Algorithmic construction of static perfect fluid spheres

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Perfect fluid spheres, either Newtonian or relativistic, are the first step in developing realistic stellar models (or models for fluid planets). Despite the importance of these models, explicit and fully general solutions of the perfect fluid constraint in general relativity have only very recently been developed. In this paper we present a variant of Lake's algorithm wherein (1) we recast the algorithm in terms of variables with a clear physical meaning—the average density and the locally measured acceleration due to gravity, (2) we present explicit and fully general formulas for the mass profile and pressure profile, and (3) we present an explicit closed-form expression for the central pressure. Furthermore we can then use the formalism to easily understand the pattern of interrelationships among many of the previously known exact solutions, and generate several new exact solutions.

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I. INTRODUCTION

Perfect fluid spheres, both Newtonian [1] and relativistic [2,3], have attracted and continue to attract considerable attention as the first step in developing realistic stellar models (or models for fluid planets). Whereas some steps toward finding all possible solutions to the perfect fluid constraint in the absence of a specific equation of state were presented in early work of Wyman and Hojman *et al.* [4,5], *explicit* and fully general solutions of the perfect fluid constraint have only very recently been developed [6,7]. In this paper we present a variant of Lake's algorithm [7] using curvature coordinates. (1) We recast the algorithm in terms of variables with a clear physical meaning—the average density and the "gravity profile," a quantity closely related to both the gravitational redshift and the locally measured acceleration due to gravity. (2) We minimize the number of differentiations and integrations by several judicious applications of integration by parts. (3) We present explicit, compact, and fully general formulas for the mass profile and pressure profile of an *arbitrary* fluid sphere. (4) We present an explicit, compact, and general formula for the central pressure of an *arbitrary* fluid sphere. (5) We compare and contrast the relativistic formulas we obtain with the much simpler Newtonian situation.

We emphasize that one of the virtues of this type of approach is that one is not fixed *a priori* in dealing with a prespecified equation of state [6]—in many interesting physical situations the equation of state is either uncertain or, because the fluid in question might be inhomogeneous, it may not even make sense to assign a single equation of state to the entire fluid sphere.

To further illustrate the formalism we show how it may be used as the basis for a partial classification scheme—there is a free parameter in the algorithm that can take simple solutions into more complicated ones. Once this is appreciated it becomes easy to see (simply by parameter counting) that certain simple solutions *must* have one-parameter generalizations. Conversely, this observation explains why so many of the earliest discovered exact solutions have one-parameter extensions.

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II. FRAMEWORK

To set the stage, consider a static spherically symmetric geometry. It is a standard result that without loss of generality we can choose coordinates to write the metric in the form

$$ds^2 = -\exp\left[-2\int_r^\infty g(\tilde{r})\,d\tilde{r}\right]dt^2 + \frac{dr^2}{1-2m(r)/r} + r^2[d\theta^2+\sin^2\theta\, d\phi^2]. \tag{1}$$

Here $g(r)$ is the "gravity profile." It is related to the gravitational redshift by

$$1+z = \exp\left[\int_r^\infty g(\tilde{r})\,d\tilde{r}\right], \tag{2}$$

and is related to the locally measured acceleration due to gravity by

$$a = \sqrt{1-\frac{2m(r)}{r}}\,g(r). \tag{3}$$

Our convention is that $g(r)$ is positive for a downward acceleration. The function $m(r)$ is the quasilocal mass. In the vacuum region beyond the surface (if any) of the starlike object, the Schwarzschild solution yields $g(r)=(M/r^2)/(1-2M/r)$ and $m(r)=M$. We find it more convenient to write the metric in the form

$$ds^2 = -\exp\left[-2\int_r^\infty g(\tilde{r})\,d\tilde{r}\right]dt^2 + \frac{dr^2}{1-2\mu(r)r^2} + r^2[d\theta^2+\sin^2\theta\, d\phi^2], \tag{4}$$

where $\mu(r)=(4\pi/3)\overline{\rho}(r)$ is proportional to the average density inside radius r. In terms of these variables, the Einstein equations are

