

# Tests and applications of Migdal's particle path-integral representation for the Dirac propagator

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We derive some nonperturbative results in  $1+1$  and  $2+1$  dimensions within the context of the particle path-integral representation for a Dirac field propagator in the presence of an external field, in a formulation introduced by Migdal. We consider the specific properties of the path-integral expressions corresponding to the  $(1+1)$ - and  $(2+1)$ -dimensional cases, presenting a derivation of the chiral anomaly in the former and of the Chern-Simons current in the latter. We also discuss particle propagation in constant electromagnetic field backgrounds.

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## I. INTRODUCTION

Particlelike path-integral representations have been used in quantum field theory for a long time, starting with the pioneering work of Schwinger [1]. In this approach, the objects of interest are expressed in terms of path integrals over particle trajectories in proper time, something which is closer in spirit to Feynman's propagator approach [2] than to standard quantum field theory methods. Indeed, a propagator is always susceptible to a dual interpretation: it can be understood as the result of an average over field configurations, but also as a sum over proper-time "first-quantized" (i.e., particlelike) paths.

This representation has been more recently applied to the derivation of many interesting results, since its use provides a framework which often becomes convenient for the introduction of nonstandard calculation techniques [3]. One of the reasons for this is that the interaction term appears in an exponential form, and this can make it possible, sometimes, to integrate out the field that mediates the interaction.

For the case of nonzero spin fields, different proposals for the integral over first-quantized trajectories have been advanced. Since they usually involve different sets of variables, the task of relating them is far from trivial, unless it is undertaken at a purely formal level. Concrete calculations, on the other hand, are always useful in order to understand the properties of each formulation on a deeper level.

With that in mind, in this article, we consider the particular case of the path-integral representation for Dirac fields introduced by Migdal in Ref. [4], and apply it as a tool for the derivation of some nonperturbative results in some quantum field theory models in  $1+1$  and  $2+1$  dimensions.

This article is organized as follows: in Sec. II we present a detailed derivation of Migdal's representation, in a way which is adapted to the applications that we consider afterwards. The  $(1+1)$ - and  $(2+1)$ -dimensional cases are dis-

cussed in more detail, and a local action representation suitable for them is introduced. At the end of this section, the equivalence between the path-integral representation and the standard formulation is explicitly shown to be true order by order in perturbation theory.

Going beyond the perturbative expansion, in Sec. III we present derivations of the chiral anomaly in  $1+1$  dimensions and of the Chern-Simons term in  $2+1$  dimensions; these are two nonperturbative tests that, as we shall see, reproduce the proper results.

The propagation in a constant electromagnetic field background in  $2+1$  dimensions is discussed in Sec. IV, by evaluating the exact Fermionic determinant in the present formulation. Finally, in Sec. V we present our conclusions.

## II. PATH-INTEGRAL REPRESENTATION FOR THE PROPAGATOR

We shall present here, for the sake of completeness, a derivation of the particle path-integral representation for the fermion propagator in an external Abelian gauge field. Besides, the procedure will emphasize some specific aspects of the  $2+1$ - and  $1+1$ -dimensional cases, such as, for example, the realization of the spin degrees of freedom. We shall also obtain the standard perturbative expansion within this framework.

### A. Derivation of the general formula

The propagator for a massive Dirac field in  $d$  Euclidean dimensions, in an Abelian gauge field background, is of course determined by the (Euclidean) action  $S_f$ ,

$$S_f(\bar{\psi}, \psi, A) = \int d^d x \bar{\psi} (\not{D} + m) \psi, \quad (1)$$

where the  $D = -i\partial + eA$  and the  $\gamma$  matrices are Hermitian and verify the Clifford algebra:

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \quad (2)$$

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$A_\mu$  is an Abelian gauge field, regarded here as external, and  $e$  is a coupling constant with the dimensions of  $[\text{mass}]^{(4-d)/2}$ .

The Dirac propagator  $G(x, y)$  is the kernel of the inverse of the operator defining the quadratic form in  $S_f$ , namely

$$G(x, y) = \langle \psi(x) \bar{\psi}(y) \rangle = \langle x | (\not{D} + m)^{-1} | y \rangle, \quad (3)$$

where we have adopted Schwinger's convention:  $\langle x | K | y \rangle$  for  $K(x, y)$ , the kernel of an operator  $K$  in coordinate space, and we have omitted the spinorial indices, although it should be evident from the context that  $\langle x | K | y \rangle$  is a  $2 \times 2$  matrix for  $d=2$  and  $d=3$ , and a  $4 \times 4$  matrix when  $d=4$ .

Assuming  $m > 0$ , we may introduce the exponential representation:

$$\langle x | (\not{D} + m)^{-1} | y \rangle = \int_0^\infty dT \langle x | U(T) | y \rangle \quad (4)$$

defined by the operator

$$U(T) = \exp[-T(\not{D} + m)], \quad (5)$$

which acts on functional and spinorial spaces. Note that the presence of a strictly positive mass  $m$  is required for Eq. (4) to be correct, since we are implicitly assuming the boundary condition  $U(+\infty) = 0$ , at least in the weak limit sense.

A functional integral representation can be naturally introduced to deal with the operator  $U(T)$ , in spite of the fact that  $U(T)$  is not the exponential of a (constant times) self-adjoint operator. As usual, in a first step, one splits up the “time”  $T$  into a number  $N$  of intervals of size  $\epsilon$ , with  $T = N\epsilon$ . Namely,

$$\langle x | U(T) | y \rangle = \langle x | \{ \exp[-\epsilon(\not{D} + m)] \}^N | y \rangle, \quad (6)$$

and then one introduces spectral resolutions of the identity at the intermediate points  $x_1, x_2, \dots, x_{N-1}$ ,

$$\begin{aligned} \langle x | U(T) | y \rangle &= \int \left( \prod_{k=1}^{N-1} d^d x_k \right) \langle x | e^{-\epsilon(\not{D} + m)} | x_{N-1} \rangle \\ &\quad \times \langle x_{N-1} | e^{-\epsilon(\not{D} + m)} | x_{N-2} \rangle \cdots \\ &\quad \times \langle x_2 | e^{-\epsilon(\not{D} + m)} | x_1 \rangle \langle x_1 | e^{-\epsilon(\not{D} + m)} | y \rangle. \end{aligned} \quad (7)$$

It should be kept in mind that the matrix elements on the right-hand side of Eq. (7) do not commute with each other; the reason is of course that each factor is a *matrix* in spinorial space rather than a number (as the abbreviated notation might suggest). Those factors *can*, however, be regarded as commuting objects, if they are put inside a “chronological” ordering symbol  $\mathcal{P}$ , and the  $\gamma$  matrices are simultaneously given an (auxiliary) dependence on a discrete time index  $k$ , that keeps track of their relative positions:

$$\begin{aligned} \langle x | U(T) | y \rangle &= \int \left( \prod_{k=1}^{N-1} d^d x_k \right) \mathcal{P} \left[ \langle x | e^{-\epsilon[\gamma(N) \cdot D + m]} | x_{N-1} \rangle \right. \\ &\quad \times \langle x_{N-1} | e^{-\epsilon[\gamma(N-1) \cdot D + m]} | x_{N-2} \rangle \cdots \\ &\quad \times \langle x_2 | e^{-\epsilon[\gamma(2) \cdot D + m]} | x_1 \rangle \\ &\quad \left. \times \langle x_1 | e^{-\epsilon[\gamma(1) \cdot D + m]} | y \rangle \right]. \end{aligned} \quad (8)$$

Then we may write the exact equation:

$$\begin{aligned} \langle x | U(T) | y \rangle &= \int \left( \prod_{k=1}^{N-1} d^d x_k \right) \\ &\quad \times \mathcal{P} \left[ \prod_{l=1}^N \langle x_l | \exp\{-\epsilon[\gamma(l) \cdot D + m]\} | x_{l-1} \rangle \right], \end{aligned} \quad (9)$$

where we have defined  $x_N \equiv x$  and  $x_0 \equiv y$ . For each of the factors under the scope of the ordering operator, we see that

$$\begin{aligned} \langle x_l | e^{-\epsilon[\gamma(l) \cdot D + m]} | x_{l-1} \rangle \\ = \int d^d p_l \langle x_l | p_l \rangle \langle p_l | e^{-\epsilon[\gamma(l) \cdot D + m]} | x_{l-1} \rangle \end{aligned} \quad (10)$$

which, for  $N \gg 1$ , may be approximated by

$$\begin{aligned} \langle x_l | e^{-\epsilon[\gamma(l) \cdot D + m]} | x_{l-1} \rangle \\ \simeq \int \frac{d^d p_l}{(2\pi)^d} e^{ip_l \cdot (x_l - x_{l-1})} e^{-\epsilon\{i\gamma(l) \cdot [p_l + eA(x_{l-1}) + m]\}}, \end{aligned} \quad (11)$$

where we have ignored terms which give no contribution in the  $N \rightarrow \infty$  limit.

