Quantum fluctuations of a "constant" gauge field

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It is argued here that the quantum computation of the vacuum pressure must take into account the contribution of zero-point oscillations of a rank-three gauge field. The field $A_{\mu\nu\rho}$ possesses no radiative degrees of freedom, its sole function being that of polarizing the vacuum through the formation of *finite* domains characterized by a non-vanishing, constant, but otherwise arbitrary pressure. This extraordinary feature, rather unique among quantum fields, is exploited to associate the $A_{\mu\nu\rho}$ field with the "bag constant" of the hadronic vacuum, or with the cosmological term in the cosmic case. We find that the quantum fluctuations of $A_{\mu\nu\rho}$ are inversely proportional to the confinement volume and interpret the result as a Casimir effect for the hadronic vacuum. With these results in hands and by analogy with the electromagnetic and string case, we proceed to calculate the Wilson loop of the three-index potential coupled to a "test" relativistic bubble. From this calculation we extract the static potential between two opposite points on the surface of a spherical bag and find it to be proportional to the enclosed volume.

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I. INTRODUCTION

It is well known that the cosmological term introduced in general relativity can be expressed as the vacuum expectation value of the energy-momentum tensor, as one might expect on the basis of relativistic covariance

$$\langle T^{\mu\nu}\rangle = \frac{\Lambda}{8\pi G} g^{\mu\nu}. \tag{1}$$

It is less well known that the same cosmological term can be formulated as the gauge theory of a rank-three antisymmetric tensor gauge potential $A_{\mu\nu\rho}$ [1–4] with an associated field strength

$$F_{\mu\nu\rho\sigma} = \nabla_{[\mu} A_{\nu\rho\sigma]} \tag{2}$$

invariant under the tensor gauge transformation

$$A_{\mu\nu\rho} \to A_{\mu\nu\rho} + \nabla_{[\mu}\lambda_{\nu\rho]}. \tag{3}$$

Indeed, one readily verifies that the classical action

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R - \frac{1}{2\times 4!}\int d^4x\sqrt{-g}F^{\lambda\mu\nu\rho}F_{\lambda\mu\nu\rho} \tag{4}$$

leads to the familiar Einstein equations in the presence of a cosmological term [4,5].

Equation (1) suggests that the cosmological term is associated with the zero-point energy of the cosmic vacuum. Then, in view of the *equivalence* stated above, we are naturally led to question the calculability of the zero-point energy due to the quantum fluctuations of the $A_{\mu\nu\rho}$ field. As a matter of fact, we shall argue in the following sections that there are non-trivial volume effects due to the quantum fluctuations of the A field.

Let us switch now from the cosmological case to the hadronic case and consider the implications of quantum vacuum energy in connection with the outstanding problem of color confinement in the theory of strong interactions. Somewhat surprisingly, perhaps, the formal connection between the two extreme cases, cosmological and hadronic, is provided by the same three-index potential $A_{\mu\nu\rho}$ introduced earlier.

Quantum chromodynamics is universally accepted as the fundamental gauge theory of quarks and gluons. Equally accepted, however, is the view that QCD is still poorly understood in the non-perturbative regime where the problem of color confinement sets in. On the other hand, the phenomenon of quark confinement is accounted for, as an input, by the phenomenological "bag models," with or without surface tension [6]. In some such models it is assumed, for instance, that the normal vacuum is a color magnetic conductor characterized by an infinite value of the color magnetic permeability while the interior of the bag, even an empty one, is characterized by a finite color magnetic permeability. In the interior of the bag the vacuum energy density acts as a hadronic "cosmological constant" originating from zero-point energy due to quantum fluctuations inside the bag. *This is a type of Casimir effect for the hadronic vacuum*. To our knowledge, in spite of the fairly large amount of literature on the subject [7], this effect has never been discussed before in terms of the $A_{\mu\nu\rho}$ field. Ultimately, the origin of this effect, and therefore of the cosmological bag constant, should be traced back to the fundamental dynamics of the Yang-Mills field.

Our suggestion, to be discussed in detail in a forthcoming publication, is that the link between the $A_{\mu\nu\rho}$ field and the

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fundamental variables of QCD is given by the "topological density" $\mathrm{Tr}\,\mathbf{F}^{\mu\nu}{}^*F_{\mu\nu}$ through the specific identification

$$A_{\mu\nu\rho}=\frac{1}{16\pi^2\Lambda_{QCD}^2}\mathrm{Tr}(\mathbf{A}_{[\mu}\partial_\nu\mathbf{A}_{\rho]}+\mathbf{A}_{[\mu}\mathbf{A}_\nu\mathbf{A}_{\rho]}). \quad (5)$$

In support of this identification, notice that a Yang-Mills gauge transformation in Eq. (5) induces an *Abelian* gauge transformation of the type (3)

$$\delta A_{\mu\nu\rho}=\frac{1}{g\Lambda_{QCD}}\mathrm{Tr}[(\mathbf{D}_{[\mu}\boldsymbol{\Lambda})\mathbf{F}_{\nu\rho]}]\equiv\frac{1}{g}\partial_{[\mu}\lambda_{\nu\rho]} \quad (6)$$

where Λ_{QCD} is the energy scale at which *QCD* becomes intrinsically non-perturbative.

Against this background, this paper is the second in a series dealing with the hadronic and cosmological implications of the vacuum quantum energy associated with the three-index potential $A_{\mu\nu\rho}$. In view of the chain of arguments offered above, we shall refer to that field as the "cosmological field" or "topological field," depending on the specific application under consideration. Some such applications in the cosmological case, in particular in connection with the problem of dark energy and dark matter in the Universe have been discussed in the first paper of the series [5].

Rank three gauge potentials also appear in different sectors of high energy theoretical physics, e.g., supergravity [2], cosmology [8], and both gauge theory of gravity [9] and of extended objects [10]. As argued above, a central role is played by this kind of gauge field in connection with the problem of confinement [11].

The present paper focuses on the general properties of the topological field as an *Abelian gauge field of higher rank* but with an eye on the future discussion of the problem of confinement in *QCD*. *Ultimately, we wish to calculate the Wilson loop for the three-index potential coupled to the three-dimensional world history of a spherical bubble.* To our knowledge, this calculation has never been done before and represents the preparatory ground for the inclusion of fermions in the model [12].

Our calculations are performed in the Euclidean regime and represent a generalization of the more conventional calculations for the Wilson loop in the case of quantum chromodynamic strings leading to the so called "area law" that is taken as a signature of color confinement [13]. From the Wilson loop we extract the static potential between two antipodal points on the surface of the bag and find it to be proportional to the volume enclosed by the surface. This is consistent with the basic underlying idea of confinement that it would require an infinite amount of energy to separate the two points. This calculation is performed in Sec. IV.

