Gauge theory on a noncommutative supersphere from a supermatrix model

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We construct a supermatrix model that has a classical solution representing the noncommutative (fuzzy) two-supersphere. Expanding supermatrices around the classical background, we obtain a gauge theory on a noncommutative superspace on a sphere. This theory has $osp(1|2)$ supersymmetry and $u(2L+1|2L)$ gauge symmetry. We also discuss a commutative limit of the model keeping the radius of the supersphere fixed.

I. INTRODUCTION

The deformation of superspace by introducing noncommutativity has attracted much interest recently. It is suggested that nonanticommutativity of fermionic coordinates of superspace appears in superstring theory in the background of the Ramond-Ramond (RR) or graviphoton field strength [1–3]. This phenomenon is similar to the well-known case of string theory in the Neveu-Schwarz–Neveu-Schwarz (NS-NS) two-form B background, where the bosonic space-time coordinates become noncommutative [4,5]. The supersymmetric gauge theories and Wess-Zumino model on noncommutative superspace are under active study, and various aspects of those field theories, which include renormalizability in perturbations, UV-IR mixing, etc., have been discussed [6–27]. There are also earlier works where noncommutative superspace was studied [28–32].

There are also analyses of noncommutative superspace using supermatrices [33–38]. Supersymmetric actions for scalar multiplets on the fuzzy two-supersphere were constructed in [33] based on the $osp(1|2)$ graded Lie algebra. Furthermore, a graded differential calculus on the fuzzy supersphere is discussed in [34]. Supersymmetric gauge theories on this noncommutative superspace were studied in [35] by using differential forms on it. In [36], noncommutative superspaces and their flat limits were studied by using the graded Lie algebras $osp(1|2)$, $osp(2|2)$, and $psu(2|2)$. Recently, the concept of noncommutative superspace based on a supermatrix was also introduced in proving the Dijkgraaf-Vafa conjecture as a large N reduction [39]. The supermatrix model was also studied from the viewpoint of background independent formulations of matrix models, which are expected to give constructive definitions of string theories [40].

In this paper we construct a supersymmetric gauge theory on the fuzzy two-supersphere based on a supermatrix model. This is a natural extension of constructing gauge theories on the bosonic noncommutative space in matrix models. In ordinary matrix models of the Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) type [41], background space-time appears as a classical background of matrices A_i, and their fluctuations around the classical solution are interpreted as gauge

fields on this space-time. If the classical solutions are noncommutative, we can obtain noncommutative gauge theories [42]. In this approach, the constructions of the open Wilson lines and the background independence of the noncommutative gauge theories become manifest [42–44]. The construction of supermatrix models whose classical backgrounds represent a noncommutative superspace will similarly play an important role in understanding various properties of field theories on the noncommutative superspace. In this paper we investigate in particular a supersymmetric gauge theory on the fuzzy two-supersphere by using a simple supermatrix model based on the $osp(1|2)$ graded Lie algebra. Noncommutative superspace coordinates (x_i, θ_α) and gauge superfields $(\tilde{a}_i, \varphi_\alpha)$ on it are combined as single supermatrices $A_i \sim x_i + \tilde{a}_i$ and $\psi_\alpha \sim \theta_\alpha + \varphi_\alpha$. Our formulation of a supersymmetric gauge theory on the fuzzy supersphere has some similarities to the covariant superspace approach in ordinary supersymmetric gauge theories [45]. In this approach, connection superfields on the superspace are introduced and constraints are imposed on them to eliminate extra degrees of freedom. It turns out that supermatrices in our model correspond to connection superfields on the noncommutative supersphere.

This paper is organized as follows. In Sec. II, we first review the construction of the fuzzy two-supersphere based on the $osp(1|2)$ graded Lie algebra. The representations of $osp(1|2)$ are explained, and fields on the fuzzy space are introduced as polynomials of the representation matrices of the $osp(1|2)$ generators. In Sec. III, we construct a supermatrix model which has a classical solution corresponding to the fuzzy two-supersphere. Expanding a supermatrix around this classical background, we obtain a supersymmetric gauge theory on the fuzzy supersphere. The action has $osp(1|2)$ supersymmetry and $u(2L+1|2L)$ gauge symmetry. Then it is shown that in the commutative limit this model gives the $U(1)$ gauge theory on a commutative sphere. Conclusions and a discussion are given in Sec. IV. Brief explanations of the graded Lie algebra and supermatrix are given in the Appendix.

II. FUZZY TWO-SUPERSPHERE

In this section we review the construction of supermatrix models and field theories on the fuzzy two-supersphere based on $osp(1|2)$ algebra. This was first studied in [33].

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The notation and definitions used in this paper are given in the Appendix.