The integration variable  $p_l$  is then shifted:  $p_l \rightarrow p_l - eA(x_{l-1})$ , with the effect of disentangling the gauge field from the  $\gamma$  matrices:

$$\begin{aligned} \langle x_l | e^{-\epsilon[\gamma(l) \cdot D + m]} | x_{l-1} \rangle \\ \simeq \int \frac{d^d p_l}{(2\pi)^d} e^{i[p_l - eA(x_l)] \cdot (x_l - x_{l-1})} e^{-\epsilon[i\gamma(l) \cdot p_l + m]}. \end{aligned} \quad (12)$$

Inserting this expression into Eq. (9), one sees that

$$\begin{aligned} \langle x | U(T) | y \rangle &\simeq \int \left( \prod_{k=1}^{N-1} d^d x_k \right) \left( \prod_{l=1}^N \frac{d^d p_l}{(2\pi)^d} \right) e^{\epsilon \sum_{l=1}^N [ip_l \cdot \dot{x}_l - m]} \\ &\quad \times \mathcal{P} \left[ e^{-\epsilon \sum_{l=1}^N i\gamma(l) \cdot p_l} e^{-i\epsilon e \sum_{l=1}^N \dot{x}_l \cdot A(x_{l-1})} \right], \end{aligned} \quad (13)$$

where  $\dot{x}_l \equiv (x_l - x_{l-1})/\epsilon$ . The time dependence of the  $\gamma$  matrices may now be ignored, since the ordering along the subdivisions of  $T$  is fully determined by the label “ $l$ ” of the  $p_l$  which is adjoined to  $\gamma_l$ .

Taking the continuum limit,  $N \rightarrow \infty$  with  $T = N\epsilon$  fixed, one obtains the exact (albeit formal) expression

$$\langle x|U(T)|y\rangle = \int \mathcal{D}p \mathcal{D}x e^{\int_0^T d\tau [ip \cdot \dot{x} - m]} \times \mathcal{P}[e^{-i\int_0^T d\tau \phi}] e^{-ie\int_0^T d\tau \dot{x}(\tau) \cdot A[x(\tau)]}, \quad (14)$$

where

$$\mathcal{D}p \mathcal{D}x \equiv \frac{d^d p(T)}{(2\pi)^d} \prod_{0 < \tau < T} \frac{d^d x(\tau) d^d p(\tau)}{(2\pi)^d}. \quad (15)$$

When used in combination with Eq. (4), Eq. (14) yields the representation for the fermion propagator we were looking for:

$$\langle x|(\not{D} + m)^{-1}|y\rangle = \int_0^\infty dT \int_{x(0)=y}^{x(T)=x} \mathcal{D}p \mathcal{D}x e^{\int_0^T d\tau [ip \cdot \dot{x} - m]} \times \mathcal{P}[e^{-i\int_0^T d\tau \phi}] e^{-ie\int_0^T d\tau \dot{x}(\tau) \cdot A[x(\tau)]}, \quad (16)$$

where we have indicated explicitly the boundary conditions satisfied by the paths that have to be integrated out.

It is worth noting the role played by the extra  $d^d p$  integration in the measure, Eq. (15): each phase-space volume factor  $d^d p d^d x$  is dimensionless, thus the mass dimension of the measure is determined by the extra  $d^d p$  factor. Hence the measure has units of  $[\text{mass}]^d$ . Combining this fact with the property [self-evident in Eq. (16)] that  $T$  has dimensions of  $[\text{mass}]^{-1}$ , we see that the propagator has the dimensions of a  $[\text{mass}]^{d-1}$ , as it should be (twice the mass dimensions of a fermion field).

### B. Adiabatic approximation and spin degrees of freedom

The fact that the functional integral describes the propagation of a spinning particle manifests itself in the existence of a path-ordered factor,

$$\Phi(T) = \mathcal{P}[e^{-i\int_0^T d\tau \phi(\tau)}], \quad (17)$$

whose properties we shall discuss now.

The  $d=3$  case is very special, since  $\Phi(T)$  allows for a quite straightforward interpretation as the (quantum) evolution operator for a spin-1/2 in three *spatial* dimensions, in the presence of a time-dependent homogeneous “magnetic field”  $p_\mu(\tau)$ . Of course, “evolution” is here understood to mean evolution in the fictitious time  $\tau$ . The three components of  $p_\mu$  are then regarded as the spatial components of a magnetic field  $\mathbf{B} = (B_1, B_2, B_3)$ , with  $B_1 = p_1$ ,  $B_2 = p_2$ ,  $B_3 = p_0$ .

It should be obvious that, within the crudest infrared approximation where only constant  $p_\mu$  trajectories contribute,  $\Phi(T)$  will not exhibit any interesting behavior regarding its spin aspect. Indeed, for a constant magnetic field, one knows that

$$\Phi(T) = e^{-iT p \cdot \gamma}, \quad (18)$$

which has the eigenvalues  $e^{\mp i T |p|}$ , where  $|p| \equiv \sqrt{p_\mu p_\mu}$ .

On the other hand, even for a slowly varying  $p_\mu$ , interesting effects may, and indeed do, arise as a consequence of the existence of nonintegrable Berry's phases, which is a way this representation has for displaying the nontrivial spin of the field, in the adiabatic approximation.

For a slowly varying  $p_\mu(\tau)$ , and assuming  $p_\mu(\tau) \neq 0$  to avoid degeneracy, the adiabatic approximation can be applied to obtain an expression for  $\Phi(T)$ . If the initial (two-component) state is an eigenstate of  $\not{p}(0)$ , it will, in this approximation, remain an instantaneous eigenstate during the evolution. At this point, we introduce an explicit convention for the  $d=3$   $\gamma$  matrices:  $\gamma_0 = \sigma_3$ ,  $\gamma_1 = \sigma_1$ , and  $\gamma_2 = \sigma_2$ , where  $\sigma_j$ , with  $j=1,2,3$ , denote the usual Pauli matrices. With this convention, they verify the relation:

$$\gamma_\mu \gamma_\nu = \delta_{\mu\nu} I + i \epsilon_{\mu\nu\lambda} \gamma_\lambda. \quad (19)$$

Denoting by  $|v_\pm(\tau)\rangle$  the instantaneous eigenstates at time  $\tau$ , with the eigenvalues  $\pm |p(\tau)|$ , respectively, we then have the adiabatic  $\Phi(T)$ ,

$$\Phi(T) \simeq e^{i[\gamma_+(T) - \int_0^T d\tau |p(\tau)|]} |v_+(\tau)\rangle \langle v_+(0)| + e^{i[\gamma_-(T) + \int_0^T d\tau |p(\tau)|]} |v_-(\tau)\rangle \langle v_-(0)|, \quad (20)$$

where  $\gamma_\pm(T)$  denotes the nonintegrable phase corresponding to each state.

The normalized instantaneous eigenstates  $|v_\pm(\tau)\rangle$  can, with suitable phase conventions, be written as

$$|v_+(\tau)\rangle = \begin{pmatrix} \cos \frac{\theta(\tau)}{2} e^{-i[\phi(\tau)/2]} \\ \sin \frac{\theta(\tau)}{2} e^{i[\phi(\tau)/2]} \end{pmatrix} \\ |v_-(\tau)\rangle = \begin{pmatrix} -\sin \frac{\theta(\tau)}{2} e^{-i[\phi(\tau)/2]} \\ \cos \frac{\theta(\tau)}{2} e^{i[\phi(\tau)/2]} \end{pmatrix}, \quad (21)$$

where  $|p|$ ,  $\theta$ , and  $\phi$  are the spherical coordinates of the vector  $p_\mu$ ,

$$p_0(\tau) = |p(\tau)| \cos \theta(\tau), \\ p_1(\tau) = |p(\tau)| \sin \theta(\tau) \cos \phi(\tau), \\ p_2(\tau) = |p(\tau)| \sin \theta(\tau) \sin \phi(\tau). \quad (22)$$

Then, the spinning nature of the field is evident from the phases  $\gamma_\pm(T)$ , which are given explicitly by

$$\begin{aligned}\gamma_{\pm}(T) &= i \int_0^T d\tau \langle v_{\pm}(\tau) | \frac{d}{d\tau} | v_{\pm}(\tau) \rangle \\ &= \pm \frac{1}{2} \int_0^T d\tau \frac{d\phi(\tau)}{d\tau} \cos[\theta(\tau)].\end{aligned}\quad (23)$$

For a closed path  $\mathcal{C}$  in the evolution of  $p_{\mu}(\tau)$ ,

$$\gamma_{\pm}(T) = \pm \frac{1}{2} \int_{\mathcal{C}} d\phi \cos \theta = \pm \frac{1}{2} \int_{S(\mathcal{C})} d\phi \wedge d \cos \theta, \quad (24)$$

where  $S(\mathcal{C})$  is a regular surface with  $\mathcal{C}$  as the boundary. Note that a closed path appears when the integral is evaluated with boundary conditions for the momenta, typically periodic, rather than the coordinates.