As a stepping stone toward that calculation, we investigate in Sec. III what amounts to the Casimir effect for the $A_{\mu\nu\rho}$ field. Section II discusses some of the unique properties of the $A_{\mu\nu\rho}$ field that are manifest even at the classical level but are instrumental for our discussion of the effect of quantum fluctuations of the A field. Some concluding remarks are offered in Sec. V. Finally, Appendix A contains some technical details of the regularization procedure and the counting of degrees of freedom of the A field, while Appendix B describes the mathematical steps involved in the calculation of the Wilson factor and the static potential that follows from it.

II. CLASSICALLY "TRIVIAL" DYNAMICS VS NON-TRIVIAL QUANTUM EFFECTS

Rank-three potentials $A_{\mu\nu\rho}(x)$ were introduced as a generalization of the electromagnetic potential and of the Kalb-Ramond potential in string theory [14–16].

In the free case, unlike the electromagnetic case, the classical dynamics described by the Lagrangian density

$$L_0\equiv\frac{1}{2\cdot 4!}(\partial_{[\mu}A_{\nu\rho\sigma]})^2, \quad (7)$$

is *exactly solvable*: the field strength $F_{\mu\nu\rho\sigma}\equiv\partial_{[\mu}A_{\nu\rho\sigma]}$ that solves the generalized Maxwell equations

$$\partial_\mu F^{\mu\nu\rho\sigma}=0 \quad (8)$$

describes a constant background field, $F_{\mu\nu\rho\sigma}=f\epsilon_{\mu\nu\rho\sigma}$, where f is an arbitrary integration constant.

The physical meaning of the constant f is most simply understood in terms of two unique properties of the three-index potential.

(i) The energy momentum tensor derived from Eq. (4) in the limit of flat spacetime

$$T_{\mu\nu}\equiv\left(\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}\right)_{g=\delta} \to\frac{1}{3!}F_{\mu\alpha\beta\gamma}F_\nu^{\alpha\beta\gamma}-\frac{1}{2\cdot 4!}\delta_{\mu\nu}F^{\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta} \quad (9)$$

reduces to the following simple form:

$$T_{\mu\nu}=\frac{f^2}{2}\delta_{\mu\nu}. \quad (10)$$

At first sight, quantizing A seems to be meaningless because there are no dynamical degrees of freedom carried by A. However, the similarity between Eqs. (10) and (1) suggests that, in spite of the "triviality" of the classical field equation for the three-index potential, the constant of integration f may be related, at a quantum-mechanical level, to the vacuum expectation value of the energy momentum tensor arising from the zero-point energy due to the quantum fluctuations of the A field. We shall confirm this expectation and calculate the quantum corrections to the energy-momentum tensor in the following section.

(ii) The gauge transformation property, Eq. (3), requires that even in the absence of gravity, the $A_{\mu\nu\rho}(x)$ potential be coupled to a rank-three current density $J^{\mu\nu\rho}(x)$ with support over the spacetime history of a relativistic membrane, or 2-brane [8].

Since the latter property is instrumental for our subsequent discussion, it may be helpful to elaborate briefly on it. For later convenience, in this paper we work with Euclidean, or Wick rotated, quantities. Then, the Lagrangian density under consideration is as follows:

$$L=\frac{1}{2\cdot 4!}(\partial_{[\mu}A_{\nu\rho\sigma]})^2-\frac{\kappa}{3!}J^{\mu\nu\rho}A_{\mu\nu\rho}$$
$$=-\frac{1}{2\cdot 4!}F^{\lambda\mu\nu\rho}F_{\lambda\mu\nu\rho}+\frac{1}{4!}F^{\lambda\mu\nu\rho}\partial_{[\lambda}A_{\mu\nu\rho]}$$
$$-\frac{\kappa}{3!}J^{\mu\nu\rho}A_{\mu\nu\rho} \tag{11}$$

$$J^{\mu\nu\rho}(x;Y)\equiv\int_H \delta[x-Y]dY^\mu\wedge dY^\nu\wedge dY^\rho$$
$$=\int_\Sigma d^3\sigma\,\delta^{4)}[x-Y]\epsilon^{mnr}\partial_m Y^\mu\partial_n Y^\nu\partial_r Y^\rho \tag{12}$$

where H is the target spacetime image of the world-manifold Σ through the embedding $Y:\Sigma\rightarrow H$. It is important to keep in mind that in the *first order* formulation adopted here, $F_{\lambda\mu\nu\rho}$ and $A_{\mu\nu\rho}$ are treated as independent variables [17]. In this formulation, the F-field equation is algebraic rather than differential, and this provides the link between first and second order formulation:

$$\frac{\delta L}{\delta F^{\lambda\mu\nu\rho}}=0\rightarrow F_{\lambda\mu\nu\rho}=\partial_{[\lambda}A_{\mu\nu\rho]} \tag{13}$$

$$\frac{\delta L}{\delta A^{\mu\nu\rho}}=0\rightarrow \partial_\lambda F^{\lambda\mu\nu\rho}=\kappa J^{\mu\nu\rho}(x). \tag{14}$$

The model Lagrangian, Eq. (11), is the basis for classical and quantum "membrane dynamics," CMD and QMD, respectively. Provided that the current is divergence-free, the model is invariant under the extended gauge transformation:

$$\delta A_{\mu\nu\rho}=\partial_{[\mu}\lambda_{\nu\rho]}\leftrightarrow\partial_\mu J^{\mu\nu\rho}(x)=0. \tag{15}$$

The divergence free condition (15) is satisfied whenever the membrane history has no boundary, which means either (a) spatially closed, real membranes, whose world track is infinitely extended along the timelike direction, or (b) spatially closed, virtual branes emerging from the vacuum and recollapsing into the vacuum after a finite interval of proper time [18].

This property is central to our subsequent discussion in Secs. III and IV. Thus, in order to prove that this is the case, let us compute the divergence of the current:

$$\partial_\mu J^{\mu\nu\rho}(x)=\int_\Sigma d^3\sigma\left(\frac{\partial}{\partial x^\mu}\delta^{4)}[x-Y]\right)\epsilon^{mnr}\partial_m Y^\mu\partial_n Y^\nu\partial_r Y^\rho$$
$$=\int_\Sigma d^3\sigma\left(\frac{\partial}{\partial Y^\mu}\delta^{4)}[x-Y]\right)\epsilon^{mnr}\partial_m Y^\mu\partial_n Y^\nu\partial_r Y^\rho$$
$$=\int_\Sigma d^3\sigma(\partial_m\delta^{4)}[x-Y])\epsilon^{mnr}\partial_n Y^\nu\partial_r Y^\rho$$
$$=\int_\Sigma d^3\sigma\,\epsilon^{mnr}\partial_m(\delta^{4)}[x-Y]\partial_n Y^\nu\partial_r Y^\rho)$$
$$=\int_H d(\delta^{4)}[x-Y]dY^\nu\wedge dY^\rho)$$
$$=\int_{\partial H=\emptyset}\delta^{4)}[x-y]dy^\nu\wedge dy^\rho=0. \tag{16}$$

Thus, $\partial_\mu J^{\mu\nu\rho}(x)=0\leftrightarrow\partial H=\emptyset$.