$$8\pi\rho = G_{\hat{t}\hat{t}} = 2m'(r)/r^2 = 2[r\mu'(r) + 3\mu(r)], \tag{5}$$

$$8\pi p = G_{\hat{r}\hat{r}} = 2\left\{\frac{g(r)[1-2\mu(r)r^2]}{r} - \mu(r)\right\}, \tag{6}$$

$$8\pi p = G_{\hat{\theta}\hat{\theta}} = -r[1+rg(r)]\frac{d\mu(r)}{dr} - 2\left\{[1+rg(r)]^2 + r^2\frac{\mathrm{d}g(r)}{\mathrm{d}r}\right\}\mu(r) + \left[\frac{\mathrm{d}g(r)}{\mathrm{d}r} + \frac{g(r)}{r} + g(r)^2\right]. \tag{7}$$

The first of these equations integrates to

$$\mu(r) = \frac{1}{r^3}\int_0^r 4\pi\rho(\tilde{r})\tilde{r}^2\mathrm{d}\tilde{r}, \tag{8}$$

which justifies the choice of notation $m(r) = \mu(r)r^3$.

III. GENERAL SOLUTION AND GENERATING FUNCTION

By demanding the isotropy condition $G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}}$ and algebraically solving for dg/dr we obtain

$$\frac{\mathrm{d}g}{\mathrm{d}r} = -g^2 + \frac{1+\mu' r^3}{r(1-2\mu r^2)}g + \frac{r\mu'}{1-2\mu r^2}. \tag{9}$$

This is a Riccati equation, for which there is no general analytic solution. If, on the other hand, we take this *same* equation and rearrange it algebraically to extract $\mathrm{d}\mu/\mathrm{d}r$ we find

$$\frac{\mathrm{d}\mu}{\mathrm{d}r} = -\frac{2r(g^2+g')}{1+rg}\mu + \frac{(g/r)' + g^2/r}{1+rg}. \tag{10}$$

But this is a simple first-order linear ordinary differential equation (ODE) and hence explicitly solvable. A symbolic manipulation program such as MAPLE, or a slightly tedious hand computation, easily yields the general solution

$$\mu(r) = \exp\left[-2\int\frac{r[g^2(r)+g'(r)]}{1+rg(r)}dr\right] \times\left\{C_1 + \int\frac{-g(r)+rg'(r)+rg(r)^2}{r^2[1+rg(r)]} \times\exp\left[+2\int\frac{r[g^2(r)+g'(r)]}{1+rg(r)}\mathrm{d}r\right]\right\}. \tag{11}$$

This statement is equivalent to the algorithm presented by Lake [7]: Given a prescribed gravity profile $g(r)$ (the "generating function") and the knowledge that we are dealing with a perfect fluid, the mass profile $m(r) = \mu(r)r^3$ is deduced in closed form. The algorithm (11) is also equivalent to that presented by Rahman and Visser [6], after a change of coordinates (from isotropic to curvature coordinates) and a change of variables. A particularly nice feature of the present version of the algorithm is that the generating function $g(r)$ has a clear physical interpretation in terms of the gravitational field. Now the above is by no means the most useful form in which $\mu(r)$ can be presented. An integration by parts permits us to simplify the appearance of the integrating factor

$$\exp\left[+2\int\frac{r[g^2(r)+g'(r)]}{1+rg(r)}\mathrm{d}r\right] = [1+rg(r)]^2\exp\left[-2\int g(r)\frac{1-rg(r)}{1+rg(r)}\mathrm{d}r\right]. \tag{12}$$

It is now extremely useful to introduce the notation

$$\vartheta(r) = \int g(r)\frac{1-rg(r)}{1+rg(r)}\mathrm{d}r \quad \text{and}$$

$$\vartheta(r_1;r_2) = \int_{r_1}^{r_2} g(r)\frac{1-rg(r)}{1+rg(r)}\mathrm{d}r. \tag{13}$$

