative terms, we see that in this limit matter on the positive tension brane couples to the brane geometry by means of the conventional four-dimensional Einstein equations, whereas the geometry on the negative tension brane is dominated by its coupling to matter on the positive tension brane, and will only start to look like conventional Einstein gravity if a ‘‘stabilization’’ mechanism exists which freezes ϕ to a constant value. In the latter case, stress energy on each brane acts like a dark matter source for gravity on the other brane.

These equations (A15) describe matter interacting in an unconventional way with gravity, and yield a more complicated perturbation theory than usual. Our approach makes it clear that it is simpler to work with the effective four-dimensional theory in Einstein frame with a scalar field with a canonical kinetic term, and then simply to use the map $g_{\mu\nu}^+ = [\cosh(\phi/\sqrt{6})]^2 g_{\mu\nu}$ and $g_{\mu\nu}^- = [-\sinh(\phi/\sqrt{6})]^2 g_{\mu\nu}$ to infer the brane geometries. The only sense in which this theory differs from conventional four-dimensional physics is that the different forms of matter couple non-minimally to gravity through the scalar field.

APPENDIX B: GAUGE INVARIANT VARIABLES

As in four dimensions the cosmological symmetry of the background metric allows us to find a set of gauge invariant variables, which facilitates the comparison of two different gauges. What the natural gauge invariant variables are depends on the form of the background and our definition closely follows, but is not identical to, those in Ref. [44].

We begin with the background metric written in the form

$$ds^2 = n^2(t, y) (-dt^2 + t^2 dy^2) + b^2(t, y) \delta_{ij} dx^i dx^j. \quad (\text{B1})$$

We shall only consider spatially flat cosmologies for simplicity but the generalization to closed and open universes is easy. The most general scalar metric perturbation can be written as

$$ds^2 = n^2 [-(1+2\Phi) dt^2 - 2W dt dy + t^2 (1-2\Gamma) dy^2] - 2\nabla_i \alpha dx^i dt + 2t^2 \nabla_i \beta dy dx^i + b^2 [(1-2\Psi) \delta_{ij} - 2\nabla_i \nabla_j \chi] dx^i dx^j, \quad (\text{B2})$$

writing the perturbed metric as $g_{AB} + h_{AB}$ where g_{AB} is the background metric, then under a gauge transformation $x^A \rightarrow x^A + \xi^A$ the metric perturbation transforms as

$$h_{AB} \rightarrow h_{AB} - g_{AC} \partial_B \xi^C - g_{BC} \partial_A \xi^C - \xi^C \partial_C g_{AB}. \quad (\text{B3})$$

Since a five-vector ξ_A has three scalar degrees of freedom ξ^t , ξ^y and $\xi^i = \nabla_i \xi^s$, only four of the seven functions $(\Phi, \Gamma, W, \alpha, \beta, \Psi, \chi)$ are physical. This immediately tells us that we expect to be able to define four gauge invariant variables constructed from the metric alone. Let \dot{A} denote $\partial A / \partial t$ and A' denote $\partial A / \partial y$. Under a gauge transformation each of the variables transforms as

$$\begin{aligned} \Phi &\rightarrow \Phi - \xi^t - \xi^t \frac{\dot{n}}{n} - \xi^y \frac{n'}{n}, \\ \Gamma &\rightarrow \Gamma + \xi'^y + \frac{1}{t} \xi^t + \xi^t \frac{\dot{n}}{n} + \xi^y \frac{n'}{n}, \\ W &\rightarrow W - \xi'^t + t^2 \xi^y, \\ \alpha &\rightarrow \alpha - \xi^t + \frac{b^2}{n^2} \xi^s, \\ \beta &\rightarrow \beta - \xi^y - \frac{b^2}{n^2 t^2} \xi'^s, \\ \Psi &\rightarrow \Psi + \xi^t \frac{\dot{b}}{b} + \xi^y \frac{b'}{b}, \\ \chi &\rightarrow \chi + \xi^s. \end{aligned} \quad (\text{B4})$$

