# Inverse scattering method, Lie-Bäcklund transformations, and solitons for low-energy effective field equations of 5D string theory

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In the framework of the 5D low-energy effective field theory of the heterotic string with no vector fields excited, we combine two nonlinear methods in order to construct a solitonic field configuration. We first apply the inverse scattering method on a trivial vacuum solution and obtain a stationary axisymmetric two-soliton configuration consisting of a massless gravitational field coupled to a nontrivial chargeless dilaton and to an axion field endowed with charge. The implementation of this method was done following a scheme previously proposed by Yurova. We also show that within this scheme it is not possible to get massive gravitational solitons at all. We then apply a nonlinear Lie-Bäcklund matrix transformation of Ehlers type on this massless solution and get a massive rotating axisymmetric gravitational soliton coupled to axion and dilaton fields endowed with charges. We study as well some physical properties of the constructed massless and massive solitons and discuss the effect of the generalized solution generating technique on the seed solution and its further generalizations.

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## I. INTRODUCTION

The aim of this paper is to apply two nonlinear methods for constructing solitonic solutions in the framework of the 5D truncation of the low-energy effective field theory of the heterotic string with no vector fields excited. In a first stage we implement the inverse scattering method (ISM) on a trivial (flat space-time) seed solution in order to obtain a solitonic solution. At this step we make use of a chiral $SL(4,\mathbf{R})/SO(4)$ representation of the stationary axisymmetric theory and construct a rotating massless gravitational soliton following the scheme proposed by Yurova in [1] for chiral matrices of dimension greater than 2. Afterward we endow this object with gravitational mass and dilaton charge by means of a nonlinear transformation of Lie-Bäcklund type, the so-called normalized Ehlers transformation (NET).

In the framework of general relativity, Belinski and Zakharov [2] demonstrated that the vacuum stationary axially symmetric gravitational equations written in chiral form may be integrated with the aid of the ISM. This technique allows one to obtain the $N$-soliton configuration starting from flat space-time by making use of a symmetric chiral $2\times2$ matrix which must not satisfy any group condition. It was shown, in particular, that the Kerr-NUT (Newman-Unti-Tamburino) metric can be interpreted as a two-soliton solution of 4D Einstein theory in the presence of two commuting Killing vectors. As far as we know, the ISM has not been generalized for gravitational systems involving three space-time variables, i.e., with just one Killing vector imposed. Thus, in order to apply the ISM to gravitational models and their extensions we must consider configurations that depend at most on two space-time coordinates.

In the same way, in the framework of $D$-dimensional low-energy effective string theories toroidally reduced down to two space-time dimensions, i.e., in the presence of $D-2$ Killing vectors, we hope to obtain black hole (or black brane) solitonic solutions by applying the ISM. However, in this realm the problem becomes more complicated due to the fact that, in general, the chiral representations of the reduced low-energy effective field theories have dimensions greater than 2 and must satisfy, indeed, nontrivial group conditions. For instance, the chiral model which describes the low-energy effective field theory of the heterotic string when reduced to three space-time dimensions possesses the $SO(d+1,d+n+1)/[SO(d+1)\times SO(d+n+1)]$ symmetry group [3], where $d$ is the number of compactified dimensions and $n$ is the number of Abelian vector fields of the theory.[1] Thus, the chiral representation of this theory involves symmetric matrices of dimension $(2d+n+2)$ which must satisfy orthogonal group conditions. As a consequence, the original scheme described in [2] cannot be applied anymore and must be suitably modified.

In fact, it is not so easy to overcome this difficulty since both the dimensionality of the chiral matrices and their group symmetry condition strongly restrict the solitons we can obtain, leading sometimes to rotating massless gravitational configurations [1] and, even more, to trivial solutions (see below). For example, in the framework of the scheme fol-

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[1] In fact, the further reduction of the theory to two space-time dimensions by imposing one more Killing vector does not increase the dimensionality of the chiral representation.

 

lowed in this paper, this problem reduces to the choice of five relationships between eight constant parameters. However, not every set of such conditions leads to a soliton configuration depending on three parameters. Moreover, among them there are conditions that lead to trivial objects, i.e., to flat space-time solutions. Thus, this work brings some insight into the understanding of obtaining such solitonic objects, always with the hope of clarifying all the conditions under which we can apply the ISM and construct the general $N$-soliton solution for systems represented by chiral matrices of any dimension greater than 2 which, indeed, must satisfy nontrivial symmetry group conditions (see the related work [4], where the ISM has been implemented for special systems in the framework of string theory).

Another interesting issue concerns the physical interpretation of the constructed soliton since it describes a massless gravitational object[2] coupled to nontrivial dilaton and Kalb-Ramond fields. However, despite the massless character of the gravitational configuration, it possesses angular momentum, a strange feature that must be clarified. In this context, it is also interesting to see whether or not it is possible to endow our two-solitonic configuration with gravitational mass and dilaton charge since there is no way to get a massive solution when implementing the ISM on a trivial solution within the scheme proposed by Yurova (see Sec. III). In order to achieve this goal we apply a solution generating technique based on the use of a nonlinear transformation of Lie-Bäcklund type [5], namely, we perform the NET [6] on our massless two-soliton and get a rotating gravitational configuration with mass term coupled to dilaton and axion fields endowed with their respective charges.