We warn the reader that we cannot generally assume $rg(r) \leqslant 1$, and consequently ϑ may become negative. For instance, in the physically reasonable regime $m(r)/r > 1/3$ [that is, $\mu(r)r^2 > 1/3$] and $p \geqslant 0$, it can be shown from the $G_{\hat{r}\hat{r}}$ Einstein equation that $rg(r) > 1$. All that we can safely say in general is that as long as local gravity points down we must have

$$-\int g(r)\mathrm{d}r < \vartheta < \int g(r)\mathrm{d}r. \tag{14}$$

With this notation

$$\mu(r) = \frac{\exp[+2\vartheta(r)]}{[1+rg(r)]^2}\left\{C_2 + \int[1+rg(r)] \times\frac{[-g(r)+rg'(r)+rg(r)^2]}{r^2}\exp[-2\vartheta(r)]\mathrm{d}r\right\}. \tag{15}$$

A second integration by parts now yields

$$\mu(r) = \frac{\exp[+2\vartheta(r)]}{[1+rg(r)]^2}\left\{C_3 + \frac{1}{2}[1+rg(r)]^2\frac{\exp[-2\vartheta(r)]}{r^2} + \int\frac{1+rg(r)}{r^3}\exp[-2\vartheta(r)]\mathrm{d}r\right\}. \tag{16}$$

In this version of the result we have eliminated all the derivatives of $g(r)$. A third integration by parts, using $1/r^3 = -\frac{1}{2}(1/r^2)'$, then leads to

$$\mu(r)=\frac{g(r)}{r}\frac{[1+\frac{1}{2}rg(r)]}{[1+rg(r)]^2}+\frac{\exp[+2\vartheta(r)]}{[1+rg(r)]^2}\times\left\{C_4+2\int\frac{g(r)^2}{r[1+rg(r)]}\exp[-2\vartheta(r)]\mathrm{d}r\right\}. \tag{17}$$

This final version, as we shall soon see, has nice behavior at the origin. Again, we emphasize that this is the explicit, and most general, solution to the perfect fluid constraint for *arbitrary* generating function $g(r)$. All perfect fluid spheres, no matter how derived, must satisfy this equation.

The pressure can now be determined using the $G_{\hat{r}\hat{r}}$ Einstein equation so that

$$p(r)=\frac{1}{8\pi[1+rg(r)]^2}\Bigg[-g(r)^2-2[1+2rg(r)]\times\exp[+2\vartheta(r)]\left\{C_4+2\int\frac{g(r)^2}{r[1+rg(r)]}\times\exp[-2\vartheta(r)]\mathrm{d}r\right\}\Bigg]. \tag{18}$$

This now provides for us the explicit and fully general solution to the mass profile and pressure profile, given only the gravity profile and the information that we are dealing with a static spherically symmetric perfect fluid. For a consistency check, we can compare these formulas to the much simpler result for Newtonian stars:

$$\mu(r)=\frac{g(r)}{r},\quad p(r)=\frac{1}{8\pi}\left[-g(r)^2+C_5-4\int\frac{g(r)^2}{r}\mathrm{d}r\right]. \tag{19}$$

To complete the analysis we should now impose boundary conditions. There are three natural locations to work with: (1) the center of the fluid body, (2) the surface of the fluid body [assuming it has a well-defined surface], and (3) spatial infinity. Perhaps surprisingly, the simplest results are obtained if we normalize at spatial infinity.

IV. BOUNDARY CONDITIONS AT SPATIAL INFINITY

We will now adopt the very mild condition that the total mass of the fluid sphere is finite, so that in particular $\mu(r)\to0$ as one approaches spatial infinity. We also assume that $r\,p(r)\to0$ at spatial infinity. Then (from the $G_{\hat{r}\hat{r}}$ equation) we deduce $g(r)\to0$ at spatial infinity. Physically this means that the present discussion is capable of handling situations with a tenuous atmosphere extending all the way to infinity, and that the special case where the fluid body has a sharp surface with $p(r\geqslant R)=0$ and $m(r\geqslant R)=M$ is automatically included. Then fixing boundary conditions at spatial infinity, the mass profile (17) is given by

$$\mu(r)=\frac{g(r)}{r}\frac{[1+rg(r)]}{[1+rg(r)]^2}-2\frac{\exp[-2\vartheta(r;\infty)]}{[1+rg(r)]^2}\times\int_r^\infty\frac{g(\tilde{r})^2}{\tilde{r}[1+\tilde{r}g(\tilde{r})]}\exp[+2\vartheta(\tilde{r};\infty)]\mathrm{d}\tilde{r}. \tag{20}$$