It is then relatively easy to construct the following gauge invariant quantities:

$$\Phi_{inv} = \Phi - \tilde{\alpha} - \frac{\tilde{n}}{\tilde{n}} - \tilde{\beta} \frac{n'}{n},$$

$$\Gamma_{inv} = \Gamma + \tilde{\beta}' + \frac{1}{t} \tilde{\alpha} + \frac{\tilde{n}}{\tilde{n}} + \tilde{\beta} \frac{n'}{n},$$

$$W_{inv} = W - \tilde{\alpha}' + t^2 \tilde{\beta},$$

$$\Psi_{inv} = \Psi + \frac{\dot{b}}{b} \tilde{\alpha} + \frac{b'}{b} \tilde{\beta},$$

where $\tilde{\alpha} = \alpha - (b^2/n^2) \dot{\chi}$ and $\tilde{\beta} = \beta + (b^2/n^2 t^2) \chi'$. We then see that there is a special gauge defined by $\chi = \alpha = \beta = 0$ in which

$$\Phi_{inv} = \Phi,$$

$$\Gamma_{inv} = \Gamma,$$

$$W_{inv} = W,$$

$$\Psi_{inv} = \Psi.$$

We define this to be five-dimensional longitudinal gauge and so we see that the gauge invariant variables equal the values of the metric perturbations in longitudinal gauge, in perfect analogy with four-dimensional cosmological perturbation theory. This gauge is characterized by being spatially isotropic in the x^i coordinates but in general there will be a non-zero t - y component of the metric.

Position of branes

In general, the locations in y of the perturbed branes will be different in different gauges, and it is very important to understand this location in each case. Remarkably, in the case where the brane matter has no anisotropic stress this is easy to establish. Start in the gauge $\alpha = \chi = 0$. From the above transformation rules we can see that we can always go to this gauge using only ξ^t and ξ^s transformation. This then leaves us with the freedom to perform any ξ^y transformation such that the position of each brane remains unperturbed. Then working out the Israel matching conditions we find that β on the branes is related to the anisotropic part of the brane's stress energy. So if we are considering only perfect fluids, for which the shear vanishes, then the Israel matching condition gives $\beta(y = y_{\pm}) = 0$. We can then go to longitudinal gauge ($\alpha = \beta = \chi = 0$) with the transformation $\xi^y = \beta$ alone. But since β vanishes on the branes, so does ξ^y , implying that the brane trajectories are unperturbed. So we see that for the special case of matter with no anisotropic stress the locations of the branes in longitudinal gauge are their unperturbed values $y = y_{\pm}$. We can then infer the position of the branes in an arbitrary gauge by means of the above gauge transformations to be

$$y = y_{\pm} - \tilde{\beta}, \quad (\text{B5})$$

where y_{\pm} are the background values. In particular, in the class of Milne gauges we have defined in (57) the branes are located at

$$y = y_{\pm} - \frac{q^2}{t^2} \chi'. \quad (\text{B6})$$

APPENDIX C: BIRKHOFF'S THEOREM AND THE BACKGROUND METRIC

The bulk geometry considered in this paper solves the five-dimensional Einstein's equations sourced by a pure negative cosmological constant. For the background solution we restrict to solutions possessing cosmological symmetry on three-dimensional spatial slices. In close analogy to the familiar situation for spherical symmetry in 3+1 dimensions, a Birkhoff-type theorem guarantees that, in our case, away from the branes the background must take the form of either anti-de Sitter (AdS) space-time, Schwarzschild-AdS or AdS with a naked singularity. In each case the metric may be written as

$$ds^2 = \left(\frac{r^2}{L^2} + k - \frac{\mu}{r^2} \right)^{-1} dr^2 - \left(\frac{r^2}{L^2} + k - \frac{\mu}{r^2} \right) dT^2 + r^2 \gamma_{ij} dx^i dx^j, \quad (C1)$$