The graded commutation relations of $osp(1|2)$ algebra are given by

$$[\hat{l}_i, \hat{l}_j] = i\epsilon_{ijk}\hat{l}_k,$$

$$[\hat{l}_i, \hat{v}_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}\hat{v}_\beta, \quad (2.1)$$

$$\{\hat{v}_\alpha, \hat{v}_\beta\} = \frac{1}{2}(C\sigma_i)_{\alpha\beta}\hat{l}_i,$$

where $C = i\sigma_2$ is a charge conjugation matrix. The even part of this algebra is $su(2)$, which is generated by $\hat{l}_i(i = 1,2,3)$, and the odd generators $\hat{v}_\alpha$ ($\alpha = 1,2$) are $su(2)$ spinors. Irreducible representations of $osp(1|2)$ algebra [46] are characterized by values of the Casimir operator $\hat{K}_2 = \hat{l}_i\hat{l}_i + C_{\alpha\beta}\hat{v}_\alpha\hat{v}_\beta = L(L+\frac{1}{2})$ where the quantum number L is called superspin and $L \in \mathbb{Z}_{\geq 0}/2$. Each representation consists of spin L and $L-\frac{1}{2}$ representations of $su(2)$, $|L,l_3\rangle, |L-\frac{1}{2},l_3\rangle$ and its dimension is $N \equiv (2L+1) + 2L = 4L+1$. The explicit expressions of the generators are

$$l_i^{(L)} = \begin{pmatrix} L_i^{(L)} & 0 \\ 0 & L_i^{(L-1/2)} \end{pmatrix}, \quad v_\alpha^{(L)} = \begin{pmatrix} 0 & V_\alpha^{(L,L-1/2)} \\ V_\alpha^{(L-1/2,L)} & 0 \end{pmatrix}.$$
(2.2)

The matrix elements of $L_\pm = L_1 \pm iL_2$, $V_+ = V_1$, and $V_- = V_2$ are given by

$$\langle L,l_3+1|L_+^{(L)}|L,l_3\rangle = \sqrt{(L-l_3)(L+l_3+1)},$$

$$\langle L,l_3-1|L_-^{(L)}|L,l_3\rangle = \sqrt{(L+l_3)(L-l_3+1)},$$

$$\langle L,l_3+1/2|V_+^{(L,L-1/2)}|L-1/2,l_3\rangle = -\frac{1}{2}\sqrt{L+l_3+\frac{1}{2}},$$

$$\langle L,l_3-1/2|V_-^{(L,L-1/2)}|L-1/2,l_3\rangle = -\frac{1}{2}\sqrt{L-l_3+\frac{1}{2}},$$
(2.3)

$$\langle L-1/2,l_3+1/2|V_+^{(L-1/2,L)}|L,l_3\rangle = -\frac{1}{2}\sqrt{L-l_3},$$

$$\langle L-1/2,l_3-1/2|V_-^{(L-1/2,L)}|L,l_3\rangle = \frac{1}{2}\sqrt{L+l_3}.$$

These are the superstar representations of $osp(1|2)$,

$$l_i^{(L)\ddagger} = l_i^{(L)}, \quad v_\alpha^{(L)\ddagger} = -C_{\alpha\beta}v_\beta^{(L)}. \quad (2.4)$$

See the Appendix for superstar conjugation $\ddagger$.

The condition $\hat{K}_2 = L(L+\frac{1}{2})$ defines a two-dimensional supersphere. Consider polynomials $\Phi(l_i^{(L)}, v_\alpha^{(L)})$ of the representation matrices $l_i^{(L)}$ and $v_\alpha^{(L)}$ with superspin L. Let us denote the space spanned by $\Phi(l_i^{(L)}, v_\alpha^{(L)})$ as $\mathcal{A}_L$. The

$osp(1|2)$ algebra acts on $\mathcal{A}_L$ by three kinds of action, the left action ($\hat{l}_i^L, \hat{v}_\alpha^L$), the right action ($\hat{l}_i^R, \hat{v}_\alpha^R$), and the adjoint action ($\hat{\mathcal{L}}_i \equiv \hat{l}_i^L - \hat{l}_i^R, \hat{\mathcal{V}}_\alpha = \hat{v}_\alpha^L - \hat{v}_\alpha^R$),

$$\hat{l}_i^L\Phi = l_i^{(L)}\Phi, \quad \hat{v}_\alpha^L\Phi = v_\alpha^{(L)}\Phi, \quad (2.5)$$

$$\hat{l}_i^R\Phi = \Phi l_i^{(L)}, \quad \hat{v}_\alpha^R\Phi = \Phi v_\alpha^{(L)}, \quad (2.6)$$

$$\hat{\mathcal{L}}_i\Phi = [l_i^{(L)}, \Phi], \quad \hat{\mathcal{V}}_\alpha\Phi = [v_\alpha^{(L)}, \Phi]. \quad (2.7)$$

The right action satisfies the $osp(1|2)$ algebra with a minus sign ($-l_i^R, -v_\alpha^R$). The polynomials transform as $L \otimes L$ under the left and right actions of $osp(1|2)$ and can be decomposed into irreducible representations under the adjoint action as

$$L \otimes L = 0 \oplus \frac{1}{2} \oplus 1 \oplus \cdots \oplus 2L - \frac{1}{2} \oplus 2L.$$