It is with this motivation that we perform the present investigation. The paper is organized as follows. In Sec. II we present the 5D low-energy effective action of the theory under consideration as well as the matrix Ernst potential (MEP) formulation of the theory reduced down to three dimensions. Then we recall the NET of the stationary theory in the language of the MEP and write down an alternative $SL(4,\mathbf{R})/SO(4)$ representation of the stationary theory. In Sec. III we describe the ISM and the modifications that one must perform in order to apply it to our string system. Afterward we show that, within the scheme proposed by Yurova, one can construct just massless gravitational solitons and present an explicit exact solution. We also analyze some physical properties of this two-soliton object. In Sec. IV we perform a simplified NET on a seed solution which corresponds to the solitonic configuration constructed in Sec. III and get a gravitational soliton endowed with a mass term. We study as well some physical properties and limits of the solution obtained. Finally, in Sec. V we summarize our results and discuss on the physical peculiarities of the constructed solutions and also analyze the technical details of the implemented nonlinear methods that produce them. Here we give as well some suggestions concerning the further development and generalization of the techniques applied in this work.

[2]An object without mass term in the asymptotical expansion of the $g_{tt}$ component of the metric tensor.

## II. LOW-ENERGY EFFECTIVE ACTION

We shall study the 5D low-energy effective field theory of the heterotic string with no vector fields excited. This theory is described by the following action:

$$S^{(5)}=\int d^5x|G^{(5)}|^{1/2}e^{-\phi^{(5)}}\times\left(R^{(5)}+\phi^{(5)}_{;M}\phi^{(5);M}-\frac{1}{12}H^{(5)}_{MNP}H^{(5)MNP}\right), \quad (1)$$

where $H^{(5)}_{MNP}=\partial_M B^{(5)}_{NP}+$cyclic permutations of $M,N,P$; $G^{(5)}_{MN}$ is the metric, $B^{(5)}_{MN}$ is the anti-symmetric Kalb-Ramond field, $\phi^{(5)}$ is the dilaton, and $M,N,P=1,2,\ldots,5$. After the Kaluza-Klein compactification of this model on $T^2$ [3], the resulting stationary theory possesses the $SO(3,3)/[SO(3)\times SO(3)]$ symmetry group and describes the three-dimensional dilaton field $\phi$ and the scalar $2\times 2$ matrices $G\equiv G_{pq}$ and $B\equiv B_{pq}$,

$$\phi=\phi^{(5)}-\frac{1}{2}\ln|\det G|, \quad G_{pq}=G^{(5)}_{p+3,q+3},$$
$$B_{pq}=B^{(5)}_{p+3,q+3}, \quad (2)$$

where $p,q=1,2$ label the time and extra coordinates, respectively; the vector fields represented by the $2\times 3$ matrices $(A_1)^p_\mu$ and $(A_2)^{p+2}_\mu$,

$$(A_1)^p_\mu=\frac{1}{2}(G^{-1})_{pq}G^{(5)}_{q+3,\mu}, \quad (A_2)^{p+2}_\mu=\frac{1}{2}B^{(5)}_{p+3,\mu}-B_{pq}A^q_\mu, \quad (3)$$

where $\mu,\nu=1,2,3$ are the dynamical coordinates; and the antisymmetric tensor field $B_{\mu\nu}$ (which we set to zero from now on due to its nondynamical properties in three dimensions)

$$B_{\mu\nu}=B^{(5)}_{\mu\nu}-4B_{pq}A^p_\mu A^q_\nu-2(A^p_\mu A^{p+2}_\nu-A^p_\nu A^{p+2}_\mu); \quad (4)$$

all these fields are effectively coupled to three-dimensional gravity which is described by the metric tensor

$$g_{\mu\nu}=e^{-2\phi}[G^{(5)}_{\mu,\nu}-G^{(5)}_{p+3,\mu}G^{(5)}_{q+3,\nu}(G^{-1})_{pq}]. \quad (5)$$

It turns out that for stationary configurations, the vector fields can be dualized on shell through the pseudoscalar fields $u$ and $v$ as follows:

$$\nabla\times\vec{A}_1=\frac{1}{2}e^{2\phi}G^{-1}(\nabla u+B\nabla v),$$
$$\nabla\times\vec{A}_2=\frac{1}{2}e^{2\phi}G\nabla v-B\nabla\times\vec{A}_1, \quad (6)$$

where all vector and differential operations are performed with respect to the metric $g_{\mu\nu}$.

Thus, the effective stationary theory describes gravity $g_{\mu\nu}$ coupled to the scalars $G,B,\phi$ and the pseudoscalars $u,v$.

### A. Matrix Ernst potentials

All these matter fields can be arranged in the following matrix:

$$\mathcal{X}=\begin{pmatrix} -e^{-2\phi}+v^TXv & v^TX-u^T \\ Xv+u & X \end{pmatrix}, \tag{7}$$

where $X=G+B$. This is a $3\times3$ matrix which was called the matrix Ernst potential in [7] because of the close analogy existing between the representation of the low-energy effective field theory of the heterotic string and the formulation of the stationary Einstein-Maxwell (EM) theory in terms of the complex Ernst potentials [8]. Its components have the following physical meaning: the relevant information about the gravitational field is contained in the matrix potential $X$ through the matrix $G$, whereas its rotational character is encoded in the dualized variable $u$; $X$ also parametrizes the antisymmetric Kalb-Ramond tensor field $B$, whereas its multidimensional components are dualized through $v$; finally, the 3D dilaton is $\phi$. In terms of the MEP the effective stationary theory adopts the form

$$^3S=\int d^3x|g|^{1/2}\left\{-R+\mathrm{Tr}\left[\frac{1}{4}(\nabla\mathcal{X})\mathcal{G}^{-1}(\nabla\mathcal{X}^T)\mathcal{G}^{-1}\right]\right\}, \tag{8}$$

where $\mathcal{G}=\frac{1}{2}(\mathcal{X}+\mathcal{X}^T)$ is the symmetric part of the matrix potential $\mathcal{X}$, whereas the antisymmetric part reads $\mathcal{B}=\frac{1}{2}(\mathcal{X}-\mathcal{X}^T)$. The MEP $\mathcal{X}$ can be expressed as the sum of its symmetric and antisymmetric parts $\mathcal{X}=\mathcal{G}+\mathcal{B}$, where

$$\mathcal{G}=\begin{pmatrix} -e^{-2\phi}+v^TGv & v^TG \\ Gv & G \end{pmatrix}, \quad \mathcal{B}=\begin{pmatrix} 0 & v^TB-u^T \\ Bv+u & B \end{pmatrix}. \tag{9}$$

