Note on gauge theory on a fuzzy supersphere

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We construct a supermatrix model whose classical background gives a two-dimensional noncommutative supersphere. Quantum fluctuations around it give the supersymmetric gauge theories on the fuzzy supersphere constructed by Klimčík. This model has a parameter β that can tune the masses of the particles in the model and interpolate various supersymmetric gauge theories on the sphere.

I. INTRODUCTION

Matrix models have been vigorously studied to understand the nonperturbative aspects of string theories. In matrix models of the Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) type [1], background space-time appears dynamically as a classical background of matrices and their fluctuations around the classical solution are regarded as gauge and matter fields on the space-time. In particular, matrix models describe noncommutative gauge theories when the classical background of matrices are noncommutative [2]. In this approach the construction of the open Wilson loop and background independence of the noncommutative gauge theories are clarified [3].

Noncommutative gauge theories appear on D-branes in string theories in a constant Neveu-Schwarz–Neveu-Schwarz (NS-NS) two-form B background [4,5], where the bosonic space-time coordinates become noncommutative. Recently it was suggested that the nonanticommutativity of the fermionic coordinates on the superspace appears in string theories in a background of the Ramond-Ramond (RR) or graviphoton field strength [6–8]. Since the nonanticommutative fermionic coordinates can be described by (super)matrices, it is expected that (super)matrix models will play an important role in investigating various aspects of field theories on the noncommutative superspace. There are analyses of noncommutative superspace by using supermatrices [9–15]. Supersymmetric actions for scalar multiplets on the fuzzy two-supersphere were constructed in Ref. [9] based on the $osp(1|2)$ graded Lie algebra. Furthermore, a graded differential calculus on the fuzzy supersphere is discussed in Ref. [10]. Supersymmetric gauge theories on this noncommutative superspace were studied in Ref. [11] by using differential forms on it. In Ref. [12], noncommutative superspaces and their flat limits are studied by using the graded Lie algebras $osp(1|2)$, $osp(2|2)$, and $psu(2|2)$. Recently the concept of noncommutative superspace based on a supermatrix was also introduced in proving the Dijkgraaf-Vafa conjecture as the large N reduction [16]. The supermatrix model was also studied from the viewpoint of background-independent formulations of the matrix model which are expected to give constructive definitions of string theory [17].

In the previous paper [15], we constructed a supersymmetric gauge theory on a fuzzy two-supersphere based on a supermatrix model. This model has a classical solution representing a fuzzy supersphere and we obtained a supersymmetric gauge theory on the fuzzy two-supersphere expanding the supermatrices around the classical background. In a commutative limit this model is, however, different from the ordinary gauge theory in $D=2$, e.g., the action includes higher derivative terms and the fermions transform as spin-$\frac{3}{2}$ representation under the $su(2)$ isometry group on S^2. These differences originated from the fact that we did not impose appropriate constraints on the supermatrices to eliminate extra degrees of freedom. Supermatrix formulation of supersymmetric gauge theories is similar to the covariant superspace approach for the ordinary supersymmetric gauge theories [18] because each supermatrix corresponds to the connection superfields on the superspace as we saw in our previous paper [15]. In the covariant superspace approach various constrains are imposed on the connections to eliminate redundant degrees of freedom, but in our model we could not impose appropriate conditions. After we wrote the previous paper [15], we noticed Klimčík's paper [11] where he constructed a supersymmetric gauge theory on the fuzzy supersphere by using a method of differential forms and imposing suitable constraints on the connection superfields. A crucial point of his construction is the use of enlarged $osp(2|2)$ algebra. The global $\mathcal{N}=1$ supersymmetry algebra on the fuzzy supersphere is $osp(1|2)$. By adding the covariant derivatives $osp(1|2)$ is enlarged to $osp(2|2)$ because the supersymmetry generators and the covariant derivatives do not anticommute on the fuzzy supersphere. $Osp(2|2)$ algebra can be regarded as $\mathcal{N}=2$ superalgebras on the sphere. Starting from the connection superfields on this $\mathcal{N}=2$ superspace, he found the correct constraints to obtain a supersymmetric gauge theory on the fuzzy supersphere.

In this paper we reformulate Klimčík's gauge theory on fuzzy supersphere in terms of supermatrix models. Very interestingly, Klimčík's gauge theory can be obtained from a supermatrix model whose classical solution gives noncommutative supersphere. Its quantum fluctuations become the-supersymmetric gauge theory proposed by him. In a commu-

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tative limit, by taking a Wess-Zumino-like gauge, this model becomes the ordinary supersymmetric gauge theory on S^2. In the paper we use the manifestly $SO(3)$ covariant coordinates and decompose the bosonic field a_i into the normal component ϕ and tangential component $a_i^{(2)}$ on the sphere. The

action contains a parameter which can tune masses of particles in the model.