It seems physically intuitive, at this point, that the presence of a closed membrane separates spacetime into two distinct regions, namely the interior and exterior regions of a "vacuum bag." The interesting point, however, is that the two regions are characterized by a *different* value of the vacuum energy density and pressure on either side of the domain wall [8].

Mathematically, the argument goes as follows: if J, defined as in Eq. (12), is divergence-free, then it can be written as the divergence of a rank four antisymmetric *bag current* K

$$J^{\mu\nu\rho}(x)\equiv\partial_\lambda K^{\lambda\mu\nu\rho} \tag{17}$$

where

$$K^{\lambda\mu\nu\rho}(x)\equiv\int_B\delta^{4)}[x-z]dz^\lambda\wedge dz^\mu\wedge dz^\nu\wedge dz^\rho \tag{18}$$

and $H\equiv\partial B$. On the other hand,

$$dz^\lambda\wedge dz^\mu\wedge dz^\nu\wedge dz^\rho=\epsilon^{\lambda\mu\nu\rho}d^4z, \tag{19}$$

so that one can write $K^{\lambda\mu\nu\rho}(x)$ as

$$K^{\lambda\mu\nu\rho}(x)=\epsilon^{\lambda\mu\nu\rho}\Theta_B(x) \tag{20}$$

where

$$\Theta_B(x)=\int_B d^4z\,\delta^{4)}[x-z] \tag{21}$$

is the *characteristic function* of the B manifold, i.e., a generalized unit step function:

$\Theta_B(P\in B)=1,\ \ \Theta_B(P\notin B)=0.$

One can also express the bulk current K in terms of the boundary current J by inverting Eq. (17):

$$\partial_\lambda K^{\lambda\mu\nu\rho}=J^{\mu\nu\rho}(x)\rightarrow K^{\lambda\mu\nu\rho}=\partial^{[\lambda}\frac{1}{\partial^2}J^{\mu\nu\rho]}. \tag{22}$$

Now, by solving Maxwell's field equation (14), one finds the following equivalent expressions of the F field:

$$F^{\lambda\mu\nu\rho}=f\epsilon^{\lambda\mu\nu\rho}+\kappa\partial^{[\lambda}\frac{1}{\partial^2}J^{\mu\nu\rho]}$$

$$=\epsilon^{\lambda\mu\nu\rho}[f+\kappa\Theta_B(x)] \tag{23}$$

where f is, again, the constant solution of the homogeneous equation.

The gist of the above calculations is the rather remarkable fact that the dynamics of the model Lagrangian (11) is still *exactly solvable*, as it was in the free field case: the effect of the coupling is that the A field produces at most a (constant) pressure difference between the interior and exterior of a closed 2-brane. This special static effect, namely the existence of a vacuum with two distinct phases, makes the A field a suitable candidate for providing a *gauge description of a "confined cosmological constant," or vacuum pressure in a finite region of spacetime*, that is the essential ingredient of all hadronic bag models. This property is the basis for our subsequent interpretation of the interior vacuum energy density as a *quantum Casimir energy* of a hadronic bag.

It should be clear from the above discussion that the explicit computation of the Casimir pressure requires the existence of a finite volume in which the quantum fluctuations of the A field take place. At the quantum level, one may formalize this argument as follows.

The meaning of quantization of a "constant field" is perhaps best understood using the "sum over histories approach" where one has to sum over all possible configurations of the field, constant in our case, and weigh each of them with the usual factor, namely, $\exp(-\text{Euclidean action})$. The Euclidean action is the four volume integral of the Lagrangian density evaluated on the given field configuration. In the case of the A field, the Lagrangian density is constant over all possible configurations, and the Euclidean action is simply: *Euclidean action=(four volume) $\times$ const*. Then, in the limit $V\to\infty$ all quantum fluctuations are frozen and the value $f=0$ is singled out, as one might reasonably expect in the classical limit. Indeed, in the absence of both gravity and coupling to a matter field, a classical background field constant over the whole spacetime manifold can be rescaled to zero as it cannot be distinguished from the ordinary vacuum.

By reversing the argument, at the quantum level the A field has a physical meaning only if it has a non-vanishing, constant field strength F within a finite volume space(time) region, one possibility being the one entertained before, namely, the interior of a hadronic bag. Another possibility mentioned earlier, a purely quantum mechanical one, is the formation of *virtual bubbles* whose *histories* have no boundary since they arise from the vacuum and recollapse into the vacuum after a finite proper time interval within the constraint of the uncertainty principle.

Whatever the case may be, the conclusion is the same as before. Even if non-dynamical in the usual sense, $A_{\mu\nu\rho}$ is ideally suited to describe *vacuum domains, or bags*, each domain being characterized by a vacuum energy density different from the energy density of the surrounding vacuum.

III. VACUUM FLUCTUATIONS OF $A_{\mu\nu\rho}$ AND HADRONIC CASIMIR PRESSURE

Our immediate objective here is to study the effect of quantum fluctuations of the $A_{\mu\nu\rho}$ field in a finite (four) volume along the lines suggested by the arguments of the preceding section. For the purpose of illustrating the path-integral method advocated here, the mathematical object of interest is the "*finite volume*" partition functional $Z(V)$

$$Z(V)=\int[dF][DA]\exp[-S_0(F,A)] \tag{24}$$

$$S_0(F,A)=\int_V d^4x\left[\frac{1}{2\cdot 4!}F^2_{\lambda\mu\nu\rho}-\frac{1}{4!}F^{\lambda\mu\nu\rho}\partial_{[\lambda}A_{\mu\nu\rho]}\right]. \tag{25}$$

At this stage, V represents a fiducial volume momentarily introduced by hand, according to the argument of the previous section, in order to give a definite meaning to the computation of the quantum vacuum pressure within a cavity, or spacetime domain of finite extension. The physical mechanisms that may give rise to such confined quantum configurations have been mentioned before and will be discussed in the following. In the course of this calculation one must distinguish between "volume, or bulk" effects from "surface, or boundary" effects. Accordingly, we focus first on the volume contribution to the vacuum pressure. Not surprisingly, perhaps, it will turn out that the size of the domain is characteristic of the size of the homogeneous fluctuations of the A field. Later, we shall discuss the case, anticipated in the previous section, in which the spacetime region where fluctuations take place is bounded by a closed membrane coupled to A. Ultimately, we are interested in a bag model type of confinement mechanism that can be obtained by coupling $A_{\mu\nu\rho}$ to a *fermionic* current density of the type (12) [12].