### B. The normalized Ehlers transformation

In the language of the MEP the stationary action (8) possesses a set of symmetries which have been classified according to their charging properties in [6]. Among them we find the so-called normalized Ehlers and Harrison transformations, NET and NHT, respectively, which act in a nontrivial way on the space-time when solution generating techniques are applied. For instance, in the framework of general relativity, the Ehlers transformation generates the NUT parameter when applied to both Schwarzschild and Kerr solutions, whereas the Harrison transformation endows these metrics with electromagnetic charges. We would like to mention that the NET constitutes a matrix generalization of the charging symmetry of Lie-Bäcklund type introduced by Ehlers in the framework of general relativity [9].

The matrix NET transformation reads [6]

$$\mathcal{X}\rightarrow(1+\Sigma\Lambda)(1+\mathcal{X}_0\Lambda)^{-1}\mathcal{X}_0(1-\Lambda\Sigma)+\Sigma\Lambda\Sigma, \tag{10}$$

where $\Sigma=\mathrm{diag}(-1,-1;1)$ and $\Lambda$ is an arbitrary antisymmetric constant $3\times3$ matrix parameter $\Lambda=-\Lambda^T$. In the framework of solution generating techniques, by applying the NET on a stationary seed solution we obtain a new stationary solution endowed with three more parameters introduced through the antisymmetric matrix $\Lambda$. We shall apply this technique in Sec. IV on a solitonic seed solution constructed by means of the ISM in order to further analyze the physical effect of the NET and the physical properties of the generated solution.

### C. $SL(4,\mathbf{R})/SO(4)$ chiral representation of the model

In [10] it was pointed out that apart from the $SO(3,3)/[SO(3)\times SO(3)]$ symmetry group formulation, the stationary system under consideration allows an alternative $SL(4,\mathbf{R})/SO(4)$ chiral parametrization. The latter formulation is more convenient since it is parametrized by $4\times4$ matrices instead of $6\times6$ matrices, and has to satisfy a trivial group condition. Thus, the chiral action reads

$$^3S=\int d^3x|g|^{1/2}\left[-R+\frac{1}{4}\mathrm{Tr}(J^{\mathcal{N}})^2\right], \tag{11}$$

where $J^{\mathcal{N}}=\nabla\mathcal{N}\mathcal{N}^{-1}$, the symmetric matrix $\mathcal{N}$ is

$$\mathcal{N}=(\det\mathcal{G})^{-1/2}\begin{pmatrix} \mathcal{G} & \mathcal{G}\mathcal{H} \\ \mathcal{H}^T\mathcal{G} & \det\mathcal{G}+\mathcal{H}^T\mathcal{G}\mathcal{H} \end{pmatrix}, \tag{12}$$

the $3\times1$ column $\mathcal{H}$ is determined by the relation $\mathcal{H}_k\epsilon_{mnk}=\mathcal{B}_{mn}\equiv\mathcal{B}$, the matrix potentials $\mathcal{G}$ and $\mathcal{B}$ are defined in Eq. (9), and $\epsilon_{mnk}$ is the antisymmetric tensor with $\epsilon_{123}=1$ and $m,n,k=1,2,3$. The matrix $\mathcal{N}$ is indeed unimodular and belongs to the $SL(4,\mathbf{R})/SO(4)$ group.

Let us consider an axially symmetric field configuration. In this case the spatial metric can be written in the Lewis-Papapetrou form

$$ds_3^2=g_{\mu\nu}dx^\mu dx^\nu=e^{2\gamma}(d\rho^2+dz^2)+\rho^2d\varphi^2, \tag{13}$$

the effective action of the system can be expressed as follows:

$$^2S=\frac{1}{4}\int d\rho dz\rho\,\mathrm{Tr}(J^{\mathcal{N}})^2, \tag{14}$$

and the matrix field equation reads

$$\nabla(\rho J^{\mathcal{N}})=0. \tag{15}$$

The corresponding Einstein equations determine the function $\gamma$ through the relations

$$\gamma_{,z}=\frac{\rho}{4}\mathrm{Tr}(J_\rho^{\mathcal{N}}J_z^{\mathcal{N}}), \quad \gamma_{,\rho}=\frac{\rho}{8}\mathrm{Tr}[(J_\rho^{\mathcal{N}})^2-(J_z^{\mathcal{N}})^2], \tag{16}$$

where the operator $\nabla$ is related to the flat two-metric $\delta_{ab}$ and all dynamical variables depend on $\rho$ and $z$ only. These equations are automatically satisfied once a solution for Eq. (12), or equivalently Eq. (9), is found. In the next section we shall construct a solitonic solution to this matrix equation by means of the ISM.