II. osp(1|2) AND osp(2|2) ALGEBRAS

The graded commutation relations of the $osp(2|2)$ algebra are given by

$$[\hat{l}_i,\hat{l}_j]=i\epsilon_{ijk}\hat{l}_k, \qquad [\hat{l}_i,\hat{v}_\alpha]=\frac{1}{2}(\sigma_i)_{\beta\alpha}\hat{v}_\beta, \qquad \{\hat{v}_\alpha,\hat{v}_\beta\}=\frac{1}{2}(C\sigma_i)_{\alpha\beta}\hat{l}_i,$$

$$\{\hat{v}_\alpha,\hat{d}_\beta\}=-\frac{1}{4}C_{\alpha\beta}\hat{\gamma}, \qquad [\hat{l}_i,\hat{d}_\alpha]=\frac{1}{2}(\sigma_i)_{\beta\alpha}\hat{d}_\beta, \qquad \{\hat{d}_\alpha,\hat{d}_\beta\}=-\frac{1}{2}(C\sigma_i)_{\alpha\beta}\hat{l}_i,$$

$$[\hat{\gamma},\hat{v}_\alpha]=\hat{d}_\alpha, \qquad [\hat{\gamma},\hat{d}_\alpha]=\hat{v}_\alpha, \qquad [\hat{\gamma},\hat{l}_i]=0, \qquad (2.1)$$

where $\hat{l}_i$ ($i=1,2,3$) and $\hat{\gamma}$ are bosonic generators, and $\hat{v}_\alpha$ and $\hat{d}_\alpha$ ($\alpha=1,2$) are fermionic ones. $C=i\sigma_2$ is the charge conjugation matrix. The $osp(1|2)$ subalgebra consists of the generators $\hat{l}_i$ and $\hat{v}_\alpha$. Irreducible representations of the $osp(1|2)$ algebra [19] are characterized by values of the second Casimir operator $\hat{K}_2=\hat{l}_i\hat{l}_i+C_{\alpha\beta}\hat{v}_\alpha\hat{v}_\beta=L(L+\frac{1}{2})$, where quantum number $L\in\mathbb{Z}_{\geqslant 0}/2$ is called superspin. Each representation of $osp(1|2)$ consists of spin L and $L+\frac{1}{2}$ representations of the $su(2)$ subalgebra generated by $\hat{l}_i$. The dimension of the representation is $N\equiv 4L+1$. These representations are also the so-called atypical representations of $osp(2|2)$, where $osp(1|2)$ algebra can be enlarged to $osp(2|2)$ algebra by adding extra generators with the same size matrices. The representation matrices of the generators $\hat{d}_\alpha$ and $\hat{\gamma}$ can be written as second-order polynomials of the superspin L representation matrices $l_i^{(L)}$ and $v_\alpha^{(L)}$ of the $osp(1|2)$ generators,

$$\gamma^{(L)}=-\frac{1}{L+1/4}\left[C_{\alpha\beta}v_\alpha^{(L)}v_\beta^{(L)}+2L\left(L+\frac{1}{2}\right)\right], \quad (2.2)$$

$$d_\alpha^{(L)}=\frac{1}{2(L+1/4)}(\sigma_i)_{\beta\alpha}\{l_i^{(L)},v_\beta^{(L)}\}. \quad (2.3)$$

The condition $\hat{K}_2=L(L+1/2)$ defines a two-dimensional supersphere. Consider the polynomials $\Phi(l_i^{(L)},v_\alpha^{(L)})$ of the representation matrices of $l_i^{(L)}$ and $v_\alpha^{(L)}$ with superspin L. These polynomials are $(4L+1)\times(4L+1)$ supermatrices. Let us denote the space spanned by $\Phi(l_i^{(L)},v_\alpha^{(L)})$ as $\mathcal{A}_L$. The $osp(1|2)$ algebra acts on $\mathcal{A}_L$. In particular we denote the adjoint action of the $osp(1|2)$ generators as

$$\hat{\mathcal{L}}_i\Phi=[l_i^{(L)},\Phi], \quad \hat{\mathcal{V}}_\alpha\Phi=[v_\alpha^{(L)},\Phi]. \quad (2.4)$$

$\mathcal{A}_L$ can be decomposed into irreducible representations under the adjoint action of the $osp(1|2)$ algebra as $0\oplus\frac{1}{2}\oplus 1\oplus\cdots\oplus 2L-\frac{1}{2}\oplus 2L$. The dimension of this space is $(4L+1)^2$ and thus any $(4L+1)\times(4L+1)$ matrices can be ex-

panded by these polynomials. A useful basis for the expansion are the matrix superspherical harmonics $Y_{km}^S(l_i^{(L)},v_\alpha^{(L)})$,

$$(\hat{\mathcal{L}}_i\hat{\mathcal{L}}_i+C_{\alpha\beta}\hat{\mathcal{V}}_\alpha\hat{\mathcal{V}}_\beta)Y_{km}^S(l_i^{(L)},v_\alpha^{(L)})=k\left(k+\frac{1}{2}\right)Y_{km}^S(l_i^{(L)},v_\alpha^{(L)}), \quad (2.5)$$

$$\hat{\mathcal{L}}_3Y_{km}^S(l_i^{(L)},v_\alpha^{(L)})=mY_{km}^S(l_i^{(L)},v_\alpha^{(L)}). \quad (2.6)$$