This whole approach amounts to the computation of the Casimir effect for the hadronic vacuum, a case study that has been already widely reported in the literature [7]. The novelty of our approach consists in the use of the three-index gauge potential, which, to our knowledge, has never been considered before in connection with the Casimir effect. The main difference lies in the fact that, since the F field is constant within the region of confinement, it is insensitive to the shape of the boundary, so that the resulting Casimir energy density and pressure are also independent of the shape of the boundary and are affected only by the size of the volume enclosed.

In order to substantiate the above statements, let us now turn to the technical side of our computation. Starting the calculation of $Z(V)$ with the A integration, one must keep in mind that the $A_{\mu\nu\rho}$ integration measure includes gauge fixing terms and Fadeev-Popov ghosts. Appendix A discusses in full detail the regularization procedure that is required in this case. There, we show that the calculation of $Z(V)$ boils down to computing the path integral over the field strength configurations that satisfy the "constraint" $\partial_\lambda F^{\lambda\mu\nu\rho}=0$

$$Z(V) = \int [dF] J^{1/2} \delta[\partial_\lambda F^{\lambda\mu\nu\rho}] \exp\left[-\int_V d^4x \frac{1}{2\cdot 4!} F^2_{\lambda\mu\nu\rho} \right]. \tag{26}$$

Since the constraint is nothing but the classical field equation discussed in the previous section, it is easy to implement it since in four dimensions $F_{\lambda\mu\nu\rho} = F(x)\epsilon_{\lambda\mu\nu\rho}$. Accordingly, all possible classical solutions are of the form $F(x) = \text{const} \equiv f$ where f is an arbitrary parameter. The path integral is then evaluated by replacing F with its constant value in the integrand (absorbing any field independent quantity into a global normalization constant):

$$Z(V;f) = \exp\left[-\frac{1}{2} f^2 V \right] \tag{27}$$

which is the standard result available in the literature [3]. Thus, the resulting partition function is vanishing in the limit $V \to \infty$ for any value $f \neq 0$. In other words, the only allowed value is $f = 0$ giving $Z(V\to\infty) =$ "1." This is the "trivial vacuum" corresponding to a vanishing energy density/pressure. However, when the volume is finite, one must take into account contributions from the quantum vacuum fluctuations of the F field coming from all possible, constant, values of f. Here is where we depart from the conventional formulation of the sum over histories approach. Since f is constant but arbitrary, *the sum over histories amounts to integrating over all possible values of f*

$$Z(V) = \int_{-\infty}^{\infty} \frac{df}{\mu_0^2} \int [dF] J^{1/2} [\text{Det}(-\partial^2)]^{-1/2} \times \delta[F^{\lambda\mu\nu\rho} - f\epsilon^{\lambda\mu\nu\rho}] \exp\left[-\int_V d^4x \frac{1}{2\cdot 4!} F^2_{\lambda\mu\nu\rho} \right]$$

$$= \int_{-\infty}^{\infty} \frac{df}{\mu_0^2} \exp\left[-\frac{1}{2} f^2 V \right] = \sqrt{\frac{2\pi}{V\mu_0^4}} \tag{28}$$

where μ_0 is a fixed mass scale that is required in order to keep the integration measure dimensionless and all the Jacobian factors cancel out. The final result is a field-independent, but volume-dependent, constant that is missing in the standard formulation. Incidentally, the technique outlined above is the same technique that leads to the correct expression for the particle propagator in ordinary quantum mechanics [19].

From here we can proceed in two directions. First, we can calculate the size of the quantum fluctuations of the f field; second, we can derive an expression for the vacuum energy density/pressure in the finite volume in which the quantum fluctuations of the f field are confined. With reference to the first point, since Δf is defined as

$$\Delta f \equiv \sqrt{\langle f^2 \rangle - \langle f \rangle^{\ 2}} \tag{29}$$

we need to introduce an external source j in order to calculate the average values in Eq. (29). By definition

$$\langle f \rangle = -\left(\frac{1}{Z(f,j)} \frac{\partial Z(f,j)}{\partial j} \right)_{j=0} \tag{30}$$

$$\langle f^2 \rangle = \left(\frac{1}{Z(f,j)} \frac{\partial^2 Z(f,j)}{\partial j^2} \right)_{j=0} \tag{31}$$

where we use the expression (28) in the presence of an external source

$$Z(V) \to Z(V;j) = \int_{-\infty}^{\infty} \frac{df}{\mu_0^2} \exp\left[-\frac{1}{2} f^2 V - jf \right]. \tag{32}$$

Equations (30) and (31) lead to the following results:

$$\langle f \rangle = 0 \tag{33}$$

$$\langle f^2 \rangle = \frac{1}{V} \tag{34}$$

so that the variance of f, Eq. (29), is given by

$$(\Delta f)^2 = \langle f^2 \rangle = \frac{1}{V}. \tag{35}$$

The average of the F field turns out to be zero since opposite values of f are weighed equally in the partition function (28). However, the final result (35) confirms that the quantum fluctuations of the F field are confined in a finite volume such that larger volumes are associated with smaller and smaller fluctuations.

Let us now turn back to the promised expression for the vacuum energy density/pressure. This follows from the usual definition

$$p \equiv -\frac{\partial}{\partial V} \ln Z(V). \tag{36}$$

Once we compare it with the explicit expression (28), we find

$$p = \frac{1}{2V} = \frac{1}{2} \langle f^2 \rangle \tag{37}$$

which tells us that the Casimir pressure is generated solely by the quantum fluctuations of the F field and is inversely proportional to the quantization volume V.

In closing this section, we wish to study the vacuum expectation value of the energy-momentum tensor as a check on the calculation discussed above. This study will also serve the purpose of comparing the quantum computation of $\langle T_{\mu\nu} \rangle$ with its classical counterpart already discussed in Sec. II.