### III. SOLITONS VIA ISM

In this section we shall continue to apply the inverse scattering technique in order to construct soliton solutions for the 5D low-energy effective field theory under consideration. However, because of the symmetry condition that the real chiral matrix $\mathcal{N}$ must satisfy, the original scheme of Belinski and Zakharov is not applicable anymore and the implementation of such a powerful method turns out to be not so simple. Another problem is related to the dimension of the chiral matrix for which solitons are constructed. The point here is that the matrix $\mathcal{N}$ must possess certain asymptotic properties which correspond to concrete physical values that the fields must adopt at spatial infinity. Thus, the bigger is the dimension of the matrix $\mathcal{N}$, the more boundary conditions we must impose on it. In this context we shall avoid these difficulties following the modification of the ISM proposed by Yurova. Thus, the integration of the field equations (15) is associated with the LA pair

$$D_1\psi=\frac{\rho J_z^N-\lambda J_\rho^N}{\lambda^2+\rho^2}\psi,\quad D_2\psi=\frac{\rho J_\rho^N+\lambda J_z^N}{\lambda^2+\rho^2}\psi, \tag{17}$$

where $J^N=\rho J^{\mathcal{N}}$ and the differential operators are

$$D_1=\partial_z-\frac{2\lambda^2}{\lambda^2+\rho^2}\partial_\lambda,\quad D_2=\partial_\rho+\frac{2\lambda\rho}{\lambda^2+\rho^2}\partial_\lambda, \tag{18}$$

$\lambda$ is a spectral complex parameter, and $\psi=\psi(\lambda,\rho,z)$. The solution of Eq. (15) for the symmetric matrix $\mathcal{N}$ constitutes the function $\psi$ with vanishing value of the spectral parameter, i.e.,

$$\mathcal{N}(\rho,z)=\psi(0,\rho,z). \tag{19}$$

Thus, for any known solution $\psi_0$ of the system (17),(18), the function $\psi$ can be obtained in the form

$$\psi=\chi\psi_0, \tag{20}$$

where the equations for $\chi$ are

$$D_1\chi=\frac{\rho J_z^N-\lambda J_\rho^N}{\lambda^2+\rho^2}\chi-\chi\frac{\rho(J_z^N)_0-\lambda(J_\rho^N)_0}{\lambda^2+\rho^2},$$

$$D_2\chi=\frac{\rho J_\rho^N+\lambda J_z^N}{\lambda^2+\rho^2}\chi-\chi\frac{\rho(J_\rho^N)_0+\lambda(J_z^N)_0}{\lambda^2+\rho^2}.$$

The matrix $\mathcal{N}$ must be real and symmetric; in order to ensure its real character, we shall impose the condition

$$\chi(\lambda)=\overline{\chi(\bar{\lambda})} \tag{21}$$

(see [2] for details). However, in order to apply the ISM to chiral matrices with dimension greater than 2, the symmetry requirement must be imposed after the construction of the solitonic solution. Thus, after the implementation of the ISM, the constructed matrix $\mathcal{N}$ will not be symmetric and we must impose this condition afterward.

The soliton solutions for the matrix $\mathcal{N}$ correspond to pole divergences in the spectral parameter complex plane for the matrices $\chi$ and $\chi^{-1}$. When the poles are simple, these matrices can be represented as follows:

$$\chi=I+\sum_{k=1}^{N}\frac{R_k}{\lambda-\mu_k},\quad \chi^{-1}=I+\sum_{k=1}^{N}\frac{S_k}{\lambda-\nu_k}, \tag{22}$$

where the pole trajectories for each pole $k$ are determined by

$$\mu_k(\rho,z)=w_{(\mu)}-z\pm[(w_{(\mu)}-z)^2+\rho^2]^{1/2},\quad w_{(\mu)}=\text{const} \tag{23}$$

for $\mu_k(\rho,z)$ and the same equation for $\nu_k(\rho,z)$ with constants $w_{(\nu)}$. From $\chi\chi^{-1}=I$ in the poles $\mu_k$ and $\nu_k$ it follows that

$$R_k\chi^{-1}(\mu_k)=S_k\chi(\nu_k)=0. \tag{24}$$

Hence the matrices $R_k$ and $S_k$ are degenerate and can be represented as follows:

$$(R_k)_{ab}=n_a^k m_b^k,\quad (S_k)_{ab}=p_a^k q_b^k. \tag{25}$$

By substituting Eqs. (22) and (25) into Eq. (24) we obtain

$$n_a^k=\sum_{l=1}^{N}p_a^l\Gamma_{kl}^{-1},\quad q_a^k=-\sum_{l=1}^{N}m_a^l\Gamma_{kl}^{-1}$$

where

$$\Gamma_{kl}=\frac{\sum_c p_c^k m_c^l}{\mu_l-\nu_k}, \tag{26}$$

$$m_a^k=[\psi_0^{-1}(\mu_k,\rho,z)]_{ca}m_{c0}^k,$$

and

$$p_a^k=[\psi_0(\nu_k,\rho,z)]_{ac}p_{c0}^k, \tag{27}$$

where $m_{c0}^k$ and $p_{c0}^k$ are arbitrary constants and $a,b,c=1,2,3,4$.

When considering the two-soliton case, in [1] it was shown that in order to have a unimodular matrix $\mathcal{N}$, the relation

$$\mu_1\mu_2=\nu_1\nu_2 \tag{28}$$

is really important and it constitutes the main difference with respect to the scheme proposed by Belinski and Zakharov since it is not compatible with the symmetry requirement $\mathcal{N}=\chi(-\rho^2/\lambda)\mathcal{N}_0\chi^T(\lambda)$ of [2]. As a consequence, the resulting unimodular matrix $\mathcal{N}$ will be nonsymmetric. However, the symmetry conditions may be attained by a suitable choice of the arbitrary parameters of Eq. (27).

Let us apply the modified ISM in order to construct a stationary axially symmetric two-soliton solution for the 5D string model which is described by the effective action (14) when reduced to two dimensions.