where μ is the mass of the black hole, γ_{ij} is the canonical metric on S^3 , H^3 or E^3 , with k the corresponding spatial curvature, and L is the AdS radius defined by $\Lambda = -6M_5^2/L^2$ with M_5 the five-dimensional Planck mass. We are most interested in the case $k=0$, for which it is useful to change variables from r to Y obtained by setting the first term in Eq. (C1) to equal dY^2 , obtaining

$$ds^2 = dY^2 - N(Y)^2 dT^2 + A(Y)^2 d\vec{x}^2, \quad (C2)$$

where for AdS

$$A(Y)^2 = N(Y)^2 = \exp[2Y/L], \quad (C3)$$

for Schwarzschild-AdS with a horizon at $Y=0$

$$A(Y)^2 = \cosh(2Y/L) \quad \text{and} \quad N(Y)^2 = \frac{\sinh(2Y/L)^2}{\cosh(2Y/L)}, \quad (C4)$$

and for AdS with a naked singularity at $Y=0$

$$A(Y)^2 = \sinh(2Y/L), \quad N(Y)^2 = \frac{\cosh(2Y/L)^2}{\sinh(2Y/L)}. \quad (C5)$$

For any configuration of branes possessing cosmological symmetry, even if the branes move the Birkhoff theorem guarantees that the bulk geometry takes one of the three forms above [39,40]. In our case, where the branes are Z_2 -symmetric and have their tensions tuned to allow static empty brane solutions, the only bulk solution that is consistent with moving branes is the Schwarzschild-AdS solution. Consequently this is the background five-dimensional metric we use in this paper.

Technically, in order to study the perturbations it is much simpler if one changes coordinates to those in which the branes are static and the bulk is time dependent. That it is always possible to choose such a coordinate system may be seen as follows. Start with the Birkhoff-frame metric (C2) with A and N given by Eq. (C4). First, change variables from Y to Z defined by $dZ = dY/N$, with Z chosen to be zero at the collision event, so that

$$ds^2 = N^2(-dT^2 + dZ^2) + A^2 d\vec{x}^2, \quad (C6)$$

where N and A are now functions of Z . Defining light-cone coordinates $T_{\pm} = T \pm Z$ we have

$$ds^2 = N^2(-dT_+ dT_-) + A^2 d\vec{x}^2. \quad (C7)$$

We now recognize that the form of this metric is invariant under the light-cone coordinate transformation, $\tau^{\pm} y = f_{\pm}(T \pm Z)$, which takes the metric to the form

$$ds^2 = \frac{N^2}{f'_+ f'_-} (-d\tau^2 + dy^2) + A^2 d\vec{x}^2. \quad (C8)$$

Now we set $t = \pm e^{\pm \tau}$ to describe the post-or pre-collision space-times, respectively, and define $t^2 n^2(t, y) = N^2/(f'_+ f'_-)$ and $b^2(t, y) = A^2$ to obtain

$$ds^2 = n^2(t, y) (-dt^2 + t^2 dy^2) + b^2(t, y) d\vec{x}^2, \quad (C9)$$

which is the form used in this paper.

We now show that we can always choose the functions f_{\pm} to make the branes static in the new coordinates. To see this note that the new spatial coordinate

$$y(T, Z) = \frac{1}{2} [f_+(T+Z) - f_-(T-Z)] \quad (C10)$$

itself satisfies the massless field equation in two dimensions. If the two-brane trajectories are $Z = Z_{\pm}(T)$ in the T, Z coordinates, then it follows from the general theory of the wave equation that we can always solve (C10) for arbitrary chosen $y(T, Z)$ on two specified timelike curves $Z = Z_{\pm}(T)$. In particular we are free to choose constant values $y = y_+$ on the positive tension brane and $y = y_-$ on the negative tension brane. Even after this choice there is additional coordinate freedom, since to determine the solution for $y(T, Z)$ we need to specify additional Cauchy data, for example on a $T = \text{const}$ surface.