The superspherical harmonics should be properly normalized [10]. Then $\Phi(l_i^{(L)},v_\alpha^{(L)})$ can be expressed as a series of superspherical harmonics,

$$\Phi(l_i^{(L)},v_\alpha^{(L)})=\sum_{k=0,1/2,1,\cdots}^{2L}\phi_{km}Y_{km}^S(l_i^{(L)},v_\alpha^{(L)}), \quad (2.7)$$

where the Grassmann parity of the coefficient ϕ_{km} depends on the gradings of the spherical harmonics. We can map the supermatrix $\Phi(l_i^{(L)},v_\alpha^{(L)})$ to a function on the superspace (x_i,θ_α) by

$$\Phi(l_i^{(L)},v_\alpha^{(L)})\rightarrow\phi(x_i,\theta_\alpha)=\sum_{k,m}\phi_{km}y_{km}^S(x_i,\theta_\alpha), \quad (2.8)$$

where $y_{km}^S(x_i,\theta_\alpha)$ is the ordinary superspherical function. A product of supermatrices is mapped to a noncommutative star product of functions on the fuzzy supersphere [20]. In constructions of field theories on the fuzzy supersphere, the $osp(2|2)$ generators $\hat{d}_\alpha$ and $\hat{\gamma}$ play important roles. The adjoint action of $\hat{d}_\alpha$ on supermatrices is the covariant derivative on the fuzzy supersphere. The kinetic terms for a scalar multiplet on the supersphere are constructed by using these generators [9].

III. SUPERSYMMETRIC GAUGE THEORY ON FUZZY SUPERSPHERE

In this section we reformulate Klimčík's construction of a supersymmetric gauge theory on fuzzy supersphere [11] in terms of supermatrices. We construct a supermatrix model which has a classical solution representing the fuzzy supersphere, and by expanding supermatrices around this background we will obtain a supersymmetric gauge theory given by Klimčík.

This formulation has similarities to the covariant superspace approach in the ordinary supersymmetric gauge theory. Namely, we first introduce larger degrees of freedom, corresponding to the connection superfields on the fuzzy supersphere. In order to eliminate redundant degrees of freedom, we need to impose appropriate constraints on them. After fixing extra gauges, we will obtain necessary degrees of freedom to describe a supersymmetric gauge theory on the noncommutative supersphere.

We first consider direct products of two vector spaces of supermatrices, $\mathcal{A}_L$ and $\mathcal{A}_{L'}$. L' can be taken as any superspin. On the other hand, L should be taken large in order to take a commutative limit. Each element is a $(4L+1)(4L'+1) \times (4L+1)(4L'+1)$ supermatrix. We then consider a special type of supermatrix which can be written as

$$M = \sum_A X_A \otimes T^A, \qquad (3.1)$$

where $X_A \in \mathcal{A}_L$ is a general supermatrix and T^A is the superspin L' representation matrix of the $osp(2|2)$ generators; $T^A = \{l_i^{(L')}, v_\alpha^{(L')}, d_\alpha^{(L')}, \gamma^{(L')}\}$. Among them, we can define two kinds of products, $\cdot$ and $*$,

$$(X_A \otimes T^A) \cdot (Y_B \otimes T^B) = (-1)^{|T^A||Y_B|} X_A Y_B \otimes T^A T^B, \qquad (3.2)$$

$$(X_A \otimes T^A) * (Y_B \otimes T^B) = (-1)^{|T^A||Y_B|} X_A Y_B \otimes [T^A, T^B\}, \qquad (3.3)$$

where $|T^A|$ and $|Y_B|$ are the gradings of T^A and Y_B, respectively.[1]

The supermatrix M defined above can be expanded explicitly as

$$M = A_i \otimes l_i^{(L')} + C_{\alpha\beta} \varphi_\alpha \otimes v_\alpha^{(L')} - C_{\alpha\beta} \psi_\alpha \otimes d_\beta^{(L')} - \frac{1}{4} W \otimes \gamma^{(L')}. \qquad (3.4)$$

Here A_i and W are $(4L+1) \times (4L+1)$ even supermatrices, and φ_α and ψ_α are $(4L+1) \times (4L+1)$ odd supermatrices. L' is arbitrary, thus one may set $L' = \frac{1}{2}$ for simplicity. M satisfies the graded reality condition $M^\ddagger = M$, that is, $A_i^\ddagger = A_i$, $\varphi_\alpha^\ddagger = C_{\alpha\beta} \varphi_\beta$, $\psi_\alpha^\ddagger = C_{\alpha\beta} \psi_\beta$ and $W^\ddagger = W$. For the definition of $\ddagger$, see the appendix of Ref. [15]. We denote the $osp(1|2)$ part of M as M_H,

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[1] In the paper [11], $*$ is meant for $*^G - *^H$ but in our paper we use $*$ product as $*^G$ or $*^H$ according to operated supermatrices.