In order to study vacuum expectation values we need to introduce an *external source* coupled to the selected operator and then compute the corresponding generating functional. The hadronic vacuum pressure and energy density can be extracted from the expectation value of the energy momentum tensor operator

$$\langle T_{\mu\nu}\rangle=\left\langle \frac{1}{3!}F_{\mu\alpha\beta\gamma}F_{\nu}^{\alpha\beta\gamma}-\frac{1}{2\cdot 4!}\delta_{\mu\nu}F^{\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta}\right\rangle, \tag{38}$$

$T_{\mu\nu}$ being the "current" canonically conjugated to the metric tensor. Thus, we switch-on a non-trivial background metric $g_{\mu\nu}(x)$

$$S_0\to\frac{1}{2\cdot 4!}\int_V d^4x\sqrt{g}g^{\alpha\beta}g^{\mu\gamma}g^{\nu\sigma}g^{\rho\tau}F_{\alpha\mu\nu\rho}F_{\beta\gamma\sigma\tau} \tag{39}$$

where $g\equiv\det g_{\mu\nu}(x)$. The metric $g_{\mu\nu}(x)$ plays the role of external source for $T_{\mu\nu}$, which means

$$\langle T_{\mu\nu}\rangle\equiv\left(\frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}(x)}\ln Z[g;V]\right)_{g=\delta}. \tag{40}$$

Thus, the result is formally the same as in Eq. (28) except for the presence of $\sqrt{g}$ in the expression of the volume

$$Z[g;V]=\left(\frac{2\pi}{\mu_0^4V[g]}\right)^{1/2} \tag{41}$$

$$V[g]=\int_V d^4x\sqrt{g},\qquad V[g=\delta]=V. \tag{42}$$

Thus, the expression for the vacuum expectation value of $T_{\mu\nu}$ is equivalent to

$$\langle T_{\mu\nu}\rangle\equiv\left(\frac{2}{\sqrt{g}}\frac{1}{Z[g;V]}\frac{\delta Z[g;V]}{\delta g^{\mu\nu}(x)}\right)_{g=\delta}. \tag{43}$$

Since we have

$$\frac{\delta Z[g;V]}{\delta g^{\mu\nu}(x)}=-\frac{1}{2\mu_0^2}\sqrt{\frac{2\pi}{V[g]}}\ \sqrt{g}g_{\mu\nu}\left(\frac{1}{2V[g]}\right) \tag{44}$$

combining Eq. (44) with the definition (43) we finally obtain

$$\langle T_{\mu\nu}\rangle|_{g=\delta}=\left(\frac{1}{2V}\right)\delta_{\mu\nu}. \tag{45}$$

This final expression of $T_{\mu\nu}$ confirms the previous calculation of the vacuum pressure as the quantum Casimir pressure of the hadronic vacuum and concludes our discussion of the classical and quantum effects due to the three-index potential $A_{\mu\nu\rho}$ within a finite volume of spacetime. The specific coupling to a relativistic test bubble will be the subject of the next section.

IV. HADRONIC BAGS

In this section we wish to study the behavior of a *real* test bubble surrounding a vacuum domain, or bag, characterized by the Casimir energy of the $A_{\mu\nu\rho}$ field. Within the test bubble the F field may attain any value as opposed to the exterior (infinite) region where its value is zero.

Mathematically, this new situation corresponds to taking as a new action

$$S_0\to S_0+\frac{\kappa}{3!}\int d^4xA_{\mu\nu\rho}J^{\mu\nu\rho} \tag{46}$$

where $J^{\mu\nu\rho}$ is given in Eq. (12). The finite volume partition function now reads

$$Z(V;J)=\int[dF][DA]\exp[-S(F,A)] \tag{47}$$

$$S(F,A)=\int_B d^4x\left[\frac{1}{2\cdot 4!}F_{\lambda\mu\nu\rho}^{\ 2}-\frac{1}{4!}F^{\lambda\mu\nu\rho}\partial_{[\lambda}A_{\mu\nu\rho]}-\frac{\kappa}{3!}J^{\mu\nu\rho}A_{\mu\nu\rho}\right] \tag{48}$$

$$V=\int_B d^4x. \tag{49}$$

Once again, let us start the calculation of $Z(V;J)$ with the A integration. The only difference with respect to the previous case is that a bag is endowed with a non-vanishing boundary. In this case, some care must be exercised since the action S_0 can also be written in the form

$$S_0(F,A)=\int_B d^4x\left[\frac{1}{2\cdot 4!}F^2_{\lambda\mu\nu\rho}+\frac{1}{3!}A_{\mu\nu\rho}\partial_\lambda F^{\lambda\mu\nu\rho}+\frac{1}{3!}\partial_\lambda(A_{\mu\nu\rho}F^{\lambda\mu\nu\rho})\right]. \tag{50}$$

The total divergence in Eq. (50) may induce a surface term defined over ∂B. It is customary, in this instance, to assume as boundary condition that A is a *pure gauge* on ∂B

$$\frac{1}{3!}\int_B d^4x\,\partial_\lambda(A_{\mu\nu\rho}F^{\lambda\mu\nu\rho})=\frac{1}{4!}\int_{\partial B}d^3\sigma_{[\lambda}\partial_\mu\lambda_{\nu\rho]}\hat{F}^{\lambda\mu\nu\rho}\equiv\omega(F,\partial B) \tag{51}$$

where $\hat{F}$ is the field induced on the boundary by F. Proceeding in the manner discussed in the preceding section and in Appendix A, we find

$$Z(V;J)=\int[dF]\delta[\partial_\lambda F^{\lambda\mu\nu\rho}-\kappa J^{\mu\nu\rho}]\times\exp\left[-\int_B d^4x\frac{1}{2\cdot 4!}F^2_{\lambda\mu\nu\rho}\right]\times\exp[-\omega(F,\partial B)]. \tag{52}$$

The surface term does not contribute to the calculation of $Z(V;J)$ after integration over F because of Stokes' theorem, while the effect of the current is to shift the constant background value f to $f+\kappa$ within the membrane. Thus,

$$\frac{1}{2\times 4!}\int_B d^4x F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}$$
$$=\frac{1}{2\times 4!}\int_B d^4x\left(\epsilon^{\mu\nu\rho\sigma}f-\kappa\partial^{[\lambda}\frac{1}{-\partial^2}J^{\mu\nu\rho]}\right)^2$$
$$=\frac{1}{2}\int_B d^4x f_{in}^2-\frac{f\kappa}{4!}\epsilon_{\mu\nu\rho\sigma}\int d^4x\partial^{[\lambda}\frac{1}{-\partial^2}J^{\mu\nu\rho]}$$
$$+\frac{\kappa^2}{2\times 3!}\int_B d^4x\partial^{[\lambda}\frac{1}{-\partial^2}J^{\mu\nu\rho]}\partial_{[\lambda}\frac{1}{-\partial^2}J_{\mu\nu\rho]}$$
$$=\frac{1}{2}Vf_{in}[f_{in}-2\kappa\Theta_B(x)]$$
$$+\frac{\kappa^2}{2\times 3!}\int d^4x J^{\mu\nu\rho}\frac{1}{-\partial^2}J_{\mu\nu\rho}. \tag{53}$$

The final result is obtained after integrating out f_{in}:

$$Z(V;J)=\sqrt{\frac{2\pi}{\mu_0^4 V}}\exp\left\{\frac{\kappa^2}{2}V\right\}$$
$$\times\exp\left(-\frac{\kappa^2}{2\times 3!}\int_B d^4x J^{\mu\nu\rho}\frac{1}{-\partial^2}J_{\mu\nu\rho}\right). \tag{54}$$

The above expression represents the basic generating functional in the presence of a bag with a boundary. It will be used in the next section for the purpose of computing the Wilson loop of the A field.