We can simplify this slightly to yield

$$\mu(r)=\frac{g(r)}{r}\frac{[1+\frac{1}{2}rg(r)]}{[1+rg(r)]^2}-\frac{2}{[1+rg(r)]^2}\times\int_r^\infty\frac{g(\tilde{r})^2}{\tilde{r}[1+\tilde{r}g(\tilde{r})]}\exp[-2\vartheta(r;\tilde{r})]\mathrm{d}\tilde{r}. \tag{21}$$

The pressure profile determined from (18) is then

$$p(r)=\frac{1}{8\pi[1+rg(r)]^2}\Bigg[-g(r)^2+4[1+2rg(r)]\times\int_r^\infty\frac{g(\tilde{r})^2}{\tilde{r}[1+\tilde{r}g(\tilde{r})]}\exp[-2\vartheta(r;\tilde{r})]\mathrm{d}\tilde{r}\Bigg]. \tag{22}$$

For the central pressure, $p_c=p(0)$, we find

$$p_c=\frac{1}{2\pi}\int_0^\infty\frac{g(r)^2}{r[1+rg(r)]}\exp[-\vartheta(0;r)]\mathrm{d}r. \tag{23}$$

Compare with the equivalent statement for a Newtonian fluid body in which the pressure profile is

$$p(r)=\frac{1}{8\pi}\left[-g(r)^2+4\int_r^\infty\frac{g(\tilde{r})^2}{\tilde{r}}\mathrm{d}\tilde{r}\right], \tag{24}$$

and the central pressure is

$$p_c=\frac{1}{2\pi}\int_0^\infty\frac{g(r)^2}{r}\mathrm{d}r. \tag{25}$$

V. BOUNDARY CONDITIONS AT THE CENTER OF THE FLUID SPHERE

If we apply boundary conditions at the center of the sphere then, using

$$g(r)=\frac{m(r)+4\pi p(r)r^3}{r^2[1-2m(r)/r]} \tag{26}$$

and the assumed finiteness of ρ_c and p_c, implies

$$g(r)=\frac{4\pi}{3}(\rho_c+3p_c)r+O(r^2). \tag{27}$$

The mass and pressure profiles are given by

$$\mu(r)=\frac{g(r)}{r}\frac{[1+\frac{1}{2}rg(r)]}{[1+rg(r)]^2}+\frac{\exp[+2\vartheta(0;r)]}{[1+rg(r)]^2}\Bigg\{-4\pi p_c+2\int_0^r\frac{g(\tilde{r})^2}{\tilde{r}[1+\tilde{r}g(\tilde{r})]}\exp[-2\vartheta(0;\tilde{r})]\mathrm{d}\tilde{r}\Bigg\} \tag{28}$$

and

$$p(r)=\frac{1}{8\pi[1+rg(r)]^2}\Bigg[-g(r)^2+[1+2rg(r)]\times\exp[+2\vartheta(0;r;)]\Bigg\{8\pi p_c-4\int_0^r\frac{g(\tilde{r})^2}{\tilde{r}[1+\tilde{r}g(\tilde{r})]}\times\exp[-2\vartheta(0;\tilde{r})]\mathrm{d}\tilde{r}\Bigg\}\Bigg], \tag{29}$$

compared with the Newtonian result

$$p(r)=p_c-\frac{1}{8\pi}\Bigg[g(r)^2+4\int_0^r\frac{g(\tilde{r})^2}{\tilde{r}}\mathrm{d}\tilde{r}\Bigg]. \tag{30}$$

VI. BOUNDARY CONDITIONS AT THE SURFACE OF THE FLUID SPHERE

If the fluid sphere has a sharp boundary (say at radius R, with the density and pressure identically zero outside this radius), then it can be useful to normalize at this surface. For the pressure profile we find [in terms of the "surface gravity" $g_s=(M/R^2)/\sqrt{1-2M/R}$] that

$$p(r)=\frac{1}{8\pi[1+rg(r)]^2}\Bigg[-g(r)^2+[1+2rg(r)]\times\exp[-2\vartheta(r;R)]\Bigg\{\frac{g_s^2}{1+2Rg_s}+\int_r^R\frac{g(\tilde{r})^2}{\tilde{r}[1+\tilde{r}g(\tilde{r})]}\times\exp[+2\vartheta(\tilde{r};R)]\mathrm{d}\tilde{r}\Bigg\}\Bigg]. \tag{31}$$