It is clear, either in its form (23) or (24), that the phases  $\gamma_{\pm}$  do correspond to actions that can be used for the quantization of a spin-1/2 degree of freedom [5], in the presence of an external magnetic field, in three spatial dimensions. The reason for their appearance here is of course the fact that the Lorentz group in  $2+1$  dimensions has been mapped into  $SO(3)$  by the Wick rotation. Those groups have different sets of irreducible representations. The spin- $\frac{1}{2}$  one, however, has a similar meaning and properties for both of them.

It is interesting to compare the situation here with the one in  $1+1$  dimensions, where the representation of Dirac's algebra is also constructed in terms of  $2 \times 2$  matrices, but only two of them appear in  $\not{p}(\tau)$ . It only takes a little amount of thought to see that the phases  $\gamma_{\pm}(T)$  vanish in this case.

Finally, we comment on the  $3+1$ -dimensional case. Now the  $\gamma$  matrices in the irreducible representation are of  $4 \times 4$  order, however, it is obvious that the eigenvalues  $\Phi(T)$  are still given by the expression  $e^{\mp i T |p|}$ , where  $|p| \equiv \sqrt{p_{\mu} p_{\mu}}$ . The main (and important) difference with the lower dimensional cases is that (being the  $\gamma$  matrices of order  $4 \times 4$ ), each eigenvalue is doubly degenerated.

Thus the Berry's connection shall be given by a non-Abelian  $SU(2)$  gauge field, and as a consequence the expression for the adiabatic phases cannot be given as explicitly as for the  $2+1$ -dimensional case.

### C. Local action representations in $2+1$ and $1+1$ dimensions

It may be desirable, in some contexts, to have a path-integral representation for  $\langle x | U(T) | y \rangle$  where the paths are integrated with a local weight, that can be defined in terms of an action functional. It is clear that the factor  $\Phi(T)$  is an obstruction to that goal, and that a suitable local action representation for that object would immediately solve the problem.

Recalling the magnetic field analogy already used in the previous subsection, we try to use Grassmann variables to represent the kernel for the  $\Phi(T)$  operator. That this can be done for a spin-1/2 particle in a *constant* magnetic-field background is a well-known fact. Indeed, for a Hamiltonian of the form

$$H = \mathbf{B} \cdot \boldsymbol{\sigma} \quad (25)$$

with a constant  $\mathbf{B}$ , we may take the  $x_3$  axis along the direction of  $\mathbf{B}$ , and write  $H$  as follows:

$$H = B(a^{\dagger}a - aa^{\dagger}), \quad (26)$$

where  $a$  and  $a^{\dagger}$  are Fermionic operators:  $a^2 = 0$ ,  $(a^{\dagger})^2 = 0$ ,  $\{a, a^{\dagger}\} = 1$ , and  $B = |\mathbf{B}|$ . Of course, in this two-dimensional Hilbert space, those operators may be understood as defined by the matrices

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^{\dagger} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (27)$$

The kernel of the evolution operator can be written in the holomorphic representation, where the operators defined above act on the space of “analytic” functions  $f(\xi) = a + b\xi$  where  $\xi$  is a Grassmann variable, with  $a, b \in \mathbb{C}$ , with the scalar product:

$$(f, g) = \int d\xi d\bar{\xi} e^{\bar{\xi}\xi} \overline{f(\xi)} g(\xi). \quad (28)$$

The action of the operators is  $a \rightarrow \partial_{\xi}$ ,  $a^{\dagger} \rightarrow \xi$ , and it is trivial to check that they are adjoint to each other for the scalar product defined in Eq. (28).

The kernel of  $\exp(-iTH)$  can then be represented as a functional integral,

$$\langle \xi | \exp(-iTH) | \bar{\xi} \rangle = \int \mathcal{D}\xi \mathcal{D}\bar{\xi} \exp[-S(\xi, \bar{\xi})], \quad (29)$$

where

$$S(\xi, \bar{\xi}) = \bar{\xi}(T)\xi(T) + \int_0^T d\tau [\bar{\xi}(\tau)\dot{\xi}(\tau) - 2iB\bar{\xi}(\tau)\xi(\tau) + iB], \quad (30)$$

and the paths in the functional integral (29) verify the boundary conditions  $\xi(T) = \xi$  and  $\bar{\xi}(0) = \bar{\xi}$ .

Things are different when the magnetic field depends on time, since then the Hamiltonian cannot, in general, be diagonalized by the same similarity transformation at all times. Indeed, for a general  $\tau$ -dependent  $p_{\mu}$ , we have to deal with the Hamiltonian

$$\begin{aligned}H(\tau) &= 2p_0(\tau)a^{\dagger}a - p_0(\tau) + [p_1(\tau) + ip_2(\tau)]a \\ &\quad + [p_1(\tau) - ip_2(\tau)]a^{\dagger}.\end{aligned}\quad (31)$$

If the goal is to implement the adiabatic approximation, it is then convenient to use  $H(\tau)$  in terms of its canonical diagonal form:

$$H(\tau) = |p(\tau)| V^{\dagger}(\tau) \sigma_3 V(\tau), \quad (32)$$

where

$$V(\tau) = \begin{pmatrix} \cos \frac{\theta(\tau)}{2} e^{i[\phi(\tau)/2]} & \sin \frac{\theta(\tau)}{2} e^{-i[\phi(\tau)/2]} \\ -\sin \frac{\theta(\tau)}{2} e^{i[\phi(\tau)/2]} & \cos \frac{\theta(\tau)}{2} e^{-i[\phi(\tau)/2]} \end{pmatrix} \quad (33)$$

is a unitary matrix that changes the basis to the instantaneous eigenstates. If now a functional integral representation is introduced, and the adiabatic approximation is made, it is evident to realize that the evolution operator will be similar to the one of the case (29), except for the fact that there will arise a contribution proportional to the diagonal elements of  $\partial_\tau V^\dagger(\tau)V(\tau)$ , and these are again the Berry's phases. Namely, one obtains

$$\langle \xi | \Phi(T) | \bar{\xi} \rangle = \int \mathcal{D}\xi \mathcal{D}\bar{\xi} \exp[-S_\Phi(\xi, \bar{\xi}; T)], \quad (34)$$

where

$$\begin{aligned} S_\Phi(\xi, \bar{\xi}; T) = & \bar{\xi}(T)\xi(T) + \int_0^T d\tau \left[ \bar{\xi}(\tau)\dot{\xi}(\tau) \right. \\ & - i \left( p_0(\tau) - \frac{1}{2} \frac{d\phi(\tau)}{d\tau} \cos \theta(\tau) \right) \bar{\xi}(\tau)\xi(\tau) \\ & \left. + i \left( p_0(\tau) - \frac{1}{2} \frac{d\phi(\tau)}{d\tau} \cos \theta(\tau) \right) \xi(\tau)\bar{\xi}(\tau) \right]. \end{aligned} \quad (35)$$

If no approximation is implemented, an exact path-integral representation can still be written; it corresponds to using an action

$$\begin{aligned} S_\Phi(\xi, \bar{\xi}; T) = & \bar{\xi}(T)\xi(T) + \int_0^T d\tau [\bar{\xi}\dot{\xi} - ip_0(\xi\bar{\xi} - \bar{\xi}\xi) \\ & - i(p_1 + ip_2)\xi - i(p_1 - ip_2)\bar{\xi}]. \end{aligned} \quad (36)$$

This can be inserted into Eq. (16), to derive the local action representation:

$$\begin{aligned} \langle x | (\mathcal{D} + m)^{-1} | y \rangle = & \int_0^\infty dT \int_{x(0)=y, \bar{\xi}(0)=\bar{\xi}}^{x(T)=x, \xi(T)=\xi} \mathcal{D}p \mathcal{D}x \mathcal{D}\xi \mathcal{D}\bar{\xi} \\ & \times \exp[-S(p, x, \xi, \bar{\xi}; T)], \end{aligned} \quad (37)$$

where

$$\begin{aligned} S(p, x, \xi, \bar{\xi}; T) = & \bar{\xi}(T)\xi(T) + \int_0^T d\tau [-ip \cdot \dot{x} + m + \bar{\xi}\dot{\xi} \\ & - ip_0(\xi\bar{\xi} - \bar{\xi}\xi) - i(p_1 + ip_2)\xi \\ & - i(p_1 - ip_2)\bar{\xi} + i\dot{x} \cdot A]. \end{aligned} \quad (38)$$

An important remark is in order regarding the last expression. In spite of the fact that the Grassmannian part of the action looks Gaussian, it cannot be integrated by the proce-

dure of “completing the square.” Indeed, there is a difference with the usual Gaussian integral in the fact that the source terms mix Grassmann and  $c$ -number variables. Besides, except for the case when the adiabatic approximation is used, the local action has the somewhat unpleasant property of having a nonvanishing Grassmann parity. However, that property is also present in other formulations of the particle path integral, since it is an unavoidable feature of any spinning particle propagator: the fact that it should be a matrix in some internal space means that we cannot do with a purely  $c$ -number action.