A. Wilson factor and the static potential

In this section we assume that the hadronic manifold B extends indefinitely along the Euclidean time direction and keeps the coupling term between $A_{\mu\nu\rho}$ and the boundary. Our objective is to determine the static potential between pairs of points situated on the boundary of the test bubble that we take to be a spherical two-surface of radius R.

The evolution of the two-sphere in Euclidean time is represented by a hyper-cylinder $I\times S^{(2)}$, where I is the interval $0\leqslant t^E\leqslant T$ of Euclidean time t^E. On the two-surface $S^{(2)}$ let us "mark" a pair of antipodal points and follow their (Euclidean) time evolution. The two points move along parallel segments of total length T.

The standard calculation of the static potential between charges moving along an elongated rectangular loop turns, in the case under study, into the calculation of the Wilson "loop" along the hyper-cylinder $I\times S^{(2)}$. The rectangular path is now given by the two segments of length T and diameter $2R$ of the sphere at $t^E=0$ and $t^E=T$. The corresponding static potential is given by the following generalized Wilson integral:

$$V(R)\equiv-\lim_{T\to\infty}\frac{1}{T}\ln W[\partial B]. \tag{55}$$

The path integral calculation of $W[\partial B]$ follows from the finite volume boundary functional, Eq. (54). The Wilson factor is defined as follows:

$$W[\partial B]\equiv\left\langle\exp\left[-\frac{\kappa}{3!}\int d^4x A_{\lambda\mu\nu}J^{\lambda\mu\nu}\right]\right\rangle=\frac{Z[V\ ;J]}{Z[V\ ;J=0]} \tag{56}$$

where $V<\infty$ is understood and the limit $V\to\infty$ (along the Euclidean time direction) is performed at the end of the calculations.

In order to extract the static potential $V(R)$, we compute the double integral in Eq. (54) for the currents associated with a pair of antipodal points P and $\overline{P}$

$$\int_B d^4x J^{\mu\nu\rho}\frac{1}{\partial^2}J_{\mu\nu\rho}$$
$$=\int_{\partial B}\int_{\partial B}dy^\mu\wedge dy^\nu\wedge dy^\rho\frac{1}{\partial^2}dy'_\mu\wedge dy'_\nu\wedge dy'_\rho$$
$$=\frac{1}{4\pi^2}\int_0^T d\tau\int_T^0 d\tau'\int_{S^{(2)}}d^2\sigma\int_{S^{(2)}}d^2\xi y^{\mu\nu\rho}(\tau,\sigma)$$
$$\times\frac{1}{[y(\tau,\sigma)-y(\tau',\xi)]^2}y_{\mu\nu\rho}(\tau',\xi)\delta^2[\xi-\sigma] \tag{57}$$

where (σ^1,σ^2) and (ξ^1,ξ^2) are two independent sets of world coordinates on the $S^{(2)}$ manifold and we have inserted the explicit form of the scalar Green function.

The explicit details of the computation of the above integral can be found in Appendix B. The final result of that somewhat lengthy procedure is given by the following expression:

$$\int_0^T d\tau\int_0^T d\tau'\frac{1}{(\tau-\tau')^2+4R^2}$$
$$=\frac{1}{R}\int_0^T d\tau\arctan\left(\frac{\tau}{2R}\right)$$
$$=\frac{T}{R}\arctan\left(\frac{T}{2R}\right)+2R\ln\left(1+\frac{T^2}{4R^2}\right) \tag{58}$$

which, on account of the definition (55), leads to the final result

$$V(R)\equiv-\lim_{T\to\infty}\frac{1}{T}\ln W[\partial B]=\frac{\pi\kappa^2}{96}R^3. \tag{59}$$

According to Eq. (59) the antipodal points on the spherical membrane of radius R are subject to an attractive potential varying with the volume enclosed by the membrane.

V. CONCLUSIONS

In this paper we have tried to make a case that the hadronic vacuum represents an ideal laboratory to test a new approach to the computation of the quantum vacuum pressure in terms of an antisymmetric, rank-three, tensor gauge field $A_{\mu\nu\rho}$. A vacuum with two-phases, hadronic and ordinary, is the key ingredient of all ''bag models'' of quark confinement. Presumably, this feature should eventually manifest itself in the non-perturbative regime of QCD. As we have argued in the Introduction, it is quite possible that the phenomenon of color confinement in quantum chromodynamics is due to the Abelian part of the Yang-Mills field and that the long-distance behavior of QCD can be effectively described in terms of the rank-three gauge potential (5) associated with the Yang-Mills topological density [11,20].

A consistent quantization of the Abelian gauge field $A_{\mu\nu\rho}$ can be formulated in the ''sum over histories approach.'' While the field strength F is non-dynamical in the sense that it propagates no physical quanta, it induces processes involving virtual as well as real closed membranes and gives rise to a Casimir vacuum pressure that is inversely proportional to the confinement volume. These results have been confirmed by an explicit computation of the vacuum expectation value of the energy-momentum tensor.

With the above results in hand, we have calculated the Wilson loop of the three-index potential coupled to a test spherical membrane. From the Wilson factor we have then extracted the static potential, Eq. (59), between pairs of opposite points on the membrane. The ''volume law'' encoded in Eq. (59) is a natural generalization of the well known ''area law'' for the static potential between two test charges (quarks) bound by a chromodynamic string. As a matter of fact, it may be useful to compare the result of Eq. (59) with the more familiar result for the Wilson loop of a quark-antiquark pair bounded by a string. In the latter case, the integration path is taken to be a rectangle of spatial side R elongated in the Euclidean time direction.

It is generally assumed that confinement is equivalent to

$$W \propto \exp(-\sigma A) \tag{60}$$

where $A=TR$ is the area of the rectangle and σ is a constant with dimensions of $(\text{length squared})^{-2}$. From the definition (55) one extracts a linear potential between two test quarks

$$V(R)=\sigma R. \tag{61}$$

The rising of the potential with the distance between charges corresponds to the fact that an increasing energy is necessary to separate them. In correspondence with Eq. (61) we found the expression (59) according to which the energy needed to separate diametrically opposite points on a spherical membrane rises as R^3.