The dimension of the space spanned by these polynomials is $(4L+1)^2$. Among them, we can define supersymmetrized matrix spherical harmonics $Y_{km}^S(l_i^{(L)}, v_\alpha^{(L)})$ which are generalization of the ordinary matrix spherical harmonics to the supersphere (see [34] for the details),

$$(\hat{\mathcal{L}}_i\hat{\mathcal{L}}_i + C_{\alpha\beta}\hat{\mathcal{V}}_\alpha\hat{\mathcal{V}}_\beta)Y_{km}^S(l_i^{(L)}, v_\alpha^{(L)}) = k\left(k+\frac{1}{2}\right)Y_{km}^S(l_i^{(L)}, v_\alpha^{(L)}),$$
(2.8)

$$\hat{\mathcal{L}}_3Y_{km}^S(l_i^{(L)}, v_\alpha^{(L)}) = mY_{km}^S(l_i^{(L)}, v_\alpha^{(L)}). \quad (2.9)$$

k can take either an integer or a half-integer value. Any $N \times N$ supermatrix can be expanded in terms of the super-spherical harmonics as

$$\Phi(l_i^{(L)}, v_\alpha^{(L)}) = \sum_{k=0,1/2,1,\dots}^{2L} \phi_{km}Y_{km}^S(l_i^{(L)}, v_\alpha^{(L)}), \quad (2.10)$$

where the Grassmann parity of the coefficient ϕ_{km} is determined by the grading of the spherical harmonics. Even (odd) spherical harmonics has nonvanishing values only in the diagonal (off-diagonal) blocks in its matrix form. We can map the supermatrix $\Phi(l_i^{(L)}, v_\alpha^{(L)})$ to a function on the superspace (x_i, θ_α) by

$$\Phi(l_i, v_\alpha) \rightarrow \phi(x_i, \theta_\alpha) = \sum_{k,m} \phi_{km} y_{km}^S(x_i, \theta_\alpha), \quad (2.11)$$

where $y_{km}^S(x_i, \theta_\alpha)$ are ordinary superspherical functions. A product of supermatrices is mapped to a noncommutative star product of functions. An explicit form of the star product is given in [47].

In addition to the $osp(1|2)$ generators ($\hat{l}_i, \hat{v}_\alpha$), we can define additional generators with which they form bigger algebra $osp(2|2)$. These additional generators are

$$\hat{\gamma} = -\frac{1}{L+1/4}\left[C_{\alpha\beta}\hat{v}_\alpha\hat{v}_\beta + 2L\left(L+\frac{1}{2}\right)\right], \quad (2.12)$$

$$\hat{d}_\alpha = [\hat{\gamma}, \hat{v}_\alpha] = \frac{1}{2(L+1/4)} (\sigma_i)_{\beta\alpha} (\hat{v}_\beta \hat{l}_i + \hat{l}_i \hat{v}_\beta). \quad (2.13)$$

Commutation relations for the additional generators are given by

$$[\hat{\gamma}, \hat{v}_\alpha] = \hat{d}_\alpha, \quad [\hat{\gamma}, \hat{d}_\alpha] = \hat{v}_\alpha, \quad [\hat{\gamma}, \hat{l}_i] = 0,$$

$$[\hat{l}_i, \hat{d}_\alpha] = \frac{1}{2} (\sigma_i)_{\beta\alpha} \hat{d}_\beta, \quad \{\hat{d}_\alpha, \hat{d}_\beta\} = -\frac{1}{2} (C\sigma_i)_{\alpha\beta} \hat{l}_i,$$

$$\{\hat{v}_\alpha, \hat{d}_\beta\} = -\frac{1}{4} C_{\alpha\beta} \hat{\gamma}.$$

The adjoint action of the fermionic generators $D_\alpha = \mathrm{adj}\hat{d}_\alpha$ plays the role of the covariant derivatives on the supersphere. On the other hand, the adjoint actions of the original fermionic generators $Q_\alpha = \mathrm{adj}\hat{v}_\alpha$ are interpreted as supersymmetry generators. These additional generators also play an important role in constructing kinetic terms for a scalar multiplet on the supersphere [33].

The commutative limit is discussed in [33] and the fuzzy supersphere becomes the ordinary two-dimensional supersphere with two real Grassmannian coordinates. This limit can be taken by keeping the radius of the sphere fixed and taking the large L limit.

III. GAUGE THEORY ON FUZZY SUPERSPHERE

In this section we construct a supermatrix model which has a classical solution representing the fuzzy supersphere. Expanding supermatrices around the classical solution, we obtain the action with the supersymmetry and gauge symmetry. This is a supermatrix extension of the construction of a gauge theory on a fuzzy sphere from matrix models [48].

Let us consider a supermatrix M which has the following form:

$$M = A_i \otimes t_i + C_{\alpha\beta} \psi_\alpha \otimes q_\beta, \quad (3.1)$$

where $t_i (i=1,2,3)$ and $q_\alpha (\alpha=1,2)$ are the $L=1/2$ representation matrices of the $osp(1|2)$ algebra,

$$t_i = \frac{1}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \quad q_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$q_2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}. \quad (3.2)$$

A_i and ψ_α are, respectively, even and odd $N \times N$ supermatrices with $N=4L+1$. We impose a reality condition $M^\ddagger = M$, that is, $A_i^\ddagger = A_i$ and $\psi_\alpha^\ddagger = C_{\alpha\beta}\psi_\beta$. We define a grading operator B for $N \times N$ supermatrices as

$$B = \begin{pmatrix} \mathbf{1}_{2L+1} & 0 \\ 0 & -\mathbf{1}_{2L} \end{pmatrix}. \quad (3.3)$$

It should be noted that A_i and ψ_α are $(4L+1) \times (4L+1)$ supermatrices and can also be represented as polynomials of $l_i^{(L)}$ and $v_\alpha^{(L)}$ in a similar manner to Eq. (2.10). Hence they become superfields on the fuzzy supersphere in the commutative limit.