In the simplest case, the initial values of the metric and field variables correspond to flat space-time. Thus, the seed chiral matrix adopts the form

$$\mathcal{N}_0=\mathrm{diag}(-1,-1;1,1). \tag{29}$$

We shall construct the solitonic solution using the set of coordinates of Boyer and Lindquist without mass. This is because the resulting two-soliton gravitational solution corresponds to a massless source (see below for details). Thus, we have

$$\rho=(r^2-\sigma^2)^{1/2}\sin\theta, \quad z-z_1=r\cos\theta, \tag{30}$$

where the new constants $\sigma=\frac{1}{2}(w_{(\mu)}-w_{(\nu)})$ and $z_1=\frac{1}{2}(w_{(\mu)}+w_{(\nu)})$. Consequently, the pole trajectories read

$$\mu_1=2\sin^2\frac{\theta}{2}(r+\sigma), \quad \mu_2=-2\cos^2\frac{\theta}{2}(r-\sigma),$$

$$\nu_1=-2\cos^2\frac{\theta}{2}(r+\sigma), \quad \nu_2=2\sin^2\frac{\theta}{2}(r-\sigma), \tag{31}$$

and obviously satisfy the condition (28). Since $\mathcal{N}_0=\psi_0^{-1}(\mu_k,\rho,z)=\psi_0(\nu_k,\rho,z)$, then the vectors $p_a^k$ and $m_a^k$ constitute arbitrary constants [see Eq. (27)]. Thus, by applying the scheme described above, one can construct the matrix two-soliton solution $\mathcal{N}$ for Eq. (15) depending on these vectors. Such a matrix will be unimodular, but not symmetric. The following conditions provide the symmetric character of the matrix $\mathcal{N}$:

$$m_a^k=p_a^k, \tag{32}$$

for all $a,k$; and

$$p_3^2=p_3^1, \quad p_2^2=(p_3^1)^2/p_2^1, \quad p_4^2=p_4^1=-p_1^1p_3^1/p_2^1,$$

$$p_1^2=(p_3^1/p_2^1)^2p_1^1. \tag{33}$$

Thus, only three constants survive the symmetrization of the matrix $\mathcal{N}$. However, we would like to point out that just the condition (32) is not enough to ensure the symmetry character of $\mathcal{N}$; it leaves eight arbitrary constants, but it does not lead to a symmetric matrix $\mathcal{N}$. Moreover, we must impose five more restrictions on these eight constants in order to obtain the desired symmetric matrix $\mathcal{N}$. The choice of these restrictions is not unique and the constants are not independent of each other. Moreover, some choices lead to a trivial soliton solution or to a solution that depends on just two arbitrary constants instead of three. In this respect, we observe some differences between our choice of constants and the one performed by Yurova. For example, by setting our $p_1^1$ to zero we just set to zero one parameter of the solution, whereas within the choice made in [1], the author claims that if any of the constants $p_a^k$ is set to zero, all other constants vanish as well and the constructed solution turns out to be a trivial one. This fact, in turn, leads to slightly different solitonic solutions from the physical point of view (see below).

### A. Massless character of the gravitational soliton

Now we shall show that when constructing the two-soliton solution and imposing the condition (32) toward the symmetrization of the matrix $\mathcal{N}$, we necessarily obtain a massless gravitational field configuration. Thus, after taking into account the restrictions (32), we obtain a chiral matrix $\mathcal{N}$ which has the following block structure and asymptotic behavior:

$$\mathcal{N}_{as}\equiv\begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}=\begin{pmatrix} -1 & n_{12}/r & n_{13}/r & n_{14}/r \\ n_{21}/r & -1 & n_{23}/r & n_{24}/r \\ n_{31}/r & n_{32}/r & 1 & n_{34}/r \\ n_{41}/r & n_{42}/r & n_{43}/r & 1 \end{pmatrix}, \tag{34}$$

where $N_1$ is a $3\times3$ matrix, $N_2$ is a $3\times1$ column, $N_3$ is a $1\times3$ row, and $N_4=1$,

$$n_{ac}=2\kappa\sigma\frac{p_a^1p_c^2-p_c^1p_a^2}{p_b^1p_b^2r},$$

$$a\neq c, \quad \text{and} \quad \kappa=\begin{cases} -1, & a=1,2, \\ 1, & a=3,4. \end{cases} \tag{35}$$

On the other side, the matrix $\mathcal{N}$ is defined through $\mathcal{G}$ and $\mathcal{B}$ through the relation (12). By equating both representations we can, in principle, compute the *nonsymmetric* matrix $\mathcal{G}_{as}=(N_4-N_3N_1^{-1}N_2)N_1$ which asymptotically behaves as follows:

$$\mathcal{G}_{as}=\begin{pmatrix} -1 & n_{12}/r & n_{13}/r \\ n_{21}/r & -1 & n_{23}/r \\ n_{31}/r & n_{32}/r & 1 \end{pmatrix}, \tag{36}$$

and obviously does not possess Coulomb terms in the diagonal components. Thus, when looking for constructing a massive gravitational solitonic configuration within this concrete modified version of the ISM, it does not matter what kind of relationships we choose between the remaining eight constants $p_a^k$, since the conditions (32) are sufficient to restrict us to obtaining massless solitons.

Thus, by imposing the conditions (32), it is not possible at all to obtain a massive gravitational two-soliton solution. This fact also shows how strong, from the physical point of view, are the restrictions we must impose on our matrix $\mathcal{N}$ in order to make possible the application of the ISM to chiral matrices of dimension greater than 2. It remains an open question whether a different scheme of symmetrization of the matrix $\mathcal{N}$ can lead to massive gravitational solitons in the framework of the model considered. Current research in this direction is being performed by the authors.