In practice we find it is straightforward to solve these equations as a power series in t . The Israel matching conditions on the two-branes in Birkhoff coordinates read

$$\tanh(2Y_{\pm}/L) = \left(1 \pm \frac{\rho_{\pm} L^2}{6} \right) \sqrt{1 - N^{-2}(Y_{\pm})(dY_{\pm}/dT)^2}, \quad (C11)$$

where ρ_{\pm} are the densities of matter or radiation on the branes. In our case, when only radiation is present, and we normalize the brane scale factors to be unity at collision (cf. Sec. III C), we have $\rho_{\pm} = r_{\pm}/A^4(Y_{\pm})$. Equation (C11) is a first order differential equation for the brane trajectories $Y_{\pm}(T)$, allowing them to be straightforwardly determined as Taylor series in T . Likewise we may solve explicitly for Z ,

$$Z(Y) = \frac{L}{2} \left[\tan^{-1}(x) + \frac{1}{2} \ln \left(\frac{x-1}{x+1} \right) \right], \quad (C12)$$

where $x^2 \equiv \cosh(2Y/L)$, and hence obtain $Z_{\pm}(T)$ as a Taylor series in T . From Eq. (C10) we obtain

$$y_{\pm} = \frac{1}{2} \{ f_+[T + Z_{\pm}(T)] - f_-[T - Z_{\pm}(T)] \}, \quad (C13)$$

which we may differentiate with respect to T , noting that the y_{\pm} are constant, to obtain

$$f'_+[T + Z_{\pm}(T)][1 + V_{\pm}(T)] = f'_-[T - Z_{\pm}(T)][1 - V_{\pm}(T)], \quad (C14)$$

where $V_{\pm}(T) \equiv [dZ_{\pm}(T)/dT]$ are the brane velocities. These two equations may be simultaneously solved as a power series in T with the ansatz $f'_{\pm}(z) = z^{-1} + f_{\pm}^0 + f_{\pm}^1 z + \dots$. They are both trivially satisfied at order T^{-1} . At each subsequent power T^n , $n \geq 0$ one obtains two equations which fix the two constants f_{\pm}^n . Finally, writing $f_{\pm}(z) = c_{\pm} + \ln z + f_{\pm}^0 z + f_{\pm}^1 z^2/2 + \dots$, with c_{\pm} constants, we can write the equation for y_{\pm} and take the limit $T \rightarrow 0$ on the right hand side to obtain

$$y_{\pm} = \frac{1}{2}(c_+ - c_-) + \frac{1}{2} \ln \left(\frac{1 + V_{\pm}}{1 + V_{\mp}} \right) = \frac{1}{2}(c_+ - c_-) + \theta_{\pm}^B, \quad (\text{C15})$$

where θ_{\pm}^B are the rapidities of the positive and negative tension branes in the Birkhoff frame. Likewise we obtain (for $t > 0$)

$$\tau = \frac{1}{2}(c_+ + c_-) + \ln t. \quad (\text{C16})$$

Setting $t = \pm e^{\pm \tau}$ as we do for $t > 0$ or $t < 0$, respectively, and choosing $y_+ = -y_-$ (i.e. the Lorentz frame in which the branes have equal and opposite speeds), then fixes $c_+ = -c_- = -\frac{1}{2}(\theta_+^B + \theta_-^B) \equiv -\theta^B$. Now one may invert the equations $\tau \pm y = f_{\pm}(T \pm Z)$ to express $T + Z$ as a Taylor series in te^y for $t > 0$ (or te^{-y} for $t < 0$) and similarly $T - Z$ as a Taylor series in te^{-y} (or te^y). For example, post-collision one obtains

$$T + Z = te^y e^{\theta^B} + O(t^2), \quad T - Z = te^{-y} e^{-\theta^B} + O(t^2), \quad (\text{C17})$$

equations which will be useful in Appendix D. Hence we completely determine the metric functions n^2 and b^2 as Taylor series in te^y and te^{-y} . Finally, by rescaling t and \vec{x} we can also ensure that in the new coordinates, $n(t, y) = 1 + O(t)$ and $b(t, y) = 1 + O(t)$.