The corresponding result for the propagator in 1 + 1 dimensions comes at no extra price; indeed, adopting the convention that  $\gamma_0$  is represented by  $\sigma_1$  and  $\gamma_1$  by  $\sigma_2$ , we see that the analog of Eq. (31) is now

$$H(\tau) = [p_0(\tau) + ip_1(\tau)]a + [p_0(\tau) - ip_1(\tau)]a^\dagger. \quad (39)$$

Thus, in 1 + 1 dimensions, we have

$$\begin{aligned} \langle x | (\mathcal{D} + m)^{-1} | y \rangle = & \int_0^\infty dT \int_{x(0)=y, \bar{\xi}(0)=\bar{\xi}}^{x(T)=x, \xi(T)=\xi} \mathcal{D}p \mathcal{D}x \mathcal{D}\xi \mathcal{D}\bar{\xi} \\ & \times \exp[-S(p, x, \xi, \bar{\xi}; T)], \end{aligned} \quad (40)$$

where

$$\begin{aligned} S = & \bar{\xi}(T)\xi(T) + \int_0^T d\tau [-ip \cdot \dot{x} + m + \bar{\xi}\dot{\xi} - i(p_0 + ip_1)\xi \\ & - i(p_0 - ip_1)\bar{\xi} + i\dot{x} \cdot A]. \end{aligned} \quad (41)$$

#### D. Perturbative expansion

An important check the functional representation must pass is that it should reproduce (at least) the perturbative, small- $e$ , expansion for the fermion propagator in an external field. To do that, we expand the exponential inside the functional integral of Eq. (16). Using the symbol  $\mathcal{G}(x, y)$  to denote the functional integral representation (16), we see that

$$\mathcal{G}(x, y) = \sum_{n=0}^{\infty} \mathcal{G}^{(n)}(x, y), \quad (42)$$

where  $\mathcal{G}^{(0)}$  is the zeroth-order term,

$$\begin{aligned} \mathcal{G}^{(0)}(x, y) = & \int_0^\infty dT \int_{x(0)=y}^{x(T)=x} \mathcal{D}p \mathcal{D}x e^{\int_0^T d\tau [ip \cdot \dot{x} - m]} \mathcal{P}[e^{-i\int_0^T d\tau \bar{\psi}(\tau)}], \end{aligned} \quad (43)$$

which, as shown in Ref. [7], correctly reproduces the free propagator:

$$\mathcal{G}^{(0)}(x, y) = \int \frac{d^3 p}{(2\pi)^3} \frac{e^{ip \cdot (x-y)}}{i\not{p} + m}. \quad (44)$$



There is an auxiliary identity involving the functional integral that appears in expression (43) for the free propagator that shall be useful in what follows. It can be derived from the fact that the integral

$$\int_{x(0)=y}^{x(T)=x} \mathcal{D}p \mathcal{D}x e^{\int_0^T d\tau [ip \cdot \dot{x} - m]} \mathcal{P}[e^{-i \int_0^T d\tau \phi(\tau)}] \quad (45)$$

is *independent* of the boundary values of the momentum. Thus

$$0 = \int_{x(0)=y}^{x(T)=x} \mathcal{D}p \mathcal{D}x \frac{\delta}{\delta p_\mu(T)} e^{\int_0^T d\tau [ip \cdot \dot{x} - m]} \mathcal{P} e^{-i \int_0^T d\tau \phi} \quad (46)$$

and an analogous equation for  $p(T) \leftrightarrow p(0)$ . Then we derive the identities

$$\begin{aligned} & \int_{x(0)=y}^{x(T)=x} \mathcal{D}p \mathcal{D}x \dot{x}_\mu(T) e^{\int_0^T d\tau [ip \cdot \dot{x} - m]} \mathcal{P} e^{-i \int_0^T d\tau \phi} \\ &= \gamma_\mu \int_{x(0)=y}^{x(T)=x} \mathcal{D}p \mathcal{D}x e^{\int_0^T d\tau [ip \cdot \dot{x} - m]} \mathcal{P} e^{-i \int_0^T d\tau \phi} \end{aligned} \quad (47)$$

and

$$\begin{aligned} & \int_{x(0)=y}^{x(T)=x} \mathcal{D}p \mathcal{D}x \dot{x}_\mu(0) e^{\int_0^T d\tau [ip \cdot \dot{x} - m]} \mathcal{P} e^{-i \int_0^T d\tau \phi} \\ &= \int_{x(0)=y}^{x(T)=x} \mathcal{D}p \mathcal{D}x e^{\int_0^T d\tau [ip \cdot \dot{x} - m]} \mathcal{P} e^{-i \int_0^T d\tau \phi} \gamma_\mu. \end{aligned} \quad (48)$$

It is important to remember that the functional integral has a matrixlike weight, so that  $\gamma_\mu$  cannot be freely commuted with it. Besides, in both of the previous expressions, one cannot move  $\dot{x}_\mu(T)$  and  $\dot{x}_\mu(0)$  out of the integral symbol, since their values are integrated out, because they are not fixed by the boundary conditions on  $x_\mu$ . Finally, both Eqs. (47) and (48) can also be easily proven to hold true in the safer, regulated context of the discretized path integral.

The term of order  $n$  is given by

$$\begin{aligned} \mathcal{G}^{(n)}(x, y) &= \frac{(-ie)^n}{n!} \int_0^\infty dT \int_0^T d\tau_1 \cdots \\ &\times \int_0^T d\tau_n \int_{x(0)=y}^{x(T)=x} \mathcal{D}p \mathcal{D}x e^{\int_0^T d\tau [ip \cdot \dot{x} - m]} \\ &\times \mathcal{P}[e^{-i \int_0^T d\tau \phi(\tau)}] \dot{x}_{\mu_1}(\tau_1) A_{\mu_1}[x(\tau_1)] \cdots \\ &\times \bar{x}_{\mu_n}(\tau_n) A_{\mu_n}[x(\tau_n)]. \end{aligned} \quad (49)$$

The integral over the “intermediate times”  $\tau_1, \dots, \tau_n$  is obviously symmetric under permutations of the  $\tau_i$ . For each possible ordering among them, we now select the maximum time, irrespective of the ordering among the remaining times. We then rename that maximum time as “ $\tau_1$ .” Obviously, there are  $n$  possible contributions to take into account, thus we may write  $\mathcal{G}^{(n)}$  in the equivalent way,

$$\begin{aligned} \mathcal{G}^{(n)}(x, y) &= \frac{(-ie)^n}{(n-1)!} \int_0^\infty dT \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \\ &\times \int_0^{\tau_1} d\tau_n \int_{x(0)=y}^{x(T)=x} \mathcal{D}p \mathcal{D}x e^{\int_0^T d\tau [ip \cdot \dot{x} - m]} \\ &\times \mathcal{P}(e^{-i \int_0^T d\tau \phi}) \dot{x}(\tau_1) \cdot A[x(\tau_1)] \cdots \\ &\times \bar{x}(\tau_n) \cdot A[x(\tau_n)], \end{aligned} \quad (50)$$

where the times  $\tau_i$ , with  $i \neq 1$ , have  $\tau_1$  as their new upper value.