### B. Explicit two-soliton solution

Once the matrix $\mathcal{N}$ is constructed taking into account the conditions (32) and (33), we can come back to the original variables of the theory with the aid of Eqs. (12), (9), and (2)–(6). At this stage it is convenient to introduce the following notation:

$$a=\frac{\sigma[(p_3^1)^2+(p_2^1)^2]}{2p_2^1p_3^1},\quad b=\frac{\sigma(p_1^1)^2[(p_3^1)^2-(p_2^1)^2]}{2p_2^1p_3^1[(p_1^1)^2+(p_2^1)^2]},$$

$$c=\frac{\sigma p_2^1[(p_3^1)^2-(p_2^1)^2]}{2p_3^1[(p_1^1)^2+(p_2^1)^2]},\tag{37}$$

which establishes the relation $\sigma^2=a^2-(b+c)^2$; let us define as well

$$\Delta=r^2-\sigma^2,\quad \delta^2=r^2+c^2-(b-a\cos\theta)^2.\tag{38}$$

Now we shall write down the final expression of the constructed two-soliton solution by implementation of the ISM. The 5D line element reads

$$ds_5^2=G_{pq}(dx^p-\omega_\varphi^p d\varphi)(dx^q-\omega_\varphi^q d\varphi)+e^{2\phi}ds_3^2,\tag{39}$$

where the components of the matrix $G_{pq}$ have the form

$$G_{11}=-\frac{r^2+b^2-(c-a\cos\theta)^2}{\delta^2},\quad G_{12}=\frac{2cr}{\delta^2},$$

$$G_{22}=\frac{r^2+b^2-(c+a\cos\theta)^2}{\delta^2},\tag{40}$$

the metric functions $\omega_\varphi^q$ are

$$\omega_\varphi^1=\frac{-2a\sqrt{bc}r\sin^2\theta}{\Delta+a^2\sin^2\theta},$$

$$\omega_\varphi^2=2\sqrt{bc}\left[\cos\theta+\frac{a(b+c-a\cos\theta)\sin^2\theta}{\Delta+a^2\sin^2\theta}\right],\tag{41}$$

the three-dimensional dilaton field is

$$e^{2\phi}=1-\frac{4bc}{\Delta+a^2\sin^2\theta},\tag{42}$$

and the expression for the spatial line element is the following:

$$ds_3^2=(\Delta+a^2\sin^2\theta)\left[\frac{dr^2}{\Delta}+d\theta^2\right]+\Delta\sin^2\theta d\varphi^2.\tag{43}$$

The components of the antisymmetric Kalb-Ramond tensor field are defined by the relations

$$B_{12}=\frac{2br}{\delta^2},\quad B_{4,\varphi}^{(5)}=\frac{2\sqrt{bc}r(2b\cos\theta+a\sin^2\theta)}{\delta^2},$$

$$B_{5,\varphi}^{(5)}=2\sqrt{bc}\left[\cos\theta-\frac{a(c-b+a\cos\theta)\sin^2\theta}{\delta^2}\right].\tag{44}$$

Finally, the 5D dilaton field reads

$$e^{\phi^{(5)}}=\frac{r^2+(b-c)^2-a^2\cos^2\theta}{\delta^2}.\tag{45}$$

These expressions describe a stationary axially symmetric massless gravitational field configuration coupled to a nontrivial dilaton field without charge and to an axion field endowed with charge. A novel feature of this configuration is that it possesses angular momentum. By analyzing the asymptotical behavior of the functions $\omega_\varphi^q$ we observe that $\omega_\varphi^1$ defines the angular momentum according to the following relation:

$$\omega_\varphi^1|_{r\to\infty}\sim\frac{-2a\sqrt{bc}\sin^2\theta}{r},\tag{46}$$

whereas $\omega_\varphi^2$ is not an asymptotically flat quantity since at spatial infinity it behaves as

$$\omega_\varphi^2|_{r\to\infty}\sim 2\sqrt{bc}\cos\theta\tag{47}$$

and defines a NUT-like parameter. However, it is worth noticing that in order to obtain an asymptotically flat field configuration we can set to zero either the parameter $b$ or $c$. In the first case the remaining configuration constitutes a massless static inhomogeneous gravitational field with nontrivial components of the matrix $G_{pq}$ with no dilaton and Kalb-Ramond fields excited. In the second case the truncated configuration is different; it represents a massless static inhomogeneous gravitational field with nonzero components $G_{11}$ and $G_{22}$ coupled to a nontrivial massless 5D dilaton field and endowed with an axion field which possesses a charge term. One can see another quite strange property of our soliton: as soon as we require asymptotic flatness, the solution becomes static [see Eqs. (46) and (47)]. In this respect, the soliton constructed in [1] has quite different properties since in order to get an asymptotically flat field configuration we must set to zero both parameters $b$ and $c$, obtaining a trivial solution in this way. Thus, that solitonic solution does not contain asymptotically flat field configurations.

In both of the limits considered ($b=0$ and $c=0$) we obtained static inhomogeneous field configurations even when the parameter $a$, which usually is responsible for the rotation of the gravitational field, is not vanishing. Thus, our solitonic configuration does not contain a spherically symmetric subclass of solutions (in accordance with the soliton constructed in [1]). Comparison to other results in the literature shows [11] that these solutions are not obtained by setting to zero the mass or other parameters. Of course, it is interesting to study other physical properties of these massless solitonic configurations. At first glance it seems that these solutions correspond to both rotating and static inhomogeneous black strings since we have the presence of horizons; however, this topic deserves further investigation.

## IV. MASSIVE GRAVITATIONAL SOLITONS VIA NET

Since the implemented version of the ISM cannot provide gravitational solitons with a mass term, it is an open question whether nonlinear methods can provide such objects in 5D low-energy effective string theories. In this section we shall focus on this issue. One way to approach this topic is to look for a way of endowing the gravitational and dilaton fields with mass and charge terms, respectively, and then, to see, for instance, if the obtained solution corresponds to a known class. In this way it is also possible to get new massive solutions. In order to introduce more parameters in the constructed solitonic configuration one can make use of a solution generating technique. Thus, we will apply the Lie-Bäcklund transformation NET (10) on a seed solution which corresponds to the massless gravitational soliton constructed in the previous section.