As a check of this procedure, or indeed an alternative to it, one can directly solve Einstein's equations in the frame in which the branes are static. The extrinsic curvature is given by

$$\begin{aligned} K_{\mu\nu} dx^{\mu} dx^{\nu} &= \frac{1}{2nt} \partial_y g_{\mu\nu} dx^{\mu} dx^{\nu} \\ &= \frac{1}{2nt} [-(n^2)' dt^2 + (b^2)' d\vec{x}^2], \end{aligned} \quad (\text{C18})$$

and so the Israel matching conditions

$$K_{\mu\nu} = \frac{1}{2M_5^3} \left(T_{\mu\nu} - \frac{1}{3} g_{\mu\nu} T_{\lambda}^{\lambda} \right), \quad (\text{C19})$$

tell us that

$$\frac{n'}{n^2 t} = \frac{1}{L} \mp \frac{L}{3} \rho_{\pm} \mp \frac{L}{2} p_{\pm}, \quad (\text{C20})$$

$$\frac{b'}{ntb} = \frac{1}{L} \pm \frac{L}{6} \rho_{\pm}. \quad (\text{C21})$$

For the purposes of our analysis it will be convenient to define the Lorentz frame we work in to be that in which the y coordinates of the branes (their rapidities) are $y_{\pm} = \pm y_0/2$. Recall, we also define the parameters r_{\pm} to be the densities of radiation on each brane ρ_{\pm} at collision, and we treat these as free parameters. Through a direct series solution of the five-dimensional Einstein equations, imposing the Israel matching conditions (C21) at each order in t , we obtain the following solution for the background geometry near $t = 0$:

$$\begin{aligned} b(t, y) &= 1 + (b_1 \sinh y + b_2 \cosh y) t \\ &\quad + (e_0 + e_1 \sinh 2y + e_2 \cosh 2y) t^2/2, \\ n(t, y) &= 1 + (d_1 \sinh y + d_2 \cosh y) t \\ &\quad + (k_0 + k_1 \sinh 2y + k_2 \cosh 2y) t^2/2, \end{aligned} \quad (\text{C22})$$

where the constant parameters are given by

$$b_1 = \frac{12 + L^2(r_+ - r_-)}{12L} \operatorname{sech}(y_0/2),$$

$$b_2 = \frac{L}{12} (r_+ + r_-) \operatorname{cosech}(y_0/2),$$

$$d_1 = \frac{4 - L^2(r_+ - r_-)}{4L} \operatorname{sech}(y_0/2),$$

$$d_2 = -\frac{L}{4} (r_+ + r_-) \operatorname{cosech}(y_0/2),$$

$$\begin{aligned} e_0 &= \frac{1}{36L^2} [(-6 + L^2 r_-)^2 + (6 + L^2 r_+)^2 \\ &\quad + 2(-6 + L^2 r_-)(6 + L^2 r_+) \cosh y_0 + 36(\cosh 2y_0 - 1)] \\ &\quad \times (\operatorname{cosech} y_0)^2, \end{aligned}$$

$$e_1 = \frac{1}{12} [-4 - L^2(r_+ - r_-)] (r_+ + r_-) \operatorname{cosech}(y_0),$$

$$\begin{aligned} e_2 &= -\frac{1}{12L^2} \{ (24 + 4L^2(r_- - r_+) + 2L^4 r_+ r_- \\ &\quad + [-24 - 4L^2(r_- - r_+) + L^4(r_-^4 + r_+^4)] \cosh y_0 \} \\ &\quad \times (\operatorname{cosech} y_0)^2, \end{aligned}$$