$$M_H = A_i \otimes l_i^{(L')} + C_{\alpha\beta} \varphi_\alpha \otimes v_\alpha^{(L')}, \qquad (3.5)$$

and the rest as $M_{H^\perp} \equiv M - M_H$. Since all the components A_i, ϕ_α, ψ_α, and W are supermatrices and can be expanded in terms of superspin L spherical harmonics, they can be regarded as superfields on the fuzzy supersphere. The supermatrix M is shown to correspond to the covariant derivatives on the $\mathcal{N} = 2$ extended supersphere. Then we define "field strength" as

$$F = M*M - M_H*M_H + \frac{d_G}{2} M - \frac{d_H}{2} M_H$$

$$= \left(\frac{1}{4} (\sigma_i C)_{\alpha\beta} \{\psi_\alpha, \psi_\beta\} - \frac{1}{4} A_i \right) \otimes l_i^{(L')}$$

$$+ \left(-\frac{1}{4} C_{\alpha\beta} [\psi_\alpha, W] - \frac{1}{4} C_{\alpha\beta} \varphi_\alpha \right) \otimes v_\beta^{(L')}$$

$$+ \left(\frac{1}{2} (\sigma_i C)_{\alpha\beta} [A_i, \psi_\alpha] + \frac{1}{4} C_{\alpha\beta} [\varphi_\alpha, W] - \frac{1}{2} C_{\alpha\beta} \psi_\alpha \right)$$

$$\otimes d_\beta^{(L')} + \left(-\frac{1}{4} C_{\alpha\beta} \{\varphi_\alpha, \psi_\beta\} - \frac{1}{8} W \right) \otimes \gamma^{(L')}, \qquad (3.6)$$

where $d_G = 1$ and $d_H = \frac{3}{2}$ are Dynkin numbers.

In order to eliminate redundant superfields, we follow the prescription in Ref. [11] and impose a constraint for the $osp(1|2)$ part of the field strength to vanish,

$$F|_H \equiv \left(\frac{1}{4} (\sigma_i C)_{\alpha\beta} \{\psi_\alpha, \psi_\beta\} - \frac{1}{4} A_i \right) \otimes l_i^{(L')}$$

$$+ \left(-\frac{1}{4} C_{\alpha\beta} [\psi_\alpha, W] - \frac{1}{4} C_{\alpha\beta} \varphi_\alpha \right) \otimes v_\beta^{(L')} = 0. \qquad (3.7)$$

This constraint breaks the $osp(2|2)$ covariant structure, but still preserves the covariance under $osp(1|2)$ of the model. From this constraint, A_i and φ_α can be solved in terms of ψ_α and W as

$$A_i = (\sigma_i C)_{\alpha\beta} \{\psi_\alpha, \psi_\beta\}, \qquad (3.8)$$

$$\varphi_\alpha = -[\psi_\alpha, W]. \qquad (3.9)$$

Moreover, we need to constrain further redundant degrees of freedom W by the following condition:

$$-\frac{1}{L'(L'+1/2)} \mathrm{STr}_{L'} (M_{H^\perp} \cdot M_{H^\perp})$$

$$= C_{\alpha\beta} \psi_\alpha \psi_\beta + \frac{1}{4} W^2$$

$$= L(L+1/2), \qquad (3.10)$$

where $\mathrm{STr}_{L'}$ represents taking a supertrace with respect to the superspin L' representation matrices. This constraint is

also $osp(1|2)$ invariant. In a commutative limit, this equation can be solved to eliminate W, as will be seen.

We now start from the following action for the supermatrix M:

$$S = S_{F^2} + \beta S_{CS}, \tag{3.11}$$

$$S_{F^2} = \text{STr}(F \cdot F), \tag{3.12}$$

$$S_{CS} = \text{STr}\left(\frac{2}{3} M \cdot (M*M) - \frac{2}{3} M_H \cdot (M_H*M_H) + \frac{d_G}{2} M \cdot M\right.$$
$$\left. - \frac{d_H}{2} M_H \cdot M_H\right), \tag{3.13}$$

where β is a real constant parameter and d_G and d_H were defined above. This action is invariant under the following $osp(1|2)$ supersymmetry transformation:

$$\delta M = [C_{\alpha\beta}\lambda_\beta \otimes v_\alpha^{(L')}, M], \tag{3.14}$$

where a parameter λ_α is a $(4L+1)\times(4L+1)$ grading matrix multiplied by a Grassmann parameter $\tilde{\lambda}_\alpha$,

$$\lambda_\alpha = \tilde{\lambda}_\alpha \begin{pmatrix} 1_{2L+1} & 0 \\ 0 & -1_{2L} \end{pmatrix}. \tag{3.15}$$

In addition to the supersymmetry, this action is invariant under the $U(2L+1|2L)$ gauge transformation,

$$\delta M = [M, H \otimes 1], \tag{3.16}$$

where $H \in u(2L+1|2L)$, $H^\ddagger = -H$. We note that S_{F^2} and S_{CS} are independently invariant under the supersymmetry and gauge transformations.