The corresponding seed MEP $\mathcal{X}_0 = \mathcal{G}_0 + \mathcal{B}_0$ reads

$$\mathcal{X}_0 = \delta^{-2} \begin{pmatrix} -\delta^2 & 0 & 0 \\ 4\sqrt{bc}(b+c-a\cos\theta) & -[r^2+b^2-(c-a\cos\theta)^2] & 2(b+c)r \\ 4\sqrt{bc}r & 2(c-b)r & r^2+b^2-(c+a\cos\theta)^2 \end{pmatrix}, \tag{48}$$

where $\mathcal{G}_0$ and $\mathcal{B}_0$ are the matrix potentials (9) that parametrize the constructed solitonic chiral matrix (12). Here we shall use a quite simple antisymmetric matrix $\Lambda$, namely, a matrix with just one nontrivial constant parameter

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & M \\ 0 & -M & 0 \end{pmatrix}. \tag{49}$$

Thus, after performing the nonlinear transformation (10) on $\mathcal{X}_0$ we obtain a new solution of the theory under consideration. We shall present the complete solution, omitting all lengthy intermediate calculations. Thus, the components of the transformed matrix $G_{pq}$ are

$$G_{11} = -\frac{(M^2-1)[r^2+b^2-(c-a\cos\theta)^2]-4cM(r+Ma\cos\theta)}{(M^2-1)\delta^2+4bM(r-Ma\cos\theta)},$$

$$G_{12} = \frac{2c[(M^2+1)r+2Ma\cos\theta]}{(M^2-1)\delta^2+4bM(r-Ma\cos\theta)}, \tag{50}$$

$$G_{22} = \frac{(M^2-1)[r^2+b^2-(c+a\cos\theta)^2]+4cM(r+Ma\cos\theta)}{(M^2-1)\delta^2+4bM(r-Ma\cos\theta)},$$

the transformed metric functions $\omega^q_\varphi$ are

$$\omega^1_\varphi = \frac{-2a\sqrt{bc}[(M^2-1)r+2M(b+c)]\sin^2\theta}{(M^2-1)(\Delta+a^2\sin^2\theta)},$$

$$\omega^2_\varphi = 2\sqrt{bc}\left[\cos\theta - \frac{a[(M^2+1)(b+c)+(M^2-1)a\cos\theta]\sin^2\theta}{(M^2-1)(\Delta+a^2\sin^2\theta)}\right]; \tag{51}$$

the three-dimensional dilaton field remains the same under the NET, thus

$$e^{2\phi} = 1 - \frac{4bc}{\Delta+a^2\sin^2\theta}. \tag{52}$$

The transformed components of the antisymmetric Kalb-Ramond field are the following:

$$B_{12}=\frac{-2b[(M^2+1)r-2Ma\cos\theta]}{(M^2-1)\delta^2+4bM(r-Ma\cos\theta)},$$

$$B^{(5)}_{4,\varphi}=-2\sqrt{bc}\Bigg\{\frac{[2(b+c)M-(M^2-1)r]a\sin^2\theta}{(M^2-1)(\Delta^2+a^2\sin^2\theta)}+\frac{2b[(M^2+1)r-2Ma\cos\theta][(M^2-1)\Delta-(M^2+1)(b+c)a\sin^2\theta]}{(M^2-1)(\Delta^2+a^2\sin^2\theta)[(M^2-1)\delta^2+4bM(r-Ma\cos\theta)]}\Bigg\},$$

$$B^{(5)}_{5,\varphi}=2\sqrt{bc}\Bigg\{\frac{[(M^2-1)\Delta\cos\theta+(M^2+1)(b+c)a\sin^2\theta]}{(M^2-1)(\Delta^2+a^2\sin^2\theta)}+\frac{2ab[(M^2+1)r-2Ma\cos\theta][2(b+c)M+(M^2-1)r]\sin^2\theta}{(M^2-1)(\Delta^2+a^2\sin^2\theta)[(M^2-1)\delta^2+4bM(r-Ma\cos\theta)]}\Bigg\} \tag{53}$$

and the 5D dilaton field reads

$$e^{\phi^{(5)}}=\frac{(M^2-1)[r^2+(b-c)^2-a^2\cos^2\theta]}{(M^2-1)\delta^2+4bM(r-Ma\cos\theta)}. \tag{54}$$

It is a straightforward exercise to check that, when $M$ vanishes, we recover the seed solitonic solution. By studying the asymptotic behavior of the field configuration we can obtain information about the existence of its masses and charges. Thus, for the components $G_{pq}$ we observe that there exist mass terms

$$G_{11}|_{r\to\infty}\sim-1+\frac{2m_{11}}{r},\quad G_{12}|_{r\to\infty}\sim-\frac{m_{12}}{r},$$

$$G_{22}|_{r\to\infty}\sim1-\frac{2m_{22}}{r}, \tag{55}$$

where the masses $m_{pq}$ are defined as follows:

$$m_{11}=\frac{2(b+c)M}{(M^2-1)},\quad m_{12}=\frac{2c(M^2+1)}{(M^2-1)},\quad m_{22}=\frac{2(b-c)M}{(M^2-1)}. \tag{56}$$