The paths being integrated out in the path integral can then be split at the time  $\tau_1$  by an application of the “superposition principle” for path integrals, which then requires the value of  $x(\tau)$  at the time  $\tau_1$  to be integrated over all its possible values  $x(\tau_1) = z$ . Using also Eq. (48), we see that

$$\begin{aligned} \mathcal{G}^{(n)}(x, y) &= \frac{(-ie)^n}{(n-1)!} \int_0^\infty dT \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_1} d\tau_n \int d^3z \int_{x(\tau_1)=z}^{x(T)=x} \mathcal{D}p \mathcal{D}x e^{\int_{\tau_1}^T d\tau [ip \cdot \dot{x} - m]} \\ &\times \mathcal{P}[e^{-i \int_{\tau_1}^T d\tau \phi}] A(z) \int_{x(0)=y}^{x(\tau_1)=z} \mathcal{D}p \mathcal{D}x e^{\int_0^{\tau_1} d\tau [ip \cdot \dot{x} - m]} \\ &\times \mathcal{P}[e^{-i \int_0^{\tau_1} d\tau \phi}] \dot{x}(\tau_2) \cdot A[x(\tau_2)] \cdots \dot{x}(\tau_n) \cdot A[x(\tau_n)]. \end{aligned} \quad (51)$$

Focusing now on the two time integrals which involve the variables  $T$  and  $\tau_1$ , we see that their order of integration can be interchanged, if one properly modifies the integration ranges:

$$\int_0^\infty dT \int_0^T d\tau_1 \cdots = \int_0^\infty d\tau_1 \int_{\tau_1}^\infty dT \cdots \quad (52)$$

Interchanging those two integrations, and making also some trivial rearrangements, we see that

$$\begin{aligned}
\mathcal{G}^{(n)}(x, y) = & -ie \int d^3z \int_0^\infty d\tau_1 \int_{\tau_1}^\infty dT \int_{x(\tau_1)=z}^{x(T)=x} \mathcal{D}p \mathcal{D}x e^{\int_{\tau_1}^T d\tau [ip \cdot \dot{x} - m]} \mathcal{P}(e^{-i \int_{\tau_1}^T d\tau \phi}) \\
& \times \mathcal{A}(z) \frac{(-ie)^{n-1}}{(n-1)!} \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_1} d\tau_n \int_{x(0)=y}^{x(\tau_1)=z} \mathcal{D}p \mathcal{D}x e^{\int_0^{\tau_1} d\tau [ip \cdot \dot{x} - m]} \\
& \times \mathcal{P}(e^{-i \int_0^{\tau_1} d\tau \phi}) \dot{x}(\tau_2) \cdot A[x(\tau_2)] \cdots \dot{x}(\tau_n) \cdot A[x(\tau_n)].
\end{aligned} \tag{53}$$

Noting that

$$\begin{aligned}
& \int_{\tau_1}^\infty dT \int_{x(\tau_1)=z}^{x(T)=x} \mathcal{D}p \mathcal{D}x e^{\int_{\tau_1}^T d\tau [ip \cdot \dot{x} - m]} \mathcal{P}[e^{-i \int_{\tau_1}^T d\tau \phi(\tau)}] \\
& = \mathcal{G}^{(0)}(x, z)
\end{aligned} \tag{54}$$

and recalling the expression for the order- $n$  contribution, Eq. (49), we are led to the relation

$$\mathcal{G}^{(n)}(x, y) = - \int d^3z \mathcal{G}^{(0)}(x, z) ie \mathcal{A}(z) \mathcal{G}^{(n-1)}(z, y) \tag{55}$$

( $\forall n \geq 1$ ), which is, indeed, equivalent to the usual perturbative expansion for the propagator:

$$\mathcal{G}^{(n)} = \mathcal{G}^{(0)} - ie \mathcal{G}^{(0)} \mathcal{A} \mathcal{G}^{(0)} + (ie)^2 \mathcal{G}^{(0)} \mathcal{A} \mathcal{G}^{(0)} \mathcal{A} \mathcal{G}^{(0)} + \dots \tag{56}$$

It should be evident that the fact that we have not used the “local action” representation is not crucial to the previous derivations. It is, indeed, possible to encompass all the changes that proceeding otherwise would produce. The main differences arise of course in relations (47) and (48), since one does not have the  $\gamma$  matrices. It is, however, far from difficult to see that, in the corresponding local action representation, the equivalent identities relate integrals with components of  $\dot{x}_\mu(T)$  to integrals with Grassmann variables. For example,

$$\begin{aligned}
& \int \mathcal{D}p \mathcal{D}x \mathcal{D}\xi \mathcal{D}\bar{\xi} \frac{(\dot{x}_1 + i\dot{x}_2)}{2}(T) \exp[-\mathcal{S}(p, x, \xi, \bar{\xi}; T)] \\
& = \int \mathcal{D}p \mathcal{D}x \mathcal{D}\xi \mathcal{D}\bar{\xi} \bar{\xi}(T) \exp[-\mathcal{S}(p, x, \xi, \bar{\xi}; T)],
\end{aligned} \tag{57}$$

where both integrals are evaluated with the boundary conditions  $x(0)=y$ ,  $x(T)=y$  and  $\bar{\xi}(0)=\bar{\xi}$ ,  $\xi(T)=\xi$ .

On the other hand, integrals involving  $\dot{x}_0$  will be related to Grassmann bilinears. We shall not proceed, however, with the derivation of the perturbative expansion in that setting, since it would necessarily require, at some point, to reintroduce the  $\gamma$  matrices.

We wish to stress, however, that a perfectly consistent perturbative expansion could be built in terms of the local representation, without using the  $\gamma$  matrices explicitly.

### III. CHIRAL ANOMALY AND CHERN-SIMONS CURRENT

In this section we perform another test on the method, with the derivation of two nonperturbative objects: the chiral anomaly in 1 + 1 dimensions and the Chern-Simons term in 2 + 1 dimensions. They have of course been evaluated in the particle functional integral framework [6]; our aim is to show how to obtain them directly from the functional integral representation (16), by evaluating the corresponding vacuum currents. Besides, the role of the regularization is, as we shall see, more transparent in this calculation.

We shall first deal with the chiral anomaly in 1 + 1 dimensions, since this example already exhibits all the difficulties and properties of the evaluation of topological terms in this representation. Moreover, we shall use a gauge invariant Pauli-Villars regularization, that can be introduced smoothly within the representation we are dealing with, since it requires the introduction of (just) one fermion propagator.

It is very well known [8,9] that  $\mathcal{A}$ , the anomalous divergence of the axial current  $J_\mu^5 = \bar{\psi} \gamma_\mu \gamma_5 \psi$ ,

$$\partial_\mu J_\mu^5(x) = \mathcal{A}(x), \tag{58}$$

may be obtained from the regulated trace of the  $\gamma_5$  matrix, namely,

$$\mathcal{A}(x) = \lim_{\Lambda \rightarrow \infty} \mathcal{A}_\Lambda(x), \tag{59}$$

where

$$\mathcal{A}_\Lambda(x) = -2 \operatorname{tr} \left\{ \gamma_5 \left[ \langle x | f \left( \frac{i \not{D}}{\Lambda} \right) | x \rangle \right] \right\}, \tag{60}$$

where  $f$  is a function chosen in order to tame the UV divergences, which means that it has to satisfy

$$f(0) = 1, \quad f(\pm\infty) = f'(\pm\infty) = f^{(2)}(\pm\infty) = \dots = 0. \tag{61}$$

Of course, this is a gauge-invariant regularization, and, moreover, the results, are independent of the detailed form of  $f$ , as long as it verifies the previous conditions [8]. The particular choice  $f(u) = (1 + u^2)^{-1}$  is very convenient, since we can use a simple fractions decomposition and the fact that  $\gamma_5$  anticommutes with  $\gamma_\mu$  to write Eq. (60) as follows:

$$\mathcal{A}_\Lambda(x) = -2 \operatorname{tr} \left\{ \gamma_5 \left[ \langle x | \frac{\Lambda}{\not{D} + \Lambda} | x \rangle \right] \right\}. \tag{62}$$

Then we apply the general expression (16), with  $x=y$  and  $m=\Lambda$ , to write the fermion propagator that appears in Eq. (62) as a particle path integral, obtaining

$$\begin{aligned} \mathcal{A}_\Lambda(x) = & -2\Lambda \int_0^\infty dT \int_{x(0)=x}^{x(T)=x} \mathcal{D}p \mathcal{D}x e^{\int_0^T d\tau [ip \cdot \dot{x} - \Lambda]} \\ & \times \text{tr}\{\gamma_5 \mathcal{P}[e^{-i\int_0^T d\tau \dot{p}(\tau)}]\} e^{-ie\int_0^T d\tau \dot{x}(\tau) \cdot A[x(\tau)]}. \end{aligned} \quad (63)$$

The constant  $\Lambda$  may be absorbed in a redefinition of  $T$ :  $t = \Lambda T$  is now a dimensionless “time,” while we also introduce  $s = \Lambda \tau$  for the “proper time” that appears inside the integrals. Then, expressing all the functions in terms of the new variables,

$$\begin{aligned} \mathcal{A}_\Lambda(x) = & -2 \int_0^\infty dt \int_{x(0)=x}^{x(t)=x} \mathcal{D}p \mathcal{D}x e^{\int_0^t ds [ip \cdot (dx/ds) - 1]} \\ & \times \text{tr}\{\gamma_5 \mathcal{P}[e^{-(i/\Lambda)\int_0^t ds \dot{p}(s)}]\} e^{-ie\int_0^t ds \dot{x}(s) \cdot A[x(s)]}, \end{aligned} \quad (64)$$

where the  $x$  which appears in the boundary conditions does not have to be integrated, but it is the (fixed) value corresponding to the argument of the current operator.