Let us consider the following action for M:

$$S = \frac{1}{g^2} \mathrm{Str}_{(3 \times 3, N \times N)} (M^3 + \lambda M^2), \quad (3.4)$$

where λ and g are real constants. In terms of A_i and ψ_α, it can be rewritten, by taking traces over (3×3) matrices, as

$$S = \frac{1}{g^2} \mathrm{Str}_{(N \times N)} \left(\frac{i}{4} \epsilon_{ijk} A_i A_j A_k + \frac{\lambda}{2} A_i A_i \right.$$
$$\left. - \frac{3}{16} \psi_\alpha (\sigma_i C)_{\alpha\beta} [A_i, \psi_\beta] - \frac{\lambda}{2} C_{\alpha\beta} \psi_\alpha \psi_\beta \right). \quad (3.5)$$

This action is invariant under the $osp(1|2)$ transformation

$$\delta M = i[G, M], \quad (3.6)$$

where G has the form of

$$G = u_i \otimes t_i + \epsilon_\alpha \otimes q_\alpha, \quad G^\ddagger = G. \quad (3.7)$$

u_i are Grassmann even numbers and ϵ_α are defined as $\epsilon_\alpha = \tilde{\epsilon}_\alpha B$ where $\tilde{\epsilon}_\alpha$ are Grassmann odd numbers. The parameters u_i and $\tilde{\epsilon}_\alpha$ satisfy $(u_i)^\# = u_i$ and $(\tilde{\epsilon}_\alpha)^\# = C_{\alpha\beta}\tilde{\epsilon}_\beta$. It should be noted that ϵ_α (anti)commutes with (odd) even supermatrices because of the grading operator B in ϵ_α. Furthermore, the action is invariant under the adjoint action of $u(2L+1|2L)$,

$$\delta A_i = i[H, A_i], \quad \delta \psi_\alpha = i[H, \psi_\alpha], \quad (3.8)$$

where $H^\ddagger = H$, $H \in u(2L+1|2L)$.

The equations of motion are

$$i\epsilon_{ijk} A_j A_k + \frac{4\lambda}{3} A_i + \frac{1}{4} (\sigma_i C)_{\alpha\beta} \{\psi_\alpha, \psi_\beta\} = 0, \quad (3.9)$$

$$\frac{3}{8} (\sigma_i C)_{\alpha\beta} [A_i, \psi_\beta] + \lambda C_{\alpha\beta} \psi_\beta = 0. \quad (3.10)$$

The model has a nontrivial classical solution representing the fuzzy two-supersphere,[1]

$$A_i^{cl} = \left(\frac{16}{9} \lambda \right) l_i^{(L)}, \quad \psi_\alpha^{cl} = \pm \left(\frac{16}{9} \lambda \right) d_\alpha^{(L)}. \quad (3.11)$$

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[1]There are other classical solutions, e.g., the trivial solution $A_i = \psi_\alpha = 0$ and the fuzzy sphere solution $A_i = (4\lambda/3) l_i^{(L)}, \psi_\alpha = 0$. We concentrate here on the fuzzy supersphere solution.

We can choose the $+$ sign in the classical solution of ψ_α without loss of generality because the action is invariant under $\psi_\alpha \to -\psi_\alpha$. We note that the classical background $d_\alpha^{(L)}$ of ψ_α can also be written by $l_i^{(L)}$ and $v_\alpha^{(L)}$, Eq. (2.13). Expanding A_i and ψ_α around the classical solution,

$$A_i = \frac{16}{9}\lambda(l_i^{(L)} + \tilde{a}_i), \quad \psi_\alpha = \frac{16}{9}\lambda(d_\alpha^{(L)} + \varphi_\alpha), \quad (3.12)$$

the action becomes

$$S = \left(\frac{16}{9}\right)^2 \frac{\lambda^3}{g^2} \text{Str}_{(N \times N)} \left\{ \frac{2}{3} i\epsilon_{ijk}\left(\tilde{a}_i[l_j,\tilde{a}_k] + \frac{1}{3}\tilde{a}_i[\tilde{a}_j,\tilde{a}_k]\right) \right.$$
$$+ \frac{1}{2}\tilde{a}_i\tilde{a}_i + (\sigma_i C)_{\alpha\beta}\left(\frac{2}{3}\tilde{a}_i\{d_\alpha,\varphi_\beta\} - \frac{1}{3}\varphi_\alpha[l_i + \tilde{a}_i,\varphi_\beta]\right)$$
$$\left. - \frac{1}{2}C_{\alpha\beta}\varphi_\alpha\varphi_\beta \right\} + \frac{1}{6}\left(\frac{16}{9}\right)^2 \frac{\lambda^3}{g^2}L\left(L + \frac{1}{2}\right). \quad (3.13)$$