Analogously, we see that the transformed rotation functions $\omega^q_\varphi$ and the three-dimensional dilaton maintain the same asymptotic behavior, i.e., they do not change their behavior at spatial infinity under the NET. The transformed component $B_{12}$ of the antisymmetric tensor field as well as the 5D dilaton possess Coulomb terms,

$$B_{12}|_{r\to\infty}\sim\frac{b_{12}}{r},\quad e^{\phi^{(5)}}|_{r\to\infty}\sim1+\frac{D}{r}, \tag{57}$$

where the new charges have been introduced

$$b_{12}=\frac{2b(M^2+1)}{(1-M^2)},\quad D=\frac{4bM}{(1-M^2)}. \tag{58}$$

Finally, the asymptotic behavior of the $B^{(5)}_{p+3,\varphi}$ components of the Kalb-Ramond field reads

$$B^{(5)}_{4,\varphi}|_{r\to\infty}\sim\frac{2\sqrt{bc}[-2b(M^2+1)\cos\theta+a(M^2-1)]\sin^2\theta}{(M^2-1)r},$$

$$B^{(5)}_{5,\varphi}|_{r\to\infty}\sim2\sqrt{bc}\cos\theta. \tag{59}$$

Thus, we have obtained a stationary axially symmetric massive gravitational field configuration coupled to nontrivial dilaton and axion fields endowed with their corresponding charges. Again, in order to obtain an asymptotically flat field configuration we can set to zero either $b$ or $c$. If $c$ vanishes we get a static inhomogeneous gravitational field with massive components $G_{11}$ and $G_{22}$ coupled to nontrivial Kalb-Ramond and dilaton fields endowed with their corresponding charges. In the case when $b$ is set to zero, we recover a static inhomogeneous gravitational field with massive components $G_{pq}$ and vanishing dilaton and antisymmetric fields.

Once again we observe that, when we impose the asymptotic flatness condition, we automatically get a static field configuration since by setting to zero the NUT-like parameter implies the vanishing of the whole metric function (51). Thus, if our soliton represents a rotating field configuration, it necessarily possesses the NUT-like parameter and if we search for an asymptotically flat solution, it necessarily becomes static. This feature is not shared by rotating configurations in general relativity where, for instance, one obtains the Kerr metric from the Kerr-NUT one when the NUT charge vanishes. This fact is a consequence of the relationships that take place between the constants $a$, $b$, and $c$, which are arbitrary, but not independent of each other [for instance, from Eq. (37) it can be seen that $b\sim c$]. It seems that this is, in turn, a consequence of the restrictions (32) and (33) which we imposed on the constants $p^k_a$ in order to get a symmetric chiral matrix $\mathcal{N}$. It is interesting to propose another scheme for symmetrizing the matrix $\mathcal{N}$ which would avoid this strange physical behavior of the constructed solitonic solutions and could, in principle, provide the presence of mass and charge terms for the fields of our configuration.

## V. CONCLUSIONS AND DISCUSSION

In this paper we have combined two nonlinear methods in order to construct a solitonic gravitational field configuration to the 5D low-energy bosonic sector of string theory. By following the modified version of the ISM proposed by Yurova we clarified the unavoidable massless character of the gravitational solitonic solutions obtained. Therefore, by imposing suitable conditions on the constants that parametrize the chiral matrix $\mathcal{N}$, we construct a soliton consisting of a rotating massless gravitational field configuration coupled to a chargeless dilaton and to an axion field endowed with charge. This solution has similar but different physical properties and limits when compared to the solution constructed in [1]. Afterward, we provide this field configuration with mass and charge terms by performing a simplified nonlinear NET on it. Here we would like to point out that the nontrivial physical effect of the NET is quite different in general relativity and string theory. It is well known that in the framework of general relativity the Ehlers transformation provides the presence of the NUT-like charge when applied on vacuum seed solutions [9]. However, within the framework of string theory we observe that a simplified version (with just one parameter) of this transformation provides the mass and dilaton charge when applied on a massless seed gravitational solution and does not affect the NUT parameter at all.

Let us say a few words about the physical properties of our massless and massive gravitational field configurations. As mentioned above, when looking toward a rotating black string interpretation of them, we impose the asymptotic flatness condition and set to zero the NUT parameter. However, when we drop the NUT-like charge, the remaining configuration becomes static. Thus, from one side, if our solitons are restricted to be rotating, they necessarily possess a NUT parameter, and, from the other side, if they are conditioned to be asymptotically flat, they are necessarily static. This fact is due to the overall constant factor $\sqrt{bc}$ of the functions $\omega_\varphi^p$ for both massless and massive solutions. Such functions define both the angular momentum of the configurations and their NUT-like charges. It seems that this is, in turn, a consequence of the conditions (32) and (33) we have imposed toward the symmetrization of the matrix $\mathcal{N}$. It is of interest to propose another scheme for symmetrizing the matrix $\mathcal{N}$, which would avoid this strange physical behavior of the constructed solitonic solutions. This peculiar physical property of our solutions is quite strange, but on the other side, it is quite interesting and deserves more investigation. The computation of the scalar curvature and other invariants will help in clarifying this point. A current investigation in this direction is also in progress.

Within the framework of the solution generating technique using nonlinear transformations of Lie-Bäcklund type, it is interesting to see whether the use of a full parameterized constant matrix $\Lambda$ will affect the asymptotic behavior of the transformed solution and provide more independent charges. In this way we could obtain a massive rotating solitonic object endowed with NUT-like charge that remains spinning after the vanishing of the NUT parameter. However, this transformation involves really lengthy algebraic calculations which we hope to perform in the near future. Another way of generalizing the present results is by applying this kind of nonlinear Lie-Bäcklund transformation to string systems that include vector fields. In this context it is the NHT which must be performed on our massless seed solution. Thus, within the low-energy string theory realm, these methods could, in principle, lead to the construction of new charged black hole (black brane) solutions in $D>4$ dimensions, where it is known that such objects do exist [11]. Moreover, it is interesting to apply the ISM for the whole spectrum of the low-energy effective field theory of the heterotic string reduced to two dimensions (taking into account the Abelian vector fields). This could be possible because of the abovementioned relationship between this theory and the EM theory. For a review of the ISM applied to the stationary axisymmetric EM system, see [12].

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