Following a similar technique (but a different notation) to the one used in Ref. [6] to evaluate the  $\theta$  vacua term, we introduce the change of variables:

$$x_\mu(s) = x_\mu(0) + \int_0^s d\tilde{s} \zeta_\mu(\tilde{s}), \quad x(0) \equiv x, \quad (65)$$

which has a trivial Jacobian, so that  $\mathcal{D}x = \mathcal{D}\zeta$ , and Eq. (64) becomes

$$\begin{aligned} \mathcal{A}_\Lambda(x) = & -2 \int_0^\infty dt \int \mathcal{D}p \mathcal{D}\zeta e^{\int_0^t ds [ip \cdot \zeta - 1]} \\ & \times \text{tr}\{\gamma_5 \mathcal{P}[e^{-(i/\Lambda)\int_0^t ds \dot{p}(s)}]\} \\ & \times e^{-ie\int_0^t ds \zeta(s) \cdot A[x(0) + \int_0^s d\tilde{s} \zeta(\tilde{s})]}. \end{aligned} \quad (66)$$

It is important to note that the boundary conditions for  $x$  in Eq. (63) mean that the  $\zeta$  variable has to verify the constraints

$$\int_0^t ds \zeta_\mu(s) = 0, \quad \mu = 0, 1; \quad (67)$$

the existence of those constraints will be indicated by a  $'$  in the integral symbol, when rewriting the  $x$  integral of Eq. (64) in terms of the new variables.

Then we take advantage of the fact that the result has to be a local polynomial in  $A$ , to write

$$\begin{aligned} \mathcal{A}_\Lambda(x) = & -2 \int_0^\infty dt \int \mathcal{D}p \text{tr}\{\gamma_5 \mathcal{P}[e^{-(i/\Lambda)\int_0^t ds \dot{p}(s)}]\} \\ & \times e^{-ie\int_0^t ds [\delta/i \delta p(s)] \cdot A\left[x(0) + \int_0^s d\tilde{s} [\delta/i \delta p(\tilde{s})]\right]} \\ & \times \int' \mathcal{D}\zeta e^{\int_0^t ds [ip \cdot \zeta - 1]}. \end{aligned} \quad (68)$$

The functional integral over  $\zeta$ , including the constraint, can be explicitly evaluated, for example, by including the constraint through the addition of (yet) another Lagrange multiplier, i.e.,

$$\begin{aligned} \int' \mathcal{D}\zeta e^{\int_0^t ds [ip \cdot \zeta - 1]} &= \int \mathcal{D}\zeta \int \frac{d^2 w}{(2\pi)^2} e^{i\int_0^t ds (p \cdot \zeta + w \cdot \zeta)} e^{-t} \\ &= \int \frac{d^2 w}{(2\pi)^2} \delta[p + w] e^{-t}. \end{aligned} \quad (69)$$

It should be noted that  $\delta[p - w]$  is a functional  $\delta$ , and that the integral over  $w$  is a relic of the constraint over the  $\zeta$  integration. Thus we arrive to a more tractable expression for the anomaly,

$$\begin{aligned} \mathcal{A}_\Lambda(x) = & -2 \int_0^\infty dt e^{-t} \int \frac{d^2 w}{(2\pi)^2} \int \mathcal{D}p \text{tr}(\gamma_5 \mathcal{P} e^{-(i/\Lambda)\int_0^t ds \dot{p}}) \\ & \times e^{-ie\int_0^t ds [\delta/i \delta p(s)] \cdot A\left[x(0) + \int_0^s d\tilde{s} [\delta/i \delta p(\tilde{s})]\right]} \delta[p + w]. \end{aligned} \quad (70)$$

A simple power-counting argument shows that by expanding the  $e$  dependent term in the exponential, only the first-order term will contribute when  $\Lambda \rightarrow \infty$ . Moreover, in that term just the one which is of order 2 in the functional derivative over  $p$  survives (as it is seen *a posteriori*). Thus the  $p$  integration can also be explicitly performed:

$$\mathcal{A}_\Lambda(x) \simeq \partial_\mu A_\nu(x) T_{\mu\nu} \quad (\Lambda \sim \infty), \quad (71)$$

where  $T_{\mu\nu}$  is a constant tensor, given explicitly by

$$\begin{aligned} T_{\mu\nu} = & -2ie \int \frac{d^2 w}{(2\pi)^2} \int_0^\infty dt e^{-t} \int_0^t ds \int_0^s ds' \\ & \times \frac{\delta^2 \Phi[p]}{\delta p_\nu(s) \delta p_\mu(s')} \Big|_{p=-\omega}, \end{aligned} \quad (72)$$

where

$$\Phi[p] = \text{tr}[\gamma_5 \mathcal{P} e^{-(i/\Lambda)\int_0^t ds \dot{p}(s)}], \quad (73)$$

and  $p$  in Eq. (72) is set equal to the constant  $-\omega$  after the functional differentiation.



For the evaluation of  $T_{\mu\nu}$ , we note that

$$\begin{aligned} T_{\mu\nu} &= i \frac{e}{\Lambda^2} \int \frac{d^2 w}{(2\pi)^2} \int_0^\infty dt t^2 e^{-t} \text{tr}[\gamma_5 \gamma_\mu \gamma_\nu e^{i(t/\Lambda)\phi}] \\ &= -\epsilon_{\mu\nu} \frac{e}{\Lambda^2} \int \frac{d^2 w}{(2\pi)^2} \int_0^\infty dt t^2 e^{-t} \text{tr}[e^{i(t/\Lambda)\phi}], \end{aligned} \quad (74)$$

where we have used that

$$\gamma_\mu \gamma_\nu = \delta_{\mu\nu} I + i \epsilon_{\mu\nu} \gamma_5, \quad (75)$$

with  $I$  the identity matrix. Finally,

$$T_{\mu\nu} = -\epsilon_{\mu\nu} \frac{2e}{\Lambda^2} \int \frac{d^2 w}{(2\pi)^2} \text{tr} \left[ \frac{1}{\left(1 - \frac{i}{\Lambda} \phi\right)^3} \right]. \quad (76)$$

The  $\omega$  integration is convergent and its result is proportional to  $\Lambda^2$ , which cancels out the  $\Lambda^{-2}$  factor,

$$\begin{aligned} T_{\mu\nu} &= -\epsilon_{\mu\nu} \frac{e}{\pi} \int_0^\infty dx \frac{1-3x}{(1+x)^3} \\ &= \frac{e}{\pi} \epsilon_{\mu\nu}. \end{aligned} \quad (77)$$

Then we conclude that the anomaly  $\mathcal{A}(x)$  has the proper result, namely,

$$\mathcal{A}(x) = \frac{e}{\pi} \epsilon_{\mu\nu} \partial_\mu A_\nu(x). \quad (78)$$

Note that, because of the gauge invariant regularization procedure, we already knew that  $T_{\mu\nu}$  had to be proportional to  $\epsilon_{\mu\nu}$ .

Let us conclude this section with the evaluation of a related object: the vacuum Chern-Simons current for the Abelian case, in 2+1 dimensions, as determined by the parity anomaly. The vacuum Chern-Simons current, in a Pauli-Villars-like regularization, is given by

$$J_\mu(x) = -\text{tr}[\gamma_\mu \langle x | (\mathcal{D} + M)^{-1} | x \rangle], \quad (79)$$

where  $M \rightarrow \infty$ , and all the objects are assumed to be defined in 2+1 Euclidean dimensions. An entirely analogous derivation yields

$$J_\mu(x) \simeq -R_{\mu\nu\lambda} \partial_\nu A_\lambda \quad (\Lambda \sim \infty), \quad (80)$$

where

$$R_{\mu\nu\lambda} = -i \frac{e}{\Lambda} \int \frac{d^3 w}{(2\pi)^3} \int_0^\infty dt e^{-t} \int_0^s ds' \frac{\delta^2 \Gamma_\mu[p]}{\delta p_\lambda(s) \delta p_\nu(s')} \Big|_{p=\omega} \quad (81)$$

with

$$\Gamma_\mu[p] = \text{tr}[\gamma_\mu \mathcal{P} e^{-(i/\Lambda) \int_0^t ds \dot{p}(s)}]. \quad (82)$$

More explicitly,

$$R_{\mu\nu\lambda} = -\frac{e}{\Lambda^3} \epsilon_{\mu\nu\lambda} \int \frac{d^3 w}{(2\pi)^3} \text{tr} \left[ \frac{1}{(1-i\phi)^3} \right]. \quad (83)$$

A standard evaluation of the momentum integral yields

$$R_{\mu\nu\lambda} = \frac{e}{2\pi} \epsilon_{\mu\nu\lambda}. \quad (84)$$

Finally,

$$J_\mu = \frac{e}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda, \quad (85)$$

which again reproduces the correct result.

It is worth noting that no extra regularization has been introduced in order to regularize the anomaly and the Chern-Simons term, which is different from previous calculations in this context. Some of them seem to indicate that a space resolution scale plays an important role. We have seen, however, that the usual UV regulator does the trick.