The fluctuations $\tilde{a}_i$ and φ_α are, respectively, even and odd $N \times N$ supermatrices which can be expanded in terms of polynomials of $l_i^{(L)}$ and $v_\alpha^{(L)}$. Therefore they are regarded as the superfields on the fuzzy supersphere. Although the backgrounds of A_i and ψ_α violate the $osp(1|2)$ invariance (3.6), it can be compensated by appropriate $u(2L+1|2L)$ transformations. Actually the action is invariant under the following combination of $osp(1|2)$ and $u(2L+1|2L)$ with $H = u_i l_i^{(L)} - \epsilon_\alpha d_\alpha^{(L)}$ [where u_i and ϵ_α are introduced in Eq. (3.7)]:

$$\delta\tilde{a}_i = -\epsilon_{ijk}u_j\tilde{a}_k + iu_j[l_j^{(L)},\tilde{a}_i] - \frac{i}{2}(\sigma_i)_{\beta\alpha}\epsilon_\alpha\varphi_\beta$$
$$- i\epsilon_\alpha[d_\alpha^{(L)},\tilde{a}_i],$$

$$\delta\varphi_\alpha = -\frac{i}{2}u_i(\sigma_i)_{\beta\alpha}\varphi_\beta + iu_i[l_i^{(L)},\varphi_\alpha]$$
$$- \frac{i}{2}(C\sigma_i)_{\alpha\beta}\epsilon_\beta\tilde{a}_i - i\epsilon_\beta\{d_\beta^{(L)},\varphi_\alpha\}. \quad (3.14)$$

These are the supersymmetry transformations of this model. There is also the $u(2L+1|2L)$ gauge symmetry,

$$\delta\tilde{a}_i = i[H, l_i^{(L)} + \tilde{a}_i],$$

$$\delta\varphi_\alpha = i[H, d_\alpha^{(L)} + \varphi_\alpha]. \quad (3.15)$$

Therefore the action (3.13) we obtained describes a supersymmetric gauge theory on the fuzzy supersphere.

Let us consider the field theory representation of the supermatrix model. We introduce coordinates on the supersphere (x_i, θ_α) as

$$x_i = \frac{\rho}{\sqrt{L\left(L + \frac{1}{2}\right)}}l_i^{(L)}, \quad (3.16)$$

$$\theta_\alpha = \frac{\rho}{\sqrt{L\left(L + \frac{1}{2}\right)}}v_\alpha^{(L)}, \quad (3.17)$$

where ρ is a real constant. These coordinates parameterize the noncommutative supersphere with radius ρ: $x_ix_i + C_{\alpha\beta}\theta_\alpha\theta_\beta = \rho^2$. The noncommutativity parameter is given by $\sqrt{\rho}/L$. In the $L \to \infty$ limit,[2] x_i and θ_α become commutative coordinates. The supermatrices $\tilde{a}_i$ and φ_α are mapped to superfields $\tilde{a}_i(x,\theta)$ and $\varphi_\alpha(x,\theta)$, respectively, as in Eq. (2.11). The adjoint actions of the $osp(2|2)$ generators on supermatrices become the actions of the following differential operators on superfields [33]:

$$\text{adj}(l_i) \to K_i = R_i + \frac{1}{2}\theta_\alpha(\sigma_i)_{\alpha\beta}\frac{\partial}{\partial\theta_\beta},$$

$$\text{adj}(v_\alpha) \to K_\alpha^v = \frac{1}{2}x_i(C\sigma_i)_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} - \frac{1}{2}\theta_\beta(\sigma_i)_{\beta\alpha}\partial_i,$$

$$\text{adj}(d_\alpha) \to K_\alpha^d = -\frac{r}{2}\left(1 + \frac{\theta^2}{r^2}\right)C_{\alpha\beta}\frac{\partial}{\partial\theta_\beta}$$
$$+ \frac{1}{2r}\theta_\beta(\sigma_i)_{\beta\alpha}R_i - \frac{1}{2r}\theta_\alpha x_i\partial_i, \quad (3.18)$$

$$\text{adj}(\gamma) \to K^\gamma = \frac{1}{r}x_i(\sigma_i)_{\alpha\beta}\theta_\alpha\frac{\partial}{\partial\theta_\beta},$$

where $R_i = -i\epsilon_{ijk}x_j\partial_k$. The supertrace can be replaced by the integral on the supersphere,

$$\text{Str} \to -\frac{\rho}{2\pi}\int d^3x d^2\theta \,\delta(x^2 + \theta^2 - \rho^2). \quad (3.19)$$