Then we solve the equation of motion of the above action and expand the supermatrix M around a classical solution. The classical solution represents the background space-time and fluctuations are regarded as gauge fields on the space-time. One of the classical solutions is given by

$$A_i = l_i^{(L)}, \quad \varphi_\alpha = v_\alpha^{(L)}, \quad \psi_\alpha = d_\alpha^{(L)}, \quad W = \gamma^{(L)}, \tag{3.17}$$

which represents a fuzzy supersphere. It should be noted that the solution satisfies not only the equations of motion but also the constraints (3.7) and (3.10). M can be decomposed into the classical background (3.17) and fluctuation $\tilde{M}$ as

$$M = M_{cl} + \tilde{M}, \tag{3.18}$$

$$M_{cl} = l_i^{(L)} \otimes l_i^{(L')} + C_{\alpha\beta}v_\alpha^{(L)} \otimes v_\beta^{(L')}$$
$$- C_{\alpha\beta}d_\alpha^{(L)} \otimes d_\beta^{(L')} - \frac{1}{4}\gamma^{(L)} \otimes \gamma^{(L')}. \tag{3.19}$$

Because of the $osp(2|2)$ algebra the field strength for the classical background M_{cl} vanishes,

$$F_{cl} = M_{cl}*M_{cl} - M_{Hcl}*M_{Hcl} + \frac{d_G}{2}M_{cl} - \frac{d_H}{2}M_{Hcl} = 0, \tag{3.20}$$

where M_{Hcl} is the $osp(1|2)$ part of M_{cl}. Then the field strength (3.6) becomes

$$F = \left(M_{cl}*\tilde{M} + \tilde{M}*M_{cl} + \frac{d_G}{2}\tilde{M}\right) - \left(M_{Hcl}*\tilde{M}_H + \tilde{M}_H*M_{Hcl}\right.$$
$$\left. + \frac{d_H}{2}\tilde{M}_H\right) + \tilde{M}*\tilde{M} - \tilde{M}_H*\tilde{M}_H, \tag{3.21}$$

where $\tilde{M}_H$ is the $osp(1|2)$ part of $\tilde{M}$. In this form, terms in the first and second parentheses, respectively, coincide with $\delta^G\tilde{M}$ and $\delta^H\tilde{M}_H$ in Klimčík's notation [11] and thus this is nothing but the field strength defined by him.

Expanding each component of the supermatrices around the classical background,

$$A_i = l_i^{(L)} + \tilde{A}_i, \quad \varphi_\alpha = v_\alpha^{(L)} + \tilde{\varphi}_\alpha, \quad \psi_\alpha = d_\alpha^{(L)} + \tilde{\psi}_\alpha,$$

$$W = \gamma^{(L)} + \tilde{W}, \tag{3.22}$$

$F|_{H^\perp}$ and S_{CS} become

$$F|_{H^\perp} = \left[\frac{1}{2}(\sigma_i C)_{\alpha\beta}([l_i^{(L)}, \tilde{\psi}_\alpha] - [d_\alpha^{(L)}, \tilde{A}_i])\right.$$

$$+ \frac{1}{4}C_{\alpha\beta}([v_\alpha^{(L)}, \tilde{W}] - [\gamma^{(L)}, \tilde{\varphi}_\alpha]) - \frac{1}{2}C_{\alpha\beta}\tilde{\psi}_\alpha\right] \otimes d_\beta^{(L')}$$

$$+ \left[-\frac{1}{4}C_{\alpha\beta}(\{v_\alpha^{(L)}, \tilde{\psi}_\beta\} + \{d_\beta^{(L)}, \tilde{\varphi}_\beta\}) - \frac{1}{8}\tilde{W}\right] \otimes \gamma^{(L')}, \tag{3.23}$$

$$S_{CS} = \frac{1}{4}L'(L'+1/2)\text{STr}_L\left(2(\sigma_i C)_{\alpha\beta}\tilde{A}_i\{\tilde{\psi}_\alpha, \tilde{\psi}_\beta\}\right.$$

$$- 2C_{\alpha\beta}\tilde{W}\{\tilde{\varphi}_\alpha, \tilde{\psi}_\beta\} - 2(\sigma_i C)_{\alpha\beta}\tilde{\psi}_\alpha[l_i^{(L)}, \tilde{\psi}_\beta]$$

$$+ 4(\sigma_i C)_{\alpha\beta}\tilde{A}_i\{d_\alpha^{(L)}, \tilde{\psi}_\beta\} + 2C_{\alpha\beta}\tilde{\varphi}_\alpha[\gamma^{(L)}, \tilde{\psi}_\beta]$$

$$- 2C_{\alpha\beta}\tilde{W}\{v_\alpha^{(L)}, \tilde{\psi}_\beta\} - 2C_{\alpha\beta}\tilde{W}\{d_\beta^{(L)}, \tilde{\varphi}_\alpha\} - \tilde{A}_i\tilde{A}_i$$

$$\left. - C_{\alpha\beta}\tilde{\varphi}_\alpha\tilde{\varphi}_\beta - 2C_{\alpha\beta}\tilde{\psi}_\alpha\tilde{\psi}_\beta - \frac{1}{2}\tilde{W}^2\right). \tag{3.24}$$

Note that the supermatrices A_i, ϕ_α, ψ_α, and W are covariant derivatives on $\mathcal{N}=2$ superspace since they transform covariantly under the $U(2L+1|2L)$ gauge transformations.