#### IV. PROPAGATION IN A CONSTANT ELECTROMAGNETIC FIELD

We calculate here the Fermionic determinant for a massive Dirac field in the presence of an external constant electromagnetic field, in 2+1 Euclidean dimensions. We define

$$\exp[-\Gamma(A)] = \det(\mathcal{D} + m), \quad (86)$$

where the gauge field entering the covariant derivative is such that

$$\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} = \text{const.} \quad (87)$$

In a symmetric or coordinate gauge, we can adopt the configuration

$$A_\mu(x) = -\frac{1}{2} F_{\mu\nu} x_\nu = \frac{1}{2} \epsilon_{\mu\rho\nu} \tilde{F}_\rho x_\nu. \quad (88)$$

We consider here a gauge-field configuration where only two components of  $A_\mu$  are nonvanishing, and lead to a constant electromagnetic field. The parity-odd part of the effective action is zero for this sort of configuration, since

$$A_\mu \epsilon_{\mu\nu\rho} \partial_\nu A_\rho = A_\mu \tilde{F}_\mu = 0. \quad (89)$$

To evaluate  $\Gamma(A)$ , we use an integral representation for the logarithm; as usual we apply the formula

$$\ln(a/b) = -\int_0^\infty \frac{dT}{T} (e^{-Ta} - e^{-Tb}) \quad (90)$$

( $\text{Re}(a) > 0$ ,  $\text{Re}(b) > 0$ ) so that

$$\begin{aligned}\tilde{\Gamma}(A) &\equiv \Gamma(A) - \Gamma(0) \\ &= \int_0^\infty \frac{dT}{T} \text{Tr}[e^{-T(\hat{b}+m)} - e^{-T(\hat{b}+m)}].\end{aligned}\quad (91)$$

Introducing then the particle path-integral representation, we see that

$$\begin{aligned}\Gamma(A) &= \int_0^T \frac{dT}{T} e^{-mT} \int \mathcal{D}p \text{tr}(\mathcal{P}e^{-i\int_0^T d\tau \hat{p}}) \\ &\quad \times \int \mathcal{D}x e^{(ie/2)\int_0^T d\tau \dot{x}_\mu F_{\mu\nu} \dot{x}_\nu + i\int_0^T d\tau p_\mu \dot{x}_\mu},\end{aligned}\quad (92)$$

which differs from Eq. (91) by an (infinite) field-independent constant.

We now proceed to evaluate  $I[p]$ , the Gaussian integral over periodic  $x_\mu(\tau)$  paths in Eq. (92). After an integration by parts,

$$\begin{aligned}I[p] &= \int_{x(0)=x(T)} \mathcal{D}x \exp\left(\frac{ie}{2} \int_0^T d\tau x_\mu(\tau) \epsilon_{\mu\rho\nu} \tilde{F}_\rho \dot{x}_\nu(\tau) \right. \\ &\quad \left. - i \int_0^T d\tau x(\tau) \cdot \dot{p}(\tau) + x(0) \cdot [p(T) - p(0)]\right).\end{aligned}\quad (93)$$

The role of the last term is, after integrating over  $x(0)$ , to enforce the condition  $p(0)=p(T)$ . We can then erase that term when evaluating  $I[p]$ , while keeping in mind the fact that the remaining functional integral (over  $p$ ) in Eq. (92) must be evaluated with periodic boundary conditions for  $p(\tau)$ .

We now decompose  $x_\mu(\tau)$  into parallel ( $x^\parallel$ ) and transverse ( $x^\perp$ ) components along  $\tilde{F}$ ,

$$x_\mu(\tau) = x_\mu^\parallel(\tau) + x_\mu^\perp(\tau), \quad (94)$$

where  $x_\mu^\parallel(\tau)$  is the projection of  $x_\mu(\tau)$  along the direction of  $\tilde{F}_\mu$ , and  $x_\mu^\perp(\tau)$  its orthogonal component ( $x^\perp \cdot \tilde{F} = 0$ ). It is evident that the parallel component does not appear in the quadratic part of the exponent in Eq. (93), so that its integration yields a functional  $\delta$  of the derivative of the longitudinal component of the momentum:

$$\begin{aligned}I[p] &= \delta[\dot{p}^\parallel] \frac{L}{2\pi} \int_{x^\perp(0)=x^\perp(T)} \mathcal{D}x^\perp \\ &\quad \times e^{(ie/2)\int_0^T d\tau x_\mu^\perp \epsilon_{\mu\rho\nu} \tilde{F}_\rho \dot{x}_\nu^\perp - i \int_0^T d\tau x_\mu^\perp \dot{p}_\mu^\perp},\end{aligned}\quad (95)$$

where  $L$  is the (infinite) size of the integration in the parallel direction. The Gaussian integral over the two remaining components of  $x$  can now be performed, and the result may be written in terms of the two transverse components of the momentum. Using a coordinate system such that  $p_1$  and  $p_2$  denote the transverse components, while  $\tilde{F}_\mu$  points in the 0 direction, we see that

$$\begin{aligned}I(p) &= \delta[\dot{p}^\parallel] \frac{L}{2\pi} \int \mathcal{D}x_1 \mathcal{D}x_2 \exp\left(ie\tilde{F}_0 \int_0^T d\tau \dot{x}_1 x_2 \right. \\ &\quad \left. + i \int_0^T d\tau [\dot{x}_1(\tau) p_1(\tau) + \dot{x}_2(\tau) p_2(\tau)]\right).\end{aligned}\quad (96)$$

Introducing now an auxiliary variable  $w_1$  and the velocity  $\zeta_1$  for the  $x_1$  coordinate we have

$$\begin{aligned}I(p) &= \delta[\dot{p}^\parallel] \frac{L}{2\pi} \int \frac{dw_1}{2\pi} \mathcal{D}\zeta_1 \mathcal{D}x_2 \exp\left(ie\tilde{F}_0 \int_0^T d\tau \zeta_1(\tau) x_2(\tau) \right. \\ &\quad \left. + i \int_0^T d\tau \{\zeta_1(\tau) [p_1(\tau) + w_1] + \dot{x}_2(\tau) p_2(\tau)\}\right),\end{aligned}\quad (97)$$

where  $w_1$  is independent of  $\tau$ . Integrating now over  $x_2$  and  $\zeta_1$  we arrive at

$$\begin{aligned}I(p) &= \delta[\dot{p}^\parallel] \frac{L}{2\pi} \int \frac{dw_1}{2\pi} \int dx_2(0) \exp\left(\frac{i}{e\tilde{F}_0} \int_0^T d\tau \dot{p}_1(\tau) p_2(\tau) \right. \\ &\quad \left. + i \frac{1}{e\tilde{F}_0} w_1 [p_2(T) - p_2(0)]\right).\end{aligned}\quad (98)$$

The integral over  $x_2(0)$  is proportional to the total length of the space. On the other hand, the integral over  $w_1$  gives a delta function which produces a  $e\tilde{F}_0$  factor, and an additional total length factor. So we arrive at

$$\begin{aligned}I[p] &= \delta[\dot{p}^\parallel] \frac{V}{(2\pi)^2} e\tilde{F}_0 \left[ \det\left(\frac{-e\tilde{F}_0 \epsilon_{jk} \partial_\tau}{2\pi i}\right) \right]^{-1/2} \\ &\quad \times e^{(ie/2)\int_0^T d\tau p_j \epsilon_{jk} \partial_\tau p_k},\end{aligned}\quad (99)$$

where  $V$  is the total (infinite) space time volume. For the sake of simplicity, we shall assume in what follows that  $e\tilde{F}_0 > 0$ .

Using the result (99), we may now calculate the remaining functional integral over  $p_\mu$  in Eq. (92). Indeed, the longitudinal component of  $p$  shall be constant, so it can be extracted out of the path-ordering symbol, while for the two transverse components we take advantage of the following fact: functional integrating an ordered product of transverse components with the quadratic weight above amounts to taking a trace over the Hilbert space corresponding to the two noncommuting operators  $\hat{p}_1$  and  $\hat{p}_2$ :

$$[\hat{p}_1, \hat{p}_2] = -ie\tilde{F}_0. \quad (100)$$

More explicitly, we repeatedly use the property

$$\frac{\int_{p(0)=p(T)} \mathcal{D}p^\perp \mathcal{P}[p_{i_1}(\tau_1) \cdots p_{i_n}(\tau_n)] e^{(i/2e\tilde{F}_0) \int_0^T d\tau p_j \epsilon_{jk} \partial_\tau p_k}}{\int \mathcal{D}p^\perp e^{(i/2e\tilde{F}_0) \int_0^T d\tau p_j \epsilon_{jk} \partial_\tau p_k}} = \text{Tr}[\hat{p}_{i_1}(\tau_1) \cdots \hat{p}_{i_n}(\tau_n)], \quad (101)$$

where the indices  $i_1 \cdots i_n$  can only take the values 1 and 2. A possible way to prove Eq. (101) is to write its left-hand side in operatorial form. Since the Hamiltonian that dictates the  $\tau$  evolution vanishes, the expression on the right-hand side follows.