By using the mapping rules (2.11),(3.16)–(3.19), we obtain the following action on the noncommutative supersphere:

$$S = \left(-\frac{\rho}{2\pi}\right)\left(\frac{16}{9}\right)^2\frac{\lambda^3}{g^2}\int d^3x d^2\theta \,\delta(x^2 + \theta^2 - \rho^2)$$
$$\times \left\{ \frac{2}{3}i\epsilon_{ijk}\left(\tilde{a}_iK_j\tilde{a}_k + \frac{1}{3}\tilde{a}_i[\tilde{a}_j,\tilde{a}_k]\right) + \frac{1}{2}\tilde{a}_i\tilde{a}_i \right.$$
$$+ (\sigma_i C)_{\alpha\beta}\left(\frac{2}{3}\tilde{a}_iK_\alpha^d\varphi_\beta - \frac{1}{3}\varphi_\alpha(K_i\varphi_\beta + [\tilde{a}_i,\varphi_\beta])\right)$$
$$\left. - \frac{1}{2}C_{\alpha\beta}\varphi_\alpha\varphi_\beta \right\}_* + \frac{1}{6}\left(\frac{16}{9}\right)^2\frac{\lambda^3}{g^2}L\left(L + \frac{1}{2}\right). \quad (3.20)$$

Here the asterisk indicates the star product on the fuzzy supersphere [47].

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[2]We can consider other $L \to \infty$ limits. For instance, a flat noncommutative limit with asymmetric scalings for θ_α is studied in [36].

Next we consider the commutative limit of the model. This limit is given by the $L \to \infty$ limit, keeping the radius of the supersphere fixed. The supermatrices $\tilde{a}_i$ and φ_α become superfields which can be expanded as follows:

$$\tilde{a}_i(x,\theta) = a_i(x) + \xi_{i\alpha}(x)\theta_\alpha + \left(b_i(x) + \frac{1}{2r^2}x_j\partial_j a_i(x)\right)\theta^2,$$

(3.21)

$$\varphi_\alpha(x,\theta) = \zeta_\alpha(x) + (\sigma_\mu)_{\beta\alpha}c_\mu(x)\theta_\beta$$

$$+ \left(\chi_\alpha(x) + \frac{1}{2r^2}x_j\partial_j\zeta_\alpha(x)\right)\theta^2,$$

(3.22)

where $r^2 = x_i x_i$, $\theta^2 = C_{\alpha\beta}\theta_\alpha\theta_\beta$, and $\mu = 0,1,2,3$. a_i, b_i, and c_μ are bosonic and $\xi_{i\alpha}, \zeta_\alpha$, and χ_α are fermionic fields on the supersphere. The $u(2L+1|2L)$ gauge parameter $H(l_i^{(L)}, v_\alpha^{(L)})$ becomes a superfield $H = h(x) + h_\alpha(x)\theta_\alpha + f(x)\theta^2$ where $h(x), f(x)$ are bosonic fields and $h_\alpha(x)$ are fermionic fields. We can fix the gauge degrees of freedom corresponding to $h_\alpha(x)$ and $f(x)$ by setting $C_{\alpha\beta}\theta_\alpha\varphi_\beta = 0$, which means $c_0 = \zeta_\alpha = 0$. In this gauge, we obtain the action in the commutative limit,

$$S = \left(-\frac{\rho}{2\pi}\right)\left(\frac{16}{9}\right)^2 \frac{\lambda^3}{g^2}\int d\Omega\left[-\frac{i}{3\rho}\epsilon_{ijk}a_i R_j a_k\right.$$

$$+ \frac{4i}{3}\rho\epsilon_{ijk}a_i R_j b_k + \frac{i}{3}\epsilon_{ijk}a_i R_j c_k + \frac{2}{3}\rho^2 b_i c_i$$

$$+ \frac{i}{3}\rho\epsilon_{ijk}c_i R_j c_k - \frac{1}{4\rho}a_i a_i + \rho a_i b_i + \frac{2}{3}\rho c_i c_i$$

$$- \frac{i}{3}\epsilon_{ijk}C_{\alpha\beta}\xi_{i\alpha}R_j\xi_{k\beta} + \frac{i}{6}\rho\epsilon_{ijk}(C\sigma_i)_{\alpha\beta}\xi_{j\alpha}\xi_{k\beta}$$

$$\left.- \frac{1}{4}C_{\alpha\beta}\xi_{i\alpha}\xi_{i\beta} + \frac{1}{3}\rho^2(\sigma_i)_{\alpha\beta}\xi_{i\beta}\chi_\alpha\right].$$

(3.23)

Here we have taken the $L \to \infty$ commutative limit and dropped terms like $[a_i, a_j]_*$. The auxiliary fields b_i and χ_α can be integrated out. This leads to the following constraints:

$$c_i = -\frac{3}{2\rho}a_i - \frac{2i}{\rho}\epsilon_{ijk}R_j a_k,$$

(3.24)

$$\xi_\alpha^{(1/2)} \equiv (\sigma_i)_{\alpha\beta}\xi_{i\beta} = 0.$$

(3.25)