$$\begin{aligned} k_0 &= \frac{1}{6L^2} [21 + L^4(r_+^2 + r_-^2) + 2(-12 + L^4 r_- r_+) \cosh y_0 \\ &\quad + 3 \cosh 2y_0] (\operatorname{cosech} y_0)^2, \end{aligned}$$

$$\begin{aligned}
 k_1 &= -\frac{1}{12}(r_+ + r_-)\operatorname{cosech} y_0[5L^2(r_- - r_+) + 12 \operatorname{sech} y_0], \\
 k_2 &= \frac{1}{12L^2}\{-24 + 10L^4 r_+ r_- + [24 + 5L^4(r_+^2 + r_-^2)]\cosh y_0\} \\
 &\quad \times (\operatorname{cosech} y_0)^2. \tag{C23}
 \end{aligned}$$

APPENDIX D: MEANING OF THE CONSTANTS c_1 AND c_2

The two arbitrary gauge constants c_1 and c_2 in Eq. (67) parametrizing the violation of the asymptotically Neumann boundary condition (64) have a simple geometrical interpretation: they describe the displacement of the collision event in the T, Y plane. Recall that in the Neumann gauge, discussed in Sec. V E, the brane trajectories are unperturbed and are described by the equations $y = y_{\pm} = \text{const}$. If we now gauge transform to an asymptotically Neumann gauge, in which the normal derivatives $n^{-1}t^{-1}\chi'(t, y_{\pm})$ deviate from zero at order t as in Eq. (67), we see that the gauge transformation from conformal Newtonian gauge to the Milne gauge we are in involves a divergent y coordinate displacement of $\xi^y = -q^2\chi'/t^2$, which tends to $-(c_1\sinh y + c_2\cosh y)/t$ plus a finite part as t tends to zero. If $c_1 = c_2 = 0$, then the perturbation in the brane y coordinates $\xi^y(y_{\pm})$ is finite. The rapidities of the two branes are perturbed, but the collision event itself is still simultaneous as in Neumann gauge.

In the remainder of this appendix we provide a geometrical interpretation of the two constants c_1 and c_2 , showing that they parametrize the displacement of the brane collision event away from its background location in the embedding coordinates T, Y , at each \bar{x} .

If we start from Neumann gauge with $c_1 = c_2 = 0$, we may introduce c_1 and c_2 via the following gauge transformation:

$$\xi^x = (c_1 \cosh y + c_2 \sinh y)t,$$

$$\xi^y = -\frac{1}{t}(c_1 \sinh y + c_2 \cosh y),$$

$$\xi^t = (c_1 \cosh y + c_2 \sinh y).$$

This is part of the gauge freedom described by the solutions to Eqs. (59) and (60). Although ξ^y diverges near $t=0$, this is merely a reflection of the singular nature of the Milne (t, y) coordinate system. In terms of the Birkhoff frame T, Y coordinates defined in Appendix C, we find

$$\delta T = \frac{\partial T}{\partial t} \xi^t + \frac{\partial T}{\partial y} \xi^y,$$

$$\delta Y = \frac{\partial Y}{\partial t} \xi^t + \frac{\partial Y}{\partial y} \xi^y.$$

Then using Eq. (C17) given in Appendix C and Eqs. (D1) one infers the displacement of the collision event

$$\delta T = (c_1 \cosh \bar{\theta}^B - c_2 \sinh \bar{\theta}^B),$$

$$\delta Y = N(Y_c)(c_1 \sinh \bar{\theta}^B - c_2 \cosh \bar{\theta}^B), \tag{D1}$$

independent of y and hence holding for both branes. Here $N(Y_c)$ is the value of the lapse function (given in Appendix C) at the collision value of Y in the Birkhoff frame, and $\bar{\theta}^B$ is the mean rapidity of the two branes in that frame. Therefore all the gauge transformation (D1) does is to move the collision event around by an arbitrary finite displacement in the T, Y plane.

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