Thus we arrive to the following expression for  $\Gamma(A)$ :

$$\Gamma(A) = \frac{V}{(2\pi)^2} e\tilde{F}_0 \int_{0^+}^{\infty} \frac{dT}{T} e^{-mT} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \text{Tr}(e^{-iT\hat{p}} e^{-iT\gamma_0 p_0}). \quad (102)$$

In terms of the representation  $\gamma_0 = \sigma_3$ ,  $\gamma_1 = \sigma_1$ , and  $\gamma_2 = \sigma_2$ , we have

$$\Gamma(A) = \frac{V}{(2\pi)^2} e\tilde{F}_0 \int_{0^+}^{\infty} \frac{dT}{T} e^{-mT} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \times \text{Tr}[e^{-iT\sqrt{2e\tilde{F}_0}\hat{a}} e^{-iTp_0\sigma_3}], \quad (103)$$

where

$$\hat{a} = \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^\dagger & 0 \end{pmatrix} \quad (104)$$

with  $\hat{a} = (\hat{p}_1 - i\hat{p}_2)/\sqrt{2e\tilde{F}_0}$  and  $[\hat{a}, \hat{a}^\dagger] = 1$ . Since the integration over  $p_0$  is over an interval symmetric about 0, we use the fact that

$$\int_{-\infty}^{+\infty} \frac{dp_0}{2\pi} \sin(Tp_0) = 0 \quad (105)$$

to write

$$\Gamma(A) = \frac{V}{2\pi^2} e\tilde{F}_0 \int_{0^+}^{\infty} \frac{dT}{T} e^{-mT} \int_0^{+\infty} \frac{dp_0}{2\pi} \times \cos(Tp_0) \text{Tr}[e^{-iT\sqrt{2e\tilde{F}_0}\hat{a}}]. \quad (106)$$

On the other hand, it is simple to show that the operator

$$\hat{a} = \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^\dagger & 0 \end{pmatrix} \quad (107)$$

has the eigenvalues  $\pm\sqrt{n}$ , with  $n = \pm 1, \pm 2, \dots$ , and no degeneracy. Thus

$$\Gamma(A) = \frac{V}{\pi^2} e\tilde{F}_0 \int_{0^+}^{\infty} \frac{dT}{T} e^{-mT} \int_0^{+\infty} \frac{dp_0}{2\pi} \cos(Tp_0) \times \sum_{n=1}^{\infty} \cos[T(2e\tilde{F}_0 n)^{1/2}]. \quad (108)$$

We take advantage of the explicit dependence of this result on the external field to subtract the zero-field contribution,

$$\tilde{\Gamma}(A) = \frac{V}{\pi^2} e\tilde{F}_0 \int_{0^+}^{\infty} \frac{dT}{T} e^{-mT} \int_0^{+\infty} \frac{dp_0}{2\pi} \cos(Tp_0) \times \sum_{n=1}^{\infty} \{\cos[T(2e\tilde{F}_0 n)^{1/2}] - 1\}, \quad (109)$$

and after performing the  $T$  integration,

$$\tilde{\Gamma}(A) = -\frac{V}{(2\pi)^2} e\tilde{F}_0 \sum_{n=1}^{\infty} \int_0^{+\infty} \frac{dp_0}{2\pi} \left[ \ln \left( \frac{(p_0 - \kappa_n)^2 + m^2}{p_0^2 + m^2} \right) + \ln \left( \frac{(p_0 + \kappa_n)^2 + m^2}{p_0^2 + m^2} \right) \right], \quad (110)$$

where we introduced  $\kappa_n = \sqrt{2e\tilde{F}_0 n}$ . Finally, the contributions may be rearranged into the expression

$$\tilde{\Gamma}(A) = -\frac{V}{(2\pi)^2} e\tilde{F}_0 \sum_{n=1}^{\infty} \int_0^{+\infty} \frac{dp_0}{2\pi} \times \ln \left[ \frac{(p_0^2 + m^2 + \kappa_n^2)^2 - 4p_0^2 \kappa_n^2}{(p_0^2 + m^2)^2} \right]. \quad (111)$$

The direction of the (constant) Euclidean field  $\tilde{F}_\mu$  is arbitrary; we may of course replace  $\tilde{F}_0$  everywhere by  $|\tilde{F}|$ . The momentum integral is along the “longitudinal” direction, which does not necessarily coincide with the zero (timelike) one. Then

$$\tilde{\Gamma}(A) = -\frac{V}{(2\pi)^2} e\tilde{F}_0 \sum_{n=1}^{\infty} \int_0^{+\infty} \frac{dp_{\parallel}}{2\pi} \times \ln \left[ \frac{(p_{\parallel}^2 + m^2 + 2e|\tilde{F}|n)^2 - 8p_{\parallel}^2 e|\tilde{F}|n}{(p_{\parallel}^2 + m^2)^2} \right], \quad (112)$$

which has an explicitly covariant and frame independent form. The fact that the result depends only on the square of the mass, and not on its sign, confirms that the parity-odd term vanishes for this field configuration. In other words, the spectral asymmetry is zero.

Integrating over the momentum one arrives at a sum of the form

$$\sum_{n=1}^{\infty} \sqrt{m+2e|\tilde{F}|n}, \quad (113)$$

which can be analytically continued to the Hurwitz function and so it agrees with the result of previous calculations [13]. The 1+1-dimensional case can be obtained at no cost.

The representation for the Dirac propagator in terms of path integrals used above is not the only possible one. It is possible to derive the full fermion propagator by using a set of Grassmannian variables which carry all the spin information and avoids the presence of the path ordering operator [10,6,11,12]. This will be discussed separately in detail.

## V. CONCLUSIONS

We have shown that the path-integral representation (16) is, when expanded in powers of the coupling constant, equivalent to the usual perturbative series.

Besides, we performed two different kinds of nonperturbative tests: first, we evaluated the axial anomaly in 1+1 dimensions and the Chern-Simons term in 2+1 dimensions. Amusingly enough, the path integral is particularly suited for the evaluation of those objects in a Pauli-Villars regularization scheme, and a subsequent large mass (cutoff) expansion. Our calculation focuses in the currents, so we can have perfect control of the gauge invariance of the results. For both

cases, we have seen that the exact results are obtained.

We also considered particle propagation in a constant electromagnetic field, deriving an expression for the effective action using Migdal's representation.

This kind of calculations provide, we believe, further support for the use of these representations in the derivation of quantum field theory results, either analytically or numerically. The developments presented here can also be useful for the world line in practice given, for instance, the problems with renormalization in general and the difficulties of the second-order formalism with external fermions and spectral asymmetry originated by the Dirac operator such as the Chern-Simons term.

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- [1] J. Schwinger, Phys. Rev. **82**, 664 (1951).
  - [2] R.P. Feynman, Phys. Rev. **76**, 749 (1949).
  - [3] C. Schubert, Phys. Rep. **355**, 73 (2001).
  - [4] A.A. Migdal, Nucl. Phys. **B265** [FS15], 594 (1986); See also M.B. Halpern, A. Jevicki, and P. Senjanovic, Phys. Rev. D **16**, 2476 (1977).
  - [5] See, for example, Appendix A3 of J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* 4th. Ed. (Oxford Science Publications, New York, 2002).
  - [6] A.I. Karanikas and C.N. Ktorides, Phys. Rev. D **52**, 5883 (1995).
  - [7] A.I. Karanikas, C.N. Ktorides, and N.G. Stefanis, Phys. Rev. D **52**, 5898 (1995).
  - [8] K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979); Phys. Rev. D **21**, 2848 (1980); for a recent review, see R.D. Ball, Phys. Rep. **182**, 1 (1989).
  - [9] S.R. Coleman, Ann. Phys. (N.Y.) **101**, 239 (1976).
  - [10] E.S. Fradkin, Nucl. Phys. **76**, 588 (1966).
  - [11] C. Alexandrou, R. Rosenfelder, and A.W. Schreiber, Phys. Rev. A **59**, 1762 (1999).
  - [12] M.G. Schmidt and C. Schubert, Phys. Rev. D **53**, 2150 (1996); M.J. Strassler, Nucl. Phys. **B385**, 145 (1992); D. Fliegner, M.G. Schmidt, and C. Schubert, Nucl. Phys. B (Proc. Suppl.) **51C**, 174 (1996).
  - [13] S.K. Blau, M. Visser, and A. Wipf, Int. J. Mod. Phys. A **6**, 5409 (1991); W. Dittrich and H. Gies, "Probing the vacuum: Perturbative effective action approach in quantum electrodynamics and its applications," Springer Tracts in Modern Physics No. 166, edited by G. Höhler *et al.* (Springer, Berlin, 2001); Phys. Lett. B **392**, 182 (1997); D.G.C. McKeon and A. Rebhan, Phys. Rev. D **48**, 2891 (1993).