Then the action can be simplified as

$$S = \left(-\frac{\rho}{2\pi}\right)\left(\frac{16}{9}\right)^2 \frac{\lambda^3}{g^2}\int d\Omega\left[-\frac{2}{3\rho}F_{ij}F_{ij} + \frac{2i}{3\rho}\epsilon_{ijk}(R_l F_{li})F_{jk}\right.$$

$$- \frac{i}{12\rho}(\epsilon_{ijk}a_i R_j a_k - ia_i a_i) - \frac{i}{3}\rho\epsilon_{ijk}\xi_{i\alpha}^{(3/2)}\left(C_{\alpha\beta}R_j\right.$$

$$\left.- \frac{1}{2}(C\sigma_j)_{\alpha\beta}\right)\xi_{k\beta}^{(3/2)} - \frac{1}{4}\rho C_{\alpha\beta}\xi_{i\alpha}^{(3/2)}\xi_{i\beta}^{(3/2)}\right],$$

(3.26)

where $\xi_{i\alpha}^{(3/2)} = \xi_{i\alpha} - \frac{1}{3}(\sigma_i)_{\alpha\beta}\xi_\beta^{(1/2)}$ and $F_{ij} = R_i a_j - R_j a_i - i\epsilon_{ijk}a_k$. This theory is invariant under the $U(1)$ gauge transformations

$$\delta a_i = R_i h(x), \quad \delta\psi_\alpha = 0,$$

(3.27)

where the gauge parameter $h(x)$ is a remnant of the $u(2L+1|2L)$ transformation. The supersymmetries, which are combinations of the $osp(1|2)$ and appropriate $u(2L+1|2L)$ transformations are not manifest, because we have fixed the gauge degrees of freedom corresponding to $u(2L+1|2L)$. The dynamical variables of the action are the gauge field $a_i(i=1,2,3)$ and the fermion $\xi_{i\alpha}$ with spin $\frac{3}{2}$ under $su(2)$. The normal component of a_i becomes a two-dimensional scalar on the sphere. Although this model has gauge symmetry and supersymmetry, it is different from the ordinary supersymmetric gauge theory in $D=2$ and its physical interpretation is not very clear.

Our construction of the supersymmetric gauge theory is similar to the covariant superspace approach for ordinary supersymmetric gauge theories [45]. In this approach, connections on the superspace which are described by superfields are introduced. Then the conventional constraints and the integrability conditions of the covariant derivatives are imposed in order to eliminate extra degrees of freedom. The connections on the superspace correspond to the supermatrices A_i and ψ_α in our model. However, there seems to be no appropriate condition that preserves the $osp(1|2)$ symmetry, to eliminate extra fields. Instead of these conditions the equations of motion of the auxiliary fields partially play a similar role in our case.

Although we here concentrated on the construction of the $U(1)$ gauge theory on the fuzzy supersphere, a generalization to $U(k)$ gauge theory can easily be realized by the following replacement:

$$A_i \to \sum_{a=1}^{k^2} A_i^a \otimes T^a, \quad \psi_\alpha \to \sum_{a=1}^{k^2} \psi_\alpha^a \otimes T^a,$$

(3.28)

where $T^a(a=1,2,\ldots,k^2)$ are the generators of $U(k)$.

IV. CONCLUSIONS AND DISCUSSION

In this paper, we constructed a supermatrix model which has a classical solution representing the fuzzy two-supersphere. We obtained a supersymmetric gauge theory on this noncommutative superspace by expanding supermatrices around this background. In this formulation, the supermatrices which are the fluctuations around the classical background correspond to the superfields on a fuzzy supersphere. This model has $osp(1|2)$ symmetry, which is the supersymmetry of the model, and $u(2L+1|2L)$ gauge symmetries. The classical backgrounds corresponding to the fuzzy two-supersphere violate the $osp(1|2)$ symmetry, but the action is still invariant under the $osp(1|2)$ transformations supplemented by an appropriate $u(2L+1|2L)$ transformation compensating the violation. Then we took the commutative limit, keeping the radius of the supersphere fixed. The supermatrices such as the gauge fields and the gauge parameters be-

come superfields on a commutative supersphere in this limit. After partially gauge fixing and integrating out some auxiliary fields in the superfields, we obtained a $U(1)$ gauge theory on the supersphere. In the derived action, the supersymmetry is not manifest due to our gauge fixing condition. It is easy to generalize our construction to $U(k)(k>1)$ gauge theory on the fuzzy supersphere.

The construction of a gauge theory on the fuzzy supersphere that we considered here has similarities to the covariant superspace approach in ordinary supersymmetric gauge theories. The supermatrices A_i and ψ_α in our model correspond to connection superfields on noncommutative superspace. The covariant superspace approach can be applied to supersymmetric gauge theories in higher dimensions, e.g., $D=4$, $\mathcal{N}=1$ super Yang-Mills theory. $\mathcal{N}=\frac{1}{2}$ super Yang-Mills theory [1,3] is derived by introducing noncommutativity only between chiral fermionic coordinates in the $\mathcal{N}=1$ superspace. Although this theory is not written completely by supermatrices because bosonic and half fermionic coordinates are still commutative, it can be described by an extension of the covariant superspace approach to the $\mathcal{N}=\frac{1}{2}$ noncommutative superspace. It is interesting to construct a supermatrix model whose classical solution is the four-dimensional noncommutative superspace, and quantum fluctuations around it describe the super Yang-Mills theory.