Because of the constraints (3.7) and (3.10), $\tilde{\psi}_\alpha$ is the only independent supermatrix and the others can be solved in terms of $\tilde{\psi}_\alpha$,

$$\tilde{A}_i = (\sigma_i C)_{\alpha\beta}(2\{d_\alpha^{(L)}, \tilde{\psi}_\beta\} + \{\tilde{\psi}_\alpha, \tilde{\psi}_\beta\}), \tag{3.25}$$

$$\tilde{\varphi}_\alpha = -[d_\alpha^{(L)}, \tilde{W}] + [\gamma^{(L)}, \tilde{\psi}_\alpha] - [\tilde{\psi}_\alpha, \tilde{W}], \tag{3.26}$$

$$C_{\alpha\beta}[d_\alpha^{(L)},\tilde{\psi}_\beta]+\frac{1}{4}\{\gamma^{(L)},\tilde{W}\}+C_{\alpha\beta}\tilde{\psi}_\alpha\tilde{\psi}_\beta+\frac{1}{4}\tilde{W}^2=0.$$

$$(3.27)$$

$\tilde{\psi}_\alpha$ is an odd supermatrix which can be expanded in terms of $l_i^{(L)}$ and $v_\alpha^{(L)}$. Thus it is regarded as a spinor superfield on the fuzzy supersphere. Equation (3.27) represents a rather complicated quadratic constraint for supermatrix $\tilde{W}$ which cannot be easily solved (or, eventually, implemented in numerical simulations). Below, we shall solve and investigate it in the commutative limit $L\to\infty$. For a while we treat both of $\tilde{\psi}_\alpha$ and $\tilde{W}$ as independent variables.

Though fixing the classical background as above violates the original $osp(1|2)$ supersymmetry (3.14), it can be compensated by appropriate $u(2L+1|2L)$ gauge transformations. Actually the action is invariant under the following combined transformations:

$$\delta\tilde{\psi}_\alpha=\frac{1}{4}\lambda_\alpha\tilde{W}+C_{\beta\gamma}\lambda_\gamma\{v_\beta^{(L)},\tilde{\psi}_\alpha\},\qquad(3.28)$$

$$\delta\tilde{W}=C_{\alpha\beta}\lambda_\beta\tilde{\psi}_\alpha+C_{\alpha\beta}\lambda_\beta[v_\alpha^{(L)},\tilde{W}].$$

$$(3.29)$$

The action is also invariant under gauge transformations,

$$\delta\tilde{\psi}_\alpha=[d_\alpha^{(L)}+\tilde{\psi}_\alpha,H],\qquad\delta\tilde{W}=[\gamma^{(L)}+\tilde{W},H],\quad(3.30)$$

for $H\in u(2L+1|2L)$. These transformations are compatible with the constraints. The fact that the model has the $osp(1|2)$ supersymmetry even after choosing the classical background can be also understood as the following. The elements with the form $u_i l_i^{(L)}+C_{\alpha\beta}\lambda_\alpha v_\beta^{(L)}$ in $u(2L+1|2L)$ constitute the $osp(1|2)$ subalgebra. Thus the model originally has two independent $osp(1|2)$ symmetries. The classical background preserves a half of these symmetries which is given by the transformations (3.28).

Mapping from the supermatrix model to a noncommutative field theory on the supersphere is performed by the same method as ones used in Ref. [15]. The classical solutions $l_i^{(L)}$ and $v_\alpha^{(L)}$ are mapped to coordinates (x_i,θ_α) on the supersphere,

$$x_i=\frac{1}{\sqrt{L(L+1/2)}}l_i^{(L)},\qquad(3.31)$$

$$\theta_\alpha=\frac{1}{\sqrt{L(L+1/2)}}v_\alpha^{(L)},\qquad(3.32)$$

where we set a radius of the supersphere to 1 for simplicity, that is $x_i x_i+C_{\alpha\beta}\theta_\alpha\theta_\beta=1$. And $d_\alpha^{(L)}$ and $\gamma^{(L)}$ are written as second-order polynomials of x_i and θ_α according to (2.2) and (2.3). The adjoint actions of these $osp(2|2)$ generators are mapped to the differential operators on the supersphere. The supertrace becomes an integration on the supersphere,

$$\mathrm{STr}_L\to-\frac{1}{2\pi}\int d^3x d^2\theta\,\delta(x^2+\theta^2-1),\qquad(3.33)$$

where $\theta^2=C_{\alpha\beta}\theta_\alpha\theta_\beta$. The supermatrix $\tilde{\psi}_\alpha$ is mapped to a spinor superfield that can be regarded as the spinor connection on the fuzzy supersphere. The field theory action derived by these procedures is written in terms of this spinor superfield. The construction of the supersymmetric gauge theory on the fuzzy supersphere given here is a natural extension of the covariant superspace approach for the ordinary super-Yang-Mills theory in a flat space-time.