There is a similar but uninteresting expression for $\mu(r)$. The central pressure is now

$$p_c=\frac{1}{8\pi}\Bigg\{\frac{g_s^2\exp[-2\vartheta(0;R)]}{1+2Rg_s}+4\int_0^R\frac{g(r)^2}{r[1+rg(r)]}\times\exp[-2\vartheta(0;r)]\mathrm{d}r\Bigg\}, \tag{32}$$

where we have now reduced the range of integration from $(0,\infty)$ to $(0,R)$ at the price of introducing an extra term depending explicitly on total mass and radius of the fluid sphere. This can be compared with the equivalent Newtonian results

$$p(r)=\frac{1}{8\pi}\Bigg[g_s^2-g(r)^2+4\int_r^R\frac{g(\tilde{r})^2}{\tilde{r}}\mathrm{d}\tilde{r}\Bigg]\quad\text{with}\quad g_s=\frac{M}{R^2} \tag{33}$$

and

$$p_c=\frac{1}{8\pi}\Bigg[g_s^2+4\int_0^R\frac{g(r)^2}{r}\mathrm{d}r\Bigg]. \tag{34}$$

VII. SOLUTION GENERALIZATION TECHNIQUE

One nice feature of the present analysis is that it allows one to turn simple exact solutions into more complicated ones. While the general algorithm presented above always provides an exact solution, it may not be an "elementary" solution in the sense that the integrations might not be doable in terms of either elementary or special functions. In such a situation, a simplified algorithm is sometimes useful.

Suppose that one has found, by some unspecified means, a specific exact solution for a perfect fluid sphere. Let that exact solution be given in terms of $m(r)$ [or equivalently $\mu(r)$] and $g(r)$. Then for any arbitrary constant k,

$$\mu(r)\to\mu(r)+k\exp\Bigg[-2\int\frac{r[g^2(r)+g'(r)]}{1+rg(r)}\mathrm{d}r\Bigg] \tag{35}$$

is also an exact solution for a perfect fluid sphere [with the *same* $g(r)$]. This construction may sometimes "fail" in the sense that the integral is either too trivial [returning you to the seed solution you started with] or too complicated to perform in terms of elementary or special functions. However, in very many cases this simple construction is sufficient to understand why certain broad classes of exact solution exist.

Let us start by rescaling the time variable to remove any redundancies in the number of free parameters, n, appearing in g_{tt}. If the number of free parameters appearing in g_{rr} is not at least $n+1$, then the seed solution you have *must* have a generalization. For instance, the Minkowski solution [a particularly simple fluid sphere with zero pressure and density] has exactly zero parameters appearing in both g_{tt} and g_{rr}, and so must have a one-parameter generalization. In this case, performing the integration leads to the Einstein static universe. Similarly, the exterior Schwarzschild solution (another particularly simple fluid sphere with zero pressure and density) has exactly one free parameter (the mass) appearing in both g_{tt} and g_{rr}, and so must have a one-parameter generalization. In this case, performing the integration leads to what is called the Kuch68 II solution in the Delgaty-Lake classification [8]. A slightly more complex example, using anti–de Sitter space as a seed, leads to the Tolman IV solution. A number of additional examples of this phenomena are collected in Table I.

Of course, sometimes explicit exact solutions were first discovered in their general form, in which case this algorithm provides no extra information. (This comment applies, for instance, to the Wyman IIb geometry.) Conversely, sometimes the integral is too complicated to provide a closed-

TABLE I. Seed solutions and their generalizations.

Seed	Generalization
Minkowski	Einstein static
Schwarzschild exterior	Kuch68 II
anti–de Sitter	Tolman IV
Tolman V	Kuch2 I
Tolman VI	Wyman IIa
Kuch1 Ib	appears new
M–W III	appears new
K–O III	appears new

form solution—the generalization may be exact but too complex to write down explicitly. [The same, for instance, is true when you use the Schwarzschild–de Sitter (Kottler) geometry as seed. Similarly, by parameter counting Tolman VII and Tolman VIII must have one-parameter extensions, but it seems impossible to write then down in closed form.]