It would be interesting to study the graded unitary group symmetry $U(M|N)$ possessed by supermatrix models. In the type IIB matrix model [41], $U(N)$ gauge symmetry can be regarded as a matrix regularization of the area preserving diffeomorphism in the Schild type action of the type IIB Green-Schwarz string. There is the possibility of the graded unitary symmetry appearing as a matrix regularization of a world sheet symmetry of covariant formulations of superstring theories, e.g., superembeddings.

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APPENDIX: NOTATION AND DEFINITIONS

In this appendix we briefly explain definitions and notation related to the graded Lie algebra and supermatrix. More complete explanations can be seen, e.g., in [49,50]. We denote the space of Grassmann odd numbers as **B**, a graded algebra as $\mathcal{G}$, and its even (odd) part as $\mathcal{G}_0$ ($\mathcal{G}_1$).

1. Star and superstar for Grassmann number

$$\text{Star:} (c\,\theta_i)^* = \bar{c}\,\theta_i^*, \quad \theta_i^{**} = \theta_i, \quad (\theta_i\theta_j)^* = \theta_j^*\theta_i^*,$$

$$\text{superstar:} (c\,\theta_i)^\# = \bar{c}\,\theta_i^\#, \quad \theta_i^{\#\#} = -\theta_i, \quad (\theta_i\theta_j)^\# = \theta_i^\#\theta_j^\#;$$
(A1)

where $\theta_i \in \mathbf{B}$ and $c \in \mathbb{C}$.

2. Adjoint and superadjoint for graded Lie algebra

Adjoint:

$$\text{(i)} \quad X \in \mathcal{G}_i \to X^\dagger \in \mathcal{G}_i \quad \text{for } i=0,1$$

$$\text{(ii)} \quad (aX+bY)^\dagger = \bar{a}X^\dagger + \bar{b}Y^\dagger,$$
(A2)

$$\text{(iii)} \quad [X,Y\}^\dagger = [Y^\dagger, X^\dagger\},$$

$$\text{(iv)} \quad (X^\dagger)^\dagger = X;$$

superadjoint

$$\text{(i)} \quad X \in \mathcal{G}_i \to X^\ddagger \in \mathcal{G}_i \quad \text{for } i=0,1$$

$$\text{(ii)} \quad (aX+bY)^\ddagger = \bar{a}X^\ddagger + \bar{b}Y^\ddagger,$$
(A3)

$$\text{(iii)} \quad [X,Y\}^\ddagger = (-1)^{\deg X \cdot \deg Y}[Y^\ddagger, X^\ddagger\},$$

$$\text{(iv)} \quad (X^\ddagger)^\ddagger = (-1)^{\deg X}X,$$

where $X,Y \in \mathcal{G}$, $a,b \in \mathbb{C}$.

3. Supermatrix

The $(m+n)\times(m+n)$ supermatrix M has the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$
(A4)

where A, B, C, and D are, respectively, $m \times m, m \times n, n \times m$, and $n \times n$ matrices. An even supermatrix ($\deg M = 0$) has Grassmann even components in A and D and Grassmann odd components in B and C. An odd supermatrix ($\deg M = 1$) has Grassmann odd components in A and D and Grassmann even components in B and C.

4. Transpose and supertranspose for supermatrix

Transpose:

$$M^t = \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix},$$
(A5)

where A^t denotes the ordinary transpose of A, and $(MN)^t \neq N^t M^t$.

Supertranspose:

$$M^{st} = \begin{pmatrix} A^t & (-1)^{\deg M}C^t \\ -(-1)^{\deg M}B^t & D^t \end{pmatrix},$$

$$(M^{st})^{st} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix},$$
(A6)

$$(MN)^{st} = (-1)^{\deg M \deg N}N^{st}M^{st}.$$

5. Adjoint and superadjoint for supermatrix

Adjoint:

$$M^\dagger = (M^t)^*,$$

$$(MN)^\dagger = N^\dagger M^\dagger,$$
(A7)

$$(M^\dagger)^\dagger = M.$$

Superadjoint:

$$M^{\ddagger} = (M^{st})^{\#},$$

$$(MN)^{\ddagger} = (-1)^{\deg M \deg N} N^{\ddagger} M^{\ddagger}, \quad (A8)$$

$$(M^{\ddagger})^{\ddagger} = (-1)^{\deg M} M.$$

6. Supertrace

$$\text{Str} M = \text{tr} A - (-1)^{\deg M} \text{tr} D,$$

$$\text{Str}(M^{st}) = \text{Str} M, \quad (A9)$$

$$\text{Str}(MN) = (-1)^{\deg M \deg N} \text{Str}(NM),$$

where M has the form (A4).

7. Scalar multiplication of a supermatrix by a Grassmann number

$$bM = \begin{pmatrix} b\mathbf{1} & 0 \\ 0 & (-1)^{\deg b} b\mathbf{1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (A10)$$

$$Mb = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} b\mathbf{1} & 0 \\ 0 & (-1)^{\deg b} b\mathbf{1} \end{pmatrix}, \quad (A11)$$

where b is a Grassmann number and M is a supermatrix (A4).

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