Next we consider a commutative limit to see that our model is indeed a noncommutative generalization of the ordinary supersymmetric gauge theory on sphere. A commutative limit is given by taking the $L\to\infty$ limit, keeping the radius of the supersphere fixed. In this limit superfields $\tilde{\psi}_\alpha$ and $\tilde{W}$ are expanded as

$$\tilde{\psi}_\alpha(x,\theta)=\eta_\alpha(x)+(\sigma_\mu)_{\beta\alpha}a_\mu(x)\theta_\beta$$

$$+\left(\xi_\alpha(x)+\frac{1}{2r^2}x_i\partial_i\eta_\alpha(x)\right)\theta^2,\quad(3.34)$$

$$\tilde{W}(x,\theta)=w(x)+C_{\alpha\beta}\zeta_\alpha(x)\theta_\beta$$

$$+\left(F(x)+\frac{1}{2r^2}x_i\partial_i w(x)\right)\theta^2,\qquad(3.35)$$

where $r^2=x_i x_i$ and $\mu=0,1,2,3$. a_μ, w, F are bosonic and $\eta_\alpha,\xi_\alpha,\zeta_\alpha$ are fermionic fields. The $u(2L+1|2L)$ gauge parameter is also expanded as

$$H(x,\theta)=h(x)+C_{\alpha\beta}h_\alpha(x)\theta_\beta+f(x)\theta^2.\qquad(3.36)$$

The gauge transformation generated by $h(x)$ is an ordinary gauge transformation while the others are supersymmetric extension of it. To fix these extra gauge degrees of freedom generated by h_α and f, we set the Wess-Zumino-like gauge fixing condition $C_{\alpha\beta}\theta_\alpha\tilde{\psi}_\beta=0$; thus $\eta_\alpha=a_0=0$. In the commutative limit the constraint (3.27) can be solved as $\tilde{W}=2x_i(\sigma_i C)_{\alpha\beta}\theta_\alpha\tilde{\psi}_\beta$, so that

$$w=\zeta_\alpha=0,\quad F=-2(x\cdot a)\theta^2.\qquad(3.37)$$

Here we used the fact that the fields $\tilde{\psi}_\alpha$ and $\tilde{W}$ scale as $\mathcal{O}(1)$ while $d_\alpha^{(L)}$ and $\gamma^{(L)}$ scale as $\mathcal{O}(L)$ in $L\to\infty$. By the gauge fixing condition and the constraint, the independent fields are now $a_i(x)$ and $\xi_\alpha(x)$.

We then decompose a_i into a scalar field ϕ and a gauge field $a_i^{(2)}$ on S^2,

$$a_i=n_i\phi+a_i^{(2)},\qquad(3.38)$$

where n_i is a unit normal vector on the sphere and $n_i a_i^{(2)}=0$ and $a_i^{(2)}$ is a tangential projection; $a_i^{(2)}=(\delta_{ij}-n_i n_j)a_j$. The $U(1)$ field strength for the tangential component is defined as $F_{ij}^{(2)}=R_i a_j^{(2)}-R_j a_i^{(2)}-i\epsilon_{ijk}a_k^{(2)}$, where $R_i=-i\epsilon_{ijk}n_j\partial/\partial n_i$ are derivatives on S^2. After straightforward calculations, we obtain the action in the commutative limit,

$$S = \frac{L'(L'+1/2)}{4\pi} \int d\Omega \left\{ \frac{9}{8} F_{ij}^{(2)} F_{ij}^{(2)} - \frac{i}{4} (\epsilon_{ijk} n_i F_{jk}^{(2)}) \phi \right.$$

$$+ \frac{9}{4} (R_i \phi)^2 - \frac{1}{4} \phi^2 - \frac{9}{4} (\sigma_i C)_{\alpha\beta} \xi_\alpha R_i \xi_\beta - \frac{1}{4} C_{\alpha\beta} \xi_\alpha \xi_\beta$$

$$\left. - 3\beta [i(\epsilon_{ijk} n_i F_{jk}^{(2)}) \phi + \phi^2 + C_{\alpha\beta} \xi_\alpha \xi_\beta,] \right\}. \tag{3.39}$$

This model consists of a $U(1)$ gauge field which has no local degrees of freedom, a scalar, and a Majorana fermion. There is a free tunable parameter β. The mass of the fermion is given by $\sqrt{\mu(\mu-1)}$ with $\mu = (1+12\beta)/9$ because of

$$\langle \xi_\alpha(x) \xi_\beta(y) \rangle \sim \frac{(C\sigma_i)_{\alpha\beta} R_i + (1-\mu) C_{\alpha\beta}}{R_i R_i - \mu(\mu-1)}. \tag{3.40}$$

The massless Dirac operator is given by $D = \sigma_i R_i + 1$. The second term in D comes from the spin connection on the sphere and this D anticommutes with the chirality operator $\sigma_i n_i$. By integrating out the scalar field ϕ it can be shown that the bosonic supersymmetric parter of ξ has the same mass. For $\beta = 2/3$ the action becomes the one given in Ref. [21]. On the other hand, for $\beta = -1/12$ the mixing term between ϕ and $a_i^{(2)}$ disappears. For both cases, bosonic and fermionic excitations are massless.