Note that Eq. (59) and Eq. (61) describe the same kind of geometric behavior. In both cases the static potential is proportional to the ''volume'' of the manifold connecting the two test charges. In Eq. (61), R is essentially the ''linear volume'' of the string connecting the pair of test charges. In our case, R^3 is proportional to the volume of the spherical membrane connecting the two antipodal points. Thus, we conclude that in the bag case, confinement is signaled by a ''volume-law'' extending the string case area law.

It has been noted elsewhere [11] that the properties described in this paper represent the exact counterpart, in four spacetime dimensions, of the well known properties of the two-dimensional Schwinger model that is widely believed to be the prototype model of quark confinement. The correspondence between the dynamics of the $A_{\mu\nu\rho}$ field coupled to quantum spinor fields and the dynamics of the Schwinger model is further explored in the third paper in this series [12].

APPENDIX A: REGULARIZATION PROCEDURE AND COUNTING THE NUMBER OF DEGREES OF FREEDOM OF THE *A* FIELD

As mentioned in the text in connection with the expression (25), the $A_{\mu\nu\rho}$ integration measure includes gauge fixing terms and Fadeev-Popov ghosts whose presence can be traced back to the fact that the action S_0 is invariant under the gauge transformation

$$\delta_\lambda A_{\mu\nu\rho}=\partial_{[\mu}\lambda_{\nu\rho]} \tag{A1}$$

$$\delta_\lambda F_{\mu\nu\rho\sigma}=0 \tag{A2}$$

so that the integration measure for the A field must be properly defined in order to avoid overcounting of physically equivalent field configurations. In the second order formulation, gauge invariance prevents one from inverting the kinetic operator and from computing the A-path integral (in spite of its Gaussian-looking form.)

The usual procedure is to break gauge invariance ''by hand'' and compensate for the unphysical degrees of freedom produced by gauge fixing by means of an appropriate set of ghost fields. In the Lorentz gauge one finds

$$[DA]=[dA]\delta[\partial_\mu A^{\mu\nu\rho}]\Delta_{FP} \tag{A3}$$

where the Fadeev-Popov determinant is defined through the gauge variation of the gauge fixing function

$$\Delta_{FP}\equiv \det\left[\frac{\delta}{\delta\lambda_{\mu\nu}}\partial^\rho\partial_{[\rho}\lambda_{\sigma\tau]}\right] = \det[\partial^\rho\partial_{[\rho}\delta^\mu_\sigma\delta^\nu_{\tau]}], \tag{A4}$$

where $\Delta^{\mu\nu}_{\sigma\tau}\equiv\partial^\rho\partial_{[\rho}\delta^\mu_\sigma\delta^\nu_{\tau]}$ is the covariant D'Alembertian over 2-forms. This operator introduces a new gauge invariance that must in turn be broken and compensated until all the unphysical degrees of freedom are removed [21]. This lengthy procedure is necessary in order to perform perturbative calculations and compute Feynman graphs. However, we are interested in a non-perturbative evaluation of the path integral. With this goal in mind, let us remark that in the first order formulation $A_{\mu\nu\rho}$ enters linearly into the action rather than quadratically. In other words, the non-dynamical nature of $A_{\mu\nu\rho}$ is made manifest in the first order formulation where $A_{\mu\nu\rho}$ plays the role of a Lagrange multiplier enforcing the classical field equation for $F_{\lambda\mu\nu\rho}$. Thus, instead of going through all the steps of the Fadeev-Popov procedure, it is more expedient to separate $A_{\mu\nu\rho}$ into the sum of a Goldstone term $\theta_{\nu\rho}$ and a gauge inert part (modulo a shift by a constant) $\epsilon_{\mu\nu\rho\sigma}\partial^\sigma\phi$

$$A_{\mu\nu\rho}\equiv\epsilon_{\mu\nu\rho\sigma}\partial^\sigma\phi+\partial_{[\mu}\theta_{\nu\rho]} \tag{A5}$$

$$\delta_\lambda\phi=0, \quad \delta_\lambda\theta_{\nu\rho}=\lambda_{\nu\rho}. \tag{A6}$$

Accordingly, the functional integration measure becomes

$$[dA]=J[d\phi][d\theta] \tag{A7}$$

where J is the functional Jacobi determinant induced by the change of integration variables (A5) (not to be confused with the Fadeev-Popov determinant). Explicitly, J reads as follows:

$$J=[\mathrm{Det}(-\partial^2)]^{1/2}\times\left[\mathrm{Det}\left(-\frac{1}{3!}\epsilon_{\sigma\alpha\beta\gamma}\partial^\gamma\epsilon^{\sigma\alpha\beta\rho}\partial_\rho\right)\right]^{1/2}$$
$$=[\mathrm{Det}(-\partial^2)] \tag{A8}$$

and yields the correct counting of the physical degrees of freedom. As a matter of fact, at first sight it seems that we have introduced two new degrees of freedom: θ and ϕ, while from the classical analysis of the previous section we expect A to describe a constant background field.

Let us show first how θ drops out of the path integral. The classical action is θ independent because of gauge invariance

$$S_0(F,A)\equiv S_0(F,\phi) \tag{A9}$$

and does not provide the necessary damping of gauge equivalent paths. However, the gauge fixed-compensated integration measure reads

$$[DA]\equiv[d\phi][d\theta]J\delta[\partial_\mu\partial^{[\mu}\theta^{\nu\rho]}]\Delta_{FP} \tag{A10}$$

and we can get rid of the gauge orbit volume. Since

$$\int[d\theta]\delta[\partial_\mu\partial^{[\mu}\theta^{\nu\rho]}]\Delta_{FP}=1 \tag{A11}$$

we obtain a path integral over gauge invariant degrees of freedom only:

$$Z(V)=\int[dF][d\phi]J\exp[-S_0(F,\phi)]$$

$$S_0(F,\phi)\equiv\int_V d^4x\left[\frac{1}{2\cdot4!}F^2_{\lambda\mu\nu\rho}-\frac{1}{3!}\partial_\lambda F^{\lambda\mu\nu\rho}\epsilon_{\mu\nu\rho\sigma}\partial^\sigma\phi\right]. \tag{A12}$$

At this point we have a choice. Suppose we integrate first over F. This is a Gaussian integration and we obtain

$$Z=\int[d\phi]J\exp\left[-\int d^4x\frac{1}{2}\phi(-\partial^2)(-\partial^2)\phi\right]. \tag{A13}$$

Now, integrating over the scalar field ϕ one sees that the contribution of the ϕ-field fluctuations exactly cancel against the Jacobian because of the presence of "box squared" kinetic term [22]

$$Z=[\mathrm{Det}(-\partial^2)]\times[\mathrm{Det}(-\partial^2)^2]^{-1/2}=1. \tag{A14}$$

Thus, no spurious degrees of freedom were introduced through the ansatz (A5).