Finally, we point out that there are some cases where this formalism does lead to apparently new solutions. (We again follow the Delgaty-Lake classification [8].) For instance, the Kuch1 Ib solution

$$\mathrm{d}s^2=-(Ar+Br\ln r)^2\mathrm{d}t^2+\frac{\mathrm{d}r^2}{2}+r^2\mathrm{d}\Omega^2 \tag{36}$$

generalizes to

$$\mathrm{d}s^2=(Ar+Br\ln r)^2\mathrm{d}t^2+\frac{2(2A+2B\ln(r)+B)}{(2A+2B\ln(r)+B)-kr^2}\mathrm{d}r^2 +r^2\mathrm{d}\Omega^2 \tag{37}$$

which appears to be new. Similarly, the M–W III solution, which can be cast into the form,

$$\mathrm{d}s^2=-\left(r-\frac{r^2}{a}\right)\mathrm{d}t^2+\frac{7dr^2}{4(1-r^2/a^2)}+r^2\mathrm{d}\Omega^2, \tag{38}$$

generalizes to

$$\mathrm{d}s^2=\left(r-\frac{r^2}{a}\right)dt^2+\frac{\mathrm{d}r^2}{1-2m(r)/r}+r^2\mathrm{d}\Omega^2 \tag{39}$$

with

$$m(r)=\frac{4r^2+3a^2}{14a^2}r+k\frac{(r-a)r^{10/3}}{(4r-3a)^{4/3}} \tag{40}$$

which also appears to be new. Also, the K–O III solution can be cast into the form

$$\mathrm{d}s^2=-\left(1+\frac{r^2}{a^2}\right)^2\mathrm{d}t^2+\mathrm{d}r^2+r^2\mathrm{d}\Omega^2, \tag{41}$$

which is spatially flat. It generalizes to

$$\mathrm{d}s^2=-\left(1+\frac{r^2}{a^2}\right)^2\mathrm{d}t^2+\frac{\mathrm{d}r^2}{1-kr^2(3r^2+a^2)^{-2/3}}+r^2\mathrm{d}\Omega^2, \tag{42}$$

which is contained within the new class of exact solutions briefly described by Lake [7].

VIII. DISCUSSION

As emphasised in the article by Rahman and Visser [6], and reiterated by Lake [7], while this type of algorithm guarantees a perfect fluid body it does not necessarily guarantee a ‘‘physically reasonable’’ perfect fluid body. One physically reasonable constraint that is easy to enforce in the current formulation is $g>0$; locally measured gravity should always attract towards the center of the body. A second physically reasonable constraint which is automatically satisfied is that the central pressure is positive. It is considerably more difficult to enforce $m(r)\geqslant 0$, $\rho(r)\geqslant 0$, and $p(r)\geqslant 0$. Checking these physically motivated constraints amounts to mathematically investigating a set of integral inequalities, and seems to require a case by case investigation depending on the assumed gravity profile $g(r)$. One should not, however, lose track of the significance of what has been accomplished: (1) We have derived the exact and fully general solution to the pressure isotropy condition in terms of variables that have a direct physical meaning, the gravity profile $g(r)$ and mass profile $m(r)$. (2) We have also derived an exact and fully general formula for the pressure profile $p(r)$ of a perfect fluid sphere that depends only on the gravity profile $g(r)$. (3) In particular we have an exact and fully general expression for the central pressure of a fluid sphere, again determined directly in terms of the gravity profile $g(r)$. (4) The algorithm provides a natural framework for understanding the reason for the existence of certain broad classes of exact solution, and in some cases leads to new exact solutions.

Because this algorithmic approach works directly in terms of physically meaningful quantities, with a physically meaningful ‘‘generating function’’ in the form of the gravity profile $g(r)$, the interpretation of the results is somewhat clearer than in the algorithms presented in the Rahman and Visser [6] and Lake [7] articles. We expect that this version of the algorithm for generating perfect fluid spheres will lead to additional useful ‘‘exact solutions.’’ In particular, the new class of exact solutions briefly described in Ref. [7] has a very natural representation in terms of this algorithm.

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