Since the gauge fixing condition we chose is not invariant under the supersymmetry transformations (3.28), we must compensate them by the field-dependent gauge transformation with the following gauge parameter:

$$h_\alpha = -\lambda_\alpha \phi - i\epsilon_{ijk} (\sigma_k)_{\beta\alpha} \lambda_\beta n_i a_j^{(2)}, \tag{3.41}$$

$$f = \frac{1}{2} (\sigma_i C)_{\alpha\beta} \lambda_\beta n_i \xi_\alpha. \tag{3.42}$$

Then the supersymmetry transformations become

$$\delta_\lambda \phi = -\frac{1}{2} (C_{\alpha\beta} \lambda_\beta \xi_\alpha), \tag{3.43}$$

$$\delta_\lambda a_i^{(2)} = -\frac{i}{2} \epsilon_{ijk} n_j (\sigma_k C)_{\alpha\beta} \lambda_\beta \xi_\alpha, \tag{3.44}$$

$$\delta_\lambda \xi_\alpha = -\frac{1}{2} \lambda_\alpha \left(\phi + \frac{i}{2} \epsilon_{ijk} n_i F_{jk}^{(2)} \right)$$

$$+ \frac{1}{2} (\sigma_i)_{\beta\alpha} \lambda_\beta R_i \phi. \tag{3.45}$$

The $U(1)$ gauge transformation is given by

$$\delta a_i^{(2)} = R_i h, \quad \delta \phi = \delta \xi_\alpha = 0. \tag{3.46}$$

The action (3.39) is invariant under these transformations. It can be shown that the commutation relations between two supersymmetry transformations generate the translations on S^2 up to gauge transformations,

$$[\delta_{\lambda^1}, \delta_{\lambda^2}] \phi = \Theta_i R_i \phi, \tag{3.47}$$

$$[\delta_{\lambda^1}, \delta_{\lambda^2}] a_i^{(2)} = \Theta_j R_j a_i^{(2)} + i\epsilon_{ijk} \Theta_j a_k^{(2)}$$

$$+ R_i [\Theta_j (n_j \phi - a_j^{(2)})], \tag{3.48}$$

$$[\delta_{\lambda^1}, \delta_{\lambda^2}] \xi_\alpha = \Theta_i R_i \xi_\alpha + \frac{1}{2} \Theta_i (\sigma_i)_{\beta\alpha} \xi_\beta, \tag{3.49}$$

with $\Theta_i = -\frac{1}{2} (\sigma_i C)_{\alpha\beta} \lambda_\alpha^2 \lambda_\beta^1$. We again note that the action has these symmetries independently of the parameter β.

In this paper we concentrated on the case of $U(1)$ gauge theory, but it is easily generalized to supersymmetric gauge theory with the $U(k)$ ($k>1$) group.

IV. SUMMARY AND DISCUSSIONS

In this paper, we constructed a supersymmetric gauge theory on fuzzy supersphere from a supermatrix model. Our construction of the supermatrix model is based on the prescription given by Klimčík [11]. We have shown that the fuzzy supersphere can be obtained as a classical solution of the supermatrix model and fluctuations around it become supersymmetric gauge fields on the supersphere. Namely, both of the Klimčík's gauge theory and fuzzy supersphere background are shown to be derivable from a single supermatrix. In this sense, this model has background independence as well as the other known gauge theories on noncommutative space-time.

The formulation adopted here is similar to the so-called covariant superspace approach to supersymmetric gauge theories on superspace. Namely we first introduce larger degrees of freedom, corresponding to connection superfields on the superspace, and then impose various constraints on the superfields. In our case, we follow the prescription by Klimčík and start from connection superfields on $\mathcal{N}=2$ superspace on the sphere to obtain $\mathcal{N}=1$ supersymmetric field theory. In this way, we can impose appropriate constraints to eliminate redundant fields. In our previous paper [15], we started from less superfields corresponding to connections on $\mathcal{N}=1$ but could not impose the appropriate constraint compatible with $osp(1|2)$ global supersymmetry. This is in contrast with the ordinary flat case where we can impose appropriate constraints on the connection superfields on $\mathcal{N}=1$ superspace.

Our construction may be generalizable to higher dimensional curved cases. For example, it will be interesting to consider a supermatrix model based on $psu(2|2)$ super Lie algebra to obtain a supersymmetric gauge theory on noncommutative superspace of $AdS^2 \times S^2$.

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