On the other hand, it is interesting to check on the above procedure by reversing the order of integration and start with ϕ instead of F. In this case it is more convenient to introduce the new integration variable

$$U_{\mu\nu\rho}\equiv\epsilon_{\mu\nu\rho\sigma}\partial^\sigma\phi \tag{A15}$$

and write the integration measure as follows:

$$[d\phi]=[dU][\mathrm{Det}(-\partial^2)]^{-1/2}=J^{-1/2}[dU]. \tag{A16}$$

Hence, we obtain

$$Z=\int[dU][dF]J^{1/2}\exp\left[-\int_V d^4x\left(-\frac{1}{2\cdot4!}F^2_{\lambda\mu\nu\rho}-\frac{1}{3!}\partial_\lambda F^{\lambda\mu\nu\rho}U_{\mu\nu\rho}\right)\right] \tag{A17}$$

and notice that the path integral is linear in the U variable. In order to integrate over this variable it is convenient to rotate, *temporarily*, from a Euclidean to a Minkowskian signature in such a way as to reproduce the functional expression of the Dirac delta function

$$\int[dU]\exp\left(-\frac{i}{3!}\int_V d^4x U_{\mu\nu\rho}\partial_\lambda F^{\lambda\mu\nu\rho}\right)=\delta[\partial_\lambda F^{\lambda\mu\nu\rho}]. \tag{A18}$$

Once we rotate back to the Euclidean signature we are led to the expression (26) quoted in the text of the paper.

APPENDIX B: THE COMPUTATION OF THE WILSON FACTOR

With reference to Eq. (57) in the text, let us indicate by

$$y^{\mu\nu\rho}=\epsilon^{abc}\partial_a y^\mu\partial_a y^\nu\partial_b y^\rho \tag{B1}$$

the "tangent elements" to the world history of the test bubble. The membrane world manifold is a hyper-cylinder with the Euclidean metric given by (in polar coordinates)

$$ds^2=\gamma_{ab}(\sigma)d\sigma^a d\sigma^b=d\tau^2+R^2(d\theta^2+\sin^2\theta d\phi^2) \tag{B2}$$

where $0\leqslant\phi\leqslant2\pi$, $0\leqslant\theta\leqslant\pi$, $0\leqslant\tau\leqslant T$. The embedding in target spacetime is obtained through the equations

$$y^1=R\sin\theta\sin\phi \tag{B3}$$

$$y^2=R\sin\theta\cos\phi \tag{B4}$$

$$y^3 = R\cos\theta \tag{B5}$$

$$y^4 = \tau. \tag{B6}$$

Then, with the above choice of coordinates, we find

$$y^{\mu\nu\rho} = \partial_{[\tau} y^\mu \partial_\theta y^\nu \partial_{\phi]} y^\rho \tag{B7}$$
$$y^{ijk} \equiv 0$$
$$y^{4ij} = \partial_{[\theta} y^i \partial_{\phi]} y^j.$$

The explicit expression of the tangent elements y^{ijk} evaluated at the point P can be written as follows:

$$y^{12}(\theta,\phi) \equiv \partial_{[\theta} y^1 \partial_{\phi]} y^2 = -R^2 \cos\theta \sin\theta$$
$$y^{13}(\theta,\phi) \equiv \partial_{[\theta} y^1 \partial_{\phi]} y^3 = R^2 \sin^2\theta \cos\phi$$
$$y^{23}(\theta,\phi) \equiv \partial_{[\theta} y^2 \partial_{\phi]} y^3 = -R^2 \sin^2\theta \sin\phi.$$

Then, for the antipodal point $\bar{P}$ the same expressions become

$$y^{12}(\pi-\theta,\phi+\pi) = R^2 \cos\theta \sin\theta$$
$$y^{13}(\pi-\theta,\phi+\pi) = -R^2 \sin^2\theta \cos\phi$$
$$y^{23}(\pi-\theta,\phi+\pi) = R^2 \sin^2\theta \sin\phi.$$

From the above expressions, we explicitly calculate

$$\ln W[\partial B] = \frac{\kappa^2}{48\pi^2} \int_0^T d\tau \int_T^0 d\tau' \int_0^\pi d\theta \times \int_0^{2\pi} d\phi\, y^{ij}(\theta,\phi) \frac{1}{[y(\theta,\phi) - y(\pi-\theta,\phi+\pi)]^2} \times y_{ij}(\pi-\theta,\phi+\pi) \tag{B8}$$

so that

$$\frac{1}{2} y^{ij}(\theta,\phi) y_{ij}(\pi-\theta,\phi+\pi) = -R^4 \sin^2\theta \tag{B9}$$

$$[y(\theta,\phi) - y(\pi-\theta,\phi+\pi)]^2 = (\tau-\tau')^2 + 4R^2. \tag{B10}$$

Therefore the logarithm of $W[\partial B]$ is

$$\ln W[\partial B] = -\frac{\kappa^2 R^4}{48} \int_0^T d\tau \int_0^T d\tau' \frac{1}{(\tau-\tau')^2 + 4R^2}. \tag{B11}$$

We now proceed to calculate the double integral in Eq. (B11):

$$\int_0^T d\tau \int_0^T d\tau' \frac{1}{(\tau-\tau')^2+4R^2}$$
$$= -\int_0^T d\tau \int_\tau^{\tau-T} du \frac{1}{u^2+4R^2}, \quad u \equiv \tau - \tau'$$
$$= -\frac{1}{2R} \int_0^T d\tau \int_{\tau/2R}^{(\tau-T)/2R} dy \frac{1}{1+y^2}$$
$$= -\frac{1}{2R} \int_0^T d\tau \left[\arctan\left(\frac{\tau-T}{2R}\right) - \arctan\left(\frac{\tau}{2R}\right) \right] \tag{B12}$$

$$\int_0^T d\tau \arctan\left(\frac{\tau-T}{2R}\right)$$
$$= \int_{-T}^0 ds \arctan\left(\frac{s}{2R}\right), \quad \tau - T \equiv s$$
$$= -\int_0^T ds \arctan\left(\frac{s}{2R}\right), \quad s \to -s. \tag{B13}$$

Setting together Eqs. (B12) and (B13) we obtain the expression (58) quoted in the text.

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