Fake supergravity and domain wall stability

D. Z. Freedman,[1,2,*] C. Núñez,[2,†] M. Schnabl,[2,‡] and K. Skenderis[3,§]
[1]*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[2]*Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[3]*Institute for Theoretical Physics, University of Amsterdam, Valkenierstraat 65, 1018 XE Amsterdam, The Netherlands*
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We review the generalized Witten-Nester spinor stability argument for flat domain wall solutions of gravitational theories. Neither the field theory nor the solution need be supersymmetric. Nor is the space-time dimension restricted. We develop the nontrivial extension required for AdS-sliced domain walls and apply this to show that the recently proposed ''Janus'' solution of type-IIB supergravity is stable nonperturbatively for a broad class of deformations. Generalizations of this solution to arbitrary dimension and a simple curious linear dilaton solution of type-IIB supergravity are by-products of this work.

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I. INTRODUCTION

Many domain wall solutions of supergravity theories have been studied in the literature in order to explore the AdS conformal field theory (CFT) correspondence, to find a fundamental setting for brane world cosmology, and for other reasons. In this paper we will review and extend stability arguments for domain walls based on the elegant spinor methods of the Witten positive energy theorem and its generalizations [1–6].[1]

Many solutions studied in the past are supersymmetric. One would expect these to be stable, and there are known arguments which use the transformation rules of the supergravity theory and the Killing spinors supported by the solutions. Yet these arguments do not apply to the many solutions with curvature singularities.

Nonsupersymmetric solutions are also known and might well be important since supersymmetry (SUSY) is certainly broken in our universe. Most domain wall solutions, both SUSY and non-SUSY, are planar; the isometry group of their metrics

$$ds^2 = e^{2A(r)}\eta_{ij}dx^i dx^j + e^{2h(r)}dr^2,$$
$$\eta_{ij} = \mathrm{diag}(-1,1,1,...,1) \tag{1.1}$$

is the Poincaré group in d flat space-time dimensions. [The choice $h(r)=0$ is convenient for many purposes but we keep $h(r)$ unfixed to facilitate comparison with different radial coordinates used in the literature.]

For planar domain walls there is a formal stability argument [5,8] based on what we propose to call ''fake supergravity.'' In fake supergravity one defines a spinor energy using fake transformation rules similar to those of a real supergravity theory, but containing a superpotential $W(\phi)$ which is not that of the real theory. Instead $W(\phi)$ must satisfy an equation which relates it to the scalar potential $V(\phi)$, and one can formulate certain first order equations [the fake Bogomolńyi-Prasad-Sommerfield (BPS) equations] whose solutions automatically satisfy the second order Einstein field equations for domain wall metrics (1.1) and the accompanying scalar field $\phi(r)$. If one can find a $W(\phi)$ such that the domain wall solution under test is a solution of the first order system, then that domain wall is stable, if there are no singularities.[2] One curious feature of fake supergravity is that it can work in any space-time dimension d, whereas real supergravity is limited to $d \leqslant 11$.

Domain walls with the isometry group SO$(d-1,2)$ of the space-time AdS$_d$ have also been studied [9–11]. Their metrics take the form

$$ds^2 = e^{2A(r)}g_{ij}(x)dx^i dx^j + e^{2h(r)}dr^2, \tag{1.2}$$

where $g_{ij}(x)$ is a metric on AdS$_d$ with scale L_d. A domain wall of this type was recently found [12] as a solution of type-IIB supergravity. The solution contains a flowing dilaton $\phi(r)$, but no other r-dependent matter fields, and there is an accompanying round S_5 internal space. The solution is regular if one chooses parameters such that the rate of variation of the dilaton is sufficiently slow.

In this paper we develop stability arguments for nonsingular AdS$_d$-sliced domain walls. A nontrivial extension of the fake supergravity approach, related to the work of Ref. [9] in real $D=5$ supergravity, is required for this. This argument gives a definition of energy which vanishes for the background solution itself and is positive for fluctuations about the background which obey suitable boundary conditions.

These arguments imply that the solution of Ref. [12] enjoys nonperturbative stability with respect to fluctuations of the metric and dilaton while other fields of type-IIB supergravity remain fixed at their vacuum values. The formalism we develop can accommodate additional fields, but it be-

*Email address: dzf@math.mit.edu
†Email address: nunez@lns.mit.edu
‡Email address: schnabl@lns.mit.edu
§Email address: skenderi@science.uva.nl

[1]See Ref. [7] for a recent paper with similar aims which discusses the stability of p-brane spacetimes.

[2]Further conditions are discussed in Sec. III.

comes more difficult to establish the required properties of the superpotential. To remediate this difficulty we work in the spirit of Ref. [4] and derive inequalities which show that the "Janus" solution [12] is also stable within several different consistent truncations of type-IIB supergravity. Some of these truncations include negative m^2 fields and potentials unbounded below. These indications of global stability make it more compelling to understand the AdS-CFT dual of the solution proposed in Ref. [12]. We do not discuss this here.

It is not guaranteed that a given solution of the field equations can be reproduced in the framework of fake supergravity. Indeed there are known solutions which are pure AdS metrics [those with $A(r)\equiv r/L$ in Eq. (1.1) if $h(r)=0$ and with fixed scalars] which are unstable because small fluctuations violate the stability bound of Ref. [21]. In general it is not always possible to satisfy the required conditions on the superpotential.

In Sec. II we discuss the equations of motion satisfied by domain walls and present some simple examples of nonsupersymmetric domain wall solutions of type-IIB supergravity. They involve a single flowing scalar field, the dilaton. These simple dilaton domain walls are the prototype solutions we study. The fake supergravity stability argument for planar domain walls (1.1) is reviewed in Sec. III. In Sec. IV we extend this argument to AdS_d domain walls (1.2). Section V is devoted to stability arguments for the solution of Ref. [12] for nondilatonic fluctuations. In Sec. VI we discuss a very simple and apparently new solution of type-IIB which emerged from the techniques of Sec. IV.

II. DOMAIN WALLS: BASICS AND EXAMPLES

We consider a scalar-gravity action in $d+1$ dimensions:

$$S=\int d^{d+1}x\sqrt{-g}\left[\frac{1}{2\kappa^2}R-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)\right]. \tag{2.1}$$

Such actions can arise via Kaluza-Klein reduction of a still higher dimensional theory. Although we include only one scalar explicitly, additional scalars (with σ-model interactions) and higher rank bosonic fields can be included. The equations of motion are

$$\frac{1}{\kappa^2}R_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi+\frac{2}{d-1}g_{\mu\nu}V(\phi),$$

$$\Box\phi=\partial V/\partial\phi. \tag{2.2}$$

We assume that the potential $V(\phi)$ has a critical point at $\phi=\phi_0$, with $V_0\equiv V(\phi_0)<0$. Thus one solution of Eq. (2.2) is AdS_{d+1} with scale L. In this case we have

$$R_{\mu\nu}=-\frac{d}{L^2}g_{\mu\nu},$$

$$V_0=-\frac{d(d-1)}{2L^2\kappa^2}. \tag{2.3}$$

We will introduce explicit parameterizations of the AdS metric $g_{\mu\nu}$ when needed.

A. Flat domain walls

We are more interested in domain wall solutions of Eq. (2.2) with r-dependent scalar $\phi(r)$ and metrics of the form (1.1) or (1.2), and we require that these approach the AdS_{d+1} geometry at the boundary. With the coordinate choice $h(r)=0$, the boundary occurs as $r\to+\infty$. Frame and connection one-forms and curvature tensors for our presentation of domain walls are given in Appendix A.

We first consider flat domain walls. When the metric of Eq. (1.1) and the restriction $\phi=\phi(r)$ are incorporated, the Einstein and scalar equations of motion of Eq. (2.2) reduce to ordinary differential equations in r, namely,

$$A''-A'h'=-\frac{\kappa^2}{d-1}\phi'^2,$$

$$A'^2=\frac{\kappa^2}{d(d-1)}\phi'^2-\frac{2\kappa^2}{d(d-1)}V(\phi)e^{2h},$$

$$\phi''+(dA'-h')\phi'=\frac{\partial V}{\partial\phi}e^{2h}. \tag{2.4}$$

It is quite well known [8,13] that any solution of the following first order flow equations is also a solution of Eq. (2.4):

$$A'(r)=2e^hW[\phi(r)], \tag{2.5}$$

$$\phi'(r)=-\frac{2(d-1)}{\kappa^2}e^h\partial_\phi W[\phi(r)]. \tag{2.6}$$

The superpotential $W(\phi)$ is related to the potential $V(\phi)$ by[3]

$$\kappa^2V(\phi)=2(d-1)^2\left(\frac{1}{\kappa^2}W'^2-\frac{d}{d-1}W^2\right). \tag{2.7}$$

These fake BPS equations for flat domain walls will be rederived in the next section.

The simplest example of a domain wall is the following solution of Eq. (2.4) [with $h(r)=0$] for the theory with constant potential $V(\phi)=V_0$ of Eq. (2.3). The scalar satisfies $\phi'(r)=c\exp[-dA(r)]$. After routine integration, one finds

$$\phi(r)=\phi_0+\sqrt{\frac{d-1}{d\kappa^2}}\log\frac{1-e^{-d(r-r^*)/L}}{1+e^{-d(r-r^*)/L}},$$

[3]The prime in W' denotes a derivative with respect to ϕ, whereas the prime attached to the fields ϕ, A,h denotes a derivative with respect to the radial coordinate r.

$$A(r)=A_0+\frac{r-r^*}{L}+\frac{1}{d}\log(1-e^{-2d(r-r^*)/L}), \qquad (2.8)$$

where c is related to the other integration constants by $c=[\sqrt{d(d-1)}/\kappa L]e^{dA_0}$. This gives an asymptotically AdS geometry with boundary region $r\to\infty$, but there is a curvature singularity at $r=r^*$.

When $d=4$ this is the dilaton domain wall solution of type-IIB supergravity which was found and studied [14–16] in the early period of the AdS-CFT correspondence. As a solution of type-IIB supergravity, it is not supersymmetric. There are no true Killing spinors, since the dilation condition from the type-IIB supergravity transformation rules

$$\delta\chi=\frac{i}{2}\gamma^{\mu}(\partial_\mu\phi+ie^{\phi}\partial_\mu\xi)\epsilon^* -\frac{i}{24}\gamma^{\mu\nu\rho}(e^{-\phi/2}H^{(NS)}_{\mu\nu\rho}+ie^{\phi/2}F^{(RR)}_{\mu\nu\rho})\epsilon \qquad (2.9)$$

cannot be satisfied because the axion ξ and three-forms vanish. The indices μ, ν, ρ are ten dimensional coordinate indices.

We will now show that there is a superpotential $W(\phi)$ such that Eq. (2.8) is also a solution of Eqs. (2.5)–(2.7) for any dimension d. We thus achieve fake supersymmetry, as we will confirm by exhibiting fake Killing spinors in the next section. The obvious constant $W=\frac{1}{2}L$ does not work, but with the general solution of Eq. (3.11), namely,

$$W(\phi)=\frac{1}{2L}\cosh\left(\kappa\sqrt{\frac{d}{d-1}}(\phi-\phi_0)\right) \qquad (2.10)$$

one can easily integrate Eqs. (2.5), (2.6) and find that the solution agrees with Eq. (2.8). The constant r^* arises as an integration constant.

Note that we have chosen the solution of Eq. (2.7) which is positive near the boundary value $\phi\sim\phi_0$, and we have chosen signs in Eqs. (2.5), (2.6) so that the boundary of the geometry appears as $r\to+\infty$. These conventions are natural for the extension to AdS_d domain walls in Sec. IV, but they differ from some earlier applications.

Let us use the term "adapted superpotential" to denote the particular $W(\phi)$ for which the first order flow equations produce a given domain wall solution of Eq. (2.4). For nonconstant $V(\phi)$, it may not be possible to solve Eq. (2.7) and find the superpotential $W(\phi)$ explicitly. This may be inconvenient, but to establish fake supersymmetry we need only know that the adapted superpotential exist for a given solution $A(r)$, $\phi(r)$ of Eq. (2.4). If $\phi(r)$ is monotonic, the inverse function $r(\phi)$ exists. One may then use Eq. (2.5) to *define* the adapted superpotential.

B. AdS_d-sliced domain walls

We now discuss the equations of motion for AdS_d-sliced domain walls of codimension 1. Frames, connections and curvatures for the metric (1.2) are given in Appendix A. When inserted in Eq. (2.2) one finds that wall profile $A(r)$ and scalar $\phi(r)$ obey the coupled equations which are modifications of Eq. (2.4),

$$A''-A'h'=-\frac{\kappa^2}{d-1}\phi'^2+\frac{1}{L_d^2}e^{-2A+2h},$$

$$A'^2=\frac{\kappa^2}{d(d-1)}\phi'^2-\frac{2\kappa^2}{d(d-1)}V(\phi)e^{2h}-\frac{1}{L_d^2}e^{-2A+2h},$$

$$\phi''+(dA'-h')\phi'=\frac{\partial V}{\partial\phi}e^{2h}. \qquad (2.11)$$

A set of first order equations which extend Eqs. (2.5)–(2.7) to AdS_d-sliced walls was presented in Ref. [13]. These equations are

$$A'(r)=2\gamma(r)e^{h}W[\phi(r)],$$

$$\phi'(r)=-\frac{1}{\gamma(r)}\frac{2(d-1)}{\kappa^2}e^{h}\frac{\partial W}{\partial\phi},$$

$$V(\phi)=\frac{2(d-1)^2}{\kappa^2}\left(\frac{1}{\kappa^2\gamma(r)^2}W'^2-\frac{d}{d-1}W^2\right) \qquad (2.12)$$

which differ from Eqs. (2.5)–(2.7) by the inclusion of the factor

$$\gamma(r)\equiv\sqrt{1-\frac{e^{-2A(r)}}{4L_d^2W[\phi(r)]^2}}. \qquad (2.13)$$

The constant L_d is the AdS_d scale, and one obtains the previous Eqs. (2.5)–(2.7) as $L_d\to\infty$. The system (2.12) is well posed [13], but it is rather unworkable. In Sec. IV we will derive an alternate set of first order equations which involves an su(2)-valued superpotential $\mathbf{W}(\phi)=W_a(\phi)\tau^a$, where the τ^a are the Pauli matrices. The structure of the new equations is even simpler than Eqs. (2.5)–(2.7) and they are easily solved, given $\mathbf{W}(\phi)$. However, $\mathbf{W}(\phi)$ must satisfy a nonlinear condition in addition to Eq. (2.7). We show that any solution of the new equations also satisfies Eq. (2.12).

C. Janus solutions

A simple example of an AdS_d-sliced domain wall is the extension to d dimensions of the dilaton domain wall solution of type-IIB supergravity of Ref. [12]. We take the constant potential $V(\phi)=V_0$, see Eq. (2.3), and we proceed as in Ref. [12], but use the radial coordinate r for which $h(r)=0$. We take $\phi'=c\exp[-dA(r)]$ so the scalar equation of Eq. (2.11) is satisfied. Any solution of the wall profile equation

$$A'^2=(1/L^2)[1-e^{-2A}+be^{-2dA}], \qquad (2.14)$$

will also satisfy the equation involving A'' in Eq. (2.11). The constant b is related to c by $b=[\kappa^2/d(d-1)]c^2L^2$. We have set $L_d=L$ for simplicity.

When $b=0$, the solution gives pure AdS_{d+1} in the form

$$ds^2=\cosh^2(r/L)g_{ij}(x)dx^i dx^j+dr^2. \tag{2.15}$$

For $b\neq 0$ we will not be able to solve Eq. (2.14) exactly (unless $d=2$), and we need the following argument similar to that of Ref. [12]. With $x\equiv e^{-2A}$, we consider the polynomial $P(x)\equiv bx^d-x+1$. For small b, there are exactly two real zeros (which occur for $x>1$). This continues to be true for

$$0<b<b_0\equiv\frac{1}{d}\left(\frac{d-1}{d}\right)^{d-1}. \tag{2.16}$$

At $b=b_0$ the zeros coalesce at $x_0=(b_0d)^{-1/(d-1)}$ and become complex for $b>b_0$.

This behavior is relevant to the physics, as we can see from the implicit solution of Eq. (2.14), namely,

$$r=\int_{A_0}^{A}\frac{dA}{\sqrt{1-e^{-2A}+be^{-2dA}}}. \tag{2.17}$$

The lower limit A_0 will be specified below. For $b>b_0$, there is no natural lower bound on the variable A and the geometry would be geodesically incomplete unless extended to $A\rightarrow-\infty$ where there is a curvature singularity.

Therefore we restrict to the range $0<b<b_0$ in which the minimum value of $A_{\min}$ is given by $A_{\min}=-\ln(x_{\min})/2$, where $x_{\min}$ is the smallest zero of $P(x)$. The formula (2.17), with $A_0=A_{\min}$ thus defines half the geometry, namely, the region $0\leqslant r<+\infty$, $A_{\min}<A(r)<+\infty$. This $r>0$ region is not geodesically complete. But all odd order derivatives of $A(r)$ vanish at $r=0$, so that $A(r)$ can be extended to the region $-\infty<r<0$ as even function $A(r)=A(-r)$ and the continued function is C^∞. The full geometry is geodesically complete and has two boundary regions, namely, $r\rightarrow\pm\infty$.

With $A(r)$ defined above, the dilaton is given by

$$\phi(r)=\phi_0+c\int_0^r e^{-dA(r)}dr. \tag{2.18}$$

It is monotonic, and odd in r except for the additive integration constant ϕ_0. In the boundary regions $r\rightarrow\pm\infty$, it approaches the limits

$$\phi(r)\rightarrow\phi_\pm\equiv\phi_0\pm c\int_0^\infty e^{-dA(r)}dr. \tag{2.19}$$

Our choice of radial coordinate r [with $h(r)=0$] was motivated by the fact that Eq. (2.17) can be integrated in terms of elementary functions when $d=2$. This leads to the explicit presentation of the $d=2$ solution discussed in Appendix B. However, the space-time geometry is most easily visualized using a radial coordinate of finite range. Therefore, in the rest of this section we switch to the notation of Ref. [12], with radial variable μ (corresponding to the case $h=A$ in the notation above), and we use a standard global metric on the AdS_d slices.

The metric (1.2) then takes the form

$$ds^2=\frac{L^2e^{2A(\mu)}}{\cos^2\lambda}[-dt^2+\cos^2\lambda\, d\mu^2+d\lambda^2+\sin^2\lambda\, d\Omega_{d-2}^2], \tag{2.20}$$

where the range of the principal coordinate of AdS_d is $0\leqslant\lambda<\pi/2$ for $d>2$, but $-\pi/2<\lambda<\pi/2$ when $d=2$, and $d\Omega_{d-2}^2$ is a metric on the unit sphere S_{d-2}. The wall profile $A(\mu)$ is defined implicitly by the integral

$$\mu=\int_{A_0}^{A}\frac{dA}{\sqrt{e^{2A}-1+be^{-2(d-1)A}}}. \tag{2.21}$$

It can be extended to negative μ as discussed above. The range of μ is $-\mu_0<\mu<\mu_0$. The boundary limit μ_0 can, in principle, be obtained from the integral (2.21), with upper limit $A\rightarrow+\infty$. For small b and general d, one finds the series expansion

$$\mu_0=\frac{\pi}{2}\left(1+\frac{\Gamma\left(d+\frac{1}{2}\right)}{\Gamma(d)\Gamma\left(\frac{1}{2}\right)}b+\frac{\Gamma\left(2d+\frac{1}{2}\right)}{\Gamma(2d-1)\Gamma\left(\frac{1}{2}\right)}\frac{b^2}{2!}+O(b^3)\right). \tag{2.22}$$

For $d=2$ an exact expression is given in Appendix B. It is useful to note the near-boundary behavior of the scale factor obtained in Eq. (B12):

$$e^{2A(\mu)}\underset{\mu\rightarrow\pm\mu_0}{\sim}\frac{1}{\sin^2(\mu_0\mp\mu)}\{1+O[(\mu_0\mp\mu)^{2d}]\}. \tag{2.23}$$

Thus the effect of the running dilaton on the wall profile is a b-dependent change in the boundary limit μ_0 together with an order $(\mu\pm\mu_0)^{2d}$ effect on the near boundary shape.

In Eq. (2.20), which is the same as Eq. (20) of Ref. [12], we have extracted the conformal factor $e^{2A(\mu)}/\cos^2\lambda$, so that the line element in square brackets can be viewed, at least heuristically, as a conformal compactification. As discussed in Ref. [12] this conformal metric is similar to the Einstein static universe, and would agree with the well known conformal compactification of AdS_{d+1}, in the limit $b\rightarrow0$ when $A(\mu)\rightarrow-\ln(\cos\mu)$ and $\mu_0\rightarrow\pi/2$. In this limit, the spatial metric (i.e., fixed t) is a hemisphere of S_d. For $b>0$ and $\mu_0>\pi/2$, we also have a half sphere but with angular excess as depicted in Fig. 1(a). The boundary of the conformal metric then has two parts, hemispheres of S_{d-1} at $\mu=\pm\mu_0$ which are joined at the pole(s) where $\cos\lambda=0$.

The angular coordinates μ, λ are singular at the poles. However, one may choose regular coordinates there by embedding S_d in $\mathcal{R}_{d+1}$ with cartesian coordinates. For simplicity, we discuss the case $d=2$ in which we take coordinates

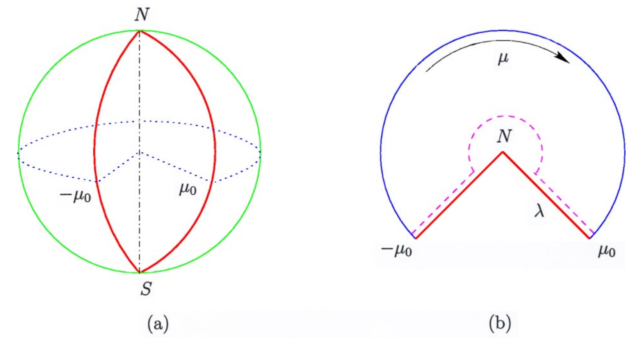

FIG. 1. (a) Conformal picture of a constant time slice of the Janusian geometry. The boundary is indicated by the bold arcs. (b) Top view of the same picture. The coordinate λ ranges from 0 at the equator to $\pi/2$ at the north pole. The dashed line indicates the ''contour'' used to evaluate $E_{\rm WN}$ in Sec. IV.

$z=\sin\lambda$, $x=\cos\lambda\cos\mu$, $y=\cos\lambda\sin\mu$. The induced metric on S_2, namely, $d\vec{s}^2=dx^2+dy^2+dz^2=d\lambda^2+\cos^2\lambda d\mu^2$, is then regular at the pole at $x=y=0$. Consider next the conformal factor $\Omega=e^{-A(\mu)}\cos\lambda$. It follows from Eq. (2.23) that its near boundary behavior is

$$\Omega\sim\sin(\mu_0\mp\mu)\cos\lambda \tag{2.24}$$

$$\sim x\sin\mu_0\mp y\cos\mu_0. \tag{2.25}$$

Thus $\nabla\Omega$, evaluated in the regular coordinates x, y is discontinuous as one continues from the boundary region $\mu=+\mu_0$, where $y/x=+\tan(\mu_0)$, to the portion where $\mu=-\mu_0$ and where $y/x=-\tan\mu_0$. This means that the factorization in Eq. (2.20) does not satisfy the strict definition of conformal compactification [17]. In practice, it means that the conformal boundary has corners at the pole(s), a geometric feature deduced by means of the regular Cartesian coordinates.[4] We will treat the corner in the boundary integral that occurs in the Witten-Nester stability analysis by deforming the contour around the corner as indicated in Fig. 1(b) and taking the limit to the corner after the integration is performed.

An alternative approach is to work with the Fefferman-Graham coordinates, i.e., to look for a coordinate system where the metric near the boundary takes the form

$$ds^2=\frac{1}{z^2}\{dz^2+[g_{(0)ij}+z^2g_{(2)ij}+\cdots+z^d(g_{(d)ij}+\log z\, h_{(d)ij})+\cdots]dx^idx^j\}. \tag{2.26}$$

Such coordinate system can always be reached [18]. In this expansion $g_{(0)}$ is the boundary metric. All coefficients in Eq. (2.26) but $g_{(d)}$ are locally related to $g_{(0)}$. $g_{(d)}$ carries information about the vacuum and correlation functions of the dual QFT, so this coordinate system is well suited for holography [19]. Transforming the Janus solution to this coordinate system appears laborious and seems to lead to singular $g_{(d)ij}$. Since we will not address the AdS-CFT duality for this solution, we will not present these results here and continue in the rest of the paper with the coordinate system in Eq. (2.20).

Domain walls with AdS slicing can also be presented using a Poincaré patch metric for the AdS_d slices [12]. The scale factor $A(\mu)$ and local aspects of the discussion above are not changed, but the global structure is affected. In particular the metric is geodesically incomplete, and one needs its global extension. For this reason we formulate our stability study using the global version. The patch version may well be appropriate for the AdS-CFT dual.

III. STABILITY OF FLAT DOMAIN WALLS

We review in this section the stability argument [5,8] for asymptotically AdS (AAdS) flat domain walls with metric in the form (1.1) and accompanying scalar $\phi(r)$. The purpose of the argument is to show that the energy of deformed solutions of the equations of motion which approach the domain wall at large distance is higher than the energy of the wall itself. We use the following notation for the background fields and deviations:

$$ds^2=[\bar{g}_{\mu\nu}+h_{\mu\nu}]dx^\mu dx^\nu, \tag{3.1}$$

$$\phi=\bar{\phi}+\varphi. \tag{3.2}$$

The fluctuations $h_{\mu\nu}$, φ are treated in full nonlinear fashion in the interior of the spacetime, but they vanish on the boundary. We will not state definite conditions on the boundary asymptotics in this section, but we will be quite specific when we discuss the extension to AdS_d-sliced domain walls in Sec. IV.

The spinor formalism of Witten and Nester provides a generalized ''energy'' $E_{\rm WN}$ with the following properties.

- It computes a linear combination of the conserved Killing charges of the isometry group of the background, specifically the subalgebra contained in the SUSY anticommutator $\{Q,\bar{Q}\}$. Thus we expect to find spatial translations in d dimensions for flat walls and the charges of the algebra $\mathrm{SO}(d-1,2)$ for the AdS_d-sliced walls we treat in Sec. IV.
- The charges vanish for the background solution under study, i.e., when $h_{\mu\nu}$ and φ vanish.
- The energy, defined with respect to the Killing vector $\partial/\partial t$ of the background, is positive for all fluctuations which are suitably damped on the boundary.

A. The Witten-Nester energy and positivity

To define $E_{\rm WN}$, we consider a d-dimensional spacelike surface Σ, which can be thought of as the initial value surface for the Cauchy problem of the deformed domain wall spacetime. Denote its boundary by $\partial\Sigma$. Then $E_{\rm WN}$ is defined by the boundary integral

$$E_{\rm WN}=\int_{\partial\Sigma}*\hat{E} \tag{3.3}$$

[4]We are very grateful to Gary Gibbons for patient and useful discussions of the geometry and its conformal compactification.

of the Hodge dual of the Nester two-form $\hat{E} = \frac{1}{2}\hat{E}_{\mu\nu}dx^{\mu}dx^{\nu}$, defined by

$$\hat{E}^{\mu\nu} = \overline{\varepsilon_1}\Gamma^{\mu\nu\rho}\hat{\nabla}_{\rho}\varepsilon_2 - \overline{\hat{\nabla}_{\rho}\varepsilon_2}\Gamma^{\mu\nu\rho}\varepsilon_1. \tag{3.4}$$

The covariant derivative is

$$\hat{\nabla}_{\mu} = \nabla_{\mu} + W(\phi)\Gamma_{\mu}, \tag{3.5}$$

where $W(\phi)$ is any function of ϕ which satisfies

$$W(\phi) \underset{\phi\to\bar{\phi}}{\sim} \frac{1}{2L} + O(\varphi^2). \tag{3.6}$$

The value of the integral over $\partial\Sigma$ depends only on the behavior of spinors $\varepsilon_1(x)$, $\varepsilon_2(x)$ near the boundary. Thus, at this stage, the spinors can be arbitrary in the interior, but must approach a background Killing spinor on the boundary. In general one must take independent spinors $\varepsilon_1(x)$, $\varepsilon_2(x)$ in order that E_{WN} contain the full set of background charges.

As in other treatments of gravitational energy, the surface integral form of E_{WN} is linear in the fluctuations $h_{\mu\nu}$, φ and thus not manifestly positive. To establish positivity we will use Stokes' theorem to rewrite Eq. (3.3) as an integral over Σ and then impose more specific conditions on $W(\phi)$ and $\varepsilon_i(x)$. Stokes' theorem gives

$$E_{\mathrm{WN}} = \int_{\Sigma} d\Sigma_{\mu}\nabla_{\nu}\hat{E}^{\mu\nu}. \tag{3.7}$$

Note that this step requires that the background and deformed solutions are nonsingular. There would otherwise be an additional surface contribution from the singularity or the horizon which shields it. There are known methods [20] to extend the treatment to include horizons, but naked singularities present substantial new problems which are beyond the scope of the present work. One should note also that the effective current $J^{\mu} = \nabla_{\nu}E^{\mu\nu}$ is identically conserved, so E_{WN} defines a conserved quantity provided that boundary asymptotics of the integrand is suitably restricted.

We now take $\varepsilon_1 = \varepsilon_2$ in Eq. (3.4) because we are interested in demonstrating positivity. The integrand of Eq. (3.7) may now be manipulated as in Ref. [5] using the equations of motion (2.2) and regrouping of terms to obtain

$$\begin{aligned} E_{\mathrm{WN}} = \int_{\Sigma} d\Sigma_{\mu}\Bigg\{ & 2\,\overline{\delta\psi_{\nu}}\Gamma^{\mu\nu\rho}\delta\psi_{\rho} - \frac{\kappa^2}{2}\overline{\delta\chi}\Gamma^{\mu}\delta\chi \\ & + \bar{\varepsilon}\Gamma^{\mu}\varepsilon\left[-\kappa^2 V(\phi) + 2(d-1)^2\left(\frac{1}{\kappa^2}W'^2 - \frac{d}{d-1}\right)\right]\Bigg\}, \end{aligned} \tag{3.8}$$

where $W' = \partial_{\phi}W$ and we have *defined*

$$\delta\psi_{\mu} = \hat{\nabla}_{\mu}\varepsilon, \tag{3.9}$$

$$\delta\chi = \left(\Gamma^{\mu}\nabla_{\mu}\phi - \frac{2(d-1)}{\kappa^2}W'\right)\varepsilon. \tag{3.10}$$

The condition that E_{WN} vanish in the undeformed background is satisfied if we impose the following conditions.

The last term in Eq. (3.8) is cancelled, both with and without fluctuations, if we require that

$$V(\phi) = \frac{2(d-1)^2}{\kappa^2}\left(\frac{1}{\kappa^2}W'^2 - \frac{d}{d-1}W^2\right). \tag{3.11}$$

We further require that (with no fluctuations)

$$[\overline{\nabla}_{\mu} + \overline{\Gamma}_{\mu}W(\phi)]\varepsilon = 0, \tag{3.12}$$

$$\left(\overline{\Gamma}^{\mu}\overline{\nabla}_{\mu}\bar{\phi} - \frac{2(d-1)}{\kappa^2}W'(\bar{\phi})\right)\varepsilon = 0. \tag{3.13}$$

An "overbar" on any quantity indicates that it is to be evaluated in the background geometry.

The integrability conditions of the equations (3.12), expressed in the coordinates of Eq. (1.1), give the first order flow equations (2.5), (2.6) [8]. The spinor solutions of Eq. (3.12) are the background Killing spinors

$$\varepsilon = e^{A(r)/2}\varepsilon_0,$$

$$\Gamma^{\hat{r}}\varepsilon_0 = \varepsilon_0, \tag{3.14}$$

where ε_0 is a constant spinor which is chiral with respect to the radial component of Γ^a.

The superpotential $W(\phi)$ must satisfy Eq. (3.11) and the boundary condition (3.6). This guarantees that the scalar profile $\bar{\phi}(r)$ and scale factor $A(r)$ obtained from Eqs. (2.6), (2.5) are also solutions of the field equations (2.4).

Positivity with fluctuations is now a relatively simple matter, since there are only two terms left in Eq. (3.8) and the second one is manifestly positive. We still have the freedom to modify the definition of the spinor $\varepsilon(x)$ for deformed solutions, and we impose the Witten condition

$$\Gamma^{k}\nabla_{k}\varepsilon(x) = 0, \tag{3.15}$$

where the time coordinate is omitted in the sum over k. One must choose a solution which approaches an arbitrary background Killing spinor on the boundary. We do not discuss the existence of Witten spinors here. In a frame where $E^{\hat{t}}$ is orthogonal to the surface E_{WN} reduces to the positive semidefinite form

$$E_{\mathrm{WN}} = \int_{\Sigma} d^{d-1}xe\left[2(\hat{\nabla}^{k}\varepsilon)^{\dagger}\hat{\nabla}_{k}\varepsilon + \frac{\kappa^2}{2}\delta\chi^{\dagger}\delta\chi\right]. \tag{3.16}$$

This energy functional vanishes if and only if

$$\hat{\nabla}_{k}\varepsilon = 0, \quad \delta\chi = 0. \tag{3.17}$$

The set of solutions to these equations is given by the space of solutions of the first order equations (2.5), (2.6). If we impose that the solution should satisfy the boundary condition set by the undeformed solution, then $E_{\rm WN}$ vanishes only for the undeformed background. The existence of other solution that have zero energy but different boundary conditions may be considered as an indication of marginal stability, but it is unclear whether we should allow such configurations. We leave this issue open.

We can now state a sufficient condition for the stability of a domain wall of the form (1.1) which is an asymptotically AdS solution of Eq. (2.4) and involves a scalar whose mass satisfies $m_{\rm BF}^2 \equiv -d^2/4 \leqslant m^2 \leqslant 0$. The scalar profile then satisfies $\bar{\phi}'=0$ on the boundary. If there is a superpotential $W(\phi)$, satisfying Eq. (3.11), such that the domain wall is a solution of Eqs. (2.6), (2.5), then it is stable. It then follows from Eq. (2.6) that W' vanishes on the boundary. For analytic $W(\phi)$ this is equivalent to Eq. (3.6). It is not guaranteed that the required adapted superpotential exist. As discussed at the end of Sec. II A, if $\bar{\phi}(r)$ is monotonic, then $W(\phi)$ is defined implicitly. If it does not exist, then one may suspect instability, but instability does not follow from this framework.

The roots of the argument above lie in supergravity, as the matrix structure of the Nester two-form (3.3) and the form of Eqs. (3.9), (3.10) clearly show. But the argument can be applied to any model of gravity and scalar fields, in any space-time dimension, provided that the required adapted superpotential exists.

B. $E_{\rm WN}$ and conserved charges

Our next goal is to obtain a concrete formula for the boundary integral form of $E_{\rm WN}$ and to show that it indeed gives a combination of the translation Killing charges of flat domain walls. Because we work at the boundary, linearized expressions for the connection and frames of the deformed metric are appropriate. Linearization in the scalar fluctuation is valid for single-scalar models where the scalar mass satisfies $m^2 > m_{\rm BF}^2$. However, terms of order φ^2 can contribute to $E_{\rm WN}$ when the scalar mass saturates the BF bound [21] and in other situations. An example was recently discussed in Ref. [22].

Let $\bar{E}_\mu^a$ denote a vielbein of the background metric in Eq. (3.1). The linearized spin connection is then given by

$$\delta\omega_{\mu ab} = \frac{1}{2}[\bar{E}_a^\nu \nabla_b h_{\mu\nu} - \bar{E}_b^\nu \nabla_a h_{\mu\nu}], \tag{3.18}$$

where ∇ is a background covariant derivative. It is most convenient to use background Killing spinors to compute $E_{\rm WN}$. We insert Eq. (3.18) in Eq. (3.3) and obtain, using Eq. (3.12) and some Dirac algebra,

$$\begin{aligned} E_{\rm WN} = & -\frac{1}{8}\int_{\partial\Sigma}\{\overline{\varepsilon_1}\Gamma_\mu\varepsilon_2[\nabla^\rho h_{\rho\nu} - \nabla_\nu(g^{\rho\sigma}h_{\rho\sigma})] \\ & -\overline{\varepsilon_1}\Gamma_\nu\varepsilon_2[\nabla^\rho h_{\rho\mu} - \nabla_\mu(g^{\rho\sigma}h_{\rho\sigma})] \\ & +\overline{\varepsilon_1}\Gamma^\rho(\nabla_\nu h_{\rho\mu} - \nabla_\mu h_{\rho\nu})\} d\Sigma^{\mu\nu} + \frac{1}{4}\int_{\partial\Sigma}[\overline{\varepsilon_1}(g^{\mu\sigma}\Gamma^{\nu\rho} \\ & + g^{\nu\sigma}\Gamma^{\rho\mu} + g^{\rho\sigma}\Gamma^{\mu\nu})h_{\rho\sigma}W(\bar{\phi})\varepsilon_2] d\Sigma_{\mu\nu} \\ & + \frac{d-1}{2}\int_{\partial\Sigma}\overline{\varepsilon_1}\Gamma^{\mu\nu}W'(\bar{\phi})\varepsilon_2\varphi d\Sigma_{\mu\nu} + \text{H.c.} \end{aligned} \tag{3.19}$$

All quantities in this equation, except $h_{\mu\nu}$ and φ refer to the background. Our computation used only the general background-fluctuation split in Eq. (3.1). It is thus valid both for flat domain walls and for other situations in which the Witten-Nester approach to stability has been applied. For example, it is applicable to asymptotically flat metrics in which $W(\bar{\phi})$ and $W'(\bar{\phi})$ vanish. In this case it is quite straightforward to show that Eq. (3.19) yields the same expressions for energy and momentum given in Eq. (70) of Ref. [1].

Let us discuss the formula (3.19) in more detail for flat domain walls. First we find from Eq. (3.14) that the bilinears $\overline{\varepsilon_1}\Gamma^\mu\varepsilon_2$ do span the expected set of translation Killing vectors.[5] However, the role of tensor bilinears $\overline{\varepsilon_1}\Gamma^{\mu\nu}\varepsilon_2$ is far from clear. To discuss them, we distinguish between components $\overline{\varepsilon_1}\Gamma^{ri}\varepsilon_2$ with one radial index, and components $\overline{\varepsilon_1}\Gamma^{ij}\varepsilon_2$ with both indices along the domain wall. The latter vanish due to the chirality properties of the Killing spinors (3.14), while the former are proportional to translation Killing vectors. Thus $E_{\rm WN}$ indeed produces a combination of the translation Killing charges of the deformed domain wall. We note further that $W'(\bar{\phi})$ vanishes, so that the last term in Eq. (3.19) is absent for flat domain walls.

As a final check let us note that the boundary volume element has components $d\Sigma_{tr}$ where t is the time coordinate of Eq. (1.1). We now use radial coordinate r for which $h(r)=0$ in Eq. (1.1). In that case, $A(r) \sim r/L$ at the boundary. It is also known that normalizable metric fluctuations vanish at the rate $h_{\mu\nu} \sim \exp(-dr/L)$. Putting things together we see that the terms in the first three lines of Eq. (3.19) are generically finite on the boundary.

We conclude this section with an illustration of one of the subtleties of the argument, namely, that the existence of an adapted superpotential satisfying Eq. (3.11) is not sufficient to guarantee stability. In addition one needs Eq. (3.6) which implies that the AdS critical point of the potential V is also a critical point of W. To illustrate this issue we consider the following superpotential:

$$W = w_0 + w_1\phi + \frac{d}{2(d-1)}\kappa^2 w_0\phi^2 + w_3\phi^3. \tag{3.20}$$

[5]The case $d=2$ is exceptional. Due to chirality, the Killing spinors have effectively only one component, so $\overline{\varepsilon_1}\Gamma^\mu\varepsilon_2$ has vanishing spatial component and gives only the time translation or energy Killing vector.

The corresponding potential from Eq. (3.11) is

$$\kappa^2 V(\phi)=2(d-1)^2\left(\frac{w_1^2}{\kappa^2}-\frac{d}{(d-1)}w_0^2\right)-2(d-1)\left(dw_1^2-\frac{6}{\kappa^2}w_1w_3(d-1)\right)\phi^2+O(\phi^3). \tag{3.21}$$

This potential has a critical point at $\phi=0$ which is AdS provided

$$w_1^2<\frac{d}{(d-1)}\kappa^2 w_0^2. \tag{3.22}$$

This critical point however is not a critical point of W. If the product w_1w_3 is sufficiently large, the mass of the scalar lies below m_{BF}^2 and the perturbative argument [21] for instability applies. We may apply the Witten-Nester argument to investigate stability of the AdS solution of the theory (2.1) with potential above. The argument does not apply if one uses the covariant derivative (3.5) with W above because AdS spacetime is not a solution[6] of the flow equations (2.6), (2.5). Nor can there be any other superpotential, satisfying both Eqs. (3.11) and (3.6) because it is known [5] that this implies that $m^2\geqslant m_{\mathrm{BF}}^2$. Thus the perturbative and nonperturbative analysis are compatible. This example illustrates the importance of the condition (3.6) for stability.

IV. STABILITY OF AdS$_d$ DOMAIN WALLS

In this section we extend the argument of Sec. III to cover AdS$_d$-sliced domain walls. The springboard for our approach was the study of AdS$_4$-sliced walls in genuine $D=5$, $N=2$ supergravity in Ref. [9]. The natural spinors in this theory are a symplectic-Majorana doublet, and the superpotential appears as the su(2)-valued matrix $\mathbf{W}(\phi)=W_a(\phi)\tau^a$, where the τ^a are the three Pauli matrices. In genuine $D=5$, $N=2$ supergravity, the matrix superpotential is determined by the gaugings of R symmetry and isometries of the internal geometry [23–25]. The internal space is the product of a very special manifold (for scalars in vector and tensor multiplets) and a quaternionic manifold (for scalars in hypermultiplets). The superpotential is given by the product of the embedding coordinates h^I of the very special manifold and a triplet of Killing prepotentials P_{Iij} depending on the scalars of the hypermultiplets. In the absence of hypermultiplets, a matrix superpotential is still possible[7] and it is determined in terms of Fayet-Iliopoulos constants and the h^I.

None of this technical detail need concern us in fake supergravity, which works in any dimension and with any number of real scalars. We simply double the spinors used in Sec. III, taking ϵ^α, $\alpha=1$, 2 as a pair of Dirac spinors in dimension $d+1$. The matrix $\mathbf{W}(\phi)$ acts on the index α, but we can usually suppress it in explicit formulas. Many previous formulas remain valid when understood as extensions to the doubled spin space, with the replacement $W(\phi)\to\mathbf{W}(\phi)$. Note that quadratic quantities such as $\mathbf{W}^2$ and $\{\mathbf{W},\mathbf{W}'\}$ are proportional to the unit matrix. When they appear in our equations below they should be interpreted as scalar valued.

The energy of any perturbation of an AdS$_d$-sliced wall is contained in the Nester two-form (3.4) with an su(2) extension of the covariant derivative (3.5). All formal manipulations which lead to the volume form (3.8) of the energy also have obvious su(2) extensions. With an su(2)-extended Witten spinor (3.15), the energy becomes manifestly nonnegative. The nontrivial task now is to establish the consistency of the formalism by showing that there are fake Killing spinors so that the energy vanishes for domain wall backgrounds of the form (1.2). We use the frames and spin connections given in Appendix A.

A. Killing spinor consistency conditions and the new flow equations

The su(2) extension of the argument of Sec. III requires that the fake Killing spinors satisfy the following conditions:[8]

$$\left[\nabla_i^{\mathrm{AdS}_d}+\Gamma_i\left(\frac{1}{2}A'e^{-h}\Gamma^{\hat r}+\mathbf{W}\right)\right]\varepsilon=0, \tag{4.1}$$

$$[\partial_r+\Gamma^{\hat r}e^h\mathbf{W}]\varepsilon=0, \tag{4.2}$$

$$\left[\Gamma^{\hat r}e^{-h}\phi'-\frac{2(d-1)}{\kappa^2}\mathbf{W}'\right]=0. \tag{4.3}$$

In addition $\mathbf{W}(\phi)$ must be related to the potential $V(\phi)$ by

$$\kappa^2V(\phi)=2(d-1)^2\left(\frac{1}{\kappa^2}\mathbf{W}'^2-\frac{d}{d-1}\mathbf{W}^2\right). \tag{4.4}$$

In Eq. (4.1), the covariant derivative contains the connection of an AdS$_d$ metric with scale L_d.

We now extract the integrability/consistency conditions for Eqs. (4.1)–(4.3) and show that they imply that the background metric and scalar satisfy the original Euler-Lagrange equations (2.11). We also obtain a constraint on $\mathbf{W}(\phi)$.

Consider first the fake dilatino condition (4.3) which can be rewritten as the chirality condition

$$\Gamma^{\hat r}\varepsilon=\frac{2(d-1)}{\kappa^2}e^h\frac{\mathbf{W}'}{\phi'}\varepsilon \tag{4.5}$$

on fake Killing spinors. The square of this gives the scalar condition

[6]A preliminary study indicates that the flow equations can be integrated, but give a pathological geometry.

[7]We thank Antoine Van Proeyen for correspondence on this issue.

[8]As in Sec. III, $\mathbf{W}'$ and $\mathbf{W}''$ denote derivatives with respect to ϕ.

$$\phi'^2 - \left(\frac{2(d-1)}{\kappa^2}\right)^2 e^{2h}\mathbf{W}'^2 = 0, \tag{4.6}$$

which shows that the matrix on the right side of Eq. (4.5) has eigenvalues ± 1, as required for the consistency of Eq. (4.5).

The integrability condition for Eq. (4.1) is

$$\frac{1}{L_d^2} + A'^2 e^{2A-2h} - 4e^{2A}\mathbf{W}^2 = 0, \tag{4.7}$$

while the compatibility of Eqs. (4.1) and (4.3) requires [after use of Eq. (4.5)]

$$A'\phi' + \frac{2(d-1)}{\kappa^2} e^{2h}\{\mathbf{W},\mathbf{W}'\} = 0. \tag{4.8}$$

The mutual integrability condition for Eqs. (4.1), (4.2) directly gives the $A'' - A'h'$ field equation of Eq. (2.11) after Eqs. (4.6) and (4.8) are used. The remaining compatibility condition between Eqs. (4.2), (4.3) will be discussed below. It is an important constraint on $\mathbf{W}(\phi)$.

We can now easily recover the other equations of motion in Eq. (2.11). First we combine Eqs. (4.6) and (4.7) and use Eq. (4.4) to obtain the A'^2 equation from Eq. (2.11). Next we take the r derivative of Eq. (4.6) and find

$$\phi'' - h'\phi' = \frac{2(d-1)^2}{\kappa^4} e^{2h}\{\mathbf{W}',\mathbf{W}''\}. \tag{4.9}$$

The sum of this plus d times Eq. (4.8) yields exactly the scalar equation in Eq. (2.11). Our formalism is thus consistent with the field equations of AdS_d-sliced domain walls.

The next step is to extract from the information above a small set of equations which determine $\phi(r)$, $A(r)$. The first equation is just the square root of Eq. (4.6) with the sign chosen to make $\phi(r)$ monotonically increasing:

$$\phi'(r) = \frac{2(d-1)}{\kappa^2} e^h \sqrt{\mathbf{W}'^2}. \tag{4.10}$$

The second equation is a purely algebraic equation for $A(r)$, obtained by equating the expressions for A'^2 obtained from Eq. (4.7) and from Eqs. (4.8), (4.10):

$$\frac{e^{-2A}}{L_d^2} = \frac{4\mathbf{W}^2\mathbf{W}'^2 - \{\mathbf{W},\mathbf{W}'\}^2}{\mathbf{W}'^2}. \tag{4.11}$$

The right side is non-negative by the Schwarz inequality.

We now show that Eqs. (4.10), (4.11) are equivalent to the first order set (2.12) provided that $\mathbf{W}$ satisfies Eq. (4.4) and a further condition given below. This then guarantees that the new system gives a solution of the original field equations (2.11). In making the comparison with Eq. (2.12), we interpret $W = \sqrt{\mathbf{W}^2}$ and $W' = (d/d\phi)W$. First we must require that the relations between $\mathbf{W}$ and the potential $V(\phi)$ in Eqs. (2.11) and (4.4) are equivalent. Thus we identify

$$\gamma^2 = \frac{\{\mathbf{W},\mathbf{W}'\}^2}{4\mathbf{W}^2\mathbf{W}'^2}. \tag{4.12}$$

The algebraic equation (4.11) then implies Eq. (2.13).[9] This also shows that Eq. (4.10) is equivalent to the ϕ' equation in Eq. (2.12).

It is also easy to obtain the A' equation in Eq. (2.12). Substitute Eq. (4.10) into Eq. (4.8) which gives

$$A' = -\frac{\{\mathbf{W},\mathbf{W}'\}}{\sqrt{\mathbf{W}'^2}} e^h. \tag{4.13}$$

We then use Eq. (4.12) to recover the form in Eq. (2.12). However, there is a subtlety here. Namely, Eq. (4.13) is compatible with the expression for A' obtained from the logarithmic derivative of Eq. (4.11) combined with Eq. (4.10) only if $\mathbf{W}(\phi)$ satisfies the constraint

$$\frac{\mathrm{Tr}\mathbf{W}\cdot\mathbf{W}'\,\mathrm{Tr}\mathbf{W}'\cdot\mathbf{W}'' - \mathrm{Tr}\mathbf{W}'^2\,\mathrm{Tr}\mathbf{W}\cdot\mathbf{W}''}{\mathrm{Tr}\mathbf{W}^2\,\mathrm{Tr}\mathbf{W}'^2 - (\mathrm{Tr}\mathbf{W}\cdot\mathbf{W}')^2} = \frac{\kappa^2}{d-1}. \tag{4.14}$$

The compatibility condition between Eqs. (4.2) and (4.3) provides a simple direct constraint on the superpotential $\mathbf{W}(\phi)$ which supersedes Eq. (4.14). After use of Eqs. (4.9) and (4.5), we find that $\mathbf{W}$ must obey the following consistency condition:

$$\left[\mathbf{W}', \frac{d-1}{\kappa^2}\mathbf{W}'' + \mathbf{W}\right] = 0. \tag{4.15}$$

This condition, which must hold for any potential, is a necessary condition for the existence of fake Killing spinors and will be important in their construction below.

Since the Cartan subalgebra of su(2) is one dimensional,

$$\mathbf{W}'' = \alpha(\phi)\mathbf{W}' - \frac{\kappa^2}{d-1}\mathbf{W}, \tag{4.16}$$

where $\alpha(\phi)$ is a real function of the scalar field. One can see that Eq. (4.14) is trivially satisfied if Eq. (4.16) is inserted. By taking the anticommutator of both sides of Eq. (4.16) with $\mathbf{W}'$, one finds that

$$\alpha(\phi) = \frac{\kappa^2}{2(d-1)\mathbf{W}'^2}\left[(d+1)\{\mathbf{W},\mathbf{W}'\} + \frac{\kappa^2}{2(d-1)}\frac{\partial V}{\partial\phi}\right]. \tag{4.17}$$

Equation (4.16) implies that the matrix $\mathbf{W}''$ lies in the vector space spanned by matrices $\mathbf{W}$ and $\mathbf{W}'$. Taking further derivatives one can see that actually all derivatives lie in the same two-dimensional vector space. Thus, assuming analyt-

[9]For the Janus solution discussed further in Sec. IV C the factor γ appearing in Eq. (2.13) vanishes at $r=0$ and has to be therefore extended as an odd function to negative r. This amounts to setting $\gamma = -\{\mathbf{W},\mathbf{W}'\}/2\sqrt{\mathbf{W}^2\mathbf{W}'^2}$.

icity, the superpotential $\mathbf{W}(\phi)$ remains in a fixed subspace for all values of ϕ. This allows us to make the convenient gauge choice

$$\mathbf{W}=\begin{pmatrix} 0 & \bar{\omega} \\ \omega & 0 \end{pmatrix}. \tag{4.18}$$

In this gauge the consistency condition (4.15) reduces to

$$\frac{\bar{\omega}'\omega''-\omega'\bar{\omega}''}{\bar{\omega}\omega'-\omega\bar{\omega}'}=\frac{\kappa^2}{d-1}. \tag{4.19}$$

It is quite remarkable that we have replaced the system (2.12) by the simpler set (4.10), (4.11) in which only one integration is required given the superpotential $\mathbf{W}(\phi)$. However, the conditions (4.4), (4.16), (4.17) which determine $\mathbf{W}(\phi)$ from $V(\phi)$ are not necessarily easy to solve, as we discuss below. It appears possible to shift the strategy as follows. First obtain a superpotential which satisfies Eq. (4.16) and use Eq. (4.17) to define a potential. The AdS_d-sliced domain wall then obtained from Eqs. (4.10), (4.11) will be stable.

We may summarize the results above as follows. If the matrix superpotential $\mathbf{W}(\phi)$ satisfies Eqs. (4.15) and (4.4), then any solution of Eqs. (4.10), (4.11) satisfies the field equations (2.11) for AdS_d-sliced domain walls. The Killing spinor equations (4.1)–(4.3) are then mutually consistent and we should be able to find the Killing spinors.

B. Explicit Killing spinors

Let ε_K denote a conventional Killing spinor of AdS_d which satisfies[10]

$$\left[\nabla_i^{\mathrm{AdS}_d}+\frac{1}{2L_d}\Gamma_i\right]\varepsilon_K=0. \tag{4.20}$$

For $d=4$ there are eight independent ε_K. For each independent ε_K, there is an su(2) fake Killing spinor of the form

$$\varepsilon=e^{(1/2)A}\begin{pmatrix} \frac{\bar{\omega}'}{\omega'} & 0 \\ 0 & \frac{\omega'}{\bar{\omega}'} \end{pmatrix}^{1/4}\begin{pmatrix} (i+\Gamma^{\hat{r}})\varepsilon_K \\ -(1+i\Gamma^{\hat{r}})\varepsilon_K \end{pmatrix}. \tag{4.21}$$

One can check directly that the defining conditions (4.1)–(4.3) are satisfied. For this purpose one needs the following formulas:

$$\frac{2A'}{\phi'}=-\frac{\kappa^2}{d-1}\left(\frac{\omega}{\omega'}+\frac{\bar{\omega}}{\bar{\omega}'}\right), \tag{4.22}$$

$$\partial_\phi\left(\log\frac{\omega'}{\bar{\omega}'}\right)=-\frac{\kappa^2}{d-1}\left(\frac{\omega}{\omega'}-\frac{\bar{\omega}}{\bar{\omega}'}\right), \tag{4.23}$$

$$0=-i\frac{e^{-A}}{L_d}+A'e^{-h}-2\bar{\omega}\sqrt{\frac{\omega'}{\bar{\omega}'}} \tag{4.24}$$

which follow easily from Eqs. (4.6)–(4.8) and (4.19). Note that the prime on ω and $\bar{\omega}$ means a derivative with respect to ϕ, whereas the prime on A or ϕ means a derivative with respect to r. The fake Killing spinor bilinears $\bar{\varepsilon}_1\gamma^i\varepsilon_2$ (with $\varepsilon_1\neq\varepsilon_2$) span the set of Killing vectors of the AdS_d isometry group $\mathrm{SO}(d-1,2)$, as they should.

C. W(ϕ) for the Janus solution

In this subsection we analyze the conditions which determine $\mathbf{W}(\phi)$ in more detail and show that there is a solution which generates the solution of Ref. [12] and thus establishes its stability. Inserting the ansatz

$$\omega(\phi)=w(\phi)e^{i\theta(\phi)} \tag{4.25}$$

into Eqs. (4.19) and (4.4) one finds

$$w'^2+w^2\theta'^2-\frac{d\kappa^2}{d-1}w^2=\frac{\kappa^4}{2(d-1)^2}V(\phi),$$

$$\frac{2w'^2}{w^2}+\frac{\theta''}{\theta'}\frac{w'}{w}-\frac{w''}{w}+\theta'^2=\frac{\kappa^2}{d-1}\quad\text{or}\quad\theta'=0. \tag{4.26}$$

Eliminating θ from the system of equations we find

$$XX''-\frac{d+1}{2d}X'^2+\frac{d+2}{2d}\kappa^2V'X'-\kappa^2\left(V''+\frac{2\kappa^2}{d-1}V\right)X$$
$$-2\kappa^2X^2=\frac{\kappa^4}{2d}V'^2, \tag{4.27}$$

where we have introduced

$$X(\phi)=2d(d-1)w^2+\kappa^2V. \tag{4.28}$$

For a *constant* potential $V=V_0$ this is an autonomous differential equation which can be solved by standard methods.[11] One takes as a new independent variable X and new dependent variable $u=X'$. Then using $d/d\phi=u(d/dX)$ we find a first order linear ordinary differential equation (ODE) for u^2,

$$\frac{X}{2}\frac{d}{dX}(u^2)-\frac{d+1}{2d}u^2-\frac{2\kappa^4}{d-1}V_0X-2\kappa^2X^2=0. \tag{4.29}$$

Solving this equation and passing back to the original variables we find $X(\phi)$ defined implicitly by

[10]In this equation $\Gamma_i=\bar{e}_{i\hat{a}}\Gamma^{\hat{a}}$ is an AdS_d gamma matrix.

[11]For an exponential potential, we can write $X=VY$ and again obtain an autonomous equation for Y.

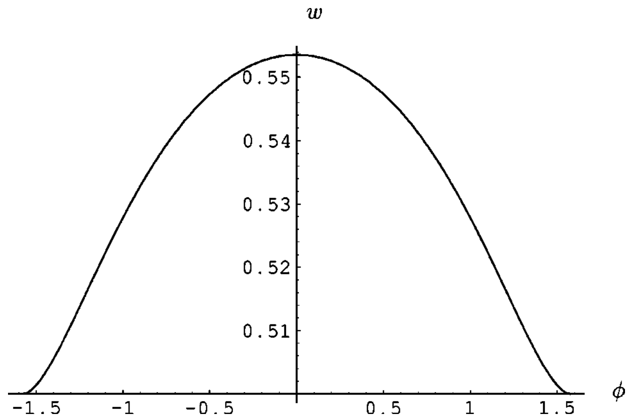

FIG. 2. Plot of the magnitude $w(\phi)$ for $d=4$, $L=1$, and $b=0.1$.

$$\sqrt{\frac{d-1}{4d\kappa^2}}\int_0^X \frac{dx}{\sqrt{x^2-\kappa^2 V_0 x-\beta x^{(d+1)/d}}}=\phi_\infty-\phi, \tag{4.30}$$

where ϕ_∞ is the boundary value of the field at $r=+\infty$. We have fixed the shift invariance of Eq. (4.29) by requiring that $X(\phi=0)=x_{\mathrm{min}}$, where x_{min} is the smallest positive root of the denominator in Eq. (4.30). Equation (4.30) thus defines $X(\phi)$ for $\phi\geqslant 0$ only. It can be continued, however, as an even C^∞ function to negative ϕ.

As we shall show below the integration constant β is related to the parameter b of the Janus solution by Eq. (4.40). Once we have obtained the magnitude w we can find the phase θ simply by an integration

$$\theta_\infty-\theta=\frac{\sqrt{\beta}}{2}\int_0^X \frac{dx\,x^{(d+1)/2d}}{(x-\kappa^2 V_0)\sqrt{x^2-\kappa^2 V_0 x-\beta x^{(d+1)/d}}}. \tag{4.31}$$

From Eqs. (4.28), (4.30), (4.31) one can find the behavior of the superpotential as $\phi\to\phi_\infty$, namely,

$$X\simeq\frac{\kappa^2 d^2}{2L^2}(\phi-\phi_\infty)^2,$$

$$w\simeq\frac{1}{2L}+\frac{\kappa^2 d}{4L(d-1)}(\phi-\phi_\infty)^2,$$

$$\theta\simeq\theta_\infty-\frac{\sqrt{\beta}d}{2d+1}\left(\frac{2L^2}{d(d-1)}\right)^{3/2}\left(\frac{\kappa^2 d^2}{2L^2}\right)^{1+1/2d}|\phi-\phi_\infty|^{2+1/d}. \tag{4.32}$$

Plots of the magnitude and the phase of the superpotential are shown in Figs. 2 and 3.

Let us now demonstrate that the above fake superpotential does indeed generate the Janus solution. From the definition (4.28) and the relation (4.4) we find easily

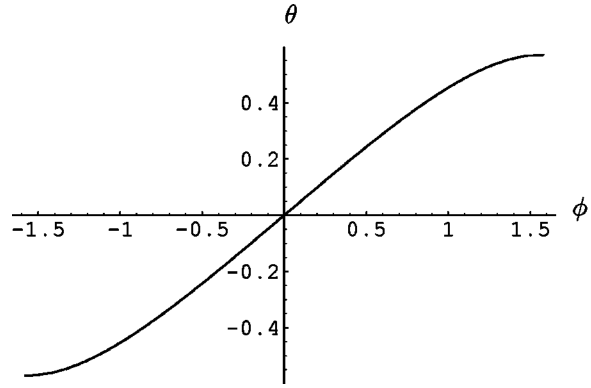

FIG. 3. Plot of the phase $\theta(\phi)$ for $d=4$ and $b=0.1$.

$$\mathbf{W}^2=\frac{1}{2d(d-1)}X+\frac{1}{4L^2}, \tag{4.33}$$

$$\mathbf{W}'^2=\frac{\kappa^2}{2(d-1)^2}X, \tag{4.34}$$

$$\{\mathbf{W},\mathbf{W}'\}=\frac{X'}{2d(d-1)}. \tag{4.35}$$

The scale factor can then be calculated from Eqs. (4.11) and (4.30):

$$e^{-2A}=L_d^2\frac{2\beta}{d(d-1)}X^{1/d}. \tag{4.36}$$

To facilitate the comparison let us choose a coordinate in which the dilaton is linear in the coordinate r. In particular we take $\phi(r)=r/\kappa L$. Clearly this can be achieved for the Janus solution since the dilaton is a monotonic function of the radial variable. Using $\phi'=1/\kappa L$ we find using Eqs. (4.10) and (4.34)

$$e^{-2h}=2L^2X. \tag{4.37}$$

From Eqs. (4.36) and (4.37) we see that $h_0\equiv h-dA$ is a constant and is given by

$$e^{2h_0}=\frac{1}{2L^2}\left(\frac{2\beta L_d^2}{d(d-1)}\right)^d. \tag{4.38}$$

Taking a logarithmic derivative of Eq. (4.36) we find

$$\begin{aligned}A'&=-\frac{1}{2d}\frac{X'}{X}\phi'\\&=\frac{1}{L}\sqrt{\frac{1}{d(d-1)}+e^{2dA+2h_0}-\left(\frac{L}{L_d}\right)^2 e^{2(d-1)A+2h_0}}.\end{aligned} \tag{4.39}$$

Comparing this first order ODE with the equation obeyed by the Janus solution following from Eq. (2.11) in the same linear dilaton coordinates, we see that they are indeed the same provided we identify

$$\beta = \frac{d(d-1)}{2L_d^2}\left(\frac{2L^2}{bd(d-1)}\right)^{1/d}. \tag{4.40}$$

Note that the coordinate independent definition of b is

$$b = \frac{\kappa^2 L^2}{d(d-1)}\phi'^2 e^{2dA-2h}, \tag{4.41}$$

which is indeed a space-time constant as follows from the equation of motion in Eq. (2.11).

Finally let us mention, that in addition to the Janus solution, there are other simpler solutions to the equations (4.26). In particular there are two solutions with constant magnitude

$$X=0, \quad w^2=\frac{1}{4L^2}, \quad \theta(\phi)=\text{const},$$

$$X=\frac{d}{2L^2}, \quad w^2=\frac{1}{4L^2}\frac{d}{d-1}, \quad \theta(\phi)=\pm\frac{\kappa}{\sqrt{d-1}}\phi. \tag{4.42}$$

The first solution is just the standard AdS_{d+1} space, whereas the second solution leads to an interesting linear dilaton background discussed further in Sec. VI. Equation (4.27) also admits a cosh-type solution

$$X=\frac{d(d-1)}{2L^2}\sinh^2\left(\kappa\sqrt{\frac{d}{d-1}}(\phi-\phi_0)\right),$$

$$w^2=\frac{1}{4L^2}\cosh^2\left(\kappa\sqrt{\frac{d}{d-1}}(\phi-\phi_0)\right),$$

$$\theta=\theta_0. \tag{4.43}$$

However, Eqs. (4.10) and (4.13) then generate the singular profiles found for flat dilaton walls in Sec. II A. This case appears to be a degenerate limit of our equations, since the right-hand side of Eq. (4.11) vanishes, implying that $L_d \to\infty$.

D. E_{WN} for deformations of the Janus solution

We have demonstrated above the existence of an su(2) superpotential $\mathbf{W}(\phi)$ for which Eqs. (4.10), (4.11) generate the AdS_4-sliced domains wall of Ref. [12] and its d-dimensional generalizations. This means that these solutions enjoy non-perturbative gravitational stability with respect to fluctuations of the metric and dilaton. To complete the discussion we now show that the surface integral (3.19) form of E_{WN} is well defined on the boundary of the coordinate chart (2.20) in Sec. II. We specify the behavior of metric and dilaton perturbations, such that E_{WN} computes a finite linear combination of charges of the AdS_d isometry group.

The treatment of Sec. III applies with few changes to AdS_d-sliced domain walls. We consider perturbed solutions of the form (3.1) with background metric (2.20) and accompanying dilaton. The background frame forms are

$$E^{\hat\mu}=Le^{A(\mu)}d\mu, \tag{4.44}$$

$$E^{\hat\lambda}=\frac{Le^{A(\mu)}}{\cos\lambda}d\lambda, \tag{4.45}$$

$$E^{\hat t}=\frac{Le^{A(\mu)}}{\cos\lambda}dt, \tag{4.46}$$

$$E^{\hat a}=\frac{Le^{A(\mu)}\sin\lambda}{\cos\lambda}e^{\hat a}, \tag{4.47}$$

where $e^{\hat a}$ is a frame on S_{d-2}, $a=1,...,d-2$.

The boundary consists of the three components shown in Fig. 1(b).

(1) The portion at $\mu=-\mu_0$ with $0<\lambda<\pi/2-\delta$ and volume form

$$d\Sigma^{t\mu}=L^2e^{-2A(\mu)}\cos\lambda E^{\hat\lambda}\wedge E^{\hat 1}\wedge\cdots\wedge E^{\widehat{d-2}}, \tag{4.48}$$

where δ is a small positive number.

(2) The keyhole surrounding the corner on which $\lambda=\pi/2-\delta$ and $-\mu_0<\mu<\mu_0$ with volume form

$$d\Sigma^{t\lambda}=-L^2e^{-2A(\mu)}\cos^2\lambda E^{\hat\mu}\wedge E^{\hat 1}\wedge\cdots\wedge E^{\widehat{d-2}}. \tag{4.49}$$

(3) The portion $\mu=\mu_0$ with $0<\lambda<\pi/2-\delta$ and volume form (4.48).

An important change is that the Killing spinors to be used in Eq. (3.19) are those given in Eq. (4.21) in which we now replace $\Gamma^{\hat r}\to\Gamma^{\hat\mu}$ and define $\xi=(\omega'/\overline{\omega}')^{1/4}$. Now let Γ denote any matrix of the Dirac (Clifford) algebra in d dimensions. It is easy to compute the Killing spinor bilinears

$$\overline{\epsilon_1}\Gamma\epsilon_2=2e^A\overline{\xi}\xi\overline{\epsilon_{K1}}(\Gamma-\Gamma^{\hat\mu}\Gamma\Gamma^{\hat\mu})\epsilon_{K2}, \tag{4.50}$$

$$\overline{\epsilon_1}\Gamma\mathbf{W}\epsilon_2=2e^A\overline{\epsilon_{K1}}\{\mathrm{Re}(\xi^2\overline{\omega})[\Gamma^{\hat\mu},\Gamma] -\mathrm{Im}(\xi^2\overline{\omega})(\Gamma+\Gamma^{\hat\mu}\Gamma\Gamma^{\hat\mu})\}\epsilon_{K2}. \tag{4.51}$$

The first equation tells us that $\overline{\epsilon_1}\Gamma^\rho\epsilon_2$ is a Killing vector of the $(d+1)$-dimensional space-time with vanishing radial component $(\rho\to\mu)$. Transverse components $(\rho\to i, i=0,...,d-1)$ are proportional to $e^A\overline{\epsilon_{K1}}\Gamma^i\epsilon_{K2}$, which is an AdS_d Killing vector, and the full set of these is spanned as we vary ϵ_{K1}, ϵ_{K2}.

Let us look first at the last term of Eq. (3.19), which involves the tensor bilinear $\overline{\epsilon_1}\Gamma^{\nu\rho}\mathbf{W}'\epsilon_2$. The second equation in Eq. (4.50) applies if we change $\mathbf{W}\to\mathbf{W}'$ on both sides. The product $\xi^2\overline{\omega}'=\sqrt{\overline{\omega}'\omega'}$ is real, so only the commutator term in Eq. (4.50) contributes. On the keyhole part of the boundary, we find $[\Gamma^{\hat\mu},\Gamma^{t\lambda}]$ which vanishes. On the boundary components at $\mu=\pm\mu_0$, we find $[\Gamma^{\hat\mu},\Gamma^{t\mu}]=-2E^{\hat\mu\mu}\Gamma^t$. The tensor bilinear thus reduces to a multiple of the energy Killing vector. Thus the last term of Eq. (3.19) certainly vanishes on the keyhole, and we now show that it vanishes on the other two boundary components by examining the behavior of the integrand as $\mu \to \pm\mu_0$. We note the behavior $\sqrt{\mathbf{W}'^2} \sim \overline{\phi'} \sim e^{-dA(\mu)}$, which follows from Eq. (4.10) and the property of dilaton in the solution of Ref. [12] noted above our Eq. (2.16). Using Eqs. (4.44), (4.50), we find that the factor $\overline{\varepsilon_1}\Gamma_{t\mu}\mathbf{W}'\varepsilon_2 d\Sigma^{t\mu}$ is constant on the boundary. However, the normalizable dilaton fluctuation vanishes on the boundary at the rate $\varphi \sim (\mu \mp \mu_0)^d \sim e^{-2A(\mu)}$. Thus the last term of Eq. (3.19) vanishes for our dilaton domain walls.

Let us look next at the terms of Eq. (3.19) involving $\overline{\varepsilon_1}\Gamma^{\rho\sigma}\mathbf{W}\varepsilon_2 d\Sigma_{\nu\tau}$ with various index assignments. On the boundary components $\mu = \pm\mu_0$, the product $\xi^2\overline{\omega}$ is real, as follows from Eq. (4.24) or (4.32). Thus only the commutator term contributes in Eq. (4.50) and it is nonvanishing for index combinations $\Gamma^{\mu i}$ only. It then follows from Eqs. (4.44), (4.50) that $\overline{\varepsilon_1}\Gamma^{\mu i}\mathbf{W}\varepsilon_2$ vanishes as $e^{-A(\mu)}$, and is proportional to an AdS_d Killing vector. Clearly, $g^{\rho\sigma} \sim e^{-2A(\mu)}$. The volume element behaves as $d\Sigma_{t\mu} \sim e^{(d+1)A}$, while normalizable metric fluctuations vanish at the rate $h_{\rho\sigma} \sim e^{(2-d)A(\mu)}$. Putting these factors together, we see that the terms under consideration give a finite contribution to the energy of a deformed domain wall.

To analyze the behavior of the tensor bilinear terms on keyhole, we must take the limit $\delta \to 0$, which is the boundary limit $\cos(\lambda) \to 0$ on the AdS_d slices. We discuss this limit first for the bulk space-time AdS_{d+1} with AdS_d slicing and then adapt the argument to the dilaton domain wall geometry.

In Sec. II of Ref. [12], global metrics for AdS_{d+1} with both standard and AdS_d slicing are both derived from the embedded hyperboloid description $X_0^2 + X_{d+1}^2 - X_1^2 - \cdots X_d^2 = L^2$. The two metrics are

$$
\begin{aligned}
ds^2 &= \frac{L^2}{\cos^2\theta}(-dt^2 + d\theta^2 + \sin^2\theta\, d\Omega_{d-1}^2) \\
&= \frac{L^2}{\cos^2\mu\cos^2\lambda}(-dt^2 + \cos^2\lambda\, d\mu^2 + d\lambda^2 \\
&\quad + \sin^2\lambda\, d\Omega_{d-2}^2).
\end{aligned} \tag{4.52}
$$

Comparison of the conformal factors yields one relation between the two sets of coordinates, namely, $\cos\theta = \cos\mu\cos\lambda$. A normalizable mode of a scalar field transforming in a representation of the isometry group $SO(d,2)$ with lowest weight Δ of the $SO(2)$ generator (the energy) vanishes at the rate $(\cos\theta)^\Delta$ on the AdS_{d+1} boundary. When expressed in terms of the coordinates for AdS_d slicing it therefore vanishes at the rate $(\cos\lambda)^\Delta$ as $\lambda \to \pi/2$. For the massless dilaton $\Delta = d$. We need the corresponding result for metric fluctuations $h_{\mu\nu}$. In the "axial gauge" $h_{\mu\mu} = h_{\mu i} = 0$, h_{ij} is related by $h_{ij} = e^{2A}\tilde{h}_{ij}$ to the field $\tilde{h}_{ij}$, whose wave equation is the same as that of a massless scalar. Thus normalizable modes of $\tilde{h}_{ij} \sim (\cos\lambda)^d$.

We use this rate to obtain the behavior of the tensor terms of Eq. (3.19) as the keyhole boundary contribution shrinks toward the corner. We need the fact that the AdS_d Killing spinors behave as $\varepsilon_K \sim (\cos\lambda)^{1/2}$, and that the volume element behaves as $d\Sigma_{t\lambda} \sim (\cos\lambda)^{-d}$. It is convenient to work in the axial gauge. Detailed inspection of the various tensor components in Eq. (3.19) shows that they vanish at least as fast as $(\cos\lambda)^3$. The analysis so far is valid for AdS_{d+1}. However, the domain wall space-time shares the isometry $SO(d-1,2)$ and may be viewed as a small distortion of AdS_{d+1} when the parameter b of Eq. (2.14) is small. Therefore we expect at most a small modification of the exponent in the behavior $h_{ij} \sim (\cos\lambda)^d$ we assumed. Thus we reach the conclusion that the contribution of tensor terms on the keyhole part of the boundary vanishes as $\delta \to 0$.

It is now straightforward to analyze the boundary behavior of the terms in Eq. (3.19) involving the Killing vector bilinears. Using the asymptotics of the metric fluctuations h_{ij} discussed above, we find a vanishing contribution from the keyhole at the rate $(\cos\lambda)^3$ as $\delta \to 0$ and a finite contribution from the boundary components at $\mu = \pm\mu_0$.

In summary, we have shown that E_{WN} computes a linear combination of the AdS_d charges for any deformation of the dilaton domain wall metric solution which satisfies the asymptotic conditions stated above. The energy of such a deformation is positive. The keyhole part of the boundary does not contribute.

V. STABILITY WITH ADDITIONAL SCALAR FIELDS

The stability argument developed in Sec. IV strictly applies to models with action (2.1) containing only a single scalar field. At the formal level it is straightforward to add additional scalars, but the equations (4.4), (4.16), (4.17) which determine the superpotential $\mathbf{W}$ become partial differential equations in field space, and it is more difficult to show that $\mathbf{W}$ exists. However, it is important to extend our results for the stability of the Janus solution of type-IIB supergravity to include the additional fields which appear in compactifications to five dimensions. In this section we develop a reasonably general stability criterion, related to the approach of Ref. [4]. We then test this criterion in several known consistent truncations of type-IIB supergravity which involve the negative m^2 scalars with potentials unbounded below. These fields are certainly the main threat to stability, and it is gratifying that the test is satisfied in all cases examined.

The new criterion applies to dilaton domain walls in theories containing the dilaton ϕ plus additional scalars ψ^a with action

$$
\begin{aligned}
S = \int d^{d+1}x\sqrt{-g}\bigg[&\frac{1}{2\kappa^2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \\
&- \frac{1}{2}\partial_\mu\psi^a\partial^\mu\psi^a - V(\psi^a)\bigg].
\end{aligned} \tag{5.1}
$$

We assume that the potential $V(\psi^a)$ does not depend on the dilaton, and that there is a scalar superpotential $U(\psi^a)$ which is related to V by (with $U_{,a} \equiv \partial U/\partial\psi^a$)

$$
V = pU_{,a}U_{,a} - qU^2. \tag{5.2}
$$

In our conventions, the constants are given by

$$p=\frac{2(d-1)^2}{\kappa^4}, \tag{5.3}$$

$$q=\frac{2d(d-1)}{\kappa^2} \tag{5.4}$$

as in Eq. (3.11), but we allow different values to facilitate comparison with models in the literature which use different conventions, but $p,q>0$ always. In the models we study below $U(\psi^a)$ is a true supergravity superpotential generated in the truncation from 10 to 5 dimensions, but it could also be a fake supergravity superpotential obtained as a solution to Eq. (5.2) viewed as a partial differential equation for $U(\psi^a)$. We also assume that

$$U_{,a}|_{\psi^b=0}=0, \tag{5.5}$$

$$V_0\equiv V(0)=-\frac{d(d-1)}{2L^2\kappa^2} \tag{5.6}$$

so that the equations of motion of the enlarged system have the same AdS_d-sliced dilaton domain wall solution discussed in Sec. II with all $\psi^a=0$. We let $\mathbf{W}(\phi)$ denote the superpotential obtained in Sec. IV for the dilaton domain wall.

Our strategy [4] is to find a new superpotential $\mathcal{W}(\phi,\psi^a)$ to be inserted in the covariant derivative (3.5) of the Witten-Nester integral. The new form should have the property that the last term in Eq. (3.8) is replaced by

$$p(\mathcal{W}_{,\phi})^2+p\mathcal{W}_{,\alpha}\mathcal{W}_{,\alpha}-q\mathcal{W}^2-V(\psi^a)\leqslant 0. \tag{5.7}$$

The last term of Eq. (3.8) will not vanish in general as it does for a true adapted superpotential, but it is non-negative. At the critical point $\psi^a=0$ it will vanish, thus guaranteeing stability.

It is quite straightforward to show that the empirically inspired form[12]

$$\mathcal{W}(\phi,\psi^a)\equiv\sqrt{\mathbf{W}(\phi)^2+U(\psi^a)^2+\frac{V_0}{q}} \tag{5.8}$$

satisfies

$$p[(\mathcal{W}_{,\phi})^2+\mathcal{W}_{,\alpha}\mathcal{W}_{,\alpha}]-q\mathcal{W}^2-V(\psi^a)$$
$$=-\frac{p\mathbf{W}'^2}{q\mathcal{W}^2}(pU_{,a}U_{,a}+qU^2+V_0). \tag{5.9}$$

Thus nonperturbative stability will hold if

$$pU_{,a}U_{,a}+qU^2+V_0\geqslant 0. \tag{5.10}$$

[12]We take the explicit matrix square root as $\mathcal{W}\equiv\mathbf{W}\sqrt{1+1/\mathbf{W}^2[U(\psi^a)^2+V_0/q]}$.

Furthermore, since $\mathcal{W}(\phi,0)\equiv\mathbf{W}(\phi)$, the energy of the dilaton domain wall background, evaluated using $E_{\rm WN}$ with the new $\hat{\nabla}$ operator, vanishes, and this background has the same AdS_d Killing spinors found in Sec. IV.

As we will see below, the inequality (5.10) is not a general property of superpotentials in supergravity. However, it is quite simple to check that it is valid in several known consistent truncations of type-IIB supergravity which involve scalars of negative m^2 and potentials unbounded below.

The simplest model contains a single scalar whose mass, namely, $m^2=-4$, saturates the BF bound. It is a special case [22] of more general models [26] considered in the framework of gauged $\mathcal{N}=8$ supergravity [27,28]. With $\kappa^2L^2=1$, the potential is

$$V(\psi)=-2e^{2\psi/\sqrt{3}}-4e^{-\psi/\sqrt{3}}, \tag{5.11}$$

and one easily finds the superpotential [using Eq. (5.3)]

$$U(\psi)=\frac{1}{3}e^{\psi/\sqrt{3}}+\frac{1}{6}e^{-2\psi/\sqrt{3}}. \tag{5.12}$$

One can check directly that Eq. (5.10) is satisfied.

The general model of this type [26] involves five independent scalars with $m^2=-4$. The potentials is a sum of exponentials of linear combinations of these fields. A special case involving two nonvanishing scalars was also derived from the viewpoint of consistent truncations of the type-IIB theory in Ref. [29]. The analysis of these models is somewhat more involved, but one can also show that Eq. (5.10) is satisfied. Since the left-hand side of Eq. (5.10) is bounded, it is enough to check the inequality for the local minima and at infinity. Given the explicit form of the superpotential U one can easily show that the matrix $pU_{,ab}+q\delta_{ab}U$ is strictly positive definite and hence all the minima are zeros of $U_{,a}$ which greatly simplifies the analysis.

A different subtheory of gauged $\mathcal{N}=8$ supergravity with potential unbounded below contains [30] scalars with masses $m^2=-4$ and $m^2=-3$. The simplest version contains two fields, called ψ_1, ψ_3 and the superpotential

$$U\sim\frac{1}{4\rho^2}[\cosh(2\psi_1)(\rho^6-2)-3\rho^6-2] \tag{5.13}$$

and $\rho=\exp(\psi_3/\sqrt{6})$. Using the conventions of Ref. [30], one also finds that Eq. (5.10) holds. A more general version with three negative m^2 scalars was studied numerically. Again Eq. (5.10) is valid.

There do not appear to be any consistent truncations of type-IIB supergravity which involve both positive and negative m^2 scalars, but several involve only positive m^2 fields. The simplest of these [31] contains the dilaton ϕ and the breathing mode ψ with $m_\psi^2=32$. The potential, which is bounded below, and superpotential are

$$V(\psi)=\frac{1}{\kappa^2L^2}[4e^{8\alpha\psi}-10e^{(16\alpha/5)\psi}]$$

$$U(\psi)=\frac{1}{3L}\left[e^{4\alpha\psi}-\frac{5}{2}e^{(8\alpha/5)\psi}\right]$$

$$\alpha=\frac{1}{2}\sqrt{\frac{5}{6}}\kappa. \tag{5.14}$$

It is easy to see that in this case the inequality (5.10) is violated for large negative ψ. However, the superpotential $\mathcal{W}(\phi,\psi)$ which provided the appropriate bound for truncations with negative m^2 need not work universally. For the breathing mode model, we can simply take the matrix superpotential $\mathbf{W}(\phi)$ of Sec. IV. The quantity

$$p\mathbf{W}'^2-q\mathbf{W}^2-V(\psi) \tag{5.15}$$

which appears in Eq. (3.8) is negative for all nonzero ψ, which is sufficient to establish stability.

It is curious to note that another simple candidate superpotential, namely the product $\mathcal{W}\equiv\mathbf{W}(\phi)U(\psi^a)/U(0)$, produces the inequality

$$p(\mathcal{W}_{,\phi})^2+p\mathcal{W}_{,\alpha}\mathcal{W}_{,\alpha}-q\mathcal{W}^2-V(\psi^a)=p\mathcal{W}'^2U^2\geqslant 0 \tag{5.16}$$

of the wrong sense for stability in all the models above.

Further improvements of the arguments above may well be possible. However, we shall be content for the present with the nonperturbative stability arguments presented for the Janus solution which involve fluctuations of the metric, the dilaton, and several examples of negative m^2 scalars.

VI. A CURIOUS LINEAR DILATON SOLUTION

In Eq. (4.42) of Sec. IV, it was noted that for constant potential $V(\phi)=V_0$ of Eq. (2.3), there is a simple su(2) superpotential

$$\mathbf{W}(\phi)=\frac{1}{2L}\sqrt{\frac{d}{d-1}}\begin{pmatrix}0 & \overline{\zeta}(\phi)\\ \zeta(\phi) & 0\end{pmatrix}, \tag{6.1}$$

$$\zeta(\phi)\equiv\exp\left(i\kappa\frac{\phi}{\sqrt{d-1}}\right) \tag{6.2}$$

which appears among more complicated implicit solutions. As a simple consistency check of our formalism we now find the solution $\phi(r)$, $A(r)$ of the first order flow equations (4.10), (4.11) for this $\mathbf{W}(\phi)$ and show that it is a solution of the second order equations of motion (2.11) or, equivalently, Eq. (2.2).

First we compute $\mathbf{W}'(\phi)$, note that $\{\mathbf{W}(\phi),\mathbf{W}'(\phi)\}=0$, and that the invariants

$$\mathbf{W}^2=\frac{d}{4L^2(d-1)}, \tag{6.3}$$

$$\mathbf{W}'^2=\frac{\kappa^2 d}{4L^2(d-1)^2} \tag{6.4}$$

are correctly related to the potential by Eq. (4.4).

The flow equation (4.10) gives the solution

$$\phi(r)=-\frac{\sqrt{d}}{\kappa L}(r-r_0). \tag{6.5}$$

The compatibility condition (4.8) implies that $A'=0$, and Eq. (4.11) then gives (for $L_d=L$)

$$e^{2A}=\frac{d-1}{d}. \tag{6.6}$$

The linear scalar obviously satisfies the scalar equation of Eq. (2.11), and it is easy to check that the second equation in Eq. (2.11) is also satisfied.

The line element (1.2) of this solution is

$$ds^2=\frac{d-1}{d}\overline{g}_{ij}(x)dx^idx^j+dr^2, \tag{6.7}$$

where $\overline{g}_{ij}(x)$ is an AdS_d metric. Thus we find the nonsingular geometry $\mathrm{AdS}_d\otimes\mathcal{R}$ with accompanying linear scalar. One can verify directly that Eq. (2.2) is satisfied.[13] It would be interesting to study the stability of this solution whose boundary structure differs from that considered in previous sections.

For $d=4$ this solution can be lifted to type IIB by adjoining an S_5 and self-dual five-form. The full system is

$$ds_{10}^2=\frac{3}{4}\overline{g}_{ij}(x)dx^idx^j+dr^2+l^2d\Omega_5^2, \tag{6.8}$$

$$\phi(r)=-\frac{2}{\kappa L}(r-r_0), \tag{6.9}$$

$$F_{\alpha\beta\gamma\delta\epsilon}=s_0\varepsilon_{\alpha\beta\gamma\delta\epsilon}, \tag{6.10}$$

where $d\Omega_5^2$ is the metric on the unit five-sphere, and $\alpha\beta\gamma\delta\epsilon$ are five-sphere coordinates. We require that this satisfy the ten-dimensional equations of motion

$$\frac{1}{\kappa^2}R_{MN}=\partial_M\phi\partial_N\phi+\frac{1}{96}F_{MPQRS}F_N^{\ PQRS}, \tag{6.11}$$

which quickly gives the scales $l=L$ and $s_0=4L^4/\kappa$. Until stability is established, it is premature to speculate about a possible physical application of this simple nonsingular solution of type-IIB supergravity.

[13] This solution was found previously in Ref. [32] where linearized stability analysis was performed. The solution was also found in Ref. [33]. We thank Alexandros Kehagias for pointing this out to us.

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APPENDIX A: CONNECTION ONE-FORMS AND CURVATURE TENSOR FOR DOMAIN WALLS

Let us start with flat domain walls in $d+1$ dimensions with the metric ansatz

$$ds^2 = e^{2A(r)}\eta_{ij}dx^i dx^j + e^{2h(r)}dr^2. \tag{A1}$$

We introduce the vielbeins

$$E^{\hat{i}} = e^A dx^i, \quad E^{\hat{r}} = e^h dr. \tag{A2}$$

The hat over an index indicates that it is a frame index. The range of indices i and j will be always taken $0,...d-1$. The spin connection one forms are given by

$$\omega^{\hat{i}\hat{r}} = A' e^{-h} E^{\hat{i}}, \quad \omega^{\hat{i}\hat{j}} = 0. \tag{A3}$$

Nonzero components of the Ricci tensor (in curved indices) are then

$$R_{ij} = -\eta_{ij}(dA'^2 + A'' - A'h')e^{2A-2h},$$
$$R_{rr} = -d(A'' + A'^2 - A'h'). \tag{A4}$$

Now, let us consider AdS_d-sliced domain walls with the metric

$$ds^2 = e^{2A(r)}\overline{g}_{ij}dx^i dx^j + e^{2h(r)}dr^2, \tag{A5}$$

where $\overline{g}_{ij}$ is a metric on the AdS_d slices. In this case our choice of vielbeins is

$$E^{\hat{i}} = e^A \overline{e}^{\hat{i}}, \quad E^{\hat{r}} = e^h dr, \tag{A6}$$

where we have denoted with $\overline{e}^{\hat{i}}$ the vielbein for AdS_d. The spin connection is now

$$\omega^{\hat{i}\hat{r}} = A' e^{-h} E^{\hat{i}}, \quad \omega^{\hat{i}\hat{j}} = \overline{\omega}^{\hat{i}\hat{j}}, \tag{A7}$$

where $\overline{\omega}^{\hat{i}\hat{j}}$ is the spin connection on the AdS_d slices, whose explicit form is not needed. Nonzero components of the Ricci tensor are given by

$$R_{ij} = \overline{R}_{ij} - \overline{g}_{ij}(dA'^2 + A'' - A'h')e^{2A-2h},$$
$$R_{rr} = -d(A'' + A'^2 - A'h'), \tag{A8}$$

where

$$\overline{R}_{ij} = -\frac{d-1}{L_d^2} g_{ij} \tag{A9}$$

is the Ricci tensor of AdS_d space of scale L_d.

APPENDIX B: FURTHER INFORMATION ON JANUS DOMAIN WALLS

1. Explicit form of solution for $d=2$

The metric for AdS_2 sliced domain walls in AdS_3 in the r coordinate takes the form

$$ds^2 = e^{2A(r)} ds^2_{AdS_2} + dr^2. \tag{B1}$$

The explicit solution of the equations of motion is

$$A(r) = \frac{1}{2}\log\left(\frac{1}{2}(1+\sqrt{1-4b}\cosh 2r)\right)$$
$$\phi(r) = \phi_0 + \frac{\sqrt{2}}{\kappa}\,\mathrm{arctanh}\,\frac{(1-\sqrt{1-4b})\tanh r}{2\sqrt{b}}. \tag{B2}$$

This is the solution for $L = L_d = 1$. To restore dependence on the scale L, one just replaces r by r/L. The relation of the constant b to c defined by $\phi' = ce^{-dA}$ is

$$b = \frac{c^2\kappa^2 L^2}{d(d-1)}. \tag{B3}$$

Two coordinate independent features are evident. First the critical value of b beyond which the geometry contains a naked singularity is $b = \frac{1}{4}$. Second the asymptotic values of ϕ on the two components of the boundary are

$$\phi_{\pm\infty} = \phi_0 \pm \frac{\mathrm{arctanh} 2\sqrt{b}}{\sqrt{2}\kappa}. \tag{B4}$$

2. Radial coordinate μ

After change of variable, the integral (2.21) which defines the wall profile can be written as

$$\mu = \int_x^{x_{\min}} dx \frac{1}{\sqrt{1-x^2+bx^{2d}}}, \tag{B5}$$

where $x_{\min}$ is the smallest positive root of the polynomial in the denominator. The maximum value of μ is

$$\mu_0 = \int_0^{x_{\min}} dx \frac{1}{\sqrt{1-x^2+bx^{2d}}}. \tag{B6}$$

Series expansion in the parameter b gives

$$x_{\min} = 1 + \frac{1}{2}b + \frac{4d-1}{8}b^2 + O(b^3). \tag{B7}$$

Calculating the expansion of μ_0 to order b^2 and arbitrary d it is easy to guess the form of the expansion to all orders in b

$$\mu_0 = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{b^n}{n!} \frac{\Gamma\left(nd+\frac{1}{2}\right)}{\Gamma[n(d-1)+1]\Gamma\left(\frac{1}{2}\right)}. \tag{B8}$$

We have verified this formula to all orders in b analytically for $d=1,2$ and numerically for $d=4$. The convenient form

$$\mu_0 - \mu = \int_0^x dx \frac{1}{\sqrt{1-x^2+bx^{2d}}} \tag{B9}$$

yields the series expansion of μ in terms of $x=e^{-A}$

$$\mu_0 - \mu = \arcsin x - b \frac{x^{2d+1}}{2(2d+1)}\, {}_2F_1\left(d+\frac{1}{2}, \frac{3}{2}, d+\frac{3}{2}, x^2\right) + O(b^2 x^{4d+1}). \tag{B10}$$

Inverting the series we find

$$e^{-A(\mu)} \equiv x = \sin(\mu_0-\mu) + \frac{b}{2(2d+1)} \sin^{2d+1}(\mu_0-\mu) \times \cos(\mu_0-\mu)\, {}_2F_1\left(d+\frac{1}{2}, \frac{3}{2}, d+\frac{3}{2}, \sin^2(\mu_0-\mu)\right) + O[b^2 \sin^{4d+2}(\mu_0-\mu)]. \tag{B11}$$

Near the boundary $\mu \approx \mu_0$ the form of the scale factor is

$$e^{2A(\mu)} \approx \frac{1}{\sin^2(\mu-\mu_0)}[1+O(\mu-\mu_0)^{2d}]. \tag{B12}$$

The equations above define $A(\mu)$ in the region $0 \leqslant \mu < \mu_0$. However, as discussed in Sec. II, it can be extended as an even C^∞ function to the full range $-\mu_0 < \mu < \mu_0$.

In the special case $d=2$ we can integrate Eq. (B9) and invert to obtain the explicit solution

$$e^{-A(\mu)} \equiv x = \gamma \mathrm{sn}\left(\frac{1}{\gamma}(\mu_0-\mu), \sqrt{b}\gamma^2\right), \tag{B13}$$

where

$$\gamma = x_{\min} = \frac{\sqrt{2}}{\sqrt{1+\sqrt{1-4b}}} \tag{B14}$$

is the smallest positive root of the equation $1-x^2+bx^4=0$ and $\mathrm{sn}(u,k)$ is the standard Jacobi elliptic function. Note that the metric is doubly periodic[14] in the coordinate μ. The real period is

$$4\gamma K(\sqrt{b}\gamma^2). \tag{B15}$$

One may easily check using the definition of the complete elliptic integral that this is the same as $4\mu_0$. The period clearly blows up as b approaches its critical value $\frac{1}{4}$ which corresponds to $\gamma = \sqrt{2}$.

[14] Real periodicity in μ can be proved to exist for all dimensions. The second complex period is special to $d=2$ and it would be interesting to see if it has a deeper